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Sharp permutation groups

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1. Introduction.

Let G be a group of permutations on a finite set Ω and θ be the permutation character; g and n denote $|G|$ and $|\Omega|$ respectively. For a set L of non-negative integers less than $n-1$, G is called an L-group if L contains $\theta(x)$ for any non-identity element x of G. The following inequality holds for an L-group:

$$(*) \qquad g \leqq \prod_{l \in L} (n-l) .$$

(This was conjectured by Bannai and Deza and was proved by Kiyota [**10**].) G is said to be *sharp* (or *L-sharp*) if the equality holds in (*). This terminology suggested by Deza can be justified by the fact that a $\{0, 1, \cdots, r-1\}$-sharp group is a sharply r-transitive group. From the literature on permutation groups, we can find many papers that deal with the classification of L-sharp groups for some particular L. For example, G has a representation as an L-group with $L=\{l\}$, $l>0$, if and only if G has a G-invariant proper partition (communicated by T. Kondo, see [8]), and such nonsolvable groups were classified by Suzuki [13]. L-sharp groups were classified for $L=\{2\}$, $\{3\}$ and $\{0, 2\}$ ([**6**], [7], [**14**], see also [**12**]). Also, the reader is referred to Deza [4] for the relevant topics.

The purpose of this paper is to determine L-sharp groups for $L=\{l, l+2\}$, $\{l, l+3\}$ and $\{l, l+1, l+2, \cdots, l+r-1\}$ with $r \geqq 2$. Let $F(G)$ be the set of points which are fixed by any element of G.

THEOREM 1. *Let G be an $\{l, l+1, l+2, \cdots, l+r-1\}$-sharp group on Ω with $r \geqq 2$. Then $|F(G)|=l$, and G is sharply r-transitive on $\Omega-F(G)$.*

THEOREM 2. *Let G be an $\{l, l+2\}$-sharp group on Ω. Then either* (i) *or* (ii) *holds:*

(i) *$|F(G)|=l$, G is transitive and is of rank* 3 *on $\Omega-F(G)$, and $G \cong D_8$, S_4, $GL(2, 3)$ or $PSL(2, 7)$, where $|\Omega-F(G)|=4, 6, 8, 14$, respectively,*

(ii) *$|F(G)|=l-1$, G has two orbits on $\Omega-F(G)$, and $G \cong S_4$ or $PSL(2, 7)$, where $|\Omega-F(G)|=7, 15$, respectively.*

In the case (i), S_4 *has two nonequivalent representations on* 6 *points as a* $\{0, 2\}$*-sharp group.*

THEOREM 3. *Let* G *be an* $\{l, l+3\}$*-sharp group on* Ω. *Then either* (i) *or* (ii) *holds:*

(i) $|F(G)|=l$, G *is transitive on* $\Omega-F(G)$, *and* $G\cong(Z_3\times Z_3)\rtimes Z_2$, $(Z_3\times Z_3)\rtimes S_3$, $(Z_3\times Z_3\times Z_3)\rtimes S_4$, $Z_3\times PSL(2, 4)$ *or* $Z_3\times PSL(2, 7)$, *where* $|\Omega-F(G)|=6, 9, 27, 15, 24$, *respectively,*

(ii) $|F(G)|=l-2$, G *has three orbits on* $\Omega-F(G)$, *and* $G\cong(Z_3\times Z_3)\rtimes Z_2$, *where* $|\Omega-F(G)|=8$.

All the semidirect products are determined uniquely except $(Z_3\times Z_3)\rtimes S_3$. $(Z_3\times Z_3)\rtimes S_3$ *has two nonequivalent representations on* 9 *points as a* $\{0, 3\}$*-sharp group; one has a trivial center and the other has a center of order* 3.

2. Reduction lemmas.

LEMMA 2.1. *Let* G *be a* $\{0, l_2, \cdots, l_r\}$*-sharp group on* Ω, *where* $0<l_2<\cdots<l_r$. *Then* G *is transitive on* Ω *and* G_α *is an* $\{l_2-1, \cdots, l_r-1\}$*-sharp group on* $\Omega-\{\alpha\}$ *for any element* α *of* Ω.

PROOF. We have $|G|=n\prod_{i=2}^{r}(n-l_i)$. Since $|G|=|G_\alpha|\cdot|\alpha^G|$, $|\alpha^G|\leqq n$ and since $|G_\alpha|\leqq\prod_{i=2}^{r}(n-l_i)$ by the inequality (*), we get that $|\alpha^G|=n$ and $|G_\alpha|=\prod_{i=2}^{r}(n-l_i)$, the desired result.

The following is the most crucial reduction lemma to treat L-sharp permutation groups with $|L|=2$.

LEMMA 2.2. *Let* G *be an* $\{l, l+s\}$*-sharp group on* Ω. *Then* $|F(G)|\geqq m$ *holds, where* $m=l+(1-s)s'+s'^2-1$ *with* $s'=\max\{1, [(s-1)/2]\}$.

PROOF. Let us decompose the permutation character θ into the sum of irreducible characters χ_i of G in the complex field: $\theta=\sum a_i\chi_i$ with χ_0 the principal character. Since each G-orbit on $\Omega-F(G)$ contributes at least one non-principal irreducible character to θ, we have

$$|F(G)|+\sum' a_i\geqq a_0, \tag{2.1}$$

the summation $\sum'$ taking over nonzero i's. Let us set $\hat{\theta}=(\theta-l\chi_0)(\theta-(l+s)\chi_0)$. Since $\hat{\theta}$ is the regular character of G [10], we have $(\hat{\theta}, \chi_0)=1$ and so

$$\sum' a_i^2=1-(a_0-l)(a_0-l-s). \tag{2.2}$$

The identity (2.2) implies $l\leqq a_0\leqq l+s$, but a_0 cannot be l because $(\theta, \chi_0)=\frac{1}{g}\sum_{x\in G}\theta(x)>l$. Therefore we get

$$l<a_0\leqq l+s. \tag{2.3}$$

By (2.1) and (2.2), we have that

$$|F(G)|\geqq a_0-1+(a_0-l)(a_0-l-s),$$

and an elementary calculation shows that

$$\min\{a_0-1+(a_0-l)(a_0-l-s)\,|\,a_0=l+1,\ l+2,\ \cdots,\ l+s\}$$
$$=l+(1-s)s'+s'^2-1,$$

where $s'=\max\{1,\ [(s-1)/2]\}$. This completes the proof.

3. Proof of Theorem 1.

LEMMA 3.1. *Let G be an $\{l,\ l+1,\ l+2,\ \cdots,\ l+r-1\}$-group on Ω. Then we have*

$$l+1\leqq k\leqq l+r-1+\frac{n-(l+r-1)}{g},$$

where k is the number of G-orbits on Ω.

PROOF. The inequality $l+1\leqq k$ is trivial from $k=\frac{1}{g}\sum_{x\in G}\theta(x)>l$. Let $\alpha_i=\#\{x\in G^{\#}\,|\,\theta(x)=l+i\}$ for $0\leqq i\leqq r-1$. Then we have

$$g=1+\sum_{i=0}^{r-1}\alpha_i$$

and

$$gk=n+\sum_{i=0}^{r-1}(l+i)\alpha_i.$$

Since

$$\sum_{i=0}^{r-1}(l+i)\alpha_i\leqq(l+r-1)\sum_{i=0}^{r-1}\alpha_i=(l+r-1)(g-1),$$

we get

$$gk-n\leqq(l+r-1)(g-1),$$

and hence the desired result.

Let $\Delta_1,\ \Delta_2,\ \cdots,\ \Delta_k$ be the G-orbits on Ω. We may assume $|\Delta_i|\geqq 2$ for all i by induction on n. Choose Δ_{i_j} and subsets Γ_{i_j} of Δ_{i_j} ($j=1,\ 2,\ \cdots,\ t$) such that

$$|\Gamma_{i_1}|+|\Gamma_{i_2}|+\cdots+|\Gamma_{i_t}|=l+r-k,$$

and

$$|\Delta_{i_j}-\Gamma_{i_j}|=1 \qquad \text{for } j=1,\ 2,\ \cdots,\ t-1,$$
$$|\Delta_{i_t}-\Gamma_{i_t}|\geqq 1.$$

This choice is possible because $\sum_{i=1}^{k}(|\Delta_i|-1)=n-k\geqq l+r-k$. Notice that $l+r-k\geqq 1$ by Lemma 3.1. By renumbering, we may assume $i_1=1,\ i_2=2,\ \cdots,\ i_t=t$.

Let H denote the pointwise stabilizer of $\Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_t$. We shall find upper and lower bounds for the order of H.

It is clear that

$$(\theta, \chi_0)_H \geqq (\theta, \chi_0)_G + |\Gamma_1| + |\Gamma_2| + \cdots + |\Gamma_t| = l + r,$$

where χ_0 is the principal character. On the other hand, we have

$$(\theta, \chi_0)_H \leqq l + r - 1 + \frac{n - (l + r - 1)}{|H|}$$

by Lemma 3.1. Therefore we get

$$|H| \leqq n - (l + r - 1). \tag{3.1}$$

Let us set $\gamma_i = |\Gamma_i|$ and $\delta_i = |\Delta_i|$. We have an inequality

$$\begin{aligned} |G : H| &= |G^{\Delta_1} : H^{\Delta_1}| \cdot |G^{\Delta_2}_{\Delta_1} : H^{\Delta_2}_{\Delta_1}| \cdot \cdots \cdot |G^{\Delta_t}_{\Delta_1 \cup \cdots \cup \Delta_{t-1}} : H^{\Delta_t}_{\Delta_1 \cup \cdots \cup \Delta_{t-1}}| \\ &\leqq \delta_1 ! \cdot \delta_2 ! \cdots \delta_{t-1} ! \cdot \delta_t (\delta_t - 1) \cdots (\delta_t - \gamma_t + 1), \end{aligned}$$

where G^{Δ_1} is the restriction of G to Δ_1, G_{Δ_1} is the pointwise stabilizer of Δ_1 and so on. Since $g = (n-l)(n-l-1) \cdots (n-l-r+1)$, we get

$$|H| \geqq \frac{(n-l)(n-l-1) \cdots (n-l-r+1)}{\delta_1 ! \cdots \delta_{t-1} ! \, \delta_t (\delta_t - 1) \cdots (\delta_t - \gamma_t + 1)}. \tag{3.2}$$

By (3.1) and (3.2), we obtain

$$\delta_t (\delta_t - 1) \cdots (\delta_t - \gamma_t + 1) \delta_{t-1} ! \cdots \delta_1 ! \geqq (n-l)(n-l-1) \cdots (n-l-r+2). \tag{3.3}$$

The right hand side of (3.3) is the product of $r-1$ consecutive integers beginning from $n-l-r+2$ ($\geqq 3$) and ending at $n-l$ ($\geqq \delta_t$); the inequality $n-l \geqq \delta_t$ comes from the inequality $\delta_t = n - \sum_{i \neq t} |\Delta_i| \leqq n - 2(k-1)$ and Lemma 3.1. Neglecting 1, the left hand side of (3.3) is a product of $\gamma_1 + \gamma_2 + \cdots + \gamma_t$ integers with $\gamma_1 + \gamma_2 + \cdots + \gamma_t \leqq r - 1$; the last inequality comes from $\gamma_1 + \gamma_2 + \cdots + \gamma_t = l + r - k$ and Lemma 3.1. Therefore (3.3) holds if and only if $t = 1$, $\delta_t = n - l$ and $\gamma_t = r - 1$. The identity $\delta_t = n - l$ implies $l = 0$. Using Lemma 2.1 repeatedly, we get the desired result.

4. Proof of Theorem 2.

We may assume $F(G) = \emptyset$ without loss of generality. The following two cases are possible by Lemma 2.2

Case I $L = \{0, 2\}$,

and

Case II $L = \{1, 3\}$.

Suppose that Case I holds. G is transitive on Ω by Lemma 2.1 and G_α has three orbits of length 1, 1, $|G_\alpha|$. Such rank 3 groups have been determined by Tuzuku [**14**], and G is one of the groups listed in Theorem 1 (i).

Suppose that Case II holds. By (2.1), (2.2) and (2.3), we have that $\sum' a_i^2=1-(a_0-1)(a_0-3)\geqq a_0$ and $2\leqq a_0\leqq 3$. Therefore we get $a_0=2$, $\sum' a_i^2=2$ and $\theta=2\chi_0+\chi_1+\chi_2$ ($\chi_1\neq\chi_2$). G has two orbits Δ_1, Δ_2 and G is 2-transitive on both Δ_1 and Δ_2.

Let us set $n_i=|\Delta_i|$ ($i=1, 2$) and $d_i=|G_{\alpha,\beta}|$ for distinct $\alpha, \beta\in\Delta_i$ ($i=1, 2$). Then we have

$$g=(n-1)(n-3)=d_i n_i(n_i-1)\,. \tag{4.1}$$

We may assume $n_1\geqq n_2$. We shall show that the solutions of (4.1) are $(d_1, d_2, n_1, n_2)=(2, 4, 4, 3)$, $(3, 4, 8, 7)$. Since $d_i(n_i-1)^2<g<(n-2)^2$ and $(n-3)^2<g<d_i\left(n_i-\frac{1}{2}\right)^2$, we get $(n_i-1)/(n-2)<1/\sqrt{d_i}<\left(n_i-\frac{1}{2}\right)\Big/(n-3)$. Therefore we have

$$1<1/\sqrt{d_1}+1/\sqrt{d_2}<1+\frac{2}{n-3}. \tag{4.2}$$

The possible values of d_1 are 1, 2 and 3, because $n_1\geqq n/2$ and $(n-1)(n-3)\geqq d_1\frac{n}{2}\frac{n-2}{2}$ by (4.1). If $d_1=1$ holds, then $n_1^2-n_1-(n-1)(n-3)=0$ by (4.1) and so $n-2<n_1<n-1$, a contradiction. If $d_1=2$ holds, then $d_2\leqq 11$ and $n\leqq 235$ by (4.2), and the solution of (4.1) is $(d_1, d_2, n_1, n_2)=(2, 4, 4, 3)$. If $d_1=3$ holds, then $d_2\leqq 5$ and $n\leqq 84$ by (4.2), and the solution of (4.1) is $(d_1, d_2, n_1, n_2)=(3, 4, 8, 7)$. The groups S_4, $PSL(2, 7)$ in the theorem come from the above parameters. This completes the proof.

5. Proof of Theorem 3.

We may assume $F(G)=\emptyset$ without loss of generality. By Lemma 2.2, the following three cases are possible:

Case I $L=\{0, 3\}$,

Case II $L=\{1, 4\}$,

and

Case III $L=\{2, 5\}$.

Case I. Suppose that Case I holds. Then G is transitive and G_α is a sharp $\{2\}$-group on $\Omega-\{\alpha\}$ by Lemma 2.1. By Iwahori [**6**],

(1) G_α fixes two points on $\Omega-\{\alpha\}$ and is regular on the remaining points,

(2) G_α is a generalized dihedral group,

or

(3) G_α is A_4, S_4 or A_5.

Suppose that the subcase (1) holds. Set $A=F(G_\alpha)$, $\Sigma=\{A^x | x\in G\}$ and $|\Sigma|=r$. Then $|A|=3$, $n=3r$, $g=9r(r-1)$ and G is doubly transitive on Σ.

For a subgroup X of G and $A, B\in\Sigma$, we use the following notation:

$$X_A=\{x\in X | \alpha^x=\alpha \quad \text{for all } \alpha\in A\},$$

$$X_A^*=\{x\in X | A^x=A\},$$

$$X_{A,B}^*=\{x\in X | A^x=A,\ B^x=B\},$$

and

$$X_{\{A,B\}}^*=\{x\in X | \{A, B\}^x=\{A, B\}\}.$$

$I(X)$ denotes the set of involutions of X.

Choose distinct blocks $A, B\in\Sigma$. Let $K=G_{A,B}^*$. Then K is of order 9, K_A and K_B are of order 3. Choose an involution t which interchanges A and B, and let $K_A=\langle a\rangle$, $K_B=\langle b\rangle$. We may assume $a^t=b$, where $a^t=t^{-1}at$. Then $K=\langle a\rangle\times\langle b\rangle$, $G_{\{A,B\}}^*=K\langle t\rangle$ and $I(K\langle t\rangle)=\{t, t^a, t^b\}$.

Let $F_\Sigma(K)=\{C\in\Sigma | C^x=C \text{ for all } x\in K\}$. We shall show $|F_\Sigma(K)|\leqq 3$. Suppose that $F_\Sigma(K)$ contains four distinct blocks A, B, C, D. Then K_A, K_B, K_C and K_D are distinct subgroups of order 3, so we may assume $K_C=\langle ab\rangle$ and $K_D=\langle a^{-1}b\rangle$. Since t normalizes $\langle ab\rangle$ and $\langle a^{-1}b\rangle$, t acts on $F(\langle ab\rangle)=C$ and $F(\langle a^{-1}b\rangle)=D$. This contradicts the fact that $G_{C,D}^*$ is order 9. Therefore $|F_\Sigma(K)|\leqq 3$.

Suppose $F_\Sigma(K)=\{A, B, C\}$. Since t normalizes K, t acts on $F_\Sigma(K)$ and so $C^t=C$. Therefore r is odd. By counting the number of

$$\{(u, \{D, E\}) | u\in I(G),\ D, E\in\Sigma,\ D\neq E,\ D^u=E\},$$

we get $|I(G)|(r-1)/2=\binom{r}{2}|I(K\langle t\rangle)|$ i.e. $|I(G)|=3r$ and so $|I(G_C^*)|=3$. Hence we have $I(G_C^*)=\{t, t^a, t^b\}$. Since $tt^a=b^{-1}a$ and $\langle tt^a\rangle$ char $\langle t, t^a\rangle=\langle I(G_C^*)\rangle\lhd G_C^*$, $\langle b^{-1}a\rangle$ is normal in G_C^*. Since G_C^* is transitive on $\Sigma-\{C\}$, $\langle b^{-1}a\rangle$ is contained in N, where N is the kernel of G on Σ. However, $b^{-1}a$ fixes each point of C, because $F_\Sigma(b^{-1}a)\ni C$, $F(t)=C$ and t inverts $b^{-1}a$. So N intersects G_D nontrivially for any $D\in\Sigma$. Since $K\supseteq N$, K intersects G_D nontrivially for any $D\in\Sigma$ and so we obtain $r=3$. We can verify directly that $G\cong(Z_3\times Z_3)\rtimes S_3$ with $|Z(G)|=3$.

Suppose $F_\Sigma(K)=\{A, B\}$. Let N be the kernel of G on Σ and $\bar{G}=G/N$. Since K is of odd order, G has a regular normal subgroup or a normal subgroup isomorphic to $PSL(2, q)$, $PSU(3, q)$ or $Sz(q)$ (Bender [2]). The 2-point stabilizers of $PSL(2, q)$, $PSU(3, q)$, $Sz(q)$ are cyclic subgroups of order $(q-1)/(2, q-1)$, $(q^2-1)/(3, q+1)$, $q-1$ respectively, whereas $K\ (=G_{A,B}^*)$ is an elementary abelian subgroup of order 9. So the possible normal subgroups are $PSL(2, 4)$ and $PSL(2, 7)$. We can verify directly that G is $Z_3\times PSL(2, 4)$ or $Z_3\times PSL(2, 7)$. (Notice that the Schur multipliers of $PSL(2, 4)$ and $PSL(2, 7)$ are both Z_2.) Therefore we may assume that G^Σ has a regular normal subgroup $\bar{R}$. $\bar{R}$ is an elementary abelian 2-group of order r, because $|F_{\Sigma}(K)|=2$. Any involution is conjugate to an element of $I(G^*_{A,B})$ $(=\{t, t^a, t^b\})$, so $I(G)$ is one class. By the same counting method in the case $|F_{\Sigma}(K)|=3$, we get $|I(G)|=3(r-1)$. Let S be a Sylow 2-subgroup of G. Suppose $r>2$. If some involution inverts the kernel N, then every involution inverts N, since $I(G)$ is one class. This is impossible. Therefore S commutes N. Since $\bar{S}\bar{N}=\bar{R}$ and N is of odd order, S is normal in G and so $|I(G)|=|S^{\#}|=r-1$, a contradiction. So $r=2$ and $G\cong(Z_3\times Z_3)\rtimes Z_2$.

Suppose that the subcase (2) holds i. e. G_α has a normal subgroup Q of index 2 such that Q has a cyclic Sylow 2-subgroup and any element of $G_\alpha-Q$ is an involution which inverts Q. G_α has four orbits $\{\alpha\}$, Γ_1, Γ_2, Γ_3 of length 1, 2, $|Q|$, $|Q|$ respectively, Q fixes Γ_1 pointwise and is regular on both Γ_2 and Γ_3, and any element of $G_\alpha-Q$ interchanges the two points of Γ_1.

Suppose $|Q|=2$. Then $n=7$, $g=7\cdot 4$, so G has an element of order 14, a contradiction. Therefore $|Q|\geqq 3$. Choose $x\in G$ and $\beta\in\Gamma_1$ such that $\beta=\alpha^x$. Since Q and Q^x are subgroups of G_β of index 2, $Q\cap Q^x$ is not trivial. Therefore $F(Q)=F(y)=F(Q^x)$ for nonidentity $y\in Q\cap Q^x$ and so $Q^x=G_{\alpha\beta}=Q$. For $\gamma\in\Gamma_2$, there exist involutions $t\in G_\alpha-Q$ and $u\in G_\beta-Q$ which fix γ. Since t and u invert Q, tu centralizes Q and so Q acts on $F(tu)$. Since $F(tu)$ contains γ and $\gamma^Q=\Gamma_2$, we have $|F(tu)|\geqq|\Gamma_2|$. Since G is a $\{0, 3\}$-group, $|Q|=|\Gamma_2|=3$ and so $n=9$. We can verify that $G\cong(Z_3\times Z_3)\rtimes S_3$ with $Z(G)=1$.

Suppose that the subcase (3) holds. We can verify by case by case argument that $G\cong(Z_3\times Z_3\times Z_3)\rtimes S_4$ with $G_\alpha\cong S_4$. Here $\varepsilon\chi$ is the character of S_4 acting on $Z_3\times Z_3\times Z_3$ in the semidirect product, where ε is the signature and $1+\chi$ is the usual 2-transitive permutation character of S_4.

REMARK. See also [**12**] section 6 for the subcase (1) and [**11**] Corollary for the subcases (2) and (3). The group $Z_3\times A_5$ is missed in the theorem 6.3 [**12**].

Case II *and Case* III. By (2.1), (2.2) and (2.3), the possible cases are

(1) G is a sharp $\{1, 4\}$ or $\{2, 5\}$-group with three orbits Δ_1, Δ_2, Δ_3 and G is 2-transitive on each orbit. For all distinct i, j, (G, Δ_i) is not isomorphic to (G, Δ_j) and G is transitive on $\Delta_i\times\Delta_j$.

(2) G is a sharp $\{1, 4\}$-group with two orbits Δ_1, Δ_2. G is 2-transitive on Δ_1 and is rank 3 on Δ_2. G is transitive on $\Delta_1\times\Delta_2$.

We first show that we may assume every orbit of G has length at least 5 (resp. 6) if G is a sharp $\{1, 4\}$ (resp. $\{2, 5\}$)-group. Suppose that G is a sharp $\{2, 5\}$-group and has an orbit Δ_1 of length 5. Let N be the kernel of G on Δ_1. Then $G/N\cong Z_5\rtimes Z_4$, A_5 or S_5 and N is a regular normal subgroup on each of the remaining orbits Δ_2, Δ_3. So N is elementary abelian, and $|N|^2$ divides $|G|$ because G is transitive on $\Delta_2\times\Delta_3$. Therefore $|N|=2, 3, 4, 5$ or 8, but this contradicts the condition $g=(n-2)(n-5)$ and $n=5+|N|+|N|$.

Suppose G is a sharp $\{2, 5\}$-group and has an orbit Δ_1 of length less than

5. For distinct $\alpha, \beta, \gamma \in \Delta_1$, $G_{\alpha,\beta,\gamma}$ is a $\{5\}$-group. So we get $(n-2)(n-5)=g \leqq |\Delta_1|(|\Delta_1|-1)(|\Delta_1|-2)|G_{\alpha,\beta,\gamma}| \leqq 4\cdot 3\cdot 2(n-5)$ i.e. $n\leqq 26$ by Kiyota's inequality (*). Since G is 2-transitive on each Δ_i and is transitive on $\Delta_i \times \Delta_j$ $(i\neq j)$, we have that $7\leqq n=n_1+n_2+n_3\leqq 26$, $n_i(n_i-1)$ divides $(n-2)(n-5)$ $(=g)$ for all i and $n_i n_j$ divides $(n-2)(n-5)$ for all distinct i, j, where $n_i=|\Delta_i|$. The (n, n_1, n_2, n_3) which satisfies the above condition is only (8, 2, 3, 3), and we get $G\cong(Z_3\times Z_3)\rtimes Z_2$.

Similarly we can show that every orbit of G has length at least 5 if G is a sharp $\{1, 4\}$-group. Therefore we may assume that G is faithful on every orbit Δ_i.

Next we show that G has no regular normal subgroup on Δ_i, if G is 2-transitive on Δ_i. Suppose that the subcase (2) holds and G has a regular normal subgroup R on Δ_1. R acts on $F(x)$ and $F(y)-F(x)$ for $x, y\in R$, since R is abelian. $|F(x)|=4$ holds and $\Delta_2\supsetneqq F(x)$ for any nonidentity $x\in R$. We can find nonidentity elements x, y in R such that $F(x)\neq F(y)$. Let R_0 be the kernel of R on $F(x)$. Then R_0 is semiregular on $F(y)-F(x)$. Therefore we get $|R_0| \leqq |F(y)-F(x)|\leqq 4$. Since $|R|=|R_0|\cdot|F(x)|$, R is of order 8 or 16.

Suppose $|R|=8$. Then $|\Delta_1|=8$ and $G_\alpha\subseteq GL(3, 2)$ for $\alpha\in\Delta_1$. Since G_α is transitive on Δ_2, $|\Delta_2|$ divides $2^3\cdot 3\cdot 7$ $(=|GL(3, 2)|)$. Since 8 divides $|\Delta_1|$ and $(n-1)(n-4)$ $(=g)$, $|\Delta_2|\equiv 1$ or 4 mod 8. Therefore $|\Delta_2|=12$, 28 or 84 and $g=(n-1)(n-4)=19\cdot 16$, $35\cdot 32$ or $91\cdot 88$. This contradicts the condition that g divides $|R|\cdot|GL(3, 2)|$. Similarly the assumption $|R|=16$ leads to a contradiction.

The subcase (1) is similar and easier to prove the nonexistence of a regular normal subgroup.

Let μ_{Δ_i} be the maximal number of fixed points of involutions on Δ_i. Then $\mu_{\Delta_i}\leqq 5$. Suppose that $\mu_{\Delta_1}=5$ with an involution u fixing 5 points on Δ_1. Then G is $\{2, 5\}$-sharp and so has two more orbits Δ_2, Δ_3. Since u has no fixed points on Δ_2 and Δ_3, $|\Delta_2|$ and $|\Delta_3|$ are even and so $\mu_{\Delta_i}\leqq 4$ $(i=2, 3)$. Therefore in the subcase (1), we may assume that $\mu_{\Delta_1}\leqq 5$, $\mu_{\Delta_2}\leqq 4$ and $\mu_{\Delta_3}\leqq 4$. Obviously in the subcase (2), $\mu_{\Delta_1}\leqq 4$.

If G is 2-transitive on Δ_i with $\mu_{\Delta_i}\leqq 4$, then G has a normal subgroup isomorphic to

(a) $PSL(2, q)$ or $Sz(q)$

or G is isomorphic to

(b) S_5, A_6, S_6 $(n_i=6, 10)$, A_7 $(n_i=7, 15)$, M_{11}, $PSL(3, 2)$, $PSL(2, 11)$ $(n_i=11)$ or $P\Gamma L(2, 8)$ $(n_i=28)$, where $n_i=|\Delta_i|$.

(All (G, Δ_i) are usual permutation representations except for S_6, A_7, $PSL(2, 11)$, $P\Gamma L(2, 8)$. See [1], [2], [3], [5], [9].) The reason why $PSU(3, q)$ is missed in (a) is that a diagonal element of $PSU(3, q)$ fixes $q+1$ points and that if $q=3$

or 4, G does not satisfy the condition $g=(n-1)(n-4)$ or $(n-2)(n-5)$.

Suppose that the subcase (1) holds. Then (G, Δ_i) is determined by the list (a), (b) for $i=2, 3$. Since (G, Δ_2) and (G, Δ_3) are not isomorphic, G is S_6, A_7, $PSL(2, 11)$ or $P\Gamma L(2, 8)$. Since these groups have at most two non-isomorphic 2-transitive representations, (G, Δ_1) is isomorphic to (G, Δ_2) or (G, Δ_3), a contradiction.

Suppose that the subcase (2) holds and G has a normal subgroup M listed in (a). First suppose that G is a Zassenhaus group on Δ_1. Let θ_i be the permutation character of G on Δ_i for $i=1, 2$, and

$$\alpha_{ij}=\#\{x\in G \mid \theta_1(x)=i,\ \theta_2(x)=j\},$$
$$\alpha_i=\#\{x\in G \mid \theta_1(x)=i\}.$$

Then, since $(\theta_1, \theta_2)=1$, we have

$$g=n_1n_2+3(\alpha_{13}+\alpha_{31})+4\alpha_{22},$$

where $n_i=|\Delta_i|$. Since $\alpha_{22}=\alpha_2=\frac{1}{2}(g-n_1^2+n_1)$, we get

$$g\geqq n_1n_2+4\alpha_2=n_1n_2+2g-2n_1(n_1-1),$$
$$\text{i.e.}\quad 2\geqq g/n_1(n_1-1)+n_2/(n_1-1).$$

Therefore $|G_{\alpha,\beta}|=g/n_1(n_1-1)=1$ for distinct α, $\beta\in\Delta_1$, a contradiction.

Next suppose that G contains an element σ $(\neq 1)$ which fixes at least 3 points on Δ_1. If M is $Sz(q)$, we may assume σ is a field automorphism, and then σ fixes at least 2^2+1 points on Δ_1, which is a contradiction. Hence M is $PSL(2, q)$. We shall show that σ is of order 2. Let H be a σ-invariant 2-point stabilizer of M on Δ_1. H is a cyclic subgroup. Let x be a generator of H and $F_{\Delta_2}(x)=\{\alpha\in\Delta_2 \mid \alpha^x=\alpha\}$. Then $|F_{\Delta_2}(x)|=2$. Since σ normalizes H $(=\langle x\rangle)$, σ acts on $F_{\Delta_2}(x)$ $(=F_{\Delta_2}(\langle x\rangle))$. Since σ^2 fixes at least 3 points of Δ_1 and the two points of $F_{\Delta_2}(x)$, we get $\sigma^2=1$. Since σ fixes at least 3 points on Δ_1, $PSL(2, q)\langle\sigma\rangle$ contains a field automorphism f of order 2. Since f fixes $\sqrt{q}+1$ points on Δ_1, G is $PSL(2, 4)\langle f\rangle$, $PSL(2, 9)\langle f\rangle$ or $PGL(2, 9)\langle f\rangle$. This, however, contradicts the condition $g=(n-1)(n-4)$.

Thus in the subcase (2), G is one of the groups listed in (b). But none of them satisfies the condition $g=(n-1)(n-4)$. This completes the proof of Theorem 3.

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