

Sharp Conditions for the Oscillation of Delay Difference Equations

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ABSTRACT

Suppose that $\{p_n\}$ is a nonnegative sequence of real numbers and let k be a positive integer. We prove that

$$\liminf_{n \rightarrow \infty} \left[\frac{1}{k} \sum_{i=n-k}^{n-1} p_i \right] > \frac{k^k}{(k+1)^{k+1}}$$

is a sufficient condition for the oscillation of all solutions of the delay difference equation

$$A_{n+1} - A_n + p_n A_{n-k} = 0, \quad n = 0, 1, 2, \dots$$

This result is sharp in that the lower bound $k^k/(k+1)^{k+1}$ in the condition cannot be improved. Some results on difference inequalities and the existence of positive solutions are also presented.

Key words: Oscillations, Difference equations, Positive solutions, Difference inequalities.

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1. INTRODUCTION AND PRELIMINARIES

Recently there has been some activity concerning the oscillation of all solutions of the delay difference equation

$$A_{n+1} - A_n + p_n A_{n-k} = 0, \quad n = 0, 1, 2, \dots \quad (1)$$

where $\{p_n\}$ is a sequence of nonnegative real numbers and k is a positive integer. See, for example, [1]–[3] and the references cited therein. Throughout this paper, the sequence $\{p_n\}$ is supposed to be defined for $n \geq 0$.

By a *solution* of Eq. (1) we mean a sequence $\{A_n\}$ which is defined for $n \geq -k$ and which satisfies Eq. (1) for $n \geq 0$. A solution $\{A_n\}$ of Eq. (1) is said to be *oscillatory* if the terms A_n of the sequence are not eventually positive or eventually negative. Otherwise, the solution is called *nonoscillatory*.

Our aim in Section 2 is to establish the following result.

Theorem 1. *Suppose that $\{p_n\}$ is a nonnegative sequence of real numbers and let k be a positive integer. Then*

$$\liminf_{n \rightarrow \infty} \left[\frac{1}{k} \sum_{i=n-k}^{n-1} p_i \right] > \frac{k^k}{(k+1)^{k+1}} \quad (2)$$

is a sufficient condition for every solution of Eq. (1) to be oscillatory.

This theorem is sharp in that the lower bound $k^k/(k+1)^{k+1}$ cannot be improved. Moreover, when

$$p_n = p \in (0, \infty) \quad \text{for } n = 0, 1, 2, \dots,$$

condition (2) reduces to

$$p > \frac{k^k}{(k+1)^{k+1}} \quad (3)$$

which is a necessary and sufficient condition for the oscillation of all solutions of the difference equation

$$A_{n+1} - A_n + pA_{n-k} = 0, \quad n = 0, 1, 2, \dots \quad (4)$$

For a proof of this result see [3]. If

$$\liminf_{n \rightarrow \infty} p_n > \frac{k^k}{(k+1)^{k+1}}, \quad (5)$$

then it follows from [2] that every solution of Eq. (1) oscillates. Clearly (2) is a substantial improvement over (5), replacing the p_n of (5) by the arithmetic mean of the terms p_{n-k}, \dots, p_{n-1} in (2).

Theorem 1 should be looked upon as a discrete analogue of the well-known theorem about the oscillation of the delay differential equation,

$$\dot{x}(t) + p(t)x(t - \tau) = 0, \quad t \geq t_0 \quad (6)$$

where

$$p \in C[[t_0, \infty), [0, \infty)] \quad \text{and} \quad \tau \in (0, \infty),$$

which states that

$$\liminf_{t \rightarrow \infty} \int_{t-\tau}^t p(s) ds > \frac{1}{e}$$

is a sufficient condition for the oscillation of all solutions of Eq. (6). See [4]. We should also remark here that it is the proof of this latter theorem (see Theorem 2.1.1 in [4]) which we used as our guide in arriving at the statement and the proof of Theorem 1. One should notice that condition (2) can be written in the form

$$\liminf_{n \rightarrow \infty} \sum_{i=n-k}^{n-1} p_i > \left(\frac{k}{k+1} \right)^{k+1} \quad (2')$$

and that

$$\lim_{k \rightarrow \infty} \left(\frac{k}{k+1} \right)^{k+1} = \lim_{k \rightarrow \infty} \left[\frac{1}{\left(1 + \frac{1}{k}\right)^k} \cdot \frac{1}{1 + \frac{1}{k}} \right] = \frac{1}{e}.$$

Finally we should mention that another sufficient condition for the oscillation of all solutions of Eq. (1) is

$$\limsup_{n \rightarrow \infty} \sum_{i=n-k}^n p_i > 1$$

which was given by Erbe and Zhang in [1].

In Section 3 we present some results about difference inequalities. In particular we prove that, under appropriate hypotheses, if the difference inequality

$$x_{n+1} - x_n + p_n x_{n-k} \leq 0, \quad n = 0, 1, 2, \dots$$

has a positive solution so does Eq. (1). Finally, we utilize this result to give a "sharp" sufficient condition for the existence of a positive solution of Eq. (1).

2. PROOF OF THEOREM 1

Assume, for the sake of contradiction, that Eq. (1) has a nonoscillatory solution $\{A_n\}$. As the opposite of a solution of Eq. (1) is also a solution, we may (and do) assume that $\{A_n\}$ is eventually positive. Then eventually

$$A_{n+1} - A_n = -p_n A_{n-k} \leq 0,$$

and so $\{A_n\}$ is an eventually decreasing sequence of positive numbers. It follows from Eq. (1) that eventually

$$A_{n+1} - A_n + p_n A_n \leq 0$$

or

$$p_n \leq 1 - \frac{A_{n+1}}{A_n}$$

and so eventually,

$$\frac{1}{k} \sum_{i=n-k}^{n-1} p_i \leq \frac{1}{k} \sum_{i=n-k}^{n-1} \left(1 - \frac{A_{i+1}}{A_i}\right). \quad (7)$$

Set

$$\alpha = \frac{k^k}{(k+1)^{k+1}}. \quad (8)$$

Then, from (2), it is clear that we can choose a constant β such that, for n sufficiently large,

$$\alpha < \beta \leq \frac{1}{k} \sum_{i=n-k}^{n-1} p_i. \quad (9)$$

Thus, in view of (7),

$$\beta \leq \frac{1}{k} \sum_{i=n-k}^{n-1} \left(1 - \frac{A_{i+1}}{A_i}\right) \quad \text{for all large } n. \quad (10)$$

By using (10) and the well-known inequality between the arithmetic and geometric means we find that for n sufficiently large,

$$\begin{aligned} \beta &\leq \frac{1}{k} \sum_{i=n-k}^{n-1} \left(1 - \frac{A_{i+1}}{A_i}\right) = 1 - \frac{1}{k} \sum_{i=n-k}^{n-1} \frac{A_{i+1}}{A_i} \\ &\leq 1 - \left(\prod_{i=n-k}^{n-1} \frac{A_{i+1}}{A_i}\right)^{1/k} = 1 - \left(\frac{A_n}{A_{n-k}}\right)^{1/k}, \end{aligned}$$

that is,

$$\left(\frac{A_n}{A_{n-k}}\right)^{1/k} \leq 1 - \beta \quad \text{for all large } n. \tag{11}$$

In particular, this implies that $0 < \beta < 1$.

Now observe that

$$\max_{0 \leq \lambda \leq 1} [(1 - \lambda)\lambda^{1/k}] = \frac{k}{(k + 1)^{1 + \frac{1}{k}}} = \alpha^{1/k}$$

where α is the positive constant defined by (8). Therefore

$$1 - \lambda \leq \alpha^{1/k} \lambda^{-1/k} \quad \text{for } 0 < \lambda \leq 1$$

and (11) yields

$$\frac{\beta}{\alpha} A_n \leq A_{n-k} \quad \text{for all large } n. \tag{12}$$

By using (12) in Eq. (1) and then by repeating the above arguments we find that

$$\left(\frac{\beta}{\alpha}\right)^2 A_n \leq A_{n-k} \quad \text{for all large } n$$

and, by induction, for every $m = 1, 2, \dots$ there exists an integer n_m such that for $n \geq N_m$,

$$\left(\frac{\beta}{\alpha}\right)^m A_n \leq A_{n-k}. \tag{13}$$

Next observe that because of (9), for n sufficiently large,

$$\sum_{i=n-k}^n p_i \geq \sum_{i=n-k}^{n-1} p_i \geq k\beta.$$

Hence, for n sufficiently large,

$$\sum_{i=n-k}^n p_i \geq M \tag{14}$$

where $M = k\beta > 0$. Choose m such that

$$\left(\frac{\beta}{\alpha}\right)^m > \left(\frac{2}{M}\right)^2. \tag{15}$$

This is possible because from (9), $\beta > \alpha$. Then for n sufficiently large, say for $n \geq n_0$, (13) is satisfied for the specific m which was chosen in (15), also

(9) and (14) hold, and $\{A_n\}$ is decreasing for $n \geq n_0$. Now in view of (14) and for $n \geq n_0 + k$, there exists an integer n^* with $n - k \leq n^* \leq n$ such that

$$\sum_{i=n-k}^{n^*} p_i \geq \frac{M}{2} \quad \text{and} \quad \sum_{i=n^*}^n p_i \leq \frac{M}{2}.$$

From Eq. (1) and the decreasing nature of $\{A_n\}$, we have

$$\begin{aligned} A_{n^*-1} - A_{n-k} &= \sum_{i=n-k}^{n^*} (A_{i+1} - A_i) \\ &= - \sum_{i=n-k}^{n^*} p_i A_{i-k} \\ &\leq - \left(\sum_{i=n-k}^{n^*} p_i \right) A_{n^*-k} \\ &\leq - \frac{M}{2} A_{n^*-k}. \end{aligned}$$

Hence,

$$\frac{M}{2} A_{n^*-k} \leq A_{n-k}. \quad (16)$$

Similarly,

$$\begin{aligned} A_{n+1} - A_{n^*} &= \sum_{i=n^*}^n (A_{i+1} - A_i) \\ &= - \sum_{i=n^*}^n p_i A_{i-k} \\ &\leq - \left(\sum_{i=n^*}^n p_i \right) A_{n-k} \\ &\leq - \frac{M}{2} A_{n-k} \end{aligned}$$

and so

$$\frac{M}{2} A_{n-k} \leq A_{n^*}. \quad (17)$$

From (16) and (17) we find

$$\left(\frac{M}{2} \right)^2 A_{n^*-k} \leq A_{n^*},$$

which in view of (13) yields

$$\left(\frac{\beta}{\alpha}\right)^m \leq \frac{A_{n^*-k}}{A_{n^*}} \leq \left(\frac{2}{M}\right)^2.$$

This contradicts (15) and so the proof of the theorem is complete.

3. DIFFERENCE INEQUALITIES

A slight modification in the proof of Theorem 1 leads to the following result about the difference inequalities,

$$x_{n+1} - x_n + p_n x_{n-k} \leq 0, \quad n = 0, 1, 2, \dots \quad (18)$$

and

$$y_{n+1} - y_n + p_n y_{n-k} \geq 0, \quad n = 0, 1, 2, \dots, \quad (19)$$

where $\{p_n\}$ is a sequence of nonnegative real numbers and k is a positive integer.

By a *solution* of (18) we mean a sequence $\{x_n\}$ which is defined for $n \geq -k$ and which satisfies (18) for $n \geq 0$. Solutions of (19) are defined in a similar manner.

Theorem 2. *Assume that $\{p_n\}$ is a nonnegative sequence of real numbers and let k be a positive integer. Suppose that (2) holds. Then (18) cannot have eventually positive solutions and (19) cannot have eventually negative solutions.*

The following result shows that, under appropriate hypotheses, if (18) has a positive solution so does Eq. (1).

Theorem 3. *Let k be a positive integer and let $\{p_n\}$ be a sequence of nonnegative real numbers such that*

$$\sum_{j=0}^{k-1} p_{n+j} > 0 \quad \text{for } n \geq 0. \quad (20)$$

Assume that $\{x_n\}$ is a solution of (18) such that

$$x_n > 0 \quad \text{for } n \geq -k.$$

Then Eq. (1) has a solution $\{A_n\}$ such that

$$0 < A_n \leq x_n \quad \text{for } n \geq -k \quad \text{and} \quad \lim_{n \rightarrow \infty} A_n = 0. \quad (21)$$

Proof. For $\tilde{n} \geq n \geq 0$ we have

$$x_{\tilde{n}+1} - x_n + \sum_{i=n}^{\tilde{n}} p_i x_{i-k} \leq 0$$

and so

$$\sum_{i=n}^{\infty} p_i x_{i-k} \leq x_n \quad \text{for } n = 0, 1, 2, \dots \quad (22)$$

Consider the space S of all sequences $\{A_n\}$ for $n \geq -k$ which are such that

$$A_n = x_n \quad \text{for } -k \leq n < 0$$

and

$$0 \leq A_n \leq x_n \quad \text{for } n \geq 0.$$

Define the operator T on S as follows. For every $A = \{A_n\} \in S$, set $TA = B = \{B_n\}$ where

$$B_n = x_n \quad \text{for } -k \leq n < 0$$

and

$$B_n = \sum_{i=n}^{\infty} p_i A_{i-k} \quad \text{for } n \geq 0. \quad (23)$$

It follows from (22) that

$$B_n \leq \sum_{i=n}^{\infty} p_i x_{i-k} \leq x_n \quad \text{for } n \geq 0$$

and so T is well-defined and $T : S \rightarrow S$.

If $A^1 = \{A_n^1\}$ and $A^2 = \{A_n^2\}$ are two sequences in S , we will say that $A^1 \leq A^2$ if and only if $A_n^1 \leq A_n^2$ for $n \geq -k$. With this definition, the operator T is monotonic in the sense that if $A^1, A^2 \in S$ with $A^1 \leq A^2$ then $TA^1 \leq TA^2$.

Next, we define the sequence $\{A^r\}$ for $r = 0, 1, 2, \dots$ of points $A^r \in S$ in the following way:

$$A^0 = \{x_n\}$$

and

$$A^{r+1} = TA^r \quad \text{for } r = 0, 1, 2, \dots \quad (24)$$

It follows by induction that

$$\dots \leq A^{r+1} \leq A^r \leq \dots \leq A^1 \leq A^0.$$

Set

$$A^r = \{A_n^r\} \text{ for } r = 0, 1, 2, \dots \text{ and } A_n = \lim_{r \rightarrow \infty} A_n^r.$$

Then we can see that, for every $n \geq 0$,

$$A_n = \sum_{i=n}^{\infty} p_i A_{i-k}$$

and so

$$A_{n+1} - A_n = -p_n A_{n-k} \text{ for } n = 0, 1, 2, \dots, \tag{25}$$

that is, $\{A_n\}$ is a solution of Eq. (1). It is also clear that

$$0 \leq A_n \leq x_n \text{ for } n \geq -k \text{ and } \lim_{n \rightarrow \infty} A_n = 0.$$

Finally, we claim that $A_n > 0$ for $n \geq -k$. Otherwise, there exists $n_0 \geq 0$ such that

$$A_n > 0 \text{ for } n = -k, \dots, n_0 - 1 \text{ and } A_{n_0} = 0.$$

Then, by summing up both sides of (25) from $n = n_0$ to $n = n_0 + k - 1$ and by taking into account (20), we find

$$\begin{aligned} 0 \leq A_{n_0+k} &= - \sum_{j=n_0}^{n_0+k-1} p_j A_{j-k} < 0 \\ &\leq - \left(\min_{n_0 \leq j \leq n_0+k-1} A_{j-k} \right) \sum_{j=n_0}^{n_0+k-1} p_j \\ &= - \left(\min_{n_0-k \leq j \leq n_0-1} A_j \right) \sum_{j=0}^{k-1} p_{n_0+j} \\ &< 0. \end{aligned}$$

This is a contradiction and the proof is complete.

The following corollary of Theorem 3 provides a sufficient condition for the existence of a positive solution of Eq. (1).

Corollary 1. *Let k be a positive integer and let $\{p_n\}$ be a sequence of nonnegative real numbers such that (20) is satisfied. Assume that there exists a number $\gamma \in (0, 1)$ such that*

$$p_n < \gamma \text{ for } n = 0, 1, 2, \dots \tag{26}$$

and

$$\prod_{i=n-k}^{n-1} \left(1 - \frac{1}{\gamma} \tilde{p}_i\right) \geq \gamma \quad \text{for } n \geq 0 \quad (27)$$

where

$$\tilde{p}_n = \begin{cases} p_n & \text{for } n \geq 0 \\ p_0 & \text{for } n < 0. \end{cases} \quad (28)$$

Then Eq. (1) has a solution $\{A_n\}$ which is positive for $n \geq -k$ and is such that

$$\lim_{n \rightarrow \infty} A_n = 0.$$

Proof. Set

$$x_n = \prod_{i=-k-1}^{n-1} \left(1 - \frac{1}{\gamma} \tilde{p}_i\right) \quad \text{for } n \geq -k.$$

Clearly, $x_n > 0$ for $n \geq -k$ and, by Theorem 3, it suffices to show that $\{x_n\}$ is a solution of the difference inequality (18). To this end, in view of (27) we have, for $n \geq 0$,

$$\begin{aligned} x_{n+1} - x_n &= \left[\left(1 - \frac{1}{\gamma} \tilde{p}_n\right) - 1 \right] \prod_{i=-k-1}^{n-1} \left(1 - \frac{1}{\gamma} \tilde{p}_i\right) \\ &= -\frac{1}{\gamma} p_n \left[\prod_{i=-k-1}^{n-k-1} \left(1 - \frac{1}{\gamma} \tilde{p}_i\right) \right] \left[\prod_{i=n-k}^{n-1} \left(1 - \frac{1}{\gamma} \tilde{p}_i\right) \right] \\ &= -p_n x_{n-k} \left[\frac{1}{\gamma} \prod_{i=n-k}^{n-1} \left(1 - \frac{1}{\gamma} \tilde{p}_i\right) \right] \\ &\leq -p_n x_{n-k} \end{aligned}$$

and the proof is complete.

We can see that (27) is a "sharp" condition for the existence of a positive solution of Eq. (1) in the sense that when p_n is a constant p then (27) becomes

$$\left(1 - \frac{1}{\gamma} p\right)^k \geq \gamma$$

or equivalently

$$p \leq \gamma(1 - \gamma^{1/k}).$$

But

$$\max_{0 \leq \gamma \leq 1} [\gamma(1 - \gamma^{1/k})] = [\gamma(1 - \gamma^{1/k})]_{\gamma=(k/k+1)^k} = \frac{k^k}{(k+1)^{k+1}}.$$

Hence with $\gamma = \left(\frac{k}{k+1}\right)^k$, (27) is satisfied provided that

$$p \leq \frac{k^k}{(k+1)^{k+1}}. \quad (29)$$

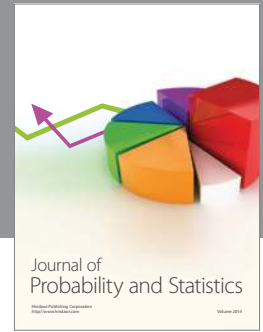
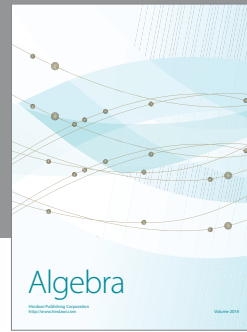
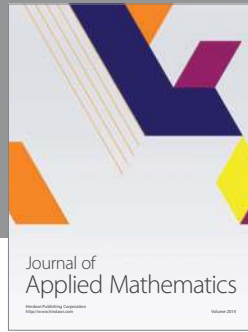
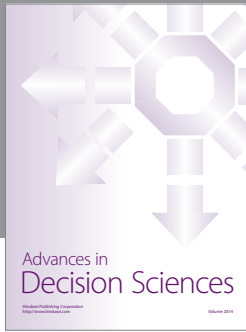
Note also that, in the case where

$$p \leq \frac{k^k}{(k+1)^{k+1}} < \left(\frac{k}{k+1}\right)^k = \gamma,$$

(26) is also satisfied. Now as we mentioned in the introduction of the paper, (3) is a necessary and sufficient condition for the oscillation of every solution of Eq. (4). Hence (29) is a necessary and sufficient condition for the existence of a positive solution of Eq. (4).

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