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SHARP-INTERFACE LIMIT OF A GINZBURG–LANDAU FUNCTIONAL WITH A RANDOM EXTERNAL FIELD*

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Abstract. We add a random bulk term, modeling the interaction with the impurities of the medium, to a standard functional in the gradient theory of phase transitions consisting of a gradient term with a double-well potential. For the resulting functional we study the asymptotic properties of minimizers and minimal energy under a rescaling in space, i.e., on the macroscopic scale. By bounding the energy from below by a coarse-grained, discrete functional, we show that for a suitable strength of the random field the random energy functional has two types of random global minimizers, corresponding to two phases. Then we derive the macroscopic cost of low energy "excited" states that correspond to a bubble of one phase surrounded by the opposite phase.

Key words. Γ-convergence, random functionals, phase segregation in disordered materials

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1. Introduction.

Models where a stochastic contribution is added to the energy of the system naturally arise in condensed matter physics, where the presence of the impurities causes the microscopic structure to vary from point to point. The starting point is a random functional which models the free energy of a two-phase material on a so-called mesoscopic scale, i.e., a scale which is much larger than the atomistic scale so that the adequate description of the state of the material is by a *continuous* scalar order parameter $m:\ D \subseteq \mathbb{R}^d \to \mathbb{R}$. The free energy functional consists of three competing parts: an "interaction term" penalizing spatial changes in m; a double-well potential $W(m)$; i.e., a nonconvex function which has exactly two minimizers (for simplicity $+1$ and -1, modeling a two-phase material); and a term which couples m to a random field $\theta g(\cdot,\omega)$, with mean zero, variance θ^2, and unit correlation length, i.e., a term which prefers at each point in space one of the two minimizers of $W(\cdot)$ and breaks the translational invariance but is "neutral" in the mean. A standard choice with the aforementioned properties is

$$\hat{G}(m,\omega) := \int_D \left(|\nabla m(y)|^2 + W(m(y)) + \theta g(y,\omega)m(y)\right) \mathrm{dy}.$$

We are, however, interested in a so-called *macroscopic* scale, which is coarser than the mesoscopic scale. Therefore we rescale space with a small parameter ϵ. If $\Lambda = \epsilon D$ and $u(x) = m(\epsilon^{-1}x)$, we obtain $\hat{G}(m,\omega) = \epsilon^{1-d} G_\epsilon(u,\omega)$, where

$$G_\epsilon(u,\omega) := \int_\Lambda \left(\epsilon|\nabla u(x)|^2 + \frac{1}{\epsilon}W(m(x)) + \frac{\theta}{\epsilon} g_\epsilon(x,\omega)m(x)\right) \mathrm{dx},$$

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where g_ϵ now has correlation length ϵ. First, we are interested in the asymptotic behavior of the minimizers, which, unlike in the case $\theta = 0$, will not be the constant functions $u(x) \equiv 1$ and $u(x) \equiv -1$, but functions varying in x and ω, and the minimal energy will be strictly negative. Second, we would like to know how functions which are not minimizers, but have energy of the same order as the minimizer, behave as $\epsilon \to 0$. This can be used to obtain information on the asymptotics of minimizers with a constraint, such as, e.g., requiring the spatial mean of u to equal a fixed value. The appropriate mathematical set-up for the second question is as follows. First we "renormalize," i.e., we subtract the energy of the minimizers (which exists by standard arguments) to obtain

$$F_\epsilon(u,\omega) = G_\epsilon(u,\omega) - \inf_{H^1(\Lambda)} G_\epsilon(\cdot,\omega),$$

and then we consider the Γ-limit of the functionals F_ϵ defined in $L^1(\Lambda)$ (with respect to the $L^1(\Lambda)$-convergence). A functional F_0 is the Γ-limit of the family $(F_\epsilon)_{\epsilon\to 0}$ with respect to the L^1-topology, if for all $u \in L^1(\Lambda)$,

(1.1)
- for all $\{u_\epsilon\} \in L^1(\Lambda)$ with $u_\epsilon \to u$ in $L^1(\Lambda)$,
$$\liminf_\epsilon F_\epsilon(u_\epsilon) \geq F_0(u),$$
- and there exists a sequence $\{v_\epsilon\} \in L^1(\Lambda)$, $v_\epsilon \to u$ in L^1 (recovery sequence or Γ-realizing sequence) such that
$$\limsup_\epsilon F_\epsilon(v_\epsilon) \leq F_0(u).$$

The Γ-limit, a notion invented by E. De Giorgi, means heuristically that $F_0(u)$ is the limit energy of the "lowest energy approximations" to u. In the the case $\theta = 0$, the minimizers are obviously the constants ± 1 with minimum energy zero, and the second question, the Γ-limit, was answered by Modica [15] and Modica and Mortola [16], who found that

$$F_0(u) = \begin{cases} \int_\Lambda \tau\left(\frac{\operatorname{grad} u}{|\operatorname{grad} u|}\right) |\operatorname{grad} u| & \text{if} \quad u \in BV(\Lambda),\ |u| = 1 \text{ a.e.}, \\ \infty & \text{else}, \end{cases} \tag{1.2}$$

$$\tau(n) = C_W = 2\int_{-1}^{1} \sqrt{W(s)}ds \quad \forall \quad n \in S^{d-1}, \tag{1.3}$$

where $S^{d-1} := \{x \in \mathbb{R}^d : \ |x| = 1\}$. The term $\operatorname{grad} u(|\operatorname{grad} u|)^{-1}$ is the generalized outward unit normal of the "jump set" of u, i.e., the set separating the region where $u = +1$ from the region where $u = -1$. (For a precise setting which uses bounded variation (BV) functions, i.e., functions such that the distributional derivative is a (vector-valued) Radon measure; see, e.g., [9].) The weight $\tau(n) = \tau(-n)$ is the *surface tension* in the language of statistical mechanics. While it is constant for $\theta = 0$, it is nonconstant (anisotropic) for g *periodic* (see [7, 6]) or for the gradient term being replaced by a bilinear form with periodic coefficients; see [2].

Note that the investigation of the limit behavior of $F_\epsilon(u,\omega)$ requires simultaneously the homogenization of a random structure and the performing of a limit of "singular" nature. Moreover, due to the nonconvexity of the double-well potential, the Euler–Lagrange equation does not have a unique solution.

The g-dependent bulk term can, because of the scaling with ϵ^{-1}, force a sequence u_ϵ to "follow" the oscillations of g. This always happens in the form of bounded oscillations around the two wells of the double-well potential. In such a situation there are still two distinct minimizers, also called "phases," adopting the language of statistical mechanics. But in principle the g-dependent term could be strong enough to enforce large oscillations, so that the minimizers will change sign and move from one "well" to the other.

In the periodic case it is possible to check on a deterministic volume with a diameter of the order of the period regardless of whether the minimizer "changes well," i.e., creates a "bubble" of the other phase; see, e.g., [7, 6].

The random case is quite different, because there is no deterministic subset of Λ such that the integral of the random field over this subset equals zero for almost all realizations of the random field—there are always *fluctuations* around the zero mean. A first consequence of this is that the approach in [6], which was successful in the periodic case, fails here: If we attempt to remove the term in G_ϵ which changes its sign by writing

$$G_\epsilon(u,\omega) = \bar{G}_\epsilon(v,\omega) - \epsilon \int_\Lambda |\nabla f_\epsilon|^2 \mathrm{dx},$$

with

$$\bar{G}_\epsilon(v,\omega) = \int_\Lambda \left(\epsilon|\nabla v|^2 + \frac{1}{\epsilon} W\left[v + f_\epsilon\right]\right)\mathrm{dx},$$

where $f_\epsilon(\cdot,\omega)$ solves

$$\Delta f_\epsilon = \frac{\theta\alpha(\epsilon)}{\epsilon^2}\left(g_\epsilon - \int_\Lambda g_\epsilon \mathrm{dx}\right)$$

with Neumann boundary conditions, then explicit computations with Green's function show that the covariance of the Gaussian random field f_ϵ is unbounded as $\epsilon \to 0$. So we cannot expect f_ϵ to be bounded as it would be in the periodic case.

Let us explain another aspect that makes the random case far more challenging than the periodic one. A set A becomes the support of a bubble of the other phase if the cost of switching to the other well, which can be estimated by the Modica–Mortola result as proportional to the boundary of A, is smaller than the integral of the random field part over A. As the correlation length is ϵ, a set $A \subseteq \Lambda$ contains roughly $|A|\epsilon^{-d}$ independent random variables, where $|\cdot|$ denotes the d-dimensional Lebesgue measure of a set. By the central limit theorem, fluctuations of order $\theta\sqrt{|A|}\epsilon^{d/2}$ are highly likely, but the probability of larger fluctuations vanishes exponentially fast. Therefore, using the isoperimetric inequality, the probability of a given set A being the support of a bubble is exponentially small if

$$c_d|A|^{(d-1)/d} \gg |A|^{1/2}\epsilon^{(d-2)/2}\theta, \tag{1.4}$$

where c_d is the isoperimetric constant. In $d \geq 3$ this is asymptotically always the case for sets of diameter of order larger than ϵ, or for sets of any size, provided $\theta \to 0$.

When determining properties of the minimizers, we are, however, not interested in whether a *single given* set A becomes the support of a bubble, but whether *there exist* "bubbles" of the other phase. In order to estimate the latter probability, we have to find a way to *count* subsets, which requires a *coarse-graining* on the scale of the correlation length. Standard coarea arguments, following the original ideas of Modica and Mortola [16], are therefore not able to avoid the need for a coarse-graining.

Let us briefly sketch how the coarse-graining is done. We define a *phase indicator* which is ± 1 if the average of u over a cube of side ϵ is close to ± 1, the minimizers of the "unperturbed" ($\theta = 0$) functional. (See (2.15).) Then we prove that the energy of a function is bounded from below by an energy that can be expressed as a function of the so-called contours (connected components of cubes where the average deviates from one of the wells) of the coarse-grained "representative" of the function. The proof of this bound does not require probabilistic arguments. The basic idea behind contours is to make explicit the region in space where the order parameter u deviates from the minimizer, which is, of course, unknown. However, one may guess that for sufficiently weak disorder (θ small) the minimizers should look almost like the ones without random field. It is thus natural to build the contour model on the basis of the ideal minimizers and to let the contours themselves keep track of the deviations of the true minimizers from these ideal minimizers. Our use of contours for functions $u : \Lambda \to \mathbb{R}$, i.e., functions taking values in continuum, has been strongly inspired by the series of papers done for Ising spin systems with Kac-type interaction by Presutti and his collaborators; see the book [19].

However, we do not impose any boundary conditions on the cube Λ because we are interested in global minimizers. This kind of free boundary condition corresponds to Neumann boundary conditions for smooth solutions of the Euler–Lagrange equations. In the "discretized" setting after "coarse-graining," the free boundary conditions will make the definition of contours more complicated than in the standard setting, where usually some type of "Dirichlet" boundary conditions are used. Additionally, the energy in [19] contains convolution terms instead of gradients, so our approach is quite different as far as the more technical parts are concerned.

These contour reduction techniques will have further applications in the analysis of random functionals which are related to a deterministic reference functional with multiple ground states (phases). The contour reduction allows us to use probabilistic techniques developed in the 1980s for the (discrete) random field Ising model. The central question heatedly discussed in the 1980s in the physics community was whether the random field Ising model would show spontaneous magnetization at low temperature and weak disorder in dimension 3. This is closely related to the question of whether there are at least two distinct minimizers, one predominantly $+$ and one predominantly $-$.

Even though (1.4) holds for one single bubble A, it is not easy to obtain from (1.4) that the probability to have *some* bubble surrounding one point, for example the origin, is exponentially small in θ when θ is small enough. Even in the simplest case in which there are no "bubbles inside bubbles," a naive subadditivity argument breaks down. Namely, the number of bubbles surrounding the origin and having surface area n grows like e^{cn} for some positive constant c; see remarks in the proof of Lemma 5.4. Of course this is not surprising since the random fields in areas with nonempty intersections are quite correlated. This problem was solved by Fisher, Fröhlich, and Spencer; see [11]. They use coarse-grained contours to take advantage of the fact that many contours enclose essentially the same volume. In an approximation in which there are no contours within contours they proved that the random field Ising model has at least two phases in $d \geq 3$ when θ and the temperature are small enough. We adapt their technique of coarse-grained contours to compare the contribution of the random magnetic field versus the surface area for any bubble uniformly in the volume Λ. This forces us to take $\theta \simeq (\log \epsilon^{-1})^{-1}$; see footnote 5. Hence in this regime the minimizers do not change sign and these estimates are sufficient for dealing with contours inside contours. We report them, adapted to our context, in

Appendix A. Later, Imbrie [14] proved that in $d = 3$ for θ fixed and small enough, there are with probability one at least two ground states of the Ising Hamiltonian, one having predominantly positive phase and the other predominantly negative phase. He used coarse-grained contours for the same reason as [11], but he dealt with contours inside contours proving, by a bootstrap strategy, that contours of a certain size are rare, assuming only that smaller contours are rare and that they can be neglected. He argues inductively from smaller to larger contours, so his proof is a sequence of suitable local minimization problems.

At positive temperature (i.e., beyond the ground state) the problem was solved by Bricmont and Kupiainen [4], who proved the existence of phase transition in $d \geq 3$ for small magnitude of the random field, and Aizenman and Wehr [1], who proved that there is no phase transition in $d = 2$ for all temperatures. We refer to the original papers and to a didactic presentation of them in [3].

After that overview, let us return to the model considered in this paper. We prove that in $d \geq 3$ and for a set of random realizations of overwhelming probability (see Theorem 2.1), there are two functions $u^+_\epsilon(\cdot,\omega)$ and $u^-_\epsilon(\cdot,\omega)$, close in L^∞ to $+1$ and -1, respectively, on which the value of the functional is close to its minimum value, and one of them is the global minimizer. The energy of these minimizers diverges as $\epsilon \to 0$, but the minimal energy is close to a deterministic sequence c_ϵ up to an error which vanishes as $\epsilon \to 0$ (see Theorem 2.2); i.e., the energy becomes deterministic in the limit by a law of large numbers.

Once this is established, the Γ-convergence of the renormalized energy F_ϵ, contained in Theorem 2.3, follows by relatively straightforward methods. We show Γ-convergence with respect to the $L^1(\Lambda)$-topology with probability one. The realization ω of the random field is treated as parameter $\mathbb{P}$ for almost all such ω.

Both Theorems 2.1 and 2.3 hold only in the case $\theta = (\log(\epsilon^{-1}))^{-1} \to 0$, while the analytic result which is crucial in obtaining these estimates, the contour reduction in Theorems 2.7 and 2.11, hold for θ which is small but does not depend on $\epsilon \downarrow 0$. The assumption $\theta \to 0$ is important because by analogy with the aforementioned Ising models with random field we expect that for θ small but finite, two (almost) minimizers exist, but they do not stay in a single "well" of the double-well potential. The + minimizer, for example, will be predominantly near $+1$, but there will be many small (diameter $\sim \epsilon$) "bubbles" where it is close to -1.

In this case a more subtle analysis is needed. One needs to find a way to deal with minimizers having contours (i.e., changing sign) as it was done for the random field Ising Hamiltonian by Imbrie [14] or Bricmont and Kupiainen [4].

In the case of "weak" disorder treated here, i.e., $\theta \to 0$, we show that the surface tension $\tau = C_W$ (see (1.2)) as in the case $\theta = 0$.

This does not mean that the disorder is too weak to have any effect: First note that the minimizers are not constants but functions depending on space and on the realization of the random field. Their energy is not zero, hence the presence of the renormalization.

Second, in Appendix C, we present a (partly heuristic) computation that indicates that minimizers in $d = 3$ are not microscopically flat, i.e., even if $S(u) = \{-1 + \delta < u < 1 - \delta\}$, the jump set of u, is a plane, an "optimal" recovery sequence u_ϵ has the property that for some $\delta > 0$ the jump set $S(u_\epsilon)$ fluctuates around the limit plane on any scale smaller than $\epsilon^{2/3}$. This is clearly not the case for $\theta = 0$, where the global minimizer has planar level sets, and in the periodic case recent results by Novaga and Valdinoci [17] indicate that $S(u_\epsilon)$ oscillates on the scale ϵ.

This paper is organized as follows. In section 2 we state the main results and define the phase indicator and our notion of contours. In section 3 we show that we can associate to each function a representative which gives rise to essentially the same coarse-grained function, but has smaller energy and is uniformly bounded and uniformly Lipschitz. This allows us to derive that such a function must be pointwise close to the minimizers if the coarse-grained function is. In section 4 we estimate the cost of a contour, i.e., a deviation of the coarse-grained function from local equilibrium. In section 5 we show the aforementioned lower bound on the energy in terms of a functional depending only on the contours of the coarse-graining. As a consequence, we prove that a minimizer stays in one single well of the double-well potential. Finally, in section 6, we use the information obtained so far to show the Γ-convergence of the renormalized functionals.

For the reader's convenience, we collect in Appendix A standard results for properties of the solution to the Euler–Lagrange equation of our random functional under the condition that the solution stays in one single well. In Appendix B we prove some of the probabilistic estimates used in this paper.

2. Notation and results.

2.1. The functional. The "macroscopic" space is given by $\Lambda := [-\frac{1}{2}, \frac{1}{2}]^d$, the d-dimensional unit cube centered at the origin. The ratio between the macroscopic and the "mesoscopic" scale is given by the small parameter $\epsilon > 0$. Hence for any ϵ the mesoscopic space is defined as $\Lambda_\epsilon := [-\frac{1}{2\epsilon}, \frac{1}{2\epsilon}]^d$. We require ϵ to be in a countable set, e.g., $\epsilon = \frac{1}{n}$, $n \in \mathbb{N}$. This choice avoids irrelevant technical difficulties.[1] The disorder or random field is constructed with the help of a family $\{g(z,\omega)\}_{z\in\mathbb{Z}^d}$ of independent and identically distributed Bernoulli random variables. The law of this family of random variables will be denoted by $\mathbb{P}$, in particular,

$$\mathbb{P}(\{g(z,\omega) = \pm 1\}) = \pm\frac{1}{2}, \qquad z \in \mathbb{Z}^d. \tag{2.1}$$

Different choices of g could be handled by minor modifications provided that g is still a random field with finite correlation length, is invariant under (integer) translations, and is such that $g(z,\omega)$ has a symmetric distribution with compact support. The disorder or random field in the functional will be obtained by a rescaling of g such that the correlation length is order ϵ and the amplitude grows as $\epsilon \to 0$. To this end, define for $x \in \Lambda$ a function $g_\epsilon(\cdot,\omega) \in L^\infty(\Lambda)$ by

$$g_\epsilon(x,\omega) := \sum_{z\in\mathbb{Z}^d} g(z,\omega) 1_{\epsilon(z+[-\frac{1}{2},\frac{1}{2}]^d)\cap\Lambda}(x), \tag{2.2}$$

where for any Borel measurable set A

$$1_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

For $u \in H^1(\Lambda)$ and any open set $A \subseteq \Lambda$, consider the following random functional:

$$G_\epsilon(A,u,\omega) := \int_A \left(\epsilon|\nabla u(x)|^2 + \frac{1}{\epsilon}W(u(x))\right)\mathrm{d}x + \frac{1}{\epsilon}\alpha(\epsilon)\theta\int_A g_\epsilon(x,\omega)u(x)\mathrm{d}x, \tag{2.3}$$

[1] It will soon become clear that this assumption simplifies some definitions; see, for example, the definition of contours given next, which avoids dealing with boundary layer problems.

where $\theta > 0$ and $0 < \alpha(\epsilon) \ll 1$ is a function of ϵ to be specified later. If $A = \Lambda$, we simply write $G_\epsilon(u, \omega)$. The potential W is a so-called double-well potential.

*Assumption H*1. $W \in C^2(\mathbb{R})$, $W \geq 0$, $W(s) = 0$ iff $s \in \{-1, 1\}$, $W(s) = W(-s)$, and $W(s)$ is strictly decreasing in $[0, 1]$. Moreover, there exist δ_0 and $C_0 > 0$ so that

$$W(s) = \frac{1}{2C_0}(s-1)^2 \qquad \forall s \in (1-\delta_0, \infty). \tag{2.4}$$

Note that W is slightly different from the standard choice $W(u) = (1-u^2)^2$. Our choice simplifies some proofs because it makes the Euler–Lagrange equation linear provided that solutions stay in one "well." These assumptions could be relaxed, but in order to keep the exposition reasonably short, we prefer to use stronger assumptions. The functional (2.3) can be extended to a lower semicontinuous functional $G_\epsilon : L^1(\Lambda) \to \mathbb{R} \cup \{+\infty\}$ by defining $G_\epsilon(v, \omega) = +\infty$ for any $v \notin H^1(\Lambda)$ and $\omega \in \Omega$. For $\epsilon > 0$ fixed and $\omega \in \Omega$, it follows in the same way as in the case without random perturbation that the functional $G_\epsilon(\cdot, \omega)$ is coercive and weakly lower semicontinuous in $H^1(\Lambda)$, so there exists at least one minimizer (see [8]), which here is a random function in $H^1(\Lambda)$, i.e., different realizations of ω will give different minimizers.

2.2. Minimizers and Γ-limit. Our first main result is the existence of two minimizing random functions $u_\epsilon^\pm$ and their properties.

THEOREM 2.1. *Let $d \geq 3$, $\theta > 0$, $\alpha(\epsilon) = \left(\ln \frac{1}{\epsilon}\right)^{-1}$ and let C_0 be the constant in* (2.4). *There exist $\epsilon_0 > 0$ and $a \equiv a(\alpha(\epsilon_0)\theta, d) > 0$ so that for all $\epsilon \leq \epsilon_0$, there exist two functions $u_\epsilon^+(\cdot, \omega)$ and $u_\epsilon^-(\cdot, \omega)$ which are almost surely in $H^1(\Lambda)$, and a set $\Omega_\epsilon \subseteq \Omega$,*

$$\mathbb{P}[\Omega_\epsilon] \geq 1 - e^{-a\left(\ln \frac{1}{\epsilon}\right)^{1+\frac{49}{50}}},$$

so that for all $\omega \in \Omega_\epsilon$, the following holds:[2]

$$\inf_{H^1(\Lambda)} G_\epsilon(\cdot, \omega) = G_\epsilon(u_\epsilon^\tau, \omega), \ \textit{where} \ \tau = -\mathrm{sign}\left(\int_\Lambda g_\epsilon\right), \tag{2.5}$$

$$\|u_\epsilon^+(\cdot, \omega) - 1\|_\infty \leq C_0\theta\alpha(\epsilon), \qquad \|u_\epsilon^-(\cdot, \omega) + 1\|_\infty \leq C_0\theta\alpha(\epsilon), \quad \omega \in \Omega_\epsilon,$$

and

$$|G_\epsilon(u_\epsilon^+, \omega) - G_\epsilon(u_\epsilon^-, \omega)| \leq \delta_\epsilon, \quad \omega \in \Omega_\epsilon \tag{2.6}$$

for some δ_ϵ with $\delta_\epsilon \to 0$ as $\epsilon \to 0$ which does not depend on ω. Moreover,

$$\mathbb{E}[u_\epsilon^\pm(r, \cdot)] = 1 \quad \forall r \in \Lambda,$$

and (decay of correlations)

$$\left|\mathbb{E}[u_\epsilon^\pm(r, \cdot)u_\epsilon^\pm(r', \cdot)] - \mathbb{E}[u_\epsilon^\pm(r, \cdot)]\mathbb{E}[u_\epsilon^\pm(r', \cdot)]\right| \leq C(d)\theta^2\alpha^2(\epsilon)e^{-\frac{1}{2\epsilon\sqrt{2C_0}}|r-r'|}. \tag{2.7}$$

In the unperturbed case $\theta = 0$ the minimum value is zero and there are two minimizers, the constant functions identical and equal to 1 or to -1. When $\theta > 0$ the

[2]The exponent $\frac{49}{50}$ is just a possible choice. The relevant issue is that for $\epsilon = \frac{1}{n}$, $\sum_n e^{f(n)}$ is finite, where here $f(n) = -a(\ln n)^{1+\frac{49}{50}}$.

infimum over $H^1(\Lambda)$ can be negative or even diverge to $-\infty$ as $\epsilon \downarrow 0$. Hence we shall introduce an additive renormalization for the functional and denote for $u \in H^1(\Lambda)$

$$F_\epsilon(u,\omega) = G_\epsilon(u,\omega) - \inf_{H^1(\Lambda)} G_\epsilon(\cdot,\omega). \tag{2.8}$$

Denote

$$c_\epsilon = \mathbb{E}\left[\inf_{H^1(\Lambda)} G_\epsilon(\cdot,\cdot)\right]. \tag{2.9}$$

We have the following result.

THEOREM 2.2. *For $d \geq 3$ and $\alpha(\epsilon) = (\ln(1/\epsilon))^{-1}$, $\theta > 0$,*

$$c_\epsilon = \mathbb{E}[G_\epsilon(u_\epsilon^+,\cdot)] = \mathbb{E}[G_\epsilon(u_\epsilon^-,\cdot)], \tag{2.10}$$

$$\mathbb{E}\left[c_\epsilon - \inf_{H^1(\Lambda)} G_\epsilon(\cdot,\cdot)\right]^2 \to 0,\ 0 < \liminf \frac{\epsilon}{\alpha(\epsilon)^2}|c_\epsilon| \leq \limsup \frac{\epsilon}{\alpha(\epsilon)^2}|c_\epsilon| < \infty. \tag{2.11}$$

The next theorem states that the renormalized functionals have a Γ-limit.

THEOREM 2.3. *For $d \geq 3$, $\epsilon = \frac{1}{n}$, $n \in \mathbb{N}$, $\alpha(\epsilon) = (\ln(1/\epsilon))^{-1}$, and $\theta > 0$, $F_\epsilon(\cdot,\omega) \to F_0(\cdot)$ in the sense of Γ-convergence (with respect to the L^1-topology) $\mathbb{P}$-almost surely, where F_0 is as in* (1.2) *and C_W is as in* (1.3).

Theorems 2.1 and 2.2 correspond to the highest order term of a so-called Γ-expansion of our functional. Their proofs are given in section 5. Theorem 2.3 characterizes the next highest order term. Its proof is given in section 6. The main problem we face is, as explained in the introduction, the characterization of the minimizers. Once we have established that each minimizer does not change sign, the proof of the Γ-convergence result (see Theorem 2.3) follows by standard arguments.

Remark 2.4 (minimizers with constraints). As a direct consequence we obtain that a sequence $u_\epsilon(\cdot,\omega)$, with

$$G_\epsilon(u_\epsilon,\omega) = \min_{\{v\in H^1:\ \int_\Lambda v=m\}} G_\epsilon(v,\omega)$$

for $m \in (-1,1)$, converges a.e. to a deterministic function $u(\cdot)$ such that

$$F_0(u) = \min_{\{v\in BV:\ \int_\Lambda v=m,\ |v|=1 \text{ a.e.}\}} F_0(v), \qquad \mathbb{P} = 1.$$

2.3. Contours and contour reduction. The proofs of Theorems 2.1 and 2.3 are based on an extension of Peierls' argument [18], to the present context using three steps: First, a reformulation of the problem in terms of contours, then an estimate of their energy, and finally an estimate of their number. As we are interested in global minimizers, we consider free boundary conditions which correspond to Neumann boundary conditions for smooth solutions of the Euler–Lagrange equations. This makes the definition of contours in the "discretized" setting more complicated. It is convenient to reformulate the problem in the *mesoscopic coordinates.* We consider $v \in H^1(\Lambda_\epsilon)$ and denote in mesoscopic coordinates

$$G_1(v,\omega) := \int_{\Lambda_\epsilon} \left(|\nabla v(x)|^2 + W(v(x))\right) \mathrm{dx} + \alpha(\epsilon)\theta \int_{\Lambda_\epsilon} g_1(x,\omega)v(x)\mathrm{dx}. \tag{2.12}$$

The relation between (2.3) and (2.12) is

$$G_\epsilon(\Lambda,u,\omega) = \epsilon^{d-1} G_1(\Lambda_\epsilon,v,\omega), \tag{2.13}$$

where $v(x) = u(\epsilon x)$ for $x \in \Lambda_\epsilon$.

2.3.1. Coarse-graining. We introduce notation for the partition of $\mathbb{R}^d$. We denote by $\mathcal{D}^{(0)} = \{C^{(0)}\}$ the partition of $\mathbb{R}^d$ into cubes of side 1, with one of them having center 0, and we denote by $C^{(0}(y)$ for $y \in \mathbb{R}^d$ the block of the partition $\mathcal{D}^{(0)}$ which contains y. Two cubes of $\mathcal{D}^{(0)}$ are *connected* if their closures have nonempty intersection. Given $m \in L^1_{\text{loc}}(\mathbb{R}^d)$ we denote for each cube $C^{(0)} \in \mathcal{D}^{(0)}$

$$m^{(0)}(y) \equiv \int_{C^{(0)}(y)} m(z)\mathrm{dz}, \tag{2.14}$$

and by

$$\eta(m,y) \equiv \eta^\zeta(m,y) = \begin{cases} 1 & \text{if } \ m^{(0)}(y) > 1-\zeta, \\ -1 & \text{if } \ m^{(0)}(y) < -1+\zeta, \\ 0 & \text{if } \ -1+\zeta \le m^{(0)}(y) \le 1-\zeta \end{cases} \tag{2.15}$$

the block variable with tolerance ζ, where $1 > \zeta > 0$. $\eta(m,\cdot)$ is constant on each cube of the partition. It determines the *sign of the cube.* We omit writing the superscript in notation (2.15) when no confusion arises.

2.3.2. Islands and contours.

Correct points. The point y is ζ-correct, or, equivalently, $C^{(0)}(y)$, the block of $\mathcal{D}^{(0)}$ containing y, is ζ-correct if $\eta^\zeta(m,y) \neq 0$ and $\eta^\zeta(m,y) = \eta^\zeta(m,y')$ on the cubes of $\mathcal{D}^{(0)}$ which are connected to $C^{(0)}(y)$. (That is, a cube is correct if its sign is nonzero and all its neighbors have the same sign.) The point y, or, equivalently, $C^{(0)}(y)$, is ζ-incorrect if it is not ζ-correct. When no confusion arises we drop the ζ- in the previous definition and we denote a point or a block only by correct or incorrect.

Correct set. The union of the correct blocks of $\mathcal{D}^{(0)}$.

Islands and signs of islands. The maximal connected components of the correct set are called *islands.* We denote them by I. In an island, $\eta(m,y)$ is constantly equal either to 1 or to -1; accordingly we define the sign of the island $\text{sign}(I) = \pm 1$.

Boundaries. The boundary $\partial^{\text{ext}} I$ of an island I is the set of cubes $C^{(0)}$ not in I but at distance 0 from I; $\partial^{\text{int}} I$ is the set of cubes $C^{(0)}$ in I and at distance 0 from $\partial^{\text{ext}} I$. The topological boundary is denoted ∂I. The definition of island ensures that $\partial^{\text{ext}} I$ is a kind of "safety zone" around I, in which $\eta(m,y)$ still has a definite sign, equal to the sign of the island.

Contours. Each maximal connected component of the incorrect set is the support of a contour. The contour is the pair $\Gamma = (\text{sp}(\Gamma), \eta_\Gamma)$, where $\text{sp}(\Gamma)$ is the spatial support of Γ, i.e., the maximal connected component of the incorrect set and η_Γ is the restriction to $\text{sp}(\Gamma)$ of $\eta(m,\cdot)$. See also Figure 1.

Boundary of a contour. The boundary $\partial^{\text{int}}(\text{sp}(\Gamma))$ of the contour Γ is the union of $\partial^{\text{ext}} I \cap \text{sp}(\Gamma)$ over the islands. The $\pm$ boundary, $\partial^{\pm}(\text{sp}(\Gamma))$, is the union of cubes in $\partial^{\text{ext}} I \cap (\text{sp}(\Gamma))$ over the $\pm$ islands I.

Contours in finite regions. When $m \in H^1(\Lambda_\epsilon)$, the block variable (see (2.15)) can be defined only for those $C^{(0)} \subset \Lambda_\epsilon$, since m has support in Λ_ϵ. The notion of correctness for a block $C^{(0)}$ needs the knowledge of the block variables of the cubes connected to $C^{(0)}$. We make the following convention.

Neumann boundary condition on Λ_ϵ. A cube $C^{(0)} \subset \Lambda_\epsilon$ is correct if $\eta^\zeta(m,y) \neq 0$ for $y \in C^{(0)}$ and $\eta^\zeta(m,y) = \eta^\zeta(m,y')$ on the cubes of $\mathcal{D}^{(0)} \subset \Lambda_\epsilon$ connected to $C^{(0)}(y)$. Contours are defined consequently and their support is contained in Λ_ϵ.

Dirichlet boundary condition on $A \subset \Lambda_\epsilon$. Let $A \subset \Lambda_\epsilon$ be a bounded, $\mathcal{D}^{(0)}$-measurable region. We say that A has boundary conditions $+1$ (or -1) when $\eta(m,y) =$

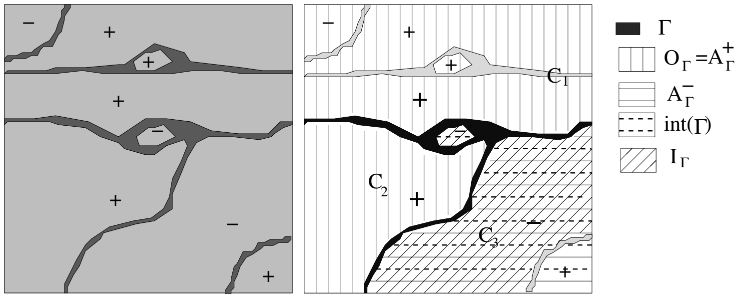

FIG. 1. *Possible types of contours and inner/outer complement.*

$+1$ (or -1) for all $y \in A^c$, $d(y, A) \leq 1$. We then use the convection that all the blocks in A^c are considered positive (negative) correct and define those inside A according to the previous rules. Contours are defined consequently and their support is contained in A.

Collection of contours and islands. Given $m \in H^1(\Lambda_\epsilon)$, $\zeta > 0$, we associate $\mathcal{G}(m) \equiv \mathcal{G}(m, \zeta) = \{\Gamma_1, \dots, \Gamma_k\}$ for $k \in \mathbb{N}$, the collection of contours according to the previous construction. This also defines the collection of islands $\mathcal{I}(m) \equiv \mathcal{I}(m, \zeta) = \{I_1, \dots, I_n\}$ for $n \in \mathbb{N}$. It is possible that there are no islands, $\mathcal{I}(m) = \emptyset$, for example, when $\mathcal{G}(m) = \{\Gamma\}$ and $\mathrm{sp}(\Gamma) = \Lambda_\epsilon$.

Outer complement of a contour Γ*.* Given a contour $\Gamma \in \mathcal{G}(m)$, consider all connected components of $\Lambda_\epsilon \setminus \mathrm{sp}(\Gamma)$, which are connected to the boundary $\partial\Lambda_\epsilon$. Denote them by $C_1, \dots, C_{K_\Gamma}$, $K_\Gamma \in \mathbb{N}$. We can associate a sign with each connected component by defining $\mathrm{sign}(C_j) := \eta(x)$ for some $x \in C_j$ with $\mathrm{dist}(x, \mathrm{sp}(\Gamma)) < 1/2$. We form the union over the positive and negative connected components, i.e.,

$$A_\Gamma^+ := \bigcup_{\mathrm{sign}(C_j)=+1} C_j, \quad A_\Gamma^- := \bigcup_{\mathrm{sign}(C_j)=-1} C_j.$$

Note that this definition does not imply that $\eta(\cdot)$ is constant on A_Γ^+. Namely, there could be contours different from Γ contained in A_Γ^+. We ignore them when assigning the sign $+$. The same applies to A_Γ^-.

We denote by O_Γ the outer complement of a contour Γ, the set

$$O_\Gamma := \begin{cases} A_\Gamma^+ & \text{if } |A_\Gamma^+| \geq |A_\Gamma^-|, \\ A_\Gamma^- & \text{if } |A_\Gamma^+| < |A_\Gamma^-|. \end{cases} \tag{2.16}$$

Inner complement of a contour Γ*.* The inner complement of a contour Γ is denoted by $\mathrm{int}(\Gamma) := \Lambda_\epsilon \setminus [\mathrm{sp}(\Gamma) \cup O_\Gamma]$.

It is convenient to define a mapping $\Gamma \in \mathcal{G}(m) \to \{I_\Gamma\} \subset \mathcal{I}(m)$, which associates with each contour Γ the corresponding set of islands $\{I_\Gamma\}$. By abuse of notation we will ignore that there may be several islands and write $\{I_\Gamma\} = I_\Gamma$ if no confusion arises.

How to associate $\{I_\Gamma\}$ *to* Γ*.* Given $\Gamma \in \mathcal{G}(m)$, let $\mathrm{int}(\Gamma)$ be the inner complement. The islands $\{I_\Gamma\}$ together with their sign are defined as follows. Fix a connected component of the inner complement $\mathrm{int}(\Gamma)$. This connected component is connected to either $\partial^+(\mathrm{sp}(\Gamma))$ or $\partial^-(\mathrm{sp}(\Gamma))$. Set $\tau = \pm 1$ accordingly. The island associated with this connected component is the union of all cubes in the considered connected component which have the same sign, i.e., $\eta(m, y) = \tau$ for all $y \in I_\Gamma$, and which are connected to $\partial^\tau(\mathrm{sp}(\Gamma))$. The sign of I_Γ equals τ. Note that the number of islands associated to Γ is equal to the number of the connected components of the inner complement and their signs can be + or −.

Virtual contour. Further, we denote

$$I_{\tilde{\Gamma}} := \Lambda_\epsilon \setminus \cup_{\Gamma\in\mathcal{G}(m)} \left(\mathrm{sp}(\Gamma) \cup \{I_\Gamma\}\right).$$

The coarse-grained phase indicator η is constant on $I_{\tilde{\Gamma}}$ (see Lemma 5.1), and we define

$$\mathrm{sign}(m) := \eta^\zeta(m,\cdot)|_{I_{\tilde{\Gamma}}}. \tag{2.17}$$

This means that $I_{\tilde{\Gamma}}$ shares this important property with the islands associated with real contours; therefore, it is justified to call it an island associated with a virtual contour $\tilde{\Gamma}$.

Remark 2.5. Note that in a finite volume with Neumann or Dirichlet boundary conditions it is always possible to divide the complement of the support of a collection of contours $\{\Gamma_1,\ldots,\Gamma_k\}$ into connected regions I_i for $i=1,\ldots,n$ so that η is constant and not zero on $\partial^{\mathrm{ext}} I_i$ (the boundary of an island).

The definitions (2.14) and (2.15) distinguish functions in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ according to their mean over unit cubes of the partition $\mathcal{D}^{(0)}$. We would like to have some control on their *pointwise* behavior on correct cubes. In the next theorem we show that, given $\zeta>0$ and $m_0\in H^1(\Lambda_\epsilon)$, we can associate a function which decreases the energy functional, has "almost" the same phase indicator η^ζ as the original function m_0, and for which positive (resp., negative) mean over correct cubes implies pointwise positivity (resp., negativity). We will refer to such a function as the ζ-*representative* of m_0.

Remark 2.6. Theorems 2.7 and 2.11 are stated for θ small and $\alpha(\epsilon)=1$. In the case $\alpha(\epsilon)\to 0$ they hold for ϵ sufficiently small.

THEOREM 2.7 (representation). *There exist $\theta_0>0$ and $0<\zeta_0<\delta_0/4$,[3] such that $\mathbb{P}$-almost surely the following holds: For all $0<\theta\le\theta_0$, $0<\zeta\le\zeta_0$, and for all $m_0\in H^1(\Lambda_\epsilon)$ we can associate $m_1\in H^1(\Lambda_\epsilon)$, $m_1\equiv m_1(\omega,m_0,\zeta)$ so that*

$$G_1(m_1,\omega)\le G_1(m_0,\omega) \tag{2.18}$$

and that m_1 has the following properties.

Let $\widehat{I}=\{x\in\Lambda_\epsilon; d(x,I)\le\frac14\}$ for $I\in\mathcal{I}(m_1,\zeta)$, and let $C_1=2C_0\|g\|_\infty$, where C_0 is the positive constant in (2.4). *Then we have the following.*

1. *If $\Gamma\in\mathcal{G}(m_0,\zeta)$, then* $\mathrm{sp}(\Gamma)\subset\mathrm{sp}(\Gamma')$ *with $\Gamma'\in\mathcal{G}(m_1,\zeta)$.*
2. *m_1 is Lipschitz continuous on $\widehat{I}$ with Lipschitz constant $L_0=L_0(d,C_1,\theta_0)$.*
3. *There exists $0<\hat\zeta<\delta_0/2$, $\hat\zeta=\hat\zeta(d,\zeta,\theta_0)$ (see* (3.4)*) so that*

$$m_1(x)\in\begin{cases}\left[1-\hat\zeta,1+C_1\theta\right], & x\in\widehat{I} \text{ and } \mathrm{sign}(I)=+1,\\ \left[-1-C_1\theta,-1+\hat\zeta\right], & x\in\widehat{I} \text{ and } \mathrm{sign}(I)=-1.\end{cases}$$

4. *$m_1(x,\omega)=\mathrm{sign}(I)+\hat v(x,\omega,\widehat{I})$ for $x\in\widehat{I}$, where $\hat v(\cdot,\omega,\widehat{I})$ is the solution of*

$$-\Delta v+\frac{1}{2C_0}v+\frac12\alpha(\epsilon)\theta g_1(\cdot,\omega)=0 \quad \text{in } \widehat{I},\qquad v=m_1-\mathrm{sign}(I) \text{ on } \partial\widehat{I}. \tag{2.19}$$

[3]The upper bound $\zeta_0<\delta_0/4$ is an immediate consequence of (3.5).

Remark 2.8. The previous theorem holds for $0 < \zeta < \zeta_0$, but it becomes meaningless for θ fixed and ζ small: In such a situation $\eta^\zeta = 0$ on too many cubes because the random field will create deviations from ± 1 which are typically larger than ζ. Theorem 2.11, stated below, holds only for an accuracy parameter $\zeta(\theta)$, not for a range reaching up to zero.

The proof of Theorem 2.7 is given at the end of section 3. It is based on several intermediate results proven in section 3.

DEFINITION 2.9. *We denote by $\mathcal{R}_{\zeta,\omega}(\Lambda_\epsilon)$ the set of the ζ-representatives of functions in $H^1(\Lambda_\epsilon)$:*

$$\mathcal{R}_{\zeta,\omega}(\Lambda_\epsilon) := \{m \in H^1(\Lambda_\epsilon) : \text{ There exists } u \in H^1(\Lambda_\epsilon) \text{ s.t. } m = m_1(\omega, u, \zeta)\}.$$

We will drop the suffix ζ, ω when no confusion arises.

For such a "representative" m_1 we can bound the energy from below in terms of contours. First we need to define two functions $u_\epsilon^+(\cdot,\omega)$ and $u_\epsilon^-(\cdot,\omega)$, which for $\theta \ll 1$ are the minimizers under the pointwise constraints $u > 0$ and $u < 0$, respectively.

DEFINITION 2.10. *Let $v_\epsilon^*(\cdot,\omega)$ be the solution of the following equation:*

$$-\epsilon\Delta v(r) + \frac{1}{2C_0}\frac{v(r)}{\epsilon} + \frac{1}{2\epsilon}\alpha(\epsilon)\theta g_\epsilon(r,\omega) = 0 \quad \text{in } \Lambda, \quad \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Lambda. \tag{2.20}$$

Let $u_\epsilon^\pm := \pm 1 + v_\epsilon^*$, and set for $x \in \Lambda_\epsilon$, $v^*(x,\omega) := v_\epsilon^*(\epsilon x, \omega)$, $u^\pm := \pm 1 + v^*$. Note that v^* depends on ϵ only through $\alpha(\epsilon)$.

The relevant properties of v_ϵ^* are summarized in Proposition B.1. The next theorem bounds from below the difference between $G_1(m,\omega)$, the functional evaluated over a generic function $m \in H^1(\Lambda_\epsilon)$, and $G_1(u^{\mathrm{sign}(m)},\omega)$, the functional evaluated at $u_\epsilon^+ := 1 + v_\epsilon^*$, when sign(m) = 1, or at $u_\epsilon^- := -1 + v_\epsilon^*$, when sign(m) = −1. This lower bound in terms of contours of m_1 holds for each realization of $\omega \in \Omega$. When the quantity which bounds this difference from below is positive, then for that realization of the random field the function $u_\epsilon^\pm := \mathrm{sign(m)} + \mathrm{v}_\epsilon^*$, having no contours, has lower energy than $G_1(m,\omega)$. Theorem 2.11 together with some probabilistic estimates is an essential step in the proof of Theorem 2.1.

THEOREM 2.11 (reduction). *Let ζ_0 and θ_0 be as in Theorem 2.7. There exists $\theta_1 > 0$ with $\theta_1 < \theta_0$ such that $\mathbb{P}$-almost surely the following holds: There exists $0 < \zeta := \zeta(\theta_0) < \zeta_0$ such that for all $0 < \theta < \theta_1$ there exists a deterministic constant $c(\theta)$ with $\liminf_{\theta\to 0} c(\theta) > 0$ such that*

$$G_1(m,\omega) - G_1(u^{\mathrm{sign}(m)},\omega) \geq \sum_{\Gamma \in \mathcal{G}(m_1,\zeta)} \left(-\mathrm{sign}(m) 2\theta \int_{I_\Gamma^{-\mathrm{sign}(m)}} g_1(x,\omega)\mathrm{dx} + \mathrm{c}(\theta) N_\Gamma \right),$$

where m_1 is a ζ-representative of m (see Theorem 2.7), $N_\Gamma = \left|\cup_{\Gamma\in\mathcal{G}(m_1,\zeta)} \mathrm{sp}(\Gamma)\right|$, and $I_\Gamma^\pm$ denotes those islands associated with Γ, where $\eta^\zeta = \pm 1$.

We show the proof of Theorem 2.11 in section 5.

Remark 2.12. Since we apply Theorems 2.7 and 2.11 to prove Theorems 2.1, 2.2, and 2.3, which hold only in $d \geq 3$, we prove Theorems 2.7 and 2.11 only for $d \geq 3$. The proof extends to $d \geq 1$ with minor modifications mainly due to the explicit representation of the solution of (2.20) in terms of the associated Green function.

In the following we denote by $\mathrm{Per}(A, Q)$ the perimeter of a set A within a set Q. When $Q = \Lambda$ we set $\mathrm{Per}(A) = \mathrm{Per}(A,\Lambda)$. Roughly speaking, $\mathrm{Per}(A,Q)$ denotes the area of that part of the boundary of A which is contained in the interior of Q. A rigorous definition uses the BV norm of the characteristic function of A, i.e., of the function which is equal to 1 in A and zero elsewhere; see, e.g., [9, 21].

3. Properties of low energy states.

3.1. Existence and properties of global minimizers. In this section we prove properties of functions with energy close to the minimal one. The statements hold either for $\alpha(\epsilon) = 1$ and θ sufficiently small, or for θ arbitrary, $\alpha(\epsilon) \to 0$, and ϵ sufficiently small. We first show that to determine the minimizers of the functional G_ϵ, it is sufficient to consider functions in $H^1(\Lambda)$ which satisfy a uniform L^∞-bound.

LEMMA 3.1. *Assume H1. We have with $\mathbb{P} = 1$ that for all $v \in H^1(\Lambda)$ and all $t > 1 + C_0\theta\alpha(\epsilon)\|g\|_\infty$,*

$$G_\epsilon(t \wedge v \vee (-t), \omega) - G_\epsilon(v, \omega) \geq \frac{1}{\epsilon}\int_{\Lambda_t} \left(C_0^{-1}(t-1) - \alpha(\epsilon)\theta\|g\|_\infty\right)(|v(y)| - t), \tag{3.1}$$

where C_0 is the constant in (2.4) *and $\Lambda_t = \{y \in \Lambda : |v(y)| > t\}$. In particular $G_\epsilon(t \wedge v \vee (-t), \omega) < G_\epsilon(v, \omega)$ unless $\Lambda_t = \emptyset$.*

Proof.

$$G_\epsilon(v, \omega) - G_\epsilon(t \wedge v \vee (-t), \omega) \geq \frac{1}{\epsilon}\int_{\Lambda_t} (W(v(y)) - W(t))\, dy$$
$$+ \frac{1}{\epsilon}\alpha(\epsilon)\theta \int_{\Lambda_t} g_\epsilon(y, \omega)[v(y) - \mathrm{sign}(v(y))t]dy,$$

and from H1 and the L^∞-bound on g we derive (3.1). ∎

This L^∞-bound on the global minimizer implies Lipschitz regularity. In order to see this, note that a global minimizer of $G_\epsilon(\cdot, \omega)$ in $H^1(\Lambda)$ is for all $\omega \in \Omega$ a weak solution of the Euler–Lagrange equation

$$\epsilon\Delta v = \frac{1}{2\epsilon}[W'(v) + \theta\alpha(\epsilon)g_\epsilon] \quad \text{in } \Lambda, \tag{3.2}$$

with homogeneous Neumann boundary conditions.

PROPOSITION 3.2. *Let*

$$L_0 = C(d)\left[\sup_{\{s:s=v(r), r\in\Lambda\}} |W'(s)| + \theta\|g\|_\infty\right]. \tag{3.3}$$

With $\mathbb{P} = 1$ it holds that the solution v of the Euler–Lagrange equation (3.2) *satisfies*

$$|v(r, \omega) - v(r', \omega)| < \frac{L_0}{\epsilon}|r - r'|, \quad r, r' \in \Lambda.$$

Proof. By Lemma 3.1, a global minimizer v satisfies the bound $|v(r, \omega)| \leq 1 + C_0\theta\|g\|_\infty\alpha(\epsilon)$ for $r \in \Lambda$ and $\omega \in \Omega$. Since $|g_\epsilon(\cdot, \omega)| \leq 1$ for all $\omega \in \Omega$, any minimizer will be a bounded solution of Poisson's equation with a bounded right-hand side.

By changing variables $y = \frac{r}{\epsilon}$, one writes (3.2) in Λ_ϵ. Denote $u(y, \omega) = v(\epsilon y, \omega)$. By the regularity theory for the Laplacian (see [12]) the solution u is Lipschitz in Λ_ϵ with a Lipschitz constant bounded by $L_0 = \sup_{\{s:s=u(x), x\in\Lambda_\epsilon\}} |W'(s)| + \theta\|g\|_\infty$ and independent of ϵ. Transforming back the solution in the old set of coordinates, one immediately obtains the result. ∎

3.2. Pointwise properties. Once the Lipschitz continuity is established, it is easy to derive pointwise properties from information about integral averages over cubes by standard estimates.

PROPOSITION 3.3. *Let* $\theta_0 > 0$ *and* $1 > \zeta_0 > 0$, $Q \in \mathcal{D}^{(0)}$ *and let*

$$k(d) = \inf_{x\in[0,1]^d} \liminf_{r\to 0} r^{-d}|B_r(x) \cap [0,1]^d|.$$

Suppose that u *is Lipschitz continuous in* Q *with Lipschitz constant* L_0, *and* $\|u\|_\infty \leq 1 + C_1\theta$ *for* $0 < \theta \leq \theta_0$. *Let*

$$\hat{\zeta}(d, L_0, \zeta_0, \theta_0) := 2\left(\frac{\zeta_0 + C_1\theta_0}{k(d)}\right)^{\frac{1}{(d+1)}} (2L_0)^{\frac{d}{(d+1)}}. \tag{3.4}$$

Then for $0 < \zeta < \zeta_0$

$$u(x) \in \begin{cases} \left[1 - \hat{\zeta}, 1 + C_1\theta\right] & \text{if } \eta^\zeta(u,x) = +1, \\ \left[-1 - C_1\theta, -1 + \hat{\zeta}\right] & \text{if } \eta^\zeta(u,x) = -1. \end{cases}$$

Proof. Suppose $\eta^\zeta(u,x) = 1$ for $x \in Q$. Let $\hat{\zeta}$ be as in (3.4) and assume there exists a point $x_0 \in Q$ such that $u(x_0) < 1 - \widehat{\zeta}$. We will show that this assumption leads to a contradiction. Let $0 < r \ll 1$. Then since u has Lipschitz constant bounded by L_0

$$u(x) < 1 - \widehat{\zeta} + L_0 r \quad \forall\, x \in B_r(x_0).$$

Moreover, we have the bound $|u| \leq 1 + C_1\theta$. Let $v_r := |B_r(x_0) \cap Q|$; then, since $\zeta \leq \zeta_0$,

$$(1 - \zeta_0) \leq (1 - \zeta) \leq \int_Q u \leq (1 - \widehat{\zeta} + L_0 r)v_r + (1 - v_r)(1 + C_1\theta_0),$$

and consequently

$$v_r(\widehat{\zeta} - L_0 r + C_1\theta_0) \leq \zeta_0 + C_1\theta_0.$$

Choose r so small that $L_0 r \leq (1/2)\widehat{\zeta}$, and let $k(d)$ be such that $v_r \geq k(d)r^d$ for $r \ll 1$. Then we derive a contradiction if $\hat{\zeta}$ is as in (3.4). Therefore x_0 cannot exist and $u(x) > 1 - \widehat{\zeta}$ for all $x \in Q$. The case $\eta^\zeta = -1$ is proven similarly. ∎

Remark 3.4. To exploit the properties of the double-well potential near the points ± 1 it is essential to require $u(x) \geq 1 - \delta_0$ for $x \in Q$, where δ_0 is the quantity defined in (2.4). Keeping in mind that by Lemma 3.1 we may assume $\|u\|_\infty \leq 1 + 2C_0\|g\|_\infty\theta$, we require

$$2\left(\frac{\zeta_0 + 2C_0\|g\|_\infty\theta_0}{k(d)}\right)^{\frac{1}{(d+1)}} (2L_0)^{\frac{d}{(d+1)}} \leq \frac{\delta_0}{2}. \tag{3.5}$$

This forces a condition on ζ_0 and θ_0 (when $\alpha(\epsilon) = 1$).

3.3. Minimizers with constraints.

DEFINITION 3.5. *Denote for $m \in H^1(\Lambda_\epsilon)$, $|m| \le 1 + C_1\theta_0$, $I \subset \Lambda_\epsilon$, a $\mathcal{D}^{(0)}$-measurable set and $\tau = \pm$, respectively,*

$$X_{I,m} = \left\{\psi \in H^1(\Lambda_\epsilon, R) : \psi = m \quad on\, (I \cup \partial^{\rm ext} I)^c\right\}, \tag{3.6}$$

$$\mathcal{A}^\tau_{I,m} = \left\{\psi \in X_{I,m} : \eta(\psi, x) = \tau \quad on\, I \cup \partial^{\rm ext} I\right\}. \tag{3.7}$$

A generic function in $\mathcal{A}^\tau_{I,m}$, e.g., an element of a recovery sequence for the Γ-convergence result in Theorem 2.3, does not need satisfy the hypothesis of Proposition 3.3. However, it will turn out that we do not need to prove that the constraint given by the mean (see (2.15)) implies a strictly pointwise constraint for a generic function in $\mathcal{A}^\tau_{I,m}$ but only for those functions minimizing the energy under the constraint to be in $\mathcal{A}^\tau_{I,m}$ (the integral constraint) and the pointwise constraint $|\psi| \le 1 + C_1\theta_0$. So we dedicate the next subsection to the proof that the minimizers of the functional (2.12), subject to the integral and the pointwise constraint just described, are, on correct cubes, Lipschitz continuous with a Lipschitz constant depending only on W, θ_0, and $\|g\|_\infty$.

DEFINITION 3.6. *Given $m_0 \in H^1(\Lambda_\epsilon)$, $\|m_0\|_{L^\infty} \le 1 + C_1\theta$, $\theta > 0$, $1 > \zeta > 0$, we define $\mathcal{S}_\epsilon(m_0) \equiv \mathcal{S}^\zeta_\epsilon(m_0)$ as follows:*

$$\begin{aligned}\mathcal{S}_\epsilon(m_0) :=& \left\{m \in H^1(\Lambda_\epsilon) :\ \|m\|_{L^\infty} \le 1 + C_1\theta\right\}\\ &\cap \left\{ m \in H^1(\Lambda_\epsilon) :\ \left\{\begin{array}{lll} \int_{C^{(0)}(x)} m \ge 1-\zeta & if & \int_{C^{(0)}(x)} m_0 > 1-\zeta, \\ \left|\int_{C^{(0)}(x)} m\right| \le 1-\zeta & if & \left|\int_{C^{(0)}(x)} m_0\right| \le 1-\zeta, \\ \int_{C^{(0)}(x)} m \le -1+\zeta & if & \int_{C^{(0)}(x)} m_0 < -1+\zeta. \end{array}\right. \right\}\end{aligned} \tag{3.8}$$

Since weak convergence in H^1 implies strong convergence in L^2, the integral constraints are preserved under weak H^1-convergence. Moreover, any sequence which converges strongly in L^2 has a subsequence which converges a.e., so that the L^∞-constraint is also preserved under weak H^1-convergence. Hence for any fixed $\epsilon > 0$ the set $S_\epsilon(m_0)$ is weakly H^1-closed and $\min_{S_\epsilon(m_0)} G_1(u,\omega)$ exists with $\mathbb{P} = 1$. Note that $m_0 \in S_\epsilon(m_0)$, so

$$\min_{S_\epsilon(m_0)} G_1(u,\omega) \le G_1(m_0,\omega). \tag{3.9}$$

Choose any $m_1 \in \mathrm{argmin}_{S_\epsilon(m_0)} G_1(u,\omega)$. We denote $m_1 \equiv m_1(\omega, m_0, \zeta)$ a *representative* of m_0. Define, as before, the block indicator $\eta^\zeta(m_1, x)$, $x \in \Lambda_\epsilon$, and the set of the associated contours $\mathcal{G}(m_1)$ and islands. Note that if $\eta^\zeta(m_0, x) = 0$, then $\eta^\zeta(m_1, x) = 0$ but, as strict inequalities are not preserved in the limit, it might happen that $\eta^\zeta(m_1, x) = 0$ even though $\eta^\zeta(m_0, x) \ne 0$.

The next lemma shows that on *correct* cubes the pointwise constraint is not active for the minimizer m_1. This is not obvious due to the simultaneous presence of both types of constraints: The one-sided integral constraint "pushes the minimizer up." Note that the integral constraint is not active on correct cubes by definition; see (2.15).

LEMMA 3.7. *Let $m_1 \in \mathrm{argmin}_{\mathcal{S}_\epsilon(m_0)} G_1(u,\omega)$, and let Q_0 be a ζ-correct cube for m_1. Define $U := \{x :\ \mathrm{dist}(x, Q_0) < 1/2\}$.*

Then there exists for any $\xi \in C_0^\infty(U)$ a $\delta_\xi > 0$ such that

$$m_1 + \delta\xi \in S_\epsilon(m_0) \quad \forall\, \delta < \delta_\xi.$$

As a simple consequence we have that the minimizer with the constraints satisfies the Euler–Lagrange equation in a weak sense.

COROLLARY 3.8. *For m_1 and ξ as in Lemma 3.7 we have that*

$$-2\int \operatorname{grad} m_1 \operatorname{grad}\xi = \int [W'(m_1)+\theta\alpha(\epsilon)g_1]\,\xi.$$

Lemma 3.7 follows from Lemmas 3.10 and 3.11 stated below in the case $\eta^\zeta(m_1,x) = 1$ on Q_0, and the obvious version of them when $\eta^\zeta(m_1,x) = -1$ on Q_0. We need the following definition.

DEFINITION 3.9. *Let $Q \subseteq \mathbb{R}^d$ be connected and $\mathcal{D}^{(0)}$-measurable, i.e., a union of translated unit cubes such that the topological interior* $\operatorname{int}(Q)$ *is connected. Moreover, let $\beta > 0$ and $C > 0$. We denote by $\Psi^{\pm}_{Q,\beta}$ the minimizer with boundary condition $\pm(1+C\theta)$, the unique element of*

$$\operatorname{argmin}_{\{v\in H^1(Q):\ v\mp(1+C\theta)\in H^1_0(Q)\}}\int_Q (|\operatorname{grad} u|^2 \pm \beta u), \tag{3.10}$$

i.e., the minimizer with boundary condition $\pm(1+C\theta)$.

To shorten notation we specialize the next lemmas to the case of positive boundary conditions and denote $\Psi^+_{Q,\beta} := \Psi_{Q,\beta}$.

LEMMA 3.10. *Let $\Psi_{Q,\beta}$ be as in Definition 3.9. Then,*

1. $-2\Delta\Psi_{Q,\beta} + \beta = 0$ *on* $\operatorname{int}(Q)$, $\Psi_{Q,\beta} = 1 + C\theta$ *on* ∂Q.
2. $1 + C\theta - C(Q)\beta \le \Psi_{Q,\beta} < 1 + C\theta$ *on* $\operatorname{int}(Q)$, *where $C(Q)$ depends only on the diameter of Q.*
3. $\int_Q |\Psi_{Q,\beta} - (1+C\theta)| \to 0$ *as* $\beta \to 0$.

Proof. Point (1) is obvious, (2) is an immediate consequence of the strong maximum principle applied to $\Psi_{Q,\beta}$ (upper bound) and the maximum principle applied to $\phi \equiv \Psi_{Q,\beta} - [\frac{\beta}{4d}|x-x_0|^2 + c_0]$, where x_0 is the center of the smallest ball containing Q and c_0 is the largest constant such that $\frac{\beta}{4d}|x-x_0|^2 + c_0 \le 1 + C\theta$ on ∂Q. Namely, ϕ is a harmonic function in Q and on the boundary of Q nonnegative, so $\phi(x) \ge 0$ for $x \in Q$. We choose $c_0 = 1 + C\theta - \frac{\beta}{4d}(\operatorname{diam}Q)^2$. This implies the lower bound in (2), setting $C(Q) = \frac{(\operatorname{diam}Q)^2}{4d}$. Finally, (3) follows from (2). ∎

LEMMA 3.11. *Let Q be connected and $\mathcal{D}^{(0)}$-measurable. Let $\Psi_{Q,\beta}$ be as in Definition 3.9 with $C \ge 2C_0\|g\|_{L^\infty}$, where C_0 is the constant in (2.4). Let $u \in H^1(Q)$ so that $\|u\|_\infty \le 1 + C\theta$. There exists $\theta_0 = \theta(W, \|g\|_\infty) > 0$ and for all $\theta \le \theta_0$ $\beta_0 = \beta_0(\theta, W, \operatorname{diam}Q)$ (see (3.14)), so that for $0 < \beta < \beta_0$ the function $\widehat{u}_\beta := u \wedge \Psi_{Q,\beta}$ satisfies the following.*

1. $G_1(Q, \widehat{u}_\beta, \omega) \le G_1(Q, u, \omega)$, *with strict inequality if* $\widehat{u}_\beta \ne u$, $\mathbb{P} = 1$.
2. $\widehat{u}_\beta < 1 + C\theta$ *in* int(Q), $\widehat{u}_\beta = u$ *on* ∂Q.
3. $|\int_{Q_i} \widehat{u}_\beta - \int_{Q_i} u| \to 0$ *as* $\beta \to 0$ *for all* $Q_i \subseteq Q$, $Q_i \in \mathcal{D}^{(0)}$.

Proof. Point (2) follows from (2) of Lemma 3.10 and the L^∞-bound on u and that, by construction, $\Psi_{Q,\beta}(\cdot) = 1 + C\theta$ on the boundary of Q. Point (3) follows from point (3) of Lemma 3.10 and the bound $u(x) \le 1 + C\theta$ a.e.

It remains to show (1). The main idea is to consider $\widetilde{\Psi} := \Psi_{Q,\beta} \vee u$ as a (compactly supported) perturbation of $\Psi := \Psi_{Q,\beta}$, thus obtaining bounds on

$$\int_{\{u(x)>\Psi(x)\}} |\operatorname{grad} u|^2.$$

These bounds, in turn, are used to obtain (1), considering $\widehat{u}_\beta \equiv \Psi_{Q,\beta} \wedge u$ as a perturbation of u. As Ψ is a minimizer (see (3.10)), we obtain

$$0 \le \int_Q \left[(|\operatorname{grad} \widetilde{\Psi}|^2 - |\operatorname{grad} \Psi|^2) + \beta(\widetilde{\Psi} - \Psi) \right]$$
$$= \int_{\{u>\Psi\}} \left[(|\operatorname{grad} u|^2 - |\operatorname{grad} \Psi|^2) + \beta(u - \Psi) \right],$$

and therefore

$$\int_{\{u>\Psi\}} (|\operatorname{grad} u|^2 - |\operatorname{grad} \Psi|^2) \ge -\beta \int_{\{u>\Psi\}} (u - \Psi). \tag{3.11}$$

Then, since by (2) of Lemma 3.10, $\Psi(\cdot) \in [1 + C\theta - C(Q)\beta, 1 + C\theta]$, and $u(\cdot) \in (1 + C\theta - C(Q)\beta, 1 + C\theta]$ for all $x \in \{u > \Psi\}$, we have

$$\begin{aligned} &G_1(Q, u, \omega) - G_1(Q, \widehat{u}_\beta, \omega) \\ &= \int_{\{u>\Psi\}} \left[(|\operatorname{grad} u|^2 - |\operatorname{grad} \Psi|^2) + \left(\frac{W(u) - W(\Psi)}{u - \Psi} + \theta g_1(\cdot, \omega) \right) (u - \Psi) \right] \\ &\ge C(\beta, \theta) \int_{\{u>\Psi\}} (u - \Psi), \end{aligned} \tag{3.12}$$

where

$$C(\beta, \theta) = \inf_{[1+C\theta - C(Q)\beta, 1+C\theta]} W'(s) - \theta \|g\|_{L^\infty} - \beta. \tag{3.13}$$

Take $\beta \le \frac{\delta_0}{C(Q)}$ so that $1 - \delta_0 < 1 + C\theta - C(Q)\beta$. By (2.4), $\inf_{[1+C\theta-C(Q)\beta,1+C\theta]} W'(s) = \frac{1}{C_0}[C\theta - C(Q)\beta]$; then, since by assumption $C \ge 2C_0\|g\|_{L^\infty}$, we obtain that

$$C(\beta, \theta) \ge \theta \|g\|_{L^\infty} - \frac{\beta}{C_0}[C_0 + C(Q)].$$

Take θ_0 and β_0 so that

$$\theta_0 \le 2\frac{\delta}{\|g\|_\infty}, \quad \beta_0 = \frac{1}{2}\theta \frac{C_0\|g\|_{L^\infty}}{C_0 + C(Q)}; \tag{3.14}$$

then $C(\beta, \theta) > \frac{1}{2}\theta\|g\|_\infty$ for all $\beta < \beta_0$.[4] ∎

Remark 3.12. For u as in Lemma 3.11 we can find $\beta \equiv \beta(u) < \beta_0$ such that $\int_{Q_i} \widehat{u}_\beta > 1 - \zeta$ if $\int_{Q_i} u > 1 - \zeta$ for all unit cubes Q_i contained in $Q = \cup_i Q_i$.

As a consequence we have that for such β, $\widehat{u}_\beta$ strictly satisfies the integral and the L^∞-constraints in Q, $G_1(Q, u, \omega) \ge G_1(Q, \hat{u}_\beta, \omega)$, with strict inequality unless $u = \hat{u}_\beta$ a.e.

Proof of Lemma 3.7. Let m_1 be a minimizer in the set $S_\epsilon(m_0)$; see (3.8) and (3.9). Let $\widehat{Q}$ be the union of Q_0 and the cubes Q_i, which are the connected neighbors of Q_0. By assumption, Q_0 is ζ-correct and we may assume that $\eta^\zeta(m_1, x) = 1$ for $x \in \widehat{Q}$. (Similar arguments hold when $\eta^\zeta(m_1, x) = -1$ for $x \in \widehat{Q}$.) By Lemma 3.11 (and its version for the negative well), there exists a $\beta > 0$ such that $|m_1(x)| \le \Psi_{\widehat{Q},\beta}(x)$ in

[4]The choices made enforce $\beta \le \frac{\delta_0}{C(Q)}$ since $C(Q) \ge 1$.

$\widehat{Q}$. This implies (see point (2) of (3.11)) that there exists a $c_0 \equiv c_0(\beta, d)$ such that $|m_1(x)| \le c_0 < 1 + C\theta$ in the set $U \subset\subset \widehat{Q}$; see the statement of the lemma. Since $\xi \in C_0^\infty(U)$, there exists δ_ξ so that for all $\delta \le \delta_\xi$, $m_1 + \delta\xi$ does not violate the pointwise constraint, i.e., $\|m_1 + \delta\xi\|_{L^\infty} < 1 + C\theta$. (Take $\delta \sup_x |\xi(x)| < 1 + C\theta - c_0$.) We may require in addition that $0 < \delta |\int_{Q_i} \xi| < \min_{Q_i \subseteq \widehat{Q}} (\int_{Q_i} m_1) - (1 - \zeta)$; then $m_1 + \delta\xi \in S_\epsilon(m_0)$. ◻

After having established that the constraint minimizer m_1 satisfies the same Euler–Lagrange as the unconstraint minimizer, we obtain Lipschitz regularity on correct cubes.

LEMMA 3.13. *With $\mathbb{P} = 1$ the following holds: Let $\theta_0 > 0$. There exists a constant $L_0 \equiv L_0(d, C_0, \theta_0, \|g\|_\infty)$ (C_0 as in (2.4)), such that for $0 < \theta < \theta_0$, $0 < \zeta < \frac{\delta_0}{4}$, the representatives $m_1 \in \operatorname{argmin}_{\mathcal{S}_\epsilon^\zeta(m_0)} G_1(\cdot, \omega)$ of any $m_0 \in H^1(\Lambda_\epsilon)$ satisfy the following on any correct cube Q_0 for $x, y \in U := \{x : \operatorname{dist}(x, Q_0) < 1/2\}$:*

$$|m_1(x) - m_1(y)| \le L_0 |x - y|.$$

Remark 3.14. Note that L_0 does not depend on ζ. This will enable us to apply Lemma 3.3 to ζ, θ that satisfy (3.5).

Proof. Let $\widehat{Q}$ be the union of Q_0 and the cubes Q_i, which are the connected neighbors of Q_0, and let $V := \{x : \operatorname{dist}(x, Q_0) < 3/4\}$. Then there exists a cutoff function $\chi \in C_0^\infty(\widehat{Q})$ such that $\|\chi\|_{W^{2,\infty}} \le K$ for some $K(d)$ independent of θ and ζ, $\chi(x) = 1$ for all $x \in U$, while $\chi(x) \equiv 0$ for $x \in \widehat{Q} \setminus V$, and $0 \le \chi(x) \le 1$ for $x \in \widehat{Q}$. Then by Corollary 3.8 we obtain that (χm_1) is a weak solution of the linear PDE

$$\Delta v = f \quad \text{on } \widehat{Q}, \quad v = 0 \quad \text{on } \partial\widehat{Q}, \tag{3.15}$$

$$f = m_1 \Delta\chi + \operatorname{grad}\chi \operatorname{grad} m_1 + \frac{1}{2}\left[W'(m_1) + \theta\alpha(\epsilon) g_1\right]\chi. \tag{3.16}$$

As the proof proceeds by standard arguments (see, e.g., [8]), we sketch it. First we show that there exists a constant depending only on W, d, K, the bound on the $W^{2,\infty}$-norm of the cutoff function, and θ_0 so that for all $\theta \le \theta_0$,

$$\int_V |\operatorname{grad} m_1|^2 \le C(W, \|g\|_\infty, d, \theta_0). \tag{3.17}$$

Now we know that f in (3.16) can be written as $f = f_1 + f_2$, $\|f_1\|_{L^\infty(\widehat{Q})} + \|f_2\|_{L^2(\widehat{Q})} \le C(W, d, \theta_0)$. By the regularity theory for weak solutions of (3.15), we obtain $v \in W^{2,2}$, hence, $\operatorname{grad} m_1 \in L^p(V')$ for a slightly smaller set V' and $p < 2d/(d-2)$. This improves the regularity of f_2 to $\|f_2\|_{L^p} < C'(W, d, \theta_0)$. This standard bootstrap procedure can be repeated until, after a number of steps depending only on the dimension, $\|f_2\|_{L^p} < C_p(W, \theta)$ for $p > d$. Then $v \in W^{2,p}$ by the L^p-regularity theory for elliptic equations and by the Sobolev embedding $v \in C^1$ with constants depending only on W, θ, $\|g\|_\infty$, and the dimension. ◻

We are now able to prove Theorem 2.7.

Proof of Theorem 2.7. Let $\zeta \le \zeta_0$ and $\mathcal{S}_\epsilon^\zeta(m_0)$ be the set defined in (3.8). The existence of a minimizer of $G_1(m, \omega)$ for $m \in \mathcal{S}_\epsilon(m_0)$ is a consequence of the fact that there exist a constant C and $C_\epsilon(\theta, \|g\|_\infty)$ so that

$$G_1(u, \omega) \ge \frac{1}{C}\left(\|\nabla u\|^2 + \|u\|^2\right) - C_\epsilon, \qquad \mathbb{P} = 1.$$

G_1 is weakly lower semicontinuous on $H^1(\Lambda_\epsilon)$ and, as pointed out before Lemma 3.7, the set $\mathcal{S}_\epsilon(m_0)$ is weakly H^1-closed. Point (1) is obvious because of the definitions of $\mathcal{S}_\epsilon(m_0)$, the block variable (see (2.15)), and the definition of contours. The Lipschitz property in point (2) is a consequence of Lemma 3.13 applied to each block in any island associated to m_1. Recall that, by definition, each island is the union of correct blocks. The positivity is a consequence of point (1) and Proposition 3.3. Further, assume without loss of generality that $\mathrm{sign}(I) = 1$; for notation see subsection 2.3.2. Set $m_1 = 1 + \hat{v}$. The functional restricted to $\widehat{I}$ can be written as the following:

$$G_1(\widehat{I}, 1+\hat{v}, \omega) = \int_{\widehat{I}} \left(|\nabla \hat{v}(y)|^2 + \frac{1}{2C_0}(\hat{v}(y))^2 \right) \mathrm{d}y + \alpha(\epsilon)\theta \int_{\widehat{I}} \mathrm{d}y\, g_1(y,\omega)(1+\hat{v}(y)).$$

The equality holds since $m_1(x) \geq 1 - \hat{\zeta}$ for $x \in \widehat{I}$ (see point (3)) and $\hat{\zeta} \leq \delta_0$ by assumption (see Remark 3.4), and by the assumption on the double-well potential (see (2.4)). Further, we proved that the constraints on m_1 are not active in $\widehat{I}$. Thus $\hat{v}$ solves the Euler–Lagrange equation (2.19). As a simple consequence of the convexity of the potential $W(s)$ when $s \geq \delta_0$ (see H1), this solution is unique. □

4. Deviations from equilibrium. In this section we estimate the cost associated with the support of a contour. We will need several lemmas for estimating the cost of a single cube which is not correct, and then conclude by a covering argument. The estimates in this section are based on standard methods, but they are complicated by the fact that we need estimates that hold even for those contours which are not "mixed," i.e., which do not separate areas where $m \sim 1$ from areas where $m \sim -1$. Let Q be a cube of sidelength ℓ. Given $m \in H^1(Q)$ and $t > 0$ we define the following.

Definition 4.1.

$$m_Q^t(x) = \begin{cases} |m(x)| \vee t & \text{if } |\{m>0\}| \geq \frac{1}{2}|Q|, \\ -(|m(x)| \vee t) & \text{if } |\{m>0\}| < \frac{1}{2}|Q|. \end{cases} \tag{4.1}$$

Lemma 4.2. *Let δ_0 and C_0 be as defined in (2.4), and let Q be a cube of sidelength ℓ. Then there exists $C_2 > 0$ (depending only on the dimension) and a t_0 with $\max\{\frac{1}{2}, 1-\delta_0\} < t_0 < 1$ so that with $D_1 := \inf_{|s|\leq \frac{1}{2}} \sqrt{2(W(s) - W(t_0))} \geq 0$ it holds that for any t with $t_0 < t < 1 - 2C_0\alpha(\epsilon)\theta\|g\|_\infty$*

$$G_1(Q,m,\omega) - G_1(Q,m_Q^t,\omega) \geq \left(D_1 - \frac{8\alpha(\epsilon)\theta\ell}{t_0 C_2} \right) \int_{-\frac{t}{2}}^{\frac{t}{2}} P\left(\{m<s\}, Q\right) ds. \tag{4.2}$$

The proof goes as in Proposition 3.6 of [7]. We will apply Lemma 4.2 together with the following isoperimetric inequality (see sections 5 and 6 of [9]):

$$\mathrm{Per}\left(\{m<s\}, Q\right) \geq \left(\min(|Q \cap \{m(x) \leq s\}|, |Q \cap \{m(x) > s\}|)\right)^{\frac{d-1}{d}}, \tag{4.3}$$

where $\mathrm{Per}(A, Q)$ is the perimeter of a set A within a set Q (see the definition at the end of section 2), and $|A|$ denotes the d-dimensional Lebesgue measure of the Borel set A. We need the following lemma, which bounds from below the cost of a zero cube.

Lemma 4.3. *Let $0 < \zeta < \frac{1}{4}$. There exist increasing and near 0 strictly increasing continuous functions $\tilde{\sigma}(\zeta) > 0$, $\tilde{\theta}(\zeta) > 0$, with $\tilde{\sigma}(0) = \tilde{\theta}(0) = 0$ which depend only* *on the double-well potential, the L^∞-norm of g, the sidelength of the cube, and the dimension, such that for $0 < \theta < \tilde{\theta}(\zeta)$ on any cube Q with*

$$-1+\zeta < \frac{1}{|Q|}\int_Q m < 1-\zeta, \quad \|m\|_{L^\infty(Q)} < 1 + 2\theta C_0,$$

it holds that

$$G_1(Q,m,\omega) - G_1(Q,u^\pm,\omega) \geq \tilde{\sigma}(\zeta)|Q|. \tag{4.4}$$

For its proof we need another lemma which we state and prove first.

LEMMA 4.4. *Let δ, ρ, t,ζ, and C_1 be positive parameters such that $0<\delta<\rho<\zeta<1/4$, and suppose that ζ and t satisfy the relation $\{\frac{1}{2}, 1-\delta_0\} < t < 1-\zeta$. Let*

$$A = \{x \in Q : -1+\zeta-\rho < m(x) < 1-\zeta+\rho\} \tag{4.5}$$

and suppose

$$|Q \cap A| < \delta|Q|. \tag{4.6}$$

Suppose, moreover, that $\theta < 1$ and let

$$\sigma_3 = \min\left\{\left(\frac{1}{2}-\delta\right), \frac{\rho-\delta}{3+C_1}\right\}. \tag{4.7}$$

Then the following implication holds.

If $|m(x)| < 1 + C_1\theta$ and $\eta^\zeta(m,x) = 0$ for all $x \in Q$, then

$$\min(|Q\cap\{m(x)>s\}|, |Q\cap\{m(x)<s\}|) \geq \sigma_3|Q| \quad \text{for } -t/2 < s < t/2. \tag{4.8}$$

Proof. We show the lemma in the case

$$\max(|Q\cap\{m(x)>0\}|, |Q\cap\{m(x)\leq 0\}|) = |Q\cap\{m(x)\geq 0\}| \tag{4.9}$$

and $s \in [0,t/2]$; the remaining case is shown similarly. By assumption, $\{\frac{1}{2}, 1-\delta_0\} < t_0 < t < 1-\zeta$. We distinguish two cases:

(a) $|Q\cap\{m(x)>s\}| \leq |Q\cap\{m(x)<s\}|$;

(b) $|Q\cap\{m(x)<s\}| \leq |Q\cap\{m(x)>s\}|$.

We start by discussing the case (a). As $s < \frac{t}{2} < 1-\zeta$, we have $|\{0<m(x)<s\}| \leq |\{0<m(x)<1-\zeta+\rho\}|$ for any $\rho > 0$ and by (4.6)

$$|Q\cap\{0<m(x)<s\}| < \delta|Q|. \tag{4.10}$$

We have

$$|Q\cap\{m(x)>s\}| = |Q| - |Q\cap\{m(x)\leq 0\}| - |Q\cap\{0<m(x)<s\}|.$$

As $|Q\cap\{m(x)\leq 0\}| \leq 1/2|Q|$ by assumptions (4.9) and (4.10), we obtain

$$|Q\cap\{m(x)>s\}| \geq \left(\frac{1}{2}-\delta\right)|Q|. \tag{4.11}$$

Take $\delta < \frac{1}{2}$ so that (4.11) is strictly positive.

In the case (b) we estimate with the help of the a priori bound $|m| \leq 1 + C_1\theta$ and $0 < s < 1-\zeta$

$$\begin{aligned}\int_Q m \geq& \int_{Q\cap\{m(x)<s\}} m + \int_{Q\cap\{s<m(x)<1-\zeta+\rho\}} m + \int_{|Q\cap\{m(x)\geq 1-\zeta+\rho\}} m \\ &\geq (-1-C_1\theta)|Q\cap\{m(x)<s\}| + (1-\zeta+\rho)(|Q\cap\{m(x)\geq 1-\zeta+\rho\}|) \\ &\geq -(1+C_1\theta)|Q\cap\{m(x)<s\}| \\ &\quad + (1-\zeta+\rho)\left[|Q| - |Q\cap\{m(x)<s\}| - |Q\cap\{-1+\zeta-\rho<m(x)<1-\zeta+\rho\}|\right].\end{aligned}$$

By $\eta^\zeta = 0$ on Q and inequality (4.6) we obtain

$$|Q|(1-\zeta) \geq -(2-\zeta+\rho+C_1\theta)|Q\cap\{m(x)<s\}| + (1-\zeta+\rho)(1-\delta)|Q|,$$

which implies for $\delta < \rho < \zeta < 1/4$

$$\frac{|Q\cap\{m(x)<s\}|}{|Q|} \geq \frac{\rho-\delta(1-\zeta+\rho)}{2-\zeta+\rho+C_1\theta} \geq \frac{\rho-\delta}{3+C_1} > 0, \tag{4.12}$$

and we obtain (4.8) for σ_3 as in (4.7). ▯

Now we show the proof of Lemma 4.3.

Proof. Assume without loss of generality that (4.9) holds. Let $\delta > 0$, $\rho > 0$, so that $0 < \delta < \rho < \zeta$. Let A be as in (4.5). We distinguish two cases.

Case 1. Suppose (4.6) does not hold, i.e., $|Q\cap A| \geq \delta|Q|$. Recall that $u^\pm = \pm 1 + v^*$ and, similarly to Proposition 3.2, one estimates $\frac{1}{|Q|}\int_Q |\nabla v^*|^2 \leq C\theta^2$, where $C = C(W, d, \|g\|_\infty)$. We have

$$\begin{aligned}G_1(Q,m,\omega) - G_1(Q,u^\pm,\omega) \geq& -\int_Q |\nabla v^*|^2 + \int_Q W(m) - \int_Q W(u^\pm) \\ &+ \theta\int_Q g_1[m\mp 1] - \theta\int_Q g_1 v^* \\ &\geq -C\theta|Q| + |Q|\frac{\delta}{2C_0}(\zeta-\rho)^2,\end{aligned} \tag{4.13}$$

since by the assumptions on the double-well potential H1

$$W(u^\pm) \leq \theta^2\|g\|_\infty^2 C_0^2, \quad \frac{1}{|Q|}\int_{Q\cap A} W(m) \geq \frac{\delta}{2C_0}(\zeta-\rho)^2.$$

Then we get (4.4) with $\tilde\sigma = \frac{\delta}{2C_0}(\zeta-\rho)^2 - C\theta$.

Case 2. Assume (4.6). We apply Lemma 4.2 to the cube Q. Again we may suppose that $|Q\cap\{m(x)>0\}| \geq 1/2|Q|$ (see (4.9)). So from Lemma 4.2, adding and subtracting, we have for $\{\frac{1}{2}, 1-\delta_0\} < t < 1-\zeta$

$$\begin{aligned}G_1(Q,m,\omega) - G_1(Q,u^\pm,\omega) \geq& [G_1(Q,m_Q^t,\omega) - G_1(Q,u^\pm,\omega)] \\ &+ \left(D_1 - \frac{8\ell\theta}{t_0 C_2}\right)\int_{-\frac{t}{2}}^{\frac{t}{2}} P(\{m<s\},Q)\,ds.\end{aligned}$$

Taking into account the assumption H1 for the potential, we estimate the first term in a straightforward manner, obtaining

$$[G_1(Q,m_Q^t,\omega) - G_1(Q,u^\pm,\omega)] \geq -C\theta|Q|,$$

where $C = C(W, d, \|g\|_\infty)$. The second term we estimate with the help of the isoperimetric inequality (4.3) together with the bound (4.8) from Lemma 4.4. In this way we obtain (4.4) with $\tilde\sigma = \sigma_3^{\frac{d-1}{d}}(D_1 - \frac{8\ell\theta}{t_0 C_2})t - C\theta$, where σ_3 and t are as in Lemma 4.4.

Finally, fix $t_0 := \frac{1}{2}(1 + \max\{1/2, 1-\delta_0\})$, $0 < \zeta < 1/4$ such that $1 - t_0 < 1 - \zeta$, $\delta = \frac{1}{4}\zeta$, and $\rho = \frac{1}{2}\zeta$, and take $\tilde\theta(\zeta)$ so that

$$\tilde\sigma = \min\left\{\sigma_3^{\frac{d-1}{d}}\left(D_1 - \frac{8\ell\theta}{t_0 C_2}\right)\frac{1}{4} - C\theta, \left(\frac{\delta}{2C_0}(\zeta-\rho)^2 - C\theta\right)\right\} \tag{4.14}$$

is strictly positive. ◻

LEMMA 4.5. *Set $0 < \zeta < \zeta_0 < 1/2$. Let $C^\pm$ be two cubes of sidelength 1, and let $z' \in \mathbb{Z}^d$ be such that $C^- \cup C^+ \subseteq Q$ for $Q := z' + 2[-\frac{1}{2}, \frac{1}{2}]^d$. Suppose that*

$$\int_{C_+} m > (1-\zeta), \ \int_{C_-} m < (-1+\zeta), \ \|m\|_{L^\infty(Q)} \le 1 + C_1\theta.$$

There exists $\theta_0 > 0$ independent of ζ *and a constant $\sigma_2 := \sigma_2(\zeta_0, \theta_0, d) > 0$ given in* (4.18) *so that for all $\theta \le \theta_0$*

$$G_1(Q, m, \omega) - G_1(Q, u^\pm, \omega) \ge \sigma_2|Q|, \qquad \mathbb{P} = 1.$$

Proof. We have

$$G_1(Q,m) - G_1(Q,u^\pm) = [G_1(Q,m) - G_1(Q,m^t)] + G_1(Q,m^t) - G_1(Q,u^\pm).$$

As in Lemma 4.3, we estimate the second addend as $G_1(Q,m^t) - G_1(Q,u^\pm) \ge -C\theta|Q|$, where $C = C(W, d, \|g\|_\infty) > 0$, and apply Lemma 4.2 and the isoperimetric inequality (4.3) for the first addend. Note that here Lemma 4.2 holds with $\ell = 2$. It remains to show that

$$\min|Q \cap \{m > s\}|, |Q \cap \{m < s\}| > \frac{1-\zeta_0}{2^d(1+C_1\theta)}|Q| \qquad \text{for any } s \in [-t/2, t/2]. \tag{4.15}$$

We obtain with the L^∞-bound on m

$$(1-\zeta) \le \int_{C_+} m \le (1 + C_1\theta)|C_+ \cap \{m > s\}| \quad \text{for } -t/2 < s < 0, \tag{4.16}$$

$$(1-\zeta) \le \int_{C_+} m \le s|C_+ \cap \{m \le s\}| + (1+C_1\theta)|C_+ \cap \{m > s\}| \quad \text{for } 0 < s < \frac{t}{2}. \tag{4.17}$$

Since $s < t/2$ and $t < 1 - \zeta_0$, we get $(1-\zeta) \le \frac{1}{2}(1-\zeta_0) + (1+C_1\theta)|C_+ \cap \{m > s\}|$ from (4.17). Then, as $\zeta < \zeta_0$, both (4.16) and (4.17) imply

$$|C_+ \cap \{m > s\}| \ge \frac{(1-\zeta_0)}{2(1+C_1\theta)}.$$

A similar estimate can be obtained for $|C_- \cap \{m < s\}|$ so we obtain (4.15). Set

$$\sigma_2 = t_0\left(D_1 - \frac{4\theta_0}{t_0 C_2}\right)\left(\frac{1-\zeta_0}{2^{d+1}(1+C_1\theta_0)}\right)^{\frac{d-1}{d}} - C\theta_0. \tag{4.18}$$

Since $\zeta_0 \leq \frac{1}{2}$ we can take θ_0 independent on ζ_0 and small enough so that $\sigma_2 > 0$. ∎

DEFINITION 4.6. *Given $m \in H^1(\mathbb{R}^d, \mathbb{R})$, $\zeta > 0$, and a $\mathcal{D}^{(0)}$-measurable region J, define*

$$\mathcal{B}_0^{(\zeta,J)}(m) \equiv \{x \in J : \eta^\zeta(m,x) = 0\},$$

$$\begin{aligned} \mathcal{B}_\pm^{(\zeta,J)}(m) = \big\{ & x \in J : \ \eta^\zeta(m,x) = \pm 1 \text{ and there is } x' \in J \text{ with} \\ & \eta^\zeta(m,x')\eta^\zeta(m,x) = -1, C^{(0)}(x') \text{ connected to } C^{(0)}(x) \big\}. \end{aligned} \tag{4.19}$$

We will show the following result.

THEOREM 4.7. *Let $0 < \zeta < \frac{1}{4}$. Given $m \in H^1_{loc}(\mathbb{R}^d, \mathbb{R})$, $\mathrm{sp}(\Gamma)$, and a bounded $\mathcal{D}^{(0)}$-measurable connected subset of ζ-incorrect cubes, there exists $\sigma_1(\zeta) > 0$ so that for all $\theta \leq \tilde{\theta}(\zeta)$, $\tilde{\theta}$ as in Lemma 4.3,*

$$G_1(\mathrm{sp}(\Gamma), m, \omega) - G_1(\mathrm{sp}(\Gamma), u^\pm, \omega) \geq \sigma_1 \left|\mathrm{sp}(\Gamma)\right|, \qquad \mathbb{P} = 1. \tag{4.20}$$

Proof. If $Q + z_0$ is an incorrect cube contained in $\mathrm{sp}(\Gamma)$, then either the cube or at least one of its connected neighbors is a zero cube, or it has a connected neighbor of opposite sign. In each of the cases it holds that the cube $3Q + z_0$ of sidelength 3 centered at the same center contains

(a) a zero cube, or
(b) a pair C_+, C_- of connected cubes with opposite sign such that C_+ or C_- is centered at z_0.

In case (a), by Lemma 4.3,

$$G_1(\mathrm{sp}(\Gamma) \cap (3Q + z_0), m) - G_1(\mathrm{sp}(\Gamma) \cap (3Q + z_0), u^\pm) \geq 3^{-d}\tilde{\sigma}(\zeta)|3Q + z_0|,$$

while in case (b), by Lemma 4.5,

$$G_1(\mathrm{sp}(\Gamma) \cap (3Q + z_0), m) - G_1(\mathrm{sp}(\Gamma) \cap (3Q + z_0), u^\pm) \geq (3/2)^{-d}\sigma_2|3Q + z_0|$$

for θ sufficiently small. Hence we have shown the following: Let $z_0 \in \mathbb{Z}^d$ be the center of a cube which is incorrect for m with accuracy ζ. Then for $\theta < \theta_0(\zeta)$ there exists $\sigma_3(\zeta)$ such that

$$G_1(\mathrm{sp}(\Gamma) \cap (3Q + z_0), m) - G_1(\mathrm{sp}(\Gamma) \cap (3Q + z_0), u^\pm) \geq \sigma_3(\zeta)|3Q + z_0|. \tag{4.21}$$

Let $\{z_1, \ldots, z_N\}$ be a collection of lattice points such that

$$z_i + Q \subseteq \mathrm{sp}(\Gamma), \ (z_i + 3Q) \cap (z_j + 3Q) = \emptyset \quad \text{for } j \neq i, \ i, j = 1, \ldots, N. \tag{4.22}$$

Then we can estimate

$$\begin{aligned} & G_1(\mathrm{sp}(\Gamma), m) - G_1(\mathrm{sp}(\Gamma), u^\pm) \\ \geq & \sum_{i=1}^N \Big(G_1(\mathrm{sp}(\Gamma) \cap (3Q + z_0), m) - G_1(\mathrm{sp}(\Gamma) \cap (3Q + z_0), u^\pm) \Big) \\ & + \ G_1\left(\mathrm{sp}(\Gamma) \setminus \left(\cup_{i=1}^N (z_i + 3Q)\right), m\right) - G_1\left(\mathrm{sp}(\Gamma) \setminus \left(\cup_{i=1}^N (z_i + 3Q)\right), u^\pm\right) \\ \geq & \ \sigma_3(\zeta)N - \theta C(\|g\|_\infty|, W)3^d|\mathrm{sp}(\Gamma)|. \end{aligned}$$

For the last estimate we used (4.21), the bound $|m| < 1 + C\theta$, and the estimates on $u^\pm = \pm 1 = v^*$ and on its gradient collected in Appendix B, Proposition B.1.

The claim of Theorem 4.7 follows by choosing θ sufficiently small, provided that we can show that there exists a constant $C(d)$ depending only on the dimension such that for any contour Γ there exists a collection of lattice sites $\{z_1,\dots,z_{N(\Gamma)}\}$ satisfying (4.22) such that $N(\Gamma)\ge C(d)^{-1}|\mathrm{sp}(\Gamma)|$. We claim that $C(d)^{-1}\ge 6^{-d}$, which is sufficient but not optimal.

The claim is shown by induction on $|\mathrm{sp}(\Gamma)|\in\mathbb{N}$. For the induction proof we will not assume that $\mathrm{sp}(\Gamma)$ is connected; the claim holds for any $\mathcal{D}^{(0)}$-measurable set. The claim is obvious with $C(d)=5^{-d}$ for $0<|\mathrm{sp}(\Gamma)|\le 5^d$.

Assume that the claim is shown for $0<|\mathrm{sp}(\Gamma)|\le n$, $n\ge 5^d$, and suppose that $|\mathrm{sp}(\Gamma)|=n+1$. Choose a cube z_0+Q in Γ and consider the nonempty set $\widehat{\Gamma}:=\mathrm{sp}(\Gamma)\setminus(5Q+z_0)$. Clearly any cube of sidelength 3 centered at any cube in $\widehat{\Gamma}$ does not intersect z_0+3Q. Therefore

$$N(\Gamma)\ge 1+N(\widehat{\Gamma})\ge 1+C^{-1}(n-5^d)=C^{-1}(n+1)+1-C^{-1}(1+5^d)\ge(n+1)C^{-1},$$

provided $1+5^d<C$. $\square$

5. Contour reduction and proof of Theorem 2.11. Take $\zeta\le\zeta_0\wedge\frac{1}{4}$, where ζ_0 is chosen according to Theorem 2.7. Let $\mathcal{G}(m,\zeta)$ be the collection of contours associated to m. First we show that the $\mathrm{sign}(m):=\eta^\zeta(m,\cdot)|_{I_{\tilde{\Gamma}}}$, defined in (2.17), is well defined.

LEMMA 5.1. *The function $\eta(m,\cdot)$ is constant on*

$$I_{\tilde{\Gamma}}:=\Lambda_\epsilon\setminus\cup_{\Gamma\in\mathcal{G}(m)}\left(\mathrm{sp}(\Gamma)\cup I_\Gamma\right).$$

Proof. By construction, $I_{\tilde{\Gamma}}\cap\mathrm{int}(\Gamma)=\emptyset$ for all $\Gamma\in\mathcal{G}(m)$; hence each cube in $I_{\tilde{\Gamma}}$ is connected to the boundary of Λ_ϵ. The function $\eta(m,\cdot)$ is constant over each connected component of $I_{\tilde{\Gamma}}$. Assume that there exist two connected components with different signs. As they are connected to the boundary of Λ_ϵ, there exist two cubes $Q^+\in I_{\tilde{\Gamma}}$ and $Q^-\in I_{\tilde{\Gamma}}$ of different sign, which touch the boundary. Hence there must be a contour $\Gamma_0\in\mathcal{G}(m)$ intersecting the boundary such that Q^+ and Q^- are in *different* connected components of O_{Γ_0}, where O_{Γ_0} is the outer complement of the contour Γ_0; see (2.16). According to our definition, either Q^+ or Q^- must be contained in I_{Γ_0}, which contradicts that both are contained in $I_{\tilde{\Gamma}}$. $\square$

Next we estimate the difference between the energy of $m\in\mathcal{R}_\zeta(\Lambda_\epsilon)$ (see Definition 2.9) and the energy of $u^\pm$ in each ζ-island of m.

LEMMA 5.2. *Let $u^\pm=\pm1+v^*$, where $v^*(x/\epsilon)$ solves* (2.20). *Let $m=\mathrm{sign}(I)+\hat{v}$ for $x\in\hat{I}$, $I\subset\subset\hat{I}$ (see Theorem* 2.7*), and let θ, ζ be as in Theorem* 2.7. *Then there exists $c=c(d,W,\|g\|_\infty)$ such that*

$$G_1(I,m,\omega)-G_1(I,u^{\mathrm{sign}(I)},\omega)\ge -c\sqrt{\theta}|\partial^{\mathrm{ext}}I|. \tag{5.1}$$

Remark 5.3. Note that for those islands that touch $\partial\Lambda_\epsilon$, in particular for $I_{\tilde{\Gamma}}$, the external boundary $\partial^{\mathrm{ext}}I$ consists of cubes contained in the support of a contour and is therefore very different from the topological boundary.

Proof of Lemma 5.2. For the sake of simplifying the presentation we prove the case $I\ne I_{\tilde{\Gamma}}$. The case $I=I_{\tilde{\Gamma}}$ is proven similarly, replacing ∂I with $\partial^{\mathrm{ext}}I$. To take advantage of the boundary influence decay, which is due to the properties of the Euler–Lagrange equation in a single "well," we separate a strip near the boundary from the rest of the island. For this purpose, let

$$I_\mu:=\{x\in I:\ \mathrm{dist}(x,\partial I)\le\mu\},$$

and choose $\mu = \sqrt{2C_0}\log\left(\theta^{-1}\right)$. We split

$$\begin{aligned} G_1(I,m,\omega) - G_1\left(I,u^{\mathrm{sign}I},\omega\right) &= \left[G_1(I_\mu,m,\omega) - G_1\left(I_\mu,u^{\mathrm{sign}I},\omega\right)\right] \\ &\quad + \left[G_1(I\setminus I_\mu,m,\omega) - G_1\left(I\setminus I_\mu,u^{\mathrm{sign}I},\omega\right)\right]. \end{aligned} \tag{5.2}$$

By the Lipschitz estimate in Proposition 3.2 and since $|v^*| \le C_0\theta$ uniformly on ϵ (see (B.3)), we obtain that

$$G_1\left(I_\mu,u^{\mathrm{sign}I},\omega\right) \le c\theta|I_\mu| \le c\theta\log\left(\theta^{-1}\right)(|\partial I|),$$

where we estimated $|I_\mu| \le c\mu|\partial I|$. Here and in what follows we adopt the constant convention and denote by c any constant depending only on $d, W, \|g\|_\infty$, even if these constants change from one inequality to another. Moreover,

$$G_1(I_\mu,m,\omega) \ge \int_{I_\mu}\theta g m \ge -2\|g\|_{L^\infty}\theta|I_\mu| \ge -2c\|g\|_{L^\infty}\sqrt{2C_0}\theta\log\left(\theta^{-1}\right)(|\partial I|),$$

hence

$$\left[G_1(I_\mu,m,\omega) - G_1\left(I_\mu,u^{\mathrm{sign}I},\omega\right)\right] \ge -c\theta\log\left(\theta^{-1}\right)(|\partial I|).$$

The remaining term in (5.2) is estimated by applying the estimate (B.11), which in mesoscopic coordinates becomes

$$|m(x) - u^{\mathrm{sign}I}(x)| \le C(d)e^{-\frac{1}{\sqrt{2C_0}}\mathrm{dist}(x,\partial I)}\|m - u^{\mathrm{sign}I}\|_{L^\infty(\partial I)} \le C(d)\theta \tag{5.3}$$

for all $x \in I\setminus I_\mu$. Denote by χ_θ a $C^\infty(\Lambda_\epsilon,[0,1])$ cutoff function so that

$$\chi_\theta(x) = \begin{cases} 1 & \text{when } x \in I\setminus(I_{\mu+\sqrt{\theta}}), \\ 0 & \text{when } x \in I_\mu, \end{cases}$$

and $\|\operatorname{grad}\chi_\theta\|_{L^\infty} \le C(d)\theta^{-1/2}$. Suppose that $\mathrm{sign}(I) = +1$. Let

$$h_\theta := \chi_\theta m + (1-\chi_\theta)u^+.$$

Then $h_\theta|_{\partial(I\setminus I_\mu)} = u^+$; hence, recalling that u^+ is a minimizer in its well,

$$G_1(I\setminus I_\mu,h_\theta,\omega) - G_1(I\setminus I_\mu,u^+,\omega) \ge 0. \tag{5.4}$$

Moreover, by Theorem 2.7 and Proposition 3.2, there exists $c \equiv c(d,W,\|g\|_\infty)$ so that $|\operatorname{grad}u^+| + |\operatorname{grad}m| \le c$. Hence, recalling (5.3), $|\operatorname{grad}h_\theta - \operatorname{grad}m| \le |\operatorname{grad}\chi_\theta||m - u^+| + |\operatorname{grad}m| + |\operatorname{grad}u^+| \le \sqrt{\theta} + c$, and therefore $|\operatorname{grad}h_\theta| < c$. By using first (5.4) and then $h_\theta = m$ on $I\setminus I_{\mu+\sqrt{\theta}}$ and the gradient bounds above, we get (for all $\omega \in \Omega$)

$$\begin{aligned} G_1(I\setminus I_\mu,m) - G_1\left(I\setminus I_\mu,u^+\right) &\ge G_1(I\setminus I_\mu,m) - G_1(I\setminus I_\mu,h_\theta) \ge -c\left|I_{\mu+\sqrt{\theta}}\setminus I_\mu\right| \\ &\ge -c\sqrt{\theta}\left|\partial I\right|. \qquad \square \end{aligned}$$

Proof of Theorem 2.11. As the proof holds for all realizations of the random field provided $\|g(\cdot,\omega)\|_\infty \le 1$, we will suppress the explicit dependence on ω. Thanks to Theorem 2.7 it is enough to show the theorem for a ζ-representative of $m \in H^1(\Lambda_\epsilon)$, $\zeta \le \zeta_0$, with ζ_0 as in Theorem 2.7. To simplify the presentation we take $\zeta = \zeta_0$.

Further, to shorten notation, we denote the representative always by m and we assume $\alpha(\epsilon) = 1$. We have

$$\begin{aligned} &G_1(m) - G_1(u^{\mathrm{sign}(m)}) = G_1(I_{\tilde\Gamma}, m) - G_1(I_{\tilde\Gamma}, u^{\mathrm{sign}(m)}) \\ &+ \sum_{\Gamma\in\mathcal{G}(m)} \left\{ \left[G_1(I_\Gamma, m) - G_1(I_\Gamma, u^{\mathrm{sign}(m)})\right] + \left[G_1(\mathrm{sp}(\Gamma), m) - G_1(\mathrm{sp}(\Gamma), u^{\mathrm{sign}(m)})\right]\right\}. \end{aligned} \tag{5.5}$$

From now on, we assume without loss of generality that the sign(m) = sign$(I_{\tilde\Gamma})$ (see (2.17)) is positive. We estimate each addend in the sum.

$$\begin{aligned} G_1(I_\Gamma, m) - G_1(I_\Gamma, u^+) &= \left[G_1(I_\Gamma, m) - G_1\left(I_\Gamma, u^{\mathrm{sign}(I_\Gamma)}\right)\right] \\ &\quad + \left[G_1(I_\Gamma, u^{\mathrm{sign}(I_\Gamma)}) - G_1(I_\Gamma, u^+)\right] \\ &\geq \left[G_1(I_\Gamma, u^{\mathrm{sign}(I_\Gamma)}) - G_1(I_\Gamma, u^+)\right] - c\sqrt{\theta}|\partial^{\mathrm{ext}} I_\Gamma| \\ &= 2\theta[\mathrm{sign}(I_{\tilde\Gamma}) - \mathrm{sign}(I_\Gamma)] \int_{I_\Gamma} g_1(x)\mathrm{d}x - c\sqrt{\theta}|\partial^{\mathrm{ext}} I_\Gamma|. \end{aligned} \tag{5.6}$$

The last inequality is a consequence of Lemma 5.2; the last equality follows from the hypothesis (2.4) and $|u^\pm - \pm 1| \leq \delta_0$. Note that the contributions of the random field on islands having the same sign as m cancel. The last term in (5.5) is estimated as

$$G_1(I_{\tilde\Gamma}, m) - G_1(I_{\tilde\Gamma}, u^+) \geq -c\sqrt{\theta}|\partial^{\mathrm{ext}} I_{\tilde\Gamma}|.$$

To estimate from below the energy of a contour we apply Theorem 4.7. Let $\theta_1 := \tilde\theta(\zeta_0)$ be as in Theorem 4.7; then, for all $\theta \leq \theta_1$, we have

$$G_1(\mathrm{sp}(\Gamma), m) - G_1(\mathrm{sp}(\Gamma), u^+) \geq \sigma_1 N_\Gamma, \tag{5.7}$$

where

$$N_\Gamma = |\mathrm{sp}(\Gamma)| = \text{the number of } \mathcal{D}^{(0)}\text{-measurable cubes in } \mathrm{sp}(\Gamma)$$

and $\sigma_1 = \sigma_1(\zeta_0)$ is the quantity defined in Theorem 4.7. The right-hand side of (5.7) is the "gain term" and the energy of a contour Γ is at least the gain term. If there are more contours in Λ_ϵ, each one will contribute proportionally to the volume of its support. Therefore from (5.5) we obtain

$$\begin{aligned} G_1(m) - G_1(u^+) &\geq \sum_{\Gamma\in\Gamma(m)} \left(2\theta \int_{I_\Gamma^-} g_1(x)\mathrm{dx} + \sigma_1 N_\Gamma - c\sqrt{\theta}|\partial^{\mathrm{ext}} I_\Gamma|\right) - c\sqrt{\theta}|\partial^{\mathrm{ext}} I_{\tilde\Gamma}| \\ &\geq \sum_{\Gamma\in\Gamma(m)} \left(2\theta \int_{I_\Gamma^-} g_1(x)\mathrm{dx} + \frac{\sigma_1}{2} N_\Gamma\right). \end{aligned} \tag{5.8}$$

To prove the last inequality we use that $N_\Gamma \geq |\partial^{\mathrm{ext}} I_\Gamma|$ and choose θ small enough. ∎

5.1. Proofs of Theorems 2.1 and 2.2. To prove Theorem 2.1 we need the properties of the "single-well minimizers" stated in Appendix B, and the probabilistic estimates stated in Appendix A. An immediate consequence of Proposition A.1 (stated in Appendix A) is the following lemma, which describes the size of the typical fluctuations of the random field in a domain R.

LEMMA 5.4. *Let $d \geq 3$ and $R \subset \Lambda_\epsilon$ denote a connected, $\mathcal{D}^{(0)}$-measurable region. Set for $\delta > 0$*

$$\Omega_{\epsilon,\delta} := \left\{ \omega \in \Omega : \ \forall R \subset \Lambda_\epsilon : \left| \int_R dy g_1(y,\omega) \right| < \frac{\delta}{\alpha(\epsilon)\theta} |\partial R| \right\}.$$

There exist $\epsilon_0 > 0$ and $a := a(\alpha(\epsilon_0)\theta, d)$ so that for $\epsilon \leq \epsilon_0$

$$\mathbb{P}[\Omega \setminus \Omega_{\epsilon,\delta}] \leq 2 \frac{|\Lambda|}{\epsilon^d} e^{-\frac{\delta^2 a}{\theta^2 \alpha^2(\epsilon)}}. \tag{5.9}$$

Further, we set [5]

$$\delta(\epsilon) = \theta(\ln(1/\epsilon))^{-\frac{1}{100}} \qquad \textit{and} \qquad \Omega_\epsilon := \Omega_{\epsilon,\delta(\epsilon)} \tag{5.10}$$

so we can find ϵ_0 such that

$$\mathbb{P}[\Omega \setminus \Omega_\epsilon] \leq e^{-a \ln \frac{1}{\epsilon} \left(\ln\left(\frac{1}{\epsilon}\right)\right)^{\frac{49}{50}}} \qquad \textit{for } 0 < \epsilon < \epsilon_0. \tag{5.11}$$

Proof. In the following we consider only regions R which are connected and $\mathcal{D}^{(0)}$-measurable, i.e., unions of unit cubes. We have

$$\begin{aligned} &\mathbb{P}\left[\exists R \subset \Lambda_\epsilon, \left| \int_R dy g_1(y,\omega) \right| \geq \frac{\delta}{\alpha(\epsilon)\theta} |\partial R| \right] \\ &= \mathbb{P}\left[\exists x_0 \in \Lambda_\epsilon, \exists R \subset \Lambda_\epsilon : x_0 \in R, \left| \int_R dy g_1(y,\omega) \right| \geq \frac{\delta}{\alpha(\epsilon)\theta} |\partial R| \right] \\ &\leq \frac{|\Lambda|}{\epsilon^d} \mathbb{P}\left[\exists R \subset \mathbb{R}^d : 0 \in R, \left| \int_R dy g_1(y,\omega) \right| \geq \frac{\delta}{\alpha(\epsilon)\theta} |\partial R| \right]. \end{aligned} \tag{5.12}$$

A naive upper bound of (5.12) (ignoring for the time being the factor $\frac{|\Lambda|}{\epsilon^d}$) is given by

$$\sum_{\{R: 0 \in R\}} \mathbb{P}\left[\left| \int_R dy g_1(y,\omega) \right| \geq \frac{\delta}{\theta\alpha(\epsilon)} |\partial R| \right] \leq \sum_{\{R: 0 \in R\}} e^{-\frac{\delta^2}{\theta^2 \alpha^2(\epsilon)} \frac{1}{2d} |\partial R|^{\frac{(d-2)}{(d-1)}}}. \tag{5.13}$$

The last inequality is obtained by the independence of the random field and then by applying the isoperimetric inequality[6] $|R| \leq 2d |\partial R|^{\frac{d}{d-1}}$; then,

$$\frac{|\partial R|^2}{|R|} \geq \frac{1}{2d} \frac{|\partial R|^2}{|\partial R|^{\frac{d}{d-1}}} = \frac{1}{2d} |\partial R|^{\frac{(d-2)}{(d-1)}}.$$

[5] Note that $\alpha(\epsilon) = (\log \epsilon^{-1})^{-1}$ is essential to control the term $\frac{|\Lambda|}{\epsilon^d}$.

[6] Note that a relative isoperimetric inequality bounds the ratio $|R|^{(d-1)/d} ||S|^{-1} \leq C(d)$ in the case where $R = I_\Gamma$ and $S = \partial^{ext}(\Gamma)$, and the island I_Γ associated with a contour is given by our definition. A proof of the relative isoperimetric inequality can be given adapting the arguments in [21, p. 230].

On the other hand (see [13]), for a fixed natural number n there are $e^{C(d)n}$ $\mathcal{D}^{(0)}$-measurable regions R which contain the origin and have surface area n. One immediately verifies that (5.13) diverges. So this analysis is inadequate.

We need to take advantage of the fact that many regions enclose essentially the same volume. In order to obtain (5.9), we apply a method we learned from [11]; see also [3, p. 115 and the following pages], reported in Proposition A.1 of Appendix A.[7]

Now choose δ as a function of ϵ, so that $\delta(\epsilon) \to 0$ sufficiently slow, e.g., as in (5.10). It is immediate to verify that there exist an ϵ_0 and a constant $a(\alpha(\epsilon_0)\theta, d)$ so that for $\epsilon \leq \epsilon_0$ the right-hand side of (5.9) is smaller than the right-hand side of (5.11). $\square$

Proof of Theorem 2.1. Applying Lemma 3.1 we get immediately that the global minimizer u_ϵ fulfills $|u_\epsilon(r,\omega)| \leq 1 + C_0\alpha(\epsilon)\theta$ for $r \in \Lambda$ and $\omega \in \Omega$. Set $u_\epsilon^+ = 1 + v_\epsilon^*$ and $u_\epsilon^- = -1 + v_\epsilon^*$, where v_ϵ^* solves (2.20) in Λ. Choose $\epsilon_0 > 0$ so that $C_0\theta\alpha(\epsilon_0) \leq \delta_0$; then by the symmetry assumption (2.4) on W we obtain for all $\epsilon < \epsilon_0$ and all $\omega \in \Omega$

$$G_\epsilon(u_\epsilon^+,\omega) - G_\epsilon(u_\epsilon^-,\omega) = \frac{2}{\epsilon}\alpha(\epsilon)\theta\int_\Lambda g_\epsilon(r,\omega)\mathrm{dr}. \tag{5.15}$$

By the Markov exponential inequality [20], one has for any $t > 0$

$$\mathbb{P}\left[\omega:\ \frac{2}{\epsilon}\alpha(\epsilon)\theta\left|\int_\Lambda g_\epsilon(r,\omega)\mathrm{dr}\right| \geq t\right] \leq 2e^{-\frac{t^2}{4\epsilon^{d-2}\theta^2\alpha^2(\epsilon)}}. \tag{5.16}$$

In dimension $d \geq 3$, for any choice $\alpha(\epsilon)$, ($\alpha(\epsilon) = 1$ suffices), we can choose $t = \delta_\epsilon$ (see (5.10)), $\lim_{\epsilon\to 0}\delta_\epsilon = 0$, sufficiently slow, so that $\frac{\delta_\epsilon^2}{4\epsilon^{d-2}\theta^2\alpha^2(\epsilon)} \to \infty$ and conclude that there exists $\Omega(\delta_\epsilon) \subset \Omega$,

$$\mathbb{P}[\Omega(\delta_\epsilon)] \geq 1 - 2e^{-\frac{\delta_\epsilon^2}{4\epsilon^{d-2}\theta^2\alpha^2(\epsilon)}}, \tag{5.17}$$

so that for $\omega \in \Omega(\delta_\epsilon)$, $|G_\epsilon(u_\epsilon^+,\omega) - G_\epsilon(u_\epsilon^-,\omega)| \leq \delta_\epsilon$. To show

$$\inf_{H^1(\Lambda)} G_\epsilon(\cdot,\omega) = \min\{G_\epsilon(u_\epsilon^+,\omega), G_\epsilon(u_\epsilon^-,\omega)\}, \qquad \omega \in \Omega_\epsilon, \tag{5.18}$$

we first prove that any $\tilde{u}$ such that

$$G_\epsilon(\tilde{u},\omega) = \inf_{H^1(\Lambda)} G_\epsilon(\cdot,\omega)$$

does not change sign, so it is in one well of the potential W. The assumption on W (see H1) and the L^∞-bound on g imply that if ϵ is small enough,

$$\inf_{u\in H^1(\Lambda):\ u>0 \text{ a.e.}} G_\epsilon(\cdot,\omega) = \inf_{u\in H^1(\Lambda):\ u>1-\delta_0 \text{ a.e.}} G_\epsilon(\cdot,\omega).$$

[7] In $d = 2$ we have

$$\mathbb{P}\left[\left|\int_R dy g_\epsilon(y,\omega)\right| \geq \epsilon\frac{\delta}{\theta\alpha(\epsilon)}|\partial R|\right] \leq 2e^{-\frac{\delta^2}{\theta^2\alpha(\epsilon)^2}}. \tag{5.14}$$

Therefore in $d = 2$, when $\alpha(\epsilon) = 1$, the upper bound in (5.14) depends only on θ. By the Borel–Cantelli lemma one sees immediately that with probability one, the event $\left|\int_R dy g_\epsilon(y,\omega)\right| \geq \frac{\delta}{\theta}\epsilon|\partial R|$ for any $\delta > 0$ occurs for a number of regions in Λ going to ∞ as $\epsilon \downarrow 0$. In $d = 2$, when $\alpha(\epsilon) = (\ln\frac{1}{\epsilon})^{-1}$ for a fixed region, the upper bound in (5.14) is small for ϵ small. Nevertheless, even in this case (see Proposition A.1), the entropic factor spoils the estimate and we are not able to show the absence of contours.

The functional G_ϵ is convex on $\{u \in H^1(\Lambda) : \ u > 1 - \delta_0 \text{ a.e.}\}$, hence it has a unique minimizer over that set. It follows easily that the constraint is not active for ϵ sufficiently small, so the minimizer solves the linear Euler–Lagrange equation. Thanks to the symmetry assumptions on W (see (2.4)), it is enough to solve the Euler–Lagrange equation in one well. In this way one obtains immediately that the two minimizers under the constraints $u > 0$ and $u < 0$, respectively, are indeed $u^*_\epsilon = \pm 1 + v^*_\epsilon$, with v^*_ϵ being the solution of (2.20).

To prove (5.18) we apply Theorem 2.11, i.e., we use the notion of contours and Theorem 2.7. It is convenient to reformulate the problem in mesoscopic coordinates and therefore study the functional (2.12) in Λ_ϵ. The idea of the proof is to show that each contour costs more than the possible gain obtained from the random field; hence a minimizer cannot have contours. Note that I_Γ need not be connected. Denote by $(I_\Gamma)_1, \dots, (I_\Gamma)_{K_\Gamma}$ its connected components, and denote by $\partial^{ext}(I_\Gamma)_j$ the exterior boundary of $(I_\Gamma)_j$; see also section 2.3.2. The connected components need not be simply connected, because there may be contours within contours. Another consequence of the presence of contours within contours is the fact that in general $\partial^{ext}(I_\Gamma)_j \not\subset \mathrm{sp}(\Gamma)$. This is a problem, because the isoperimetric inequality used in the proof of Lemma 5.4 requires us to compare a region R and its *entire* boundary, i.e., all cubes that are connected both to a cube in R and to a cube in its complement.

But we consider only $\mathcal{D}^{(0)}$ measurable sets. This implies that there exists a constant $c(d)$ such that each cube in the support of a contour is in the exterior boundary of at most $c(d)$ connected components of some islands. Therefore we can split the cost of a cube in a contour between all islands that are connected to it; i.e., for another constant $c(d)$ depending only on the dimension we get

$$N_\Gamma \geq c(d)^{-1} \sum_{\widetilde{\Gamma}} \sum_j \left| \partial^{ext}(I_{\widetilde{\Gamma}})_j \cap \mathrm{sp}(\Gamma) \right| + \frac{1}{2} N_\Gamma,$$

and we obtain that there exists a $\delta(d, \zeta)$ such that

(5.19)
$$G_1(m,\omega) - G_1(u^+,\omega) \geq \sum_{\Gamma \in \Gamma(m)} \left(\frac{\sigma_1}{2} N_\Gamma + \sum_{j=1}^{K_\Gamma} \left[2\alpha(\epsilon)\theta \int_{(I_\Gamma^-)_j} g_1(x,\omega)\mathrm{dx} + \delta \left| \partial^{\mathrm{ext}}(\mathrm{I}_\Gamma^-)_\mathrm{j} \right| \right] \right).$$

Let $\Omega_{\epsilon,\delta}$ be as in Lemma 5.4 with some $0 < \delta < 1$ to be determined later. If m is a function which has at most one block different from $\eta = 1$, then, by Theorem 4.7, the right-hand side of (5.19) is nonnegative for $\omega \in \Omega_{\epsilon,\delta}$. So for these ω the minimizer $\tilde{u}$ must have all cubes ζ-close to the sign$(\tilde{u})$ phase, i.e., $\eta^\zeta(\tilde{u}, x) = \mathrm{sign}(\tilde{u})$ for all $x \in \Lambda_\epsilon$, i.e., all blocks are correct. We strengthen the result taking $\delta = \delta(\epsilon) \downarrow 0$ for $\epsilon \downarrow 0$ as in (5.10). We can apply Proposition 3.3 to show that $|\tilde{u}(x,\omega)| > 0$ for $x \in \Lambda_\epsilon$ and $\omega \in \Omega_{\epsilon,\delta}$. From Appendix B (the minimizer in one single well) we have that the minimizer $\tilde{u}$ equals either u^+ or u^-; see Definition 2.10 in section 2. The statement (2.5) is now an immediate consequence of the symmetry of W. Obviously

$$\mathbb{E}[u^\pm_\epsilon(r,\cdot)] = 1 \qquad \forall r \in \Lambda.$$

Moreover (see (B.7)),

$$\begin{aligned} &\left| \mathbb{E}[u^\pm_\epsilon(r,\cdot) u^\pm_\epsilon(r',\cdot)] - \mathbb{E}[u^\pm_\epsilon(r,\cdot)]\mathbb{E}[u^\pm_\epsilon(r',\cdot)] \right| = \left| \mathbb{E}[v^*_\epsilon(r,\cdot) v^*_\epsilon(r',\cdot)] \right| \\ &\leq C(d)\theta^2\alpha^2(\epsilon) e^{-\frac{1}{2\epsilon\sqrt{2C_0}}|r-r'|}. \end{aligned} \tag{5.20}$$

Denote $\Omega_\epsilon := \Omega_{\epsilon,\delta(\epsilon)} \cap \Omega(\delta_\epsilon)$, where $\Omega_{\epsilon,\delta(\epsilon)}$ is the probability space defined in (5.10) and $\Omega(\delta_\epsilon)$ is the probability space defined in (5.17). We have

$$\mathbb{P}[\Omega_\epsilon] \geq 1 - 2\max\left\{2e^{-\frac{\delta_\epsilon^2}{4\epsilon^{d-2}\theta^2\alpha^2(\epsilon)}}, e^{-a\ln\frac{1}{\epsilon}\left(\ln(\frac{1}{\epsilon})\right)^{\frac{49}{50}}}\right\}.$$

Taking $\delta_\epsilon \geq \delta(\epsilon)$, we have the thesis. $\square$

In the proof of Theorem 2.1 we actually quantified the difference between the energy of a function and the energy of the minimizer. We state this for further use.

THEOREM 5.5. *There exist $\delta > 0$, $\epsilon_0 > 0$, $a := a(\epsilon_0\theta, d) > 0$ and there exists for each $\epsilon < \epsilon_0$ a set $\Omega_\epsilon \subseteq \Omega$ with $\mathbb{P}(\Omega_\epsilon) \geq 1 - e^{-a\ln\frac{1}{\epsilon}\left(\ln(\frac{1}{\epsilon})\right)^{\frac{49}{50}}}$ such that for $\omega \in \Omega_\epsilon$*

$$G_1(m,\omega) - \min\left\{G_1(u^+,\omega), G_1(u^-,\omega)\right\} \geq \delta \sum_{\Gamma\in\mathcal{G}(m)} |\mathrm{sp}(\Gamma)|. \tag{5.21}$$

Moreover, we immediately get the following corollary; for notation see (2.8).

COROLLARY 5.6. *Under the same hypothesis of Theorem 5.5, for $\omega \in \Omega_\epsilon$, we have*

$$F_\epsilon(m,\omega) \geq \epsilon^{d-1}\delta \sum_{\Gamma\in\mathcal{G}(m)} |\mathrm{sp}(\Gamma)|.$$

Proof of Theorem 2.2. From (2.5) of Theorem 2.1, the symmetry of the wells, and the fact that v_ϵ^* is solution of (2.20), one immediately obtains that

$$\inf_{H^1(\Lambda)} G_\epsilon(\cdot,\omega) = \min\left\{G_\epsilon(u^+,\omega), G_\epsilon(u^-,\omega)\right\} = \min\left\{\pm\frac{\alpha(\epsilon)\theta}{\epsilon}\int_\Lambda g_\epsilon(r,\omega)\mathrm{d}r\right\} + \mathcal{F}_\epsilon(v_\epsilon^*,\omega),$$

where $\mathcal{F}_\epsilon$ is the functional defined in (B.1). Then

$$\mathbb{E}[G_\epsilon(u_\epsilon^\pm,\cdot)] = \mathbb{E}[\mathcal{F}_\epsilon(v_\epsilon^*,\cdot)]$$

and (2.10) follows immediately. From (B.4) we have that

$$\mathcal{F}_\epsilon(v_\epsilon^*,\omega) = \frac{1}{2\epsilon}\alpha(\epsilon)\theta\int_\Lambda g_\epsilon(r,\omega)v_\epsilon^*(r,\omega)\mathrm{d}r = \frac{\alpha^2(\epsilon)}{4\epsilon^3}\theta^2\int_{\Lambda\times\Lambda} g_\epsilon(r,\omega)G_\epsilon(r,z)g_\epsilon(z,\omega)\mathrm{d}z\mathrm{d}r,$$

where $G_\epsilon(r,z)$ is the Green function solution of (B.5). Then, using the construction of g_ϵ with the help of independent and identically distributed random variables (see (2.1) and (2.2)) and the bounds on the Green function in Appendix B (see (B.8)), we have that there exists $C(d) > 0$ such that in $d \geq 3$

$$|\mathbb{E}[G_\epsilon(u_\epsilon^\pm,\cdot)]| \leq \frac{\alpha(\epsilon)^2}{4\epsilon}\theta^2 C(d)|\Lambda|, \qquad \mathbb{E}[G_\epsilon(u_\epsilon^\pm,\cdot) - c_\epsilon]^2 \leq C(d)\alpha^2(\epsilon)\theta^2\epsilon^{d-2}|\Lambda|.$$

Moreover, using the exponential decay of the Green function we obtain that for any $\delta > 0$ there exists $\epsilon(\delta) > 0$ such that $G_\epsilon(x,y) > C(d)^{-1}$ for $|x-y| < \epsilon$, $\mathrm{dist}(x,\partial\Lambda) > \delta$, $\mathrm{dist}(y,\partial\Lambda) > \delta$, and all $\epsilon < \epsilon(\delta)$. Therefore we also obtain

$$\liminf_{\epsilon\downarrow 0} \frac{4\epsilon}{\alpha(\epsilon)^2}|\mathbb{E}[G_\epsilon(u_\epsilon^\pm,\cdot)]| > 0. \qquad \square$$

6. Γ-convergence when $\alpha(\epsilon) = [\ln \frac{1}{\epsilon}]^{-1}$. We first show that passing to the representative for each function in a sequence of functions with bounded renormalized energy leaves the L^1-limit of that sequence unchanged. Although the representative depends on the realization of the random field, we will suppress this dependence in the notation when no confusion arises. Likewise we will not denote explicitly the dependence on ω of the energy.

DEFINITION 6.1. *For $m \in H^1(\Lambda)$ define $\widehat{m} : \Lambda_\epsilon \to \mathbb{R}$ by $\widehat{m}(y) := m(\epsilon y)$. Let m_1 be any ζ-representative of $\widehat{m}$ as in Theorem 2.7. Then*

$$m_{1,\epsilon}(x,\omega) := m_1(\epsilon^{-1}x,\omega), \qquad x \in \Lambda.$$

THEOREM 6.2. *Let θ_1 and ζ be as in Theorem 2.11, and let $\theta < \theta_1$. With $\mathbb{P} = 1$ the following holds: Let $(m_\epsilon)_{\epsilon\to 0} \in H^1(\Lambda)$, and let the associated representatives $(m_\epsilon)_{1,\epsilon}$ be as in Definition 6.1. Then, if*

$$\limsup_{\epsilon\to 0} F_\epsilon(m_\epsilon,\omega) < C < \infty, \quad \textit{then} \quad \int_\Lambda |m_\epsilon(x) - (m_\epsilon)_{1,\epsilon}(x,\omega)| \to 0.$$

Proof. Because of the quadratic growth of the potential and the L^∞-bound on the random field g it is easy to show that there exists a sequence $C_\epsilon \to 0$ such that for $M_\epsilon = 1 + C_\epsilon$

$$F_\epsilon((m_\epsilon \vee (-M_\epsilon)) \wedge M_\epsilon) \le F_\epsilon(m_\epsilon), \qquad \int_\Lambda |(m_\epsilon \vee (-M_\epsilon)) \wedge M_\epsilon - m_\epsilon| \mathrm{d}r \to 0.$$

Therefore we can assume that $\|m_\epsilon\|_\infty \le M$ by any constant $M > 1$ provided $\epsilon < \epsilon_0(M)$. To simplify notation we work on the rescaled cube Λ_ϵ and let (see Definition 6.1 and Theorem 2.7)

$$m(x) := m_\epsilon(\epsilon x), \quad m_1(x) := (m_\epsilon)_{1,\epsilon}(\epsilon x), \quad x \in \Lambda_\epsilon.$$

Take a smooth cutoff function $r : \Lambda_\epsilon \to [0,1]$ such that $\|\operatorname{grad} r\|_\infty < C$, $r(x) = 1$ for $x \in \bigcup_{\Gamma\in\mathcal{G}(m_1)} \mathrm{sp}(\Gamma)$, and $r(x) = 0$ for $x \in I_\Gamma \setminus \partial^{\mathrm{int}} I_\Gamma$, $\Gamma \in \mathcal{G}(m_1)$, and let

$$\tilde{m} := m(1 - r^2) + m_1 r^2.$$

This function is equal to m_1 on the contours of m_1. We immediately obtain

$$F_1(\tilde{m}) = F_1(m) + \sum_{\Gamma\in\mathcal{G}(m_1)} \left[G_1\left(\mathrm{sp}(\Gamma) \cup \partial^{\mathrm{int}} I_\Gamma, \tilde{m}\right) - G_1\left(\mathrm{sp}(\Gamma) \cup \partial^{\mathrm{int}} I_\Gamma, m\right)\right].$$

Since $r \le 1$ and m and m_1 are bounded in L^∞, we can estimate as follows:

$$\begin{aligned} &\sum_{\Gamma\in\mathcal{G}(m_1)} \left[G_1\left(\left(\mathrm{sp}(\Gamma) \cup \partial^{\mathrm{int}} I_\Gamma\right), \tilde{m}\right) - G_1\left(\left(\mathrm{sp}(\Gamma) \cup \partial^{\mathrm{int}} I_\Gamma\right), m\right)\right] \\ &\le \sum_{\Gamma\in\mathcal{G}(m_1)} \left\{c|\mathrm{sp}(\Gamma)| + \int_{\partial^{\mathrm{int}} I_\Gamma} \left[|\nabla\tilde{m}|^2 - |\nabla m|^2\right]\right\}. \end{aligned}$$

We have

$$\nabla\tilde{m} = (1 - r^2)\nabla m + r\left[2\nabla r(m_1 - m) + r\nabla m_1\right].$$

From the bound on $|\operatorname{grad} r|$ and the bound on the Lipschitz constant of m_1 we immediately get that there exists a constant C so that

$$|\nabla \tilde{m}|^2 \leq (1-r^2)^2 |\nabla m|^2 + C + r|\nabla m| C,$$

which implies

$$\left[|\nabla \tilde{m}|^2 - |\nabla m|^2\right] \leq C + r^2[r^2-1]|\nabla m|^2 + (r|\nabla m|)\left[C - (r|\nabla m|)\right] \leq C + \frac{C^2}{4},$$

as $r \leq 1$. Then we can conclude that for some constant C'

$$F_1(\tilde{m}) \leq F_1(m) + C' \sum_{\Gamma \in \mathcal{G}(m_1)} |\mathrm{sp}(\Gamma)|.$$

From Theorem 5.5 we obtain that $\sum_{\Gamma \in \mathcal{G}(m_1)} |\mathrm{sp}(\Gamma)| \leq \epsilon^{1-d} C$; hence there exists C_1 such that

$$F_1(\tilde{m}) \leq \epsilon^{1-d} C_1.$$

Therefore $\tilde{m}$ satisfies a bound on the energy of the same order as m. As m and $\tilde{m}$ are different only on $\sum_{\Gamma \in \mathcal{G}(m_1)} \left(\mathrm{sp}(\Gamma) \cup \partial^{\mathrm{int}} I_\Gamma\right)$, the L^∞-bound on both functions and the bound on the volume of the contours implies immediately that $\|\tilde{m} - m\|_1 \to 0$ as $\epsilon \to 0$.

The new function $\tilde{m}$ has an important property: On the topological boundary of an island it equals m_1 and is therefore pointwise in the well of W which corresponds to the sign of $\eta(m_1)$. This property will allow us to show that $\tilde{m}$ and m_1 are close in the islands. Note that $G_1(m_1) \geq \inf_{H^1(\Lambda_\epsilon)} G_1(\cdot)$, so we can estimate

$$\begin{aligned}
\epsilon^{1-d} C \geq G_1(\tilde{m}) - G_1(m_1) &= G_1(m_1 + (\tilde{m} - m_1)) - G_1(m_1) \\
&= \int_{\Lambda_\epsilon} [2 \operatorname{grad}(\tilde{m} - m_1) \operatorname{grad} m_1 + (W'(m_1) + \alpha(\epsilon)\theta g)(\tilde{m} - m_1)] \\
&\quad + \int_{\Lambda_\epsilon} \left(|\operatorname{grad}(\tilde{m} - m_1)|^2 + \frac{1}{2} \left(\int_0^1 W''(m_1 + s(\tilde{m} - m_1)) ds \right) (\tilde{m} - m_1)^2 \right).
\end{aligned}$$

By Corollary 3.8 (and its obvious extension to connected components of correct cubes) we get that the term in the second line equals zero since $\xi := \tilde{m} - m_1$ is an admissible test function. We have by definition of $\tilde{m}$ that $\tilde{m} - m_1 = 0$ whenever $\eta(m_1, x) = 0$, so the integration in the third line extends only over the set $\{x \in \Lambda_\epsilon : \ \eta(m_1, x) \neq 0\}$. Moreover, using the convexity of the wells (see (2.4)), we find that if m_1 and $\tilde{m}$ are both in $[1-\delta_0, \infty)$ or both in $(-\infty, -1+\delta_0]$, then $W''(m_1 + s(\tilde{m} - m_1)) \geq C$ for $s \in [0,1]$, so there exists $C'' > 0$ such that

$$\epsilon^{1-d} C \geq \int_{\Lambda_\epsilon} C(\tilde{m} - m_1)^2 - C''|\{x : \eta(m_1, x) \neq 0\} \cap \{\eta(m_1, x)\tilde{m}(x) < 1 - \delta_0\}|.$$

It remains to show that

$$|\{x : \eta(m_1, x) \neq 0\} \cap \{\eta(m_1, x)\tilde{m}(x) < 1 - \delta_0\}| \leq C\epsilon^{1-d}.$$

For $t = 1 - \delta_0$ and x in the islands of m_1, we denote

$$\tilde{m}^t := \begin{cases} |\tilde{m}(x)| \vee t & \text{if } \eta(m_1, x) = 1, \\ -(|\tilde{m}(x)| \vee t) & \text{if } \eta(m_1, x) = -1, \end{cases}$$

while $\tilde{m}^t(x) := \tilde{m}(x)$ for $x \in \mathrm{sp}(\Gamma)$, $\Gamma \in \mathcal{G}(m_1)$. Note that $\tilde{m} = m_1$ on the topological boundary of any contour, and that the representative m_1 stays pointwise in the well associated with $\eta(m_1)$ on this topological boundary of the contour; see Theorem 2.7. Therefore the function $\tilde{m}^t$ is H^1, and

$$G_1(\tilde{m}) - G_1(\tilde{m}^t) \le G_1(\tilde{m}) - \inf G_1(\cdot) < C\epsilon^{1-d}.$$

Since $\eta(m_1, x) = \eta(m, x)$ for x in the islands of m_1, by applying Lemma 4.2 and then (4.3) we obtain

$$G_1(\tilde{m}) - G_1(\tilde{m}^t) \ge C \sum_{\{z: z+Q \in I_\Gamma, \Gamma \in \mathcal{G}(m_1)\}} |(z+Q) \cap \{\eta(z,m)m < -(1-t)/2\}|^{\frac{d-1}{d}}.$$

Note that the L^∞-bound on m implies that for ζ sufficiently small $|\{m > 0\} \cap Q| > 1/2$ if $\eta(x,m) = 1$ on Q. As $(d-1)/d < 1$,

$$\left| \left(\bigcup\nolimits_{\Gamma \in \mathcal{G}(m_1)} I_\Gamma \right) \cap \{\eta(m,z)m < -(1-t)/2\} \right| \le C\epsilon^{1-d}.$$

We can easily bound $|\{-1+\delta_0 < \tilde{m} < 1-\delta_0\}|$, because on this set the double-well potential dominates the random field. So we finally obtain

$$\begin{aligned} &|\{x : \eta(m_1,x) \neq 0\} \cap \{\eta(m_1,x)\tilde{m}(x) < 1-\delta_0\}| \le |\{-1+\delta_0 < \tilde{m} < 1-\delta_0\}| \\ &+ \left| \left(\bigcup\nolimits_{\Gamma \in \mathcal{G}(m_1)} I_\Gamma \right) \cap \{\eta(m,z)m < -\delta_0/2\} \right| + \sum_{\Gamma \in \mathcal{G}(m_1)} |\mathrm{sp}(\Gamma)| \le C\epsilon^{1-d}, \end{aligned}$$

and the claim is shown. □

6.1. Identification of the Γ-limit. The proof of the lower and later of the upper bound is given in the macroscale, but still uses the notion of contours which was introduced in the mesoscale. To avoid confusion, we keep writing the contours in mesoscale and rescale $\mathrm{sp}(\Gamma)$ by ϵ when we deal with the support of the contour of the representative m_ϵ in the macroscale. Hence $m(x) := m_\epsilon(\epsilon x)$, $x \in \Lambda_\epsilon$, denotes the representative in the mesoscopic scale, and $\mathcal{G}(m) := \mathcal{G}(m, \zeta)$ the collection of contours associated to m when the chosen tolerance is ζ. We suppose $0 \le \theta < \theta_1$, with θ_1 as in Theorem 2.11, and we avoid explicitly writing the dependence on ζ, where ζ is as in Theorem 2.11.

LEMMA 6.3. *There exists a set $\widetilde{\Omega} \subseteq \Omega$ with $\mathbb{P}(\widetilde{\Omega}) = 1$ such that on $\widetilde{\Omega}$ the following holds: For any $u \in BV(\Lambda, \{-1,1\})$ and for any m_ϵ with $\|m_\epsilon - u\|_{L^1} \to 0$ we have that*

$$\liminf_{\epsilon} F_\epsilon(m_\epsilon, \omega) \ \ge C_W \int_\Lambda |\operatorname{grad} u| \qquad \text{for } C_W \text{ as in (1.3)}. \tag{6.1}$$

Proof. First fix a $\delta > 0$ independent of ω. Recall that $\epsilon = \epsilon(n) = \frac{1}{n}$ and let $\delta(\epsilon(n))$ and $\Omega_{\epsilon(n)}$ as in (5.10). We define $\widetilde{\Omega}$ by defining its complement:

$$A_n := \Omega \setminus \Omega_{\epsilon(n)}, \quad \Omega \setminus \widetilde{\Omega} = \{\omega : \ \omega \in A_n \text{ for infinitely many } n \in \mathbb{N}\}.$$

The first Borel–Cantelli lemma and the probabilistic estimates in Theorem 2.1 and in Lemma 5.4 imply that $\mathbb{P}(\Omega \setminus \widetilde{\Omega}) = 0$. By definition, for any $\omega \in \widetilde{\Omega}$, there exists $n(\omega)$ such that $\omega \in \Omega_{\epsilon(n)}$ for all $n \ge n(\omega)$. From now on we will always assume that $\omega \in \widetilde{\Omega}$

and $\epsilon(n) \le \epsilon(n(\omega))$ without stating the dependence on ω explicitly. Moreover, we will write ϵ for $\epsilon(n)$ in order to simplify notation. Note that it is sufficient to consider the case $\sup_\epsilon F_\epsilon(m_\epsilon,\omega) < \infty$.

By Theorem 6.2 we can replace m_ϵ by a representative (see Definition 6.1), which we still denote by m_ϵ for simplicity. Hence we may assume that $\|m_\epsilon\|_{L^\infty} \le 1+C_0\theta\alpha(\epsilon)$.

By Theorem 2.1 $\inf_{H^1(\Lambda)}\{G_\epsilon(\cdot,\omega)\} = \min\{G_\epsilon(u_\epsilon^+,\omega), G_\epsilon(u_\epsilon^-,\omega)\}$ and without loss of generality we suppose that $G_\epsilon(u_\epsilon^+,\omega) \le G_\epsilon(u_\epsilon^-,\omega)$. Recall that $u_\epsilon^\pm = \pm 1 + v_\epsilon^*$, and let $v_\epsilon := m_\epsilon - \mathrm{sign}(m_\epsilon)$. Due to the exponential decay of the boundary influence and the fact that the representative solves a linear PDE in the islands, one can easily show the following (see Appendix B and (B.11), (B.12)). There exist $C > 0$ and $K > 0$ such that for $\Gamma \in \mathcal{G}(m,\zeta)$ in an island I_Γ

$$(6.2)\; |u_\epsilon^\pm(r) - m_\epsilon(r)| = |v_\epsilon^*(r) - v_\epsilon(r)| < K e^{-\epsilon^{-1} C\,\mathrm{dist}(r,\epsilon\,\mathrm{sp}(\Gamma))},$$
$$(6.3)\; \left|\mathrm{grad}\left[u_\epsilon^\pm(r) - m_\epsilon(r)\right]\right| = \left|\mathrm{grad}\left[v_\epsilon^*(r) - v_\epsilon(r)\right]\right| < \epsilon^{-1} K e^{-\epsilon^{-1} C\,\mathrm{dist}(r,\epsilon\,\mathrm{sp}(\Gamma))}.$$

Given an $\delta \in (0,\delta_0)$, define

$$I_\Gamma^\delta := \left\{ y \in I_\Gamma :\ \mathrm{dist}(y,\partial I_\Gamma) > C^{-1} \ln\left(\frac{4K}{\delta}\right)\right\},$$

where C and K are the constants in (6.2), (6.3). We write $\sum_\Gamma$ for $\sum_{\Gamma\in\mathcal{G}(m)}$ and estimate

$$\begin{aligned} G_\epsilon(m_\epsilon) - G_\epsilon(u_\epsilon^+) \ge{}& \sum_\Gamma \int_{\epsilon\,(\mathrm{sp}(\Gamma)\cup(I_\Gamma\setminus I_\Gamma^\delta))} \left(2\sqrt{W}|\,\mathrm{grad}\, m_\epsilon| - 4\theta\alpha(\epsilon)\epsilon^{-1}\|g\|_\infty\right) \\ &- \sum_\Gamma \int_{\epsilon\,(\mathrm{sp}(\Gamma)\cup(I_\Gamma\setminus I_\Gamma^\delta))} \left(\epsilon|\,\mathrm{grad}\, v_\epsilon^*|^2 + \epsilon^{-1} W(1+v_\epsilon^*)\right) \\ &+ \frac{\alpha(\epsilon)}{\epsilon}\theta \sum_\Gamma \int_{\epsilon I_\Gamma^\delta} \left\{ g_\epsilon[m_\epsilon - u_\epsilon^{\mathrm{sign}(I_\Gamma)}] - g_\epsilon(1-\mathrm{sign}(I_\Gamma))\right\} \\ &+ \sum_\Gamma \int_{\epsilon I_\Gamma^\delta} \left(\epsilon(|\,\mathrm{grad}\, v_\epsilon|^2 - |\,\mathrm{grad}\, v_\epsilon^*|^2) + \frac{1}{2C_0\epsilon}(v_\epsilon^2 - (v_\epsilon^*)^2)\right). \end{aligned}$$

As $|u_\epsilon^\pm(r) - m_\epsilon(r)| < \delta/4$ on I_Γ^δ, $u^\pm \to \pm 1$ in L^∞ as $\epsilon \to 0$, so for ϵ sufficiently small $\mathrm{Per}(\{m_\epsilon < s\}) = 0$ in I_Γ^δ for $|s| < 1-\delta$. So, using the Lipschitz bounds and L^∞-bounds on v^* from Proposition B.1, we get

$$\begin{aligned} G_\epsilon(m_\epsilon) - G_\epsilon(u_\epsilon^+) \ge{}& \int_{-1+\delta}^{1-\delta} 2\sqrt{W(s)}\mathrm{Per}(\{m_\epsilon < s\})ds \\ &- C^{-1}\ln(4K/\delta)\{4\theta\alpha(\epsilon)\|g\|_\infty + C\theta^2\alpha(\epsilon)^2\} \sum_\Gamma \epsilon^{d-1}|\mathrm{sp}(\Gamma)| \\ &- \frac{\alpha(\epsilon)}{\epsilon}\theta \sum_\Gamma \int_{\epsilon(I_\Gamma\setminus I_\Gamma^\delta)} \{g_\epsilon(1-\mathrm{sign}(I_\Gamma)) + \|g\|_\infty |v_\epsilon^* - v_\epsilon|\} && (6.4) \\ &+ \sum_\Gamma \int_{\epsilon(I_\Gamma\setminus I_\Gamma^\delta)} \left(\epsilon(|\,\mathrm{grad}\, v_\epsilon|^2 - |\,\mathrm{grad}\, v_\epsilon^*|^2) + \frac{1}{2C_0\epsilon}\left(v_\epsilon^2 - (v_\epsilon^*)^2\right)\right). && (6.5) \end{aligned}$$

Now we make use of the splitting $I_\Gamma = I_\Gamma^\alpha \cup (I_\Gamma \setminus I_\Gamma^\alpha)$ with

$$I_\Gamma^\alpha := \{ y \in I_\Gamma :\ \mathrm{dist}(y,\partial I_\Gamma) > C^{-1}|\ln(\alpha(\epsilon))|\},$$

where C is the constant in (6.2), (6.3). We suppose that ϵ is so small that $I_\Gamma^\alpha \subseteq I_\Gamma^\delta$. First we are going to estimate the term in line (6.5). Define $M_\epsilon(u) := \epsilon|\operatorname{grad} u|^2 + \epsilon^{-1}u^2$. By Proposition B.1 and (6.2), (6.3), we estimate on I_Γ^α

$$|M_\epsilon(v_\epsilon(x)) - M_\epsilon(v_\epsilon^*(x))| < C\epsilon^{-1}\alpha(\epsilon)e^{-\epsilon^{-1}C\operatorname{dist}(x,\epsilon\,\partial(I_\Gamma^\alpha))}, \quad \epsilon^{-1}x \text{ in } I_\Gamma^\alpha. \tag{6.6}$$

Therefore a computation using the coarea formula yields

$$\begin{aligned}\int_{\epsilon I_\Gamma^\alpha} |M_\epsilon(v_\epsilon) - M_\epsilon(v_\epsilon^*)| &= \int_{\mathbb{R}} \left(\int |M_\epsilon(v_\epsilon) - M_\epsilon(v_\epsilon^*)| \mathrm{d}\mathcal{H}^{d-1}|_{\epsilon I_\Gamma^\alpha \cap \{x:\ \operatorname{dist}(x,\epsilon\partial I_\Gamma)=r\}} \right) \mathrm{d}r \\ &\le C\epsilon^{d-1}|\partial I_\Gamma^\alpha|\alpha(\epsilon) \le C'\epsilon^{d-1}|\operatorname{sp}(\Gamma)|\alpha(\epsilon),\end{aligned}$$

where $\mathrm{d}\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure.

Let R_ϵ be defined as the argument of the summation in (6.5). As $M_\epsilon(v_\epsilon) - M_\epsilon(v_\epsilon^*) \ge -M_\epsilon(v_\epsilon^*)$ and as $M_\epsilon(v_\epsilon^*)$ is of order $\epsilon^{-1}(\theta\alpha(\epsilon))^2$, we can estimate

$$\begin{aligned}R_\epsilon(I_\Gamma, m_\epsilon, v_\epsilon^*) &\ge \int_{\epsilon I_\Gamma^\alpha} (\dots) + \int_{\epsilon(I_\Gamma \setminus I_\Gamma^\alpha)} (\dots) \\ &\ge -C'\epsilon^{d-1}|\operatorname{sp}(\Gamma)|\alpha(\epsilon) - \|M_\epsilon(v_\epsilon^*)\|_{L^\infty}\epsilon^d|I_\Gamma \setminus I_\Gamma^\alpha| \\ &\ge -C\epsilon^{d-1}|\operatorname{sp}(\Gamma)|\Big[\alpha(\epsilon) + |\ln(\alpha(\epsilon))|(\theta\alpha(\epsilon))^2\Big].\end{aligned}$$

The term $\alpha(\epsilon)\epsilon^{-1}\|g\|_\infty|v_\epsilon^* - v_\epsilon|$ in (6.4) can be estimated in a similar way.

In order to estimate the expression in (6.4), recall that $\omega \in \Omega_{\epsilon,\delta(\epsilon)}$, hence

$$\alpha(\epsilon)\epsilon^{-1}\theta \left| \int_{\epsilon I_\Gamma} g_\epsilon(1 - \operatorname{sign}(I_\Gamma)) \right| \le \delta(\epsilon)\epsilon^{d-1}|\operatorname{sp}(\Gamma)|.$$

So far we have shown that for $\omega \in \widetilde{\Omega}$ and $\epsilon(n)$ sufficiently small

$$G_\epsilon(m_\epsilon) - G_\epsilon(u_\epsilon^+) \ge \int_{-1+\delta}^{1-\delta} 2\sqrt{W(s)}\operatorname{Per}(\{m_\epsilon < s\})ds \tag{6.7}$$

$$- \sum_\Gamma c'\big[\alpha(\epsilon)|\ln(\alpha(\epsilon)\delta)| + \delta(\epsilon)\big]\epsilon^{d-1}|\operatorname{sp}\Gamma|, \tag{6.8}$$

and, as by Corollary 5.6 for all $\omega \in \Omega_{\epsilon(n)}$,

$$\sum_{\Gamma \in \mathcal{G}(m)} \epsilon^{d-1}|\operatorname{sp}(\Gamma)| \le CF_\epsilon(m_\epsilon) < C',$$

we have that the expression in (6.8) vanishes as $\epsilon(n) \to 0$ for $\omega \in \widetilde{\Omega}$.

So it remains to bound (6.7). As $m_\epsilon \to u$ in $L^1(\Lambda)$ there exists a subsequence, denoted by m_ϵ again, which converges a.e. to u, and for this subsequence we have $1_{\{m_\epsilon < s\}}(r) \to 1_{\{u<s\}}(r)$ in $L^1(\Lambda)$. Further, it is easy to prove by applying Lemma 3.1 that $|u| = 1$ a.e. Then by lower semicontinuity of the perimeter

$$\liminf_{\epsilon \to 0} \operatorname{Per}(\{m_\epsilon < s\}) \ge \operatorname{Per}(\{u < 0\}) \quad \text{for } -1 < s < 1,$$

and, by Fatou's lemma,

$$\begin{aligned}\liminf_\epsilon \int_{-1+\delta}^{1-\delta} \Big(2\sqrt{W(s)}\operatorname{Per}(\{m_\epsilon < s\})\Big)\, ds &\ge \left(\int_{-1+\delta}^{1-\delta} 2\sqrt{W(s)}ds \right) \operatorname{Per}(\{u < 0\}) \\ &\ge (C_W - 2C\delta) \int_\Lambda |\operatorname{grad} u|.\end{aligned}$$

As $\delta > 0$ was arbitrary and independent of ω, this proves the theorem. ∎

LEMMA 6.4. *There exists a set $\widetilde{\Omega} \subseteq \Omega$ with $\mathbb{P}(\Omega \setminus \widetilde{\Omega}) = 0$ such that for any $\omega \in \widetilde{\Omega}$ the following holds: For any $u \in BV(\Lambda, \{-1,1\})$ which has the property that $E := \{x : u(x) = -1\}$ has a smooth boundary, there exists $m_\epsilon(\cdot,\omega)$ with $\|m_\epsilon(\cdot,\omega) - u\|_{L^1} \to 0$ and*

$$\limsup F_\epsilon(m_\epsilon) \le C_W \mathrm{Per}(E) \quad \text{for } C_W \text{ as in (1.3).}$$

Proof. We construct a sequence with the required properties. To this end, let $\bar{m} : \mathbb{R} \to \mathbb{R}$ be the increasing solution of

$$\bar{m}'' = W'(\bar{m}), \quad \lim_{r\to\pm\infty} \bar{m}(r) = \pm 1.$$

It is well known [10] that there exist $C,\ \lambda > 0$ such that

$$|(1 - |\bar{m}(r)|)| + \bar{m}'(r) \le C e^{-\lambda |r|}. \tag{6.9}$$

Define

$$d(x) := \begin{cases} -\mathrm{dist}(x, E) & \text{if } x \in \Lambda \setminus E, \\ \mathrm{dist}(x, \mathbb{R}^d \setminus E) & \text{if } x \in E, \end{cases} \qquad d_\epsilon(x) := \frac{d(x)}{\epsilon},$$

and

$$m_\epsilon(\cdot,\omega) := v_\epsilon^*(\cdot,\omega) + \bar{m}\left(d_\epsilon(\cdot)\right) \qquad \forall \omega \in \Omega,$$

where v_ϵ^* solves (2.20). Obviously $\|m_\epsilon(\cdot,\omega) - u\|_{L^1} \to 0$ for all $\omega \in \Omega$. To shorten notation, in what follows we avoid writing the dependence of m_ϵ and v_ϵ^* on ω. Note that $|\operatorname{grad} d(x)| = 1$; therefore,

$$|\operatorname{grad} m_\epsilon(x)|^2 \le \epsilon^{-2}[\bar{m}'(d_\epsilon(x))]^2 + 2\epsilon^{-1}|\operatorname{grad} v_\epsilon^*(x)|\bar{m}'(d_\epsilon(x)) + |\operatorname{grad} v_\epsilon^*(x)|^2,$$

and

$$G_\epsilon(m_\epsilon) - G_\epsilon(1 + v_\epsilon^*) \le \int_\Lambda \epsilon^{-1}[\bar{m}'(d_\epsilon(x))^2 + W(\bar{m}(d_\epsilon(x)))] \tag{6.10}$$

$$+ 2\int_\Lambda |\operatorname{grad} v_\epsilon^*(x)|\bar{m}'(d_\epsilon(x)) \tag{6.11}$$

$$+ \frac{1}{\epsilon}\int_\Lambda \left[W(\bar{m}(d_\epsilon(x)) + v_\epsilon^*(x)) - W(1 + v_\epsilon^*(x)) - W(\bar{m}(d_\epsilon(x)))\right] \tag{6.12}$$

$$+ \frac{\alpha}{\epsilon}\int_\Lambda (\bar{m}(d_\epsilon(x)) - 1) g_\epsilon(x). \tag{6.13}$$

It is well known (see [15]) that the expression in (6.10) converges to $C_W \mathrm{Per}(E)$. Next we show that the term in (6.11) vanishes. We obtain from Proposition B.1 for ϵ sufficiently small $|\operatorname{grad} v_\epsilon^*| \le C'\alpha(\epsilon)\epsilon^{-1}$. Hence by the coarea formula and (6.9) we estimate

$$\int_\Lambda 2|\operatorname{grad} v_\epsilon^*|\bar{m}'(d_\epsilon) \le 2C'\frac{\alpha(\epsilon)}{\epsilon}\int_{-\infty}^{\infty} \mathcal{H}^{d-1}(\{d(x) = r\}) e^{-\lambda\frac{r}{\epsilon}} dr \le C''\mathrm{Per}(E)\alpha(\epsilon) \to 0.$$

Let $\mu_\epsilon := -\epsilon \ln(\alpha(\epsilon)) = \epsilon \ln\ln(1/\epsilon) > 0$, and

$$\Lambda_{\mu_\epsilon} := \{x :\ |d(x)| < \mu_\epsilon\}.$$

Split the expression in (6.12) into an integral over Λ_{μ_ϵ} and the rest. On Λ_{μ_ϵ} we have

$$|W(\bar{m}+v_\epsilon^*)-W(\bar{m})|\le L\|v_\epsilon^*\|_\infty, \quad W(1+v_\epsilon^*)\le \frac{1}{2C_0}\|v_\epsilon^*\|_\infty^2$$

for $L:=\sup_{s\in[-2,2]}|W'(s)|$. This helps to estimate

$$\begin{aligned}\epsilon^{-1}\int_{\Lambda_{\mu_\epsilon}}(W(\bar{m}+v_\epsilon^*)-W(1+v_\epsilon^*)-W(\bar{m})) &\le \frac{|\Lambda_{\mu_\epsilon}|}{\epsilon}C\left(L+\frac{\alpha(\epsilon)}{2C_0}\right)\alpha(\epsilon)\\ &\le C'\alpha(\epsilon)\ln\left(\frac{1}{\alpha(\epsilon)}\right)\mathrm{Per}(E).\end{aligned}$$

To estimate the integral over $\Lambda\setminus\Lambda_{\mu_\epsilon}$, we use that for x so that $|d(x)|>\epsilon|\ln(\alpha)|$

$$|W(\bar{m}(d_\epsilon))|\le\frac{1}{2C_0}(\bar{m}(d_\epsilon)-1)^2\le\frac{C^2}{2C_0}e^{-2\lambda d(x)/\epsilon},$$

and then

$$|W(\bar{m}(d_\epsilon)+v_\epsilon^*)-W(1+v_\epsilon^*)|\le\left[\sup_{|s-1|\le C\alpha(\epsilon)}W'(s)\right]Ce^{-\lambda d(x)/\epsilon}\le C'\alpha(\epsilon)e^{-d(x)/\epsilon}.$$

Here the symmetry of the wells was used. The constants depend on the second fundamental form of the set E. We obtain

$$\begin{aligned}&\epsilon^{-1}\int_{\Lambda\setminus\Lambda_{\mu_\epsilon}}([W(\bar{m}+v_\epsilon^*)-W(1+v_\epsilon^*)]-W(\bar{m}))\\ &\le\epsilon^{-1}\int_{\Lambda\setminus\Lambda_{\mu_\epsilon}}\left(C'\alpha(\epsilon)e^{-d(x)/\epsilon}+\frac{C^2}{2C_0}e^{-2\lambda d(x)/\epsilon}\right).\end{aligned}$$

By the coarea formula and a change of variables $d/\epsilon=r$, this is bounded by

$$C\left(\mathrm{Per}(E)\right)\left[(\alpha(\epsilon)+1)\int_{|\ln(\alpha(\epsilon))|}^{\infty}e^{-\lambda r}dr\right]\le C'\left(\mathrm{Per}(E)\right)\alpha(\epsilon)\to 0.$$

The term in (6.13), which depends on the random field, can be bounded by

$$C'\mathrm{Per}(E)\alpha(\epsilon)+2\frac{\alpha(\epsilon)}{\epsilon}\int_E g_\epsilon.$$

Note that there exists a constant $C(d)$ depending only on the dimension, such that the following holds: There exists for any E as above an $\epsilon_0(E)$ such that for any $\epsilon<\epsilon_0(E)$ there exists a set E_ϵ which is a union of cubes of sidelength ϵ with centers on $\epsilon\mathbb{Z}^d$ and

$$C(d)^{-1}\mathrm{Per}(E_\epsilon)\le\mathrm{Per}(E)\le C(d)\mathrm{Per}(E_\epsilon)\qquad |E_\epsilon\Delta E|\to 0 \text{ as } \epsilon\to 0.$$

This can be shown, e.g., by approximating the smooth manifold ∂E by polygons and then by faces of cubes with centers on the lattice $\epsilon\mathbb{Z}^d$. Hence in arguing that the term in the fourth line vanishes we can use Lemma 5.4 with $\epsilon(n)$, $\delta(\epsilon(n))$ as in the proof of Lemma 6.3 to show that

$$\left|\frac{\alpha(\epsilon)}{\epsilon}\int_E g_\epsilon\right|\le C\delta(\epsilon)\mathrm{Per}(E).$$

Hence the lemma is proven. □

Now we prove Theorem 2.3.

Proof. From Lemma 3.1 we immediately get that $F_\epsilon \to +\infty$ if $|u|$ is different from 1 on a set of positive Lebesgue measure. By general arguments (see [15, Lemma 1]), it is sufficient to consider the upper bound in the case where E has a smooth boundary. Now the theorem follows from Lemmas 6.4 and 6.3, together with Theorem 6.2. □

Appendix A. Probabilistic estimates. Let $\mathcal{R}$ be the set of all connected unions of unit cubes such that the origin is contained in the union. We denote by R an element of $\mathcal{R}$ and by (abusing notation) $|\partial R|$ the total volume of all cubes in R which are connected to a cube not in R. We have the following result.

PROPOSITION A.1. *For $d \geq 3$, for any $S_0 > 0$, there exists $c' \equiv c'(S_0, d)$ so that for all $S > S_0$, we obtain*

$$\mathbb{P}\left[\exists R \in \mathcal{R} : 0 \in R, \left|\sum_{z\in\mathbb{Z}^d:(z+[0,1]^d)\cap R\subset R} g(z,\omega)\right| \geq S|\partial R|\right] \leq 2e^{-S^2c'}. \tag{A.1}$$

Proof. We have

$$\begin{aligned}
&\mathbb{P}\left[\exists R \in \mathcal{R} : 0 \in R, \left|\sum_{z\in\mathbb{Z}^d:(z+[0,1]^d)\cap R\subset R} g(z,\omega)\right| \geq S|\partial R|\right]\\
&\quad\leq \sum_{n\geq 1}\mathbb{P}\left[\sup_{R\in\mathcal{R}:\ |\partial R|=n,\ 0\in R}\left|\sum_{z\in\mathbb{Z}^d:(z+[0,1]^d)\cap R\subset R} g(z,\omega)\right| \geq S|\partial R|\right].
\end{aligned} \tag{A.2}$$

To estimate each addend we define a sequence of sets $R_\ell \in \mathcal{D}^{(\ell)}$, $\ell \in \mathbb{N}$, i.e., a partition of $\mathbb{R}^d$ in cubes of side 2^ℓ, with one of them having center 0. The R_ℓ, $\ell \in \mathbb{N}$, are constructed by a "coarse-graining" procedure from the original connected region $R_0 \equiv R$. We denote by $\mathcal{R}_\ell : R_0 \to R_\ell$ the map which associates to R_0 the set of cubes in $\mathcal{D}^{(\ell)}$ so that

$$|C^{(\ell)} \cap R_0| \geq \frac{1}{2}2^{d\ell},$$

where R_ℓ is the union over those cubes. Note that R_ℓ is in general not connected. One can prove (see Proposition 1 of [11]) that

$$|\partial R_\ell| \leq C(d)|\partial R_0|, \tag{A.3}$$

and that the volume of the corridor between R_ℓ and $R_{\ell-1}$ when $R_\ell \neq \emptyset$ is estimated by

$$|R_\ell \Delta R_{\ell-1}| \leq |\partial R_0|2^\ell, \tag{A.4}$$

where for two sets A and B, $A\Delta B = (A \setminus B) \cup (B \setminus A)$. Denote

$$F(R_0,\omega) = \sum_{z\in\mathbb{Z}^d:(z+[0,1]^d)\cap R_0\subset R_0} g(z,\omega).$$

Set $z = S|\partial R_0| = Sn$ and write, for any choice of $k(n) \in \mathbb{Z}$,

$$F(R_0,\omega) = F(R_{k(n)},\omega) + [F(R_{k(n)-1},\omega) - F(R_{k(n)},\omega)] + \cdots [F(R_0,\omega) - F(R_1,\omega)].$$

We have

$$\begin{aligned}\mathbb{P}\left[\sup_{|\partial R_0|=n:0\in R_0} F(R_0,\omega) > z\right] &\le \sum_{\ell=1}^{k(n)} \mathbb{P}\left[\sup_{|\partial R_0|=n:0\in R_0} \{F(R_{\ell-1},\omega)-F(R_\ell,\omega)\} > z_\ell\right]\\ &+ \mathbb{P}\left[\sup_{|\partial R_0|=n:0\in R_0} F(R_{k(n)},\omega) > z_{k(n)+1}\right]\end{aligned} \tag{A.5}$$

for any sequence z_ℓ with $\sum_{\ell=1}^{k+1} z_\ell \le z$. Since F is a sum of independent identically distributed random variables, it is immediate to see that

$$\mathbb{P}\left[\{F(R_\ell,\omega)-F(R_{\ell-1},\omega)\} > z_\ell\right] \le e^{-\frac{z_\ell^2}{|R_\ell \Delta R_{\ell-1}|}}. \tag{A.6}$$

Estimate (A.6) bounds the probability that a particular coarse-grained corridor has a large field. Therefore

$$\mathbb{P}\left[\sup_{|\partial R_0|=n:0\in R_0} \{F(R_\ell,\omega)-F(R_{\ell-1},\omega)\} > z_\ell\right] \le A_{\ell-1,n}A_{\ell,n} e^{-\frac{z_\ell^2}{\sup_{\{|\partial R_0|=n:0\in R_0\}}|R_\ell\Delta R_{\ell-1}|}}, \tag{A.7}$$

where $A_{\ell,n}$ is the number of image points in R_ℓ that are reached when mapping any of the R_0 occurring in the sup, i.e., those so that $|\partial R_0| = n$ and those that contain the origin. In [11, Proposition 2], it has been shown that there exists a constant $C = C(d)$ so that

$$A_{\ell,n} \le e^{\left(\frac{C\ell n}{2^{(d-1)\ell}}\right)}. \tag{A.8}$$

Therefore we obtain from (A.7) and (A.4)

$$\begin{aligned}\mathbb{P}\left[\sup_{|\partial R_0|=n:\ 0\in R} F(R_0,\omega) > z\right] &\le \sum_{\ell=1}^{k(n)} A_{\ell-1,n}A_{\ell,n} e^{-\frac{z_\ell^2}{n2^\ell}}\\ &+ A_{k(n),n} e^{-\frac{z_{k+1}^2}{\sup_{\{|\partial R_0|=n:0\in R_0\}}|R_{k(n)}|}}.\end{aligned} \tag{A.9}$$

By the isoperimetric inequality and (A.3) we have $\sup_{|\partial R_0|=n:0\in R_0} |R_{k(n)}| \le C(d)n^{\frac{d}{d-1}}$. By (A.8)

$$\begin{aligned}\mathbb{P}\left[\sup_{|\partial R_0|=n:0\in R_0} F(R_0,\omega) > z\right] &\le \sum_{\ell=1}^{k(n)} e^{\left(\frac{2C\ell n}{2^{(d-1)(\ell-1)}}\right)} e^{-\frac{z_\ell^2}{n2^\ell}}\\ &+ e^{\left(\frac{Ck(n)n}{2^{(d-1)k(n)}}\right)} e^{-\frac{z_{k+1}^2}{n^{\frac{d}{d-1}}}}.\end{aligned} \tag{A.10}$$

Then choose $k(n)$, the number of times one repeats the coarse-graining procedure, so that the final coarse-grained volume does not have an anomalous large total field, $R_{k(n)-1} \ne \emptyset$, and the sum on the right-hand side of (A.10) is small. Take

$$2^{k(n)} = n^{\frac{1}{3}}, \qquad z_\ell = \frac{S}{2}\frac{n}{\ell^2},$$

and note that $k(n) \simeq \log n$ and

$$\frac{S}{2}\sum_{\ell=1}^{k(n)+1}\frac{n}{\ell^2} \le \frac{S}{n}\left[1-\frac{1}{2}\frac{1}{k(n)+1}\right] \le z.$$

We obtain, since

$$z_{k(n)+1} = \frac{S}{2}\frac{n}{(k(n)+1)^2} \simeq \frac{S}{2}\frac{n}{(\ln n+1)^2}$$

and

$$\frac{k(n)n}{2^{(d-1)k(n)}} \simeq \frac{n\ln n}{n^{\frac{1}{3}(d-1)}} \qquad \text{in } d \ge 3,^{8}$$

that

$$e^{\left(\frac{Ck(n)n}{2^{(d-1)k(n)}}\right)} e^{-\frac{z_{k+1}^2}{n^{\frac{d}{d-1}}}} = e^{\left(Cn^{\frac{1}{3}(4-d)}\ln n - S^2\frac{n^{\frac{d-2}{d-1}}}{(\ln n+1)^4}\right)} \overset{n\uparrow\infty}{\to} 0.$$

For the remaining term in (A.10), when $d \ge 3$, one can choose $S_0 \equiv S_0(d) > 0$ so that for $S \ge S_0$

$$\text{(A.11)} \qquad \frac{C(\ell-1)n}{2^{(d-1)\ell}} - \frac{nS^2}{2^\ell \ell^4} = n\ell\frac{S^2}{2^\ell}\left(\frac{C}{S^2}\frac{(\ell-1)}{2^{(d-2)\ell}\ell} - \frac{1}{\ell^5}\right) < 0 \qquad \forall \ell \ge 1.$$

Then it is possible to find $c = c(S_0, d)$ so that for all $S \ge S_0$

$$\sum_{\ell=1}^{k(n)} e^{-S^2\frac{\ell}{2^\ell}nc} \le \sum_{\ell=1}^{k(n)} e^{-S^2\ell n^{\frac{2}{3}}c} \le e^{-S^2 n^{\frac{2}{3}}c}.$$

Summarizing all the estimates one immediately gets (A.1). ∎

Appendix B. Global and local minimizers in one single well.

Let

$$V(s) = \frac{1}{2C_0}s^2 \qquad \forall s \in \mathbb{R},$$

and consider for $u \in H^1(\Lambda)$ the functional

$$\text{(B.1)} \qquad \mathcal{F}_\epsilon(u,\omega) \equiv \int_\Lambda \left(\epsilon|\nabla u(y)|^2 + \frac{1}{\epsilon}V(u(y))\right)dy + \frac{1}{\epsilon}\alpha(\epsilon)\theta\int_\Lambda dy g_\epsilon(y,\omega)u(y).$$

As in Lemma 3.1, one has for all $u \in H^1(\Lambda)$

$$\text{(B.2)} \qquad \mathcal{F}_\epsilon(t\wedge u \vee (-t),\omega) < \mathcal{F}_\epsilon(u,\omega) \qquad \forall t > C_0\alpha(\epsilon)\theta, \qquad \mathbb{P}=1.$$

The minimizer of $\mathcal{F}_\epsilon(u,\omega)$ is obviously v_ϵ^*, the solution of the Euler–Lagrange equation (2.20). Next we report the properties of v_ϵ^* used throughout the paper. The proofs

[8]In $d = 2$ the choice of $k(n)$ makes the last term in the sum (A.10) diverging. Namely, we have $e^{(Cn^{\frac{2}{3}}\ln n - \frac{S^2}{(\ln n+1)^4})} \to \infty$ when $n \to \infty$. Further, in $d = 2$, the remaining term in (A.10), independently of the choice of $k(n)$, is always diverging.

use standard computations involving Green's function for (B.5) below, therefore they are omitted. For the required properties of Green's function, see, e.g., Dautray and Lions [5, p. 635].

PROPOSITION B.1. *The solution v_ϵ^* of the Euler–Lagrange equation* (2.20) *is Lipschitz continuous on Λ with Lipschitz constant bounded by*

$$\epsilon^{-1}L_0 = \epsilon^{-1}C(\|g\|_\infty)\alpha(\epsilon)\theta$$

and

$$|v_\epsilon^*(r,\omega)| \le C_0\alpha(\epsilon)\theta\|g\|_\infty, \qquad r \in \Lambda, \quad \mathbb{P} = 1. \tag{B.3}$$

It can be represented as

$$v_\epsilon^*(r,\omega) = \frac{\alpha(\epsilon)}{2\epsilon^2}\theta \int_\Lambda G_\epsilon(r,r')g_\epsilon(r',\omega)\mathrm{dr}' \qquad r \in \Lambda, \tag{B.4}$$

where $G_\epsilon(\cdot,\cdot)$ is the Green function solution of the following problem:

$$\begin{aligned} &-\Delta_r G_\epsilon(r,r') + \frac{1}{\epsilon^2}\frac{1}{2C_0}G_\epsilon(r,r') = \delta(r-r'), \qquad r,r' \in \Lambda,\\ &\frac{\partial G_\epsilon}{\partial n}(r,r') = 0, \qquad r' \in \Lambda \quad \textit{a.e. for } r \quad \partial\Lambda. \end{aligned} \tag{B.5}$$

v_ϵ^* is a Gaussian process with mean

$$\mathbb{E}[v_\epsilon^*(r,\cdot)] = 0, \qquad r \in \Lambda, \tag{B.6}$$

and covariance for $d \ge 3$

$$\mathbb{E}[v_\epsilon^*(r,\cdot)v_\epsilon^*(r',\cdot)] \le C(d)\theta^2\alpha^2(\epsilon)e^{-\frac{1}{2\epsilon\sqrt{2C_0}}|r-r'|}. \tag{B.7}$$

PROPOSITION B.2. *In $d \ge 3$ one can bound*

$$0 < G_\epsilon(r) \le C(d)\frac{1}{4\pi|r|^{d-2}}e^{-k|r|}, \qquad k = \frac{1}{\epsilon}\frac{1}{\sqrt{2C_0}}. \tag{B.8}$$

Next we consider the local minimizer in one single well with Dirichlet boundary conditions. Let $D \subset \Lambda$ and consider the following boundary value problem:

$$-\epsilon\Delta u(r) + \frac{1}{\epsilon}\frac{1}{2C_0}u(r) + \frac{1}{2\epsilon}\alpha(\epsilon)\theta g_\epsilon(r,\omega) = 0 \quad \text{in } D, \quad u = v_0 \qquad \text{on } \partial D, \tag{B.9}$$

where $v_0 \in H^1(\Lambda)$. We have the following boundary influence decay for the solution of (B.9).

PROPOSITION B.3. *For $d \ge 3$, there exists a positive constant $C(d)$ so that for $\mathbb{P} = 1$ the following holds: Let v be the solution of* (B.9). *We have*

$$|v(r,\omega)| \le C(d)\sup_{y\in\partial D}|v_0(y)|e^{-\frac{1}{\epsilon 4\sqrt{2C_0}}d(r,\partial D)} + C_0\alpha(\epsilon)\|g\|_\infty\theta, \qquad r \in D. \tag{B.10}$$

For solutions of (B.9) *with different boundary conditions we obtain*

$$|v_1(r,\omega) - v_2(r,\omega)| \le C(d)\sup_{y\in\partial D}|v_1(y)-v_2(y)|e^{-\frac{d(r,\partial D)}{4\epsilon\sqrt{2C_0}}}, \quad x \in D, \tag{B.11}$$

$$|\operatorname{grad}(v_1(r,\omega) - v_2(r,\omega))| \le \frac{\widehat{C}(d)}{\epsilon}\sup_{y\in\partial D}|v_1(y)-v_2(y)|e^{-\frac{d(r,\partial D)}{4\epsilon\sqrt{2C_0}}} \tag{B.12}$$

$$\text{for } r \in D \text{ and } d(r,\partial D) > \epsilon.$$

Appendix C. In this section we show for a simplified functional that sequences that approximate a function with a flat jump set are not microscopically flat. First we give some definitions. From now on $d = 3$, $x = (x_1, x_2, x_3)$, $\Lambda = (-1/2, 1/2)^3$. As a simplification we replace the part of the functional G_ϵ which consists of the gradient part and the double-well potential directly by its sharp-interface limit and and we restrict our attention to functions which are of bounded variation with values in $\{+1, -1\}$; i.e., we consider

$$\hat{G}_\epsilon(u, \omega) = \begin{cases} \int_\Lambda \left(|\operatorname{grad} u| + \frac{\alpha(\epsilon)}{\epsilon} g_\epsilon u \right) & \text{if } u \in BV(\Lambda, \{-1, 1\}), \\ +\infty & \text{else.} \end{cases}$$

Recall that the Heaviside function $H(x) : \mathbb{R} \to \mathbb{R}$ is defined as $H(x) = 1$ for $x > 0$, $H(0) = 0$, and $H(x) = -1$ for $x < 0$. We will show that perturbations of the "planar" function $U(x) := H(x_3)$ decrease the energy. More precisely we consider "graph-like" perturbations, i.e., functions $V : \Lambda \to \{-1, 1\}$ for which there exist functions $\varphi : (-1/2, 1/2)^2 \to (-1, 1)$ so that $\{V = -1\} = \{x : \ x_3 \le \varphi(x_1, x_2)\}$, and we will show that we can find with high probability such a φ with

$$\operatorname{osc}(\varphi) := \sup_{(-1/2,1/2)^2} \varphi - \inf_{(-1/2,1/2)^2} \varphi \gg \epsilon,$$

and lower energy than that of $\varphi(x_1, x_2) = 0$.

This indicates that the minimizer under boundary conditions that enforce a "planar" jump are not planar on small scales. We make another assumption which is not automatic because the g_ϵ here is constant on deterministic cubes.

Assumption H2. There is a $\delta > 0$ so that for any measurable set A

$$\mathbb{P}\left(\int_A g_\epsilon > \epsilon^{3/2} \sqrt{|A|} \right) \ge \frac{1}{2} \mathbb{P}\left(\left| \int_A g_\epsilon \right| > \epsilon^{3/2} \sqrt{|A|} \right) \ge \delta > 0$$

and the random variables $\int_A g_\epsilon$, $\int_{A'} g_\epsilon$ are independent and identically distributed for $\operatorname{dist}(A, \ A') > \epsilon$.

If A and A' are unions of sufficiently many cubes, then H2 is an immediate consequence of the central limit theorem.

THEOREM C.1. *Let $U(x) := H(x_3)$, $0 < \beta < 1$, $\epsilon = \frac{1}{n}$, and assume H2. Let $h_\epsilon = \alpha(\epsilon)\epsilon^{(2\beta+1)/3}$; then there exists a function $\varphi_\epsilon(\cdot, \omega) : \ [-1, 1]^2 \to [0, h_\epsilon)$, such that $\mathbb{P}$-almost surely for any $i \in \mathbb{Z}^2$*

$$\lim_{\epsilon \to 0} h_\epsilon^{-1} \left(\sup_{\epsilon^\beta(i + [-1,1]^2) \subset [-1,1]^2} (\varphi_\epsilon(\cdot, \omega)) - \inf_{\epsilon^\beta(i + [-1,1]^2) \subset [-1,1]^2} (\varphi_\epsilon(\cdot, \omega)) \right) > 0.$$

Let $V_\epsilon(\cdot, \omega) : \Lambda \to \{-1, 1\}$ be such that $\{V_\epsilon = -1\} = \{x : \ x_3 \le \varphi_\epsilon(x_1, x_2, \omega)\}$; then there exists $C > 0$ such that

$$\mathbb{P}\left[\hat{G}(U) - \hat{G}(V_\epsilon(\omega)) > C\epsilon^{2/3(1-\beta)} \alpha(\epsilon)^2 \right] \to 1.$$

Proof. Let $r_\epsilon = \epsilon^\beta$, and divide the square $(-1/2, 1/2)^2$ into cubes $Q_r(y_i)$ of sidelength $2r_\epsilon$ centered at $y_i = \epsilon^\beta i \in (-1/2, 1/2)^2$ for $i = (j_1, j_2) \in \mathbb{Z}^2$.

We denote by $P_\epsilon \subseteq \mathbb{R}^3$ the pyramid with center at the origin, base $(-r_\epsilon, r_\epsilon)^2 \times \{x_3 = 0\}$, and height h_ϵ. The excess area (surface of the pyramid minus area of the base) is $r\sqrt{r^2+h^2}-r^2$. We translate the basis of the pyramid on the plane $(-1/2,1/2)^2$ and denote it by $P_\epsilon + (y_i, 0)$ for all $i \in \mathbb{Z}^2$ so that $y_i = i\epsilon^\beta \in (-1/2,1/2)^2$. Next we define a random variable which indicates whether a perturbation is convenient.

$$\eta_i(y,\omega) = \begin{cases} 1 & \text{if} \quad \alpha(\epsilon)\epsilon^{-1}\int_{P_\epsilon+(y_i,0)} g_\epsilon > 2r_\epsilon^2(\sqrt{1+(h_\epsilon/r_\epsilon)^2}-1), \ y \in Q_r(y_i), \\ 0 & \text{else.} \end{cases}$$

Now let $\varphi_{r_\epsilon}(x_1,x_2):\ Q_{r_\epsilon}(0) \to [0,h_\epsilon]$ be such that $\varphi_{r_\epsilon}(x_1,x_2)$ is the graph of P_ϵ and denote

$$\varphi_{r_\epsilon}(x_1,x_2,\omega) = \sum_{(j_1,j_2)\in\mathbb{Z}^2 : y_{(j_1,j_2)}\in(-1/2,1/2)^2} \eta_i((x_1,x_2),\omega)\varphi_{r_\epsilon}\left((x_1,x_2)-\epsilon^\beta(j_1,j_2)\right).$$

The theorem follows immediately from a Borel–Cantelli argument if we are able to show that $1 > \mathbb{P}(\eta(0)=1) > 0$. The upper bound follows from the symmetry of the random field, which yields $\mathbb{P}(\eta(0)=1) \le 1/2$. The lower bound is a consequence of H2: The volume of the pyramid is $1/3 r_\epsilon^2 h_\epsilon^2$; i.e., H2 implies

$$\mathbb{P}\left(\alpha(\epsilon)\epsilon^{-1}\int_{P_\epsilon} g_\epsilon > \epsilon^{1/2}\alpha(\epsilon)(1/3)r_\epsilon\sqrt{h_\epsilon}\right) > \delta,$$

and for ϵ sufficiently small

$$\frac{\epsilon^{1/2}\alpha(\epsilon)\sqrt{(1/3)}r_\epsilon\sqrt{h_\epsilon}}{2r_\epsilon^2(\sqrt{1+(h_\epsilon/r_\epsilon)^2}-1)} \ge \frac{\epsilon^{1/2}\alpha(\epsilon)\epsilon^\beta}{\epsilon^{\beta+1/2}\alpha(\epsilon)^{3/2}} = \frac{1}{\alpha(\epsilon)^{\frac{1}{2}}} > 1. \qquad \square$$

Remark C.2. The error in the upper bound, Lemma 6.4, is of order $\alpha(\epsilon) \gg \epsilon^{2/3}$; therefore the error when replacing G_ϵ by the functional $\hat{G}_\epsilon$ defined in this appendix is larger than the effect described here. Hence this does not prove that minimizing sequences of F_ϵ with plane-like constraints are not flat. However, a careful analysis of the next order for the functional G_ϵ would be beyond the scope of this paper.

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