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## Research Article

# **Sharp Power Mean Bounds for Sándor Mean**

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We prove that the double inequality  $M_p(a,b) < X(a,b) < M_q(a,b)$  holds for all a,b > 0 with  $a \ne b$  if and only if  $p \le 1/3$  and  $q \ge \log 2/(1 + \log 2) = 0.4093...$ , where X(a,b) and  $M_r(a,b)$  are the Sándor and rth power means of a and b, respectively.

## 1. Introduction

Let  $p \in \mathbb{R}$  and a, b > 0 with  $a \neq b$ . Then the pth power mean  $M_p(a, b)$  of a and b is given by

$$M_{p}\left(a,b\right)=\left(\frac{a^{p}+b^{p}}{2}\right)^{1/p} \quad \left(p\neq0\right), \ M_{0}\left(a,b\right)=\sqrt{ab}. \tag{2}$$

The main properties for the power mean are given in [1]. It is well known that  $M_p(a,b)$  is strictly increasing with respect to  $p \in \mathbb{R}$  for fixed a,b>0 with  $a\neq b$ . Many classical means are the special cases of the power mean; for example,  $M_{-1}(a,b)=2ab/(a+b)=H(a,b)$  is the harmonic mean,  $M_0(a,b)=\sqrt{ab}=G(a,b)$  is the geometric mean,  $M_1(a,b)=(a+b)/2=A(a,b)$  is the arithmetic mean, and  $M_2(a,b)=\sqrt{(a^2+b^2)/2}=Q(a,b)$  is the quadratic mean.

Let  $L(a,b) = (a-b)/(\log a - \log b)$ ,  $P(a,b) = (a-b)/[2\arcsin((a-b)/(a+b))]$ ,  $I(a,b) = (a^a/b^b)^{1/(a-b)}/e$ ,  $M(a,b) = (a-b)/[2\sinh^{-1}((a-b)/(a+b))]$ , and  $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$  be the logarithmic, first Seiffert, identric, Neuman-Sándor, and second Seiffert means of two distinct positive real numbers a and b, respectively. Then it is well known that the inequalities

$$H(a,b) < G(a,b) < L(a,b) < P(a,b)$$
  
 $< I(a,b) < A(a,b) < M(a,b)$  (2)  
 $< T(a,b) < Q(a,b)$ 

hold for all a, b > 0 with  $a \neq b$ .

Recently, the bounds for certain bivariate means in terms of the power mean have been the subject of intensive research. Seiffert [2] proved that the inequalities

$$\frac{2}{\pi}M_{1}(a,b) < P(a,b) < M_{1}(a,b)$$

$$< T(a,b) < M_{2}(a,b)$$
(3)

hold for all a, b > 0 with  $a \neq b$ .

Jagers [3] proved that the double inequality

$$M_{1/2}(a,b) < P(a,b) < M_{2/3}(a,b)$$
 (4)

holds for all a, b > 0 with  $a \neq b$ .

In [4, 5], Hästö established that

$$P(a,b) > M_{\log 2/\log \pi}(a,b),$$
  
 $P(a,b) > \frac{2\sqrt{2}}{\pi} M_{2/3}(a,b)$  (5)

for all a, b > 0 with  $a \neq b$ .

Witkowski [6] proved that the double inequality

$$\frac{2\sqrt{2}}{\pi}M_{2}(a,b) < T(a,b) < \frac{4}{\pi}M_{1}(a,b)$$
 (6)

holds for all a, b > 0 with  $a \neq b$ .

In [7], Costin and Toader presented that

$$M_{\log 2/(\log \pi - \log 2)}(a, b) < T(a, b)$$
  
 $< M_{5/3}(a, b)$  (7)

for all a, b > 0 with  $a \neq b$ .

Chu and Long [8] proved that the double inequality

$$M_p(a,b) < M(a,b) < M_a(a,b)$$
 (8)

holds for all a, b > 0 with  $a \neq b$  if and only if  $p \leq \log 2 / \log[2 \log(1 + \sqrt{2})] = 1.224...$  and  $q \geq 4/3$ .

The following sharp bounds for the logarithmic and identric means in terms of the power means can be found in the literature [9–16]:

$$\begin{split} &M_{0}\left(a,b\right) < L\left(a,b\right) < M_{1/3}\left(a,b\right), \\ &M_{2/3}\left(a,b\right) < I\left(a,b\right) < M_{\log 2}\left(a,b\right), \\ &M_{0}\left(a,b\right) < L^{1/2}\left(a,b\right)I^{1/2}\left(a,b\right) < M_{1/2}\left(a,b\right), \\ &M_{\log 2/(1+\log 2)}\left(a,b\right) < \frac{L\left(a,b\right) + I\left(a,b\right)}{2} < M_{1/2}\left(a,b\right) \end{split} \tag{9}$$

for all a, b > 0 with  $a \neq b$ .

Recently, Sándor [17] introduced the Sándor mean X(a, b) of two positive real numbers a and b, which is given by

$$X(a,b) = A(a,b) e^{(G(a,b)/P(a,b))-1}.$$
 (10)

In [18], Sándor proved that

$$X(a,b) < \frac{P^{2}(a,b)}{A(a,b)},$$

$$\frac{A(a,b)G(a,b)}{P(a,b)} < X(a,b) < \frac{A(a,b)P(a,b)}{2P(a,b) - G(a,b)},$$

$$X(a,b) > \frac{A(a,b)L(a,b)}{P(a,b)}e^{(G(a,b)/L(a,b))-1},$$

$$X(a,b) > \frac{A(a,b)[P(a,b) + G(a,b)]}{3P(a,b) - G(a,b)},$$

$$\frac{A^{2}(a,b)G(a,b)}{P(a,b)L(a,b)}e^{(L(a,b)/A(a,b))-1} < X(a,b)$$

$$< A(a,b)\left[\frac{1}{e} + \left(1 - \frac{1}{e}\right)\frac{G(a,b)}{P(a,b)}\right],$$

$$A(a,b) + G(a,b) - P(a,b) < X(a,b)$$

$$< A^{-1/3}(a,b)\left[\frac{A(a,b) + G(a,b)}{2}\right]^{4/3},$$

$$(11)$$

$$P^{1/(\log \pi - \log 2)}(a, b) A^{1-1/(\log \pi - \log 2)}(a, b)$$

$$< X(a,b) < P^{-1}(a,b) \left[ \frac{A(a,b) + G(a,b)}{2} \right]^{2}$$

for all a, b > 0 with  $a \neq b$ .

In the Introduction we cite only a minor part of the existing literature on the considered means. For example, an important paper on the first Seiffert mean P(a, b) is again due to Sándor [19].

The main purpose of this paper is to present the best possible parameters p and q such that the double inequality  $M_p(a,b) < X(a,b) < M_q(a,b)$  holds for all a,b > 0 with  $a \neq b$ .

#### 2. Lemmas

In order to prove our main results we need several lemmas, which we present in this section.

**Lemma 1.** Let  $g_1:(0,1)\times\mathbb{R}\to\mathbb{R}$  be defined by

$$g_1(x,p) = \frac{\sqrt{x}(x-1)(x^{p-1}+1)}{(x+1)(x^p+1)} - \arcsin\frac{x-1}{x+1}.$$
 (12)

Then

- (1)  $g_1(x, p)$  is strictly decreasing with respect to x on (0, 1) if and only if  $p \ge 1/2$ ;
- (2)  $g_1(x, p)$  is strictly increasing with respect to x on (0, 1) if and only if  $p \le 1/3$ .

Proof. It follows from (12) that

$$\frac{\partial g_1(x,p)}{\partial x} = \frac{(1-x)x^{p-3/2}}{2(x+1)^2(x^p+1)^2}g_2(x,p), \qquad (13)$$

where

$$g_2(x, p) = -3x^{1-p} - x^{2-p} + x^p + 3x^{p+1} + (2p-1)x^2 - 2p + 1.$$
 (14)

(1) If  $g_1(x, p)$  is strictly decreasing with respect to x on (0, 1), then (13) leads to the conclusion that  $g_2(x, p) < 0$  for all  $x \in (0, 1)$ . In particular, we have  $g_2(0^+, p) \le 0$ . We assert that  $p \ge 1/2$ . Indeed, from (14) we clearly see that  $g_2(0^+, 0) = 2$ ,  $g_2(0^+, p) = +\infty$  if p < 0, and  $g_2(0^+, p) = 1 - 2p > 0$  if 0 .

If  $p \ge 1/2$ , then it follows from (14) that

$$\frac{\partial g_2(x,p)}{\partial p} = \left(3x^{p+1} + 3x^{1-p} + x^{2-p} + x^p\right)\log x$$

$$-2\left(1 - x^2\right) < 0$$
(15)

for all  $x \in (0, 1)$ .

Equation (14) and inequality (15) lead to the conclusion that

$$g_2(x, p) \le g_2(x, \frac{1}{2}) = -2\sqrt{x}(1 - x) < 0$$
 (16)

for all  $x \in (0, 1)$ .

Therefore,  $g_1(x, p)$  is strictly decreasing with respect to x on (0, 1) which follows from (13) and (16).

(2) If  $g_1(x, p)$  is strictly increasing with respect to x on (0, 1), then (13) leads to the conclusion that  $g_2(x, p) > 0$  for all  $x \in (0, 1)$ . In particular, we have

$$\lim_{x \to 1^{-}} \frac{g_2(x, p)}{1 - x} = 4 - 12p \ge 0 \tag{17}$$

and  $p \le 1/3$ .

If  $p \le 1/3$ , then (14) and (15) lead to the conclusion that

$$g_2(x, p) \ge g_2(x, \frac{1}{3}) = \frac{1}{3} (1 + x^{1/3}) (1 + 5x^{1/3} + x^{2/3})$$

$$\times (1 - x^{1/3})^3 > 0$$
(18)

for all  $x \in (0, 1)$ .

Therefore,  $g_1(x, p)$  is strictly increasing with respect to x on (0, 1) which follows from (13) and (18).

**Lemma 2.** Let  $g_1: (0,1) \times \mathbb{R} \to \mathbb{R}$  be defined by (12). Then there exists  $x_0 \in (0,1)$  such that  $g_1(x,p)$  is strictly increasing with respect to x on  $(0,x_0]$  and strictly decreasing with respect to x on  $[x_0,1)$  if 1/3 .

*Proof.* Let  $p \in (1/3, 1/2)$  and  $g_2(x, p)$  be defined by (14). Then (14) leads to

$$q_2(0, p) = 1 - 2p > 0, q_2(1, p) = 0, (19)$$

$$x^{1-p} \frac{\partial g_2(x,p)}{\partial x}$$

$$= 3(p-1)x^{1-2p} + (p-2)x^{2-2p}$$

$$+ 2(2p-1)x^{2-p} + 3(p+1)x + p := g_3(x,p),$$

$$g_3(0,p) = p > 0, \qquad g_3(1,p) = 12p - 4 > 0, \qquad (21)$$

$$x^{2p} \frac{\partial g_3(x,p)}{\partial x}$$

$$= 3(p+1)x^{2p} - 2(2p-1)(p-2)x^{1+p}$$

$$-2(p-1)(p-2)x - 3(2p-1)(p-1)$$

$$:= g_4(x,p),$$
(22)

$$g_4(0, p) = -3(1-p)(1-2p) < 0,$$
  

$$g_4(1, p) = 4(3p-1)(2-p) > 0,$$
(23)

$$\frac{\partial^{2} g_{4}(x, p)}{\partial x^{2}} = -2p(1 - 2p)(p + 1) 
\times \left[3 + (2 - p)x^{1-p}\right]x^{2p-2} < 0$$
(24)

for  $x \in (0, 1)$ .

Inequality (24) implies that  $g_4(x, p)$  is strictly convex with respect to x on (0, 1). From (22) and (23) together with the strict convexity of  $g_4(x, p)$  with respect to x on (0, 1) we clearly see that there exists  $x_1 \in (0, 1)$  such that  $g_3(x, p)$  is strictly decreasing with respect to x on  $(0, x_1]$  and strictly increasing with respect to x on  $[x_1, 1)$ . We assert that

$$g_3(x_1, p) < 0.$$
 (25)

Indeed, if  $g_3(x_1, p) \ge 0$ , then it follows from (20) and the piecewise monotonicity of  $g_3(x, p)$  with respect to x on (0, 1) that  $g_2(x, p)$  is strictly increasing with respect to x on (0, 1).

Hence, we get  $g_2(x, p) < g_2(1, p) = 0$  for all  $x \in (0, 1)$ . This conjunction with Lemma 1 and (13) leads to the conclusion that  $p \ge 1/2$ , which contradicts with 1/3 .

From (20) and (21) together with (25) and the piecewise monotonicity of  $g_3(x, p)$  with respect to x on (0, 1) we clearly see that there exist  $x_{11} \in (0, x_1)$  and  $x_{12} \in (x_1, 1)$  such that  $g_2(x, p)$  is strictly increasing with respect to x on  $(0, x_{11}] \cup [x_{12}, 1)$  and strictly decreasing with respect to x on  $[x_{11}, x_{12}]$ .

Therefore, Lemma 2 follows easily from (13) and (19) together with the piecewise monotonicity of  $g_2(x, p)$  with respect to x on (0, 1).

**Lemma 3.** Let  $g_1:(0,1)\times\mathbb{R}\to\mathbb{R}$  be defined by (12). Then the following statements are true:

- (1)  $g_1(x, p) > 0$  for all  $x \in (0, 1)$  if and only if  $p \ge 1/2$ ;
- (2)  $g_1(x, p) < 0$  for all  $x \in (0, 1)$  if and only if  $p \le 1/3$ ;
- (3) if  $1/3 , then there exists <math>\mu_0 \in (0,1)$  such that  $g_1(\mu_0, p) = 0$ ,  $g_1(x, p) < 0$  for  $x \in (0, \mu_0)$ , and  $g_1(x, p) > 0$  for  $x \in (\mu_0, 1)$ .

*Proof.* (1) If  $g_1(x, p) > 0$  for all  $x \in (0, 1)$ , then  $g_1(0^+, p) \ge 0$ . Therefore,  $p \ge 1/2$  follows from  $g_1(0^+, p) = -\infty$  for p < 1/2.

If  $p \ge 1/2$ , then Lemma 1 (1) leads to the conclusion that  $g_1(x, p) > g_1(1, p) = 0$  for all  $x \in (0, 1)$ .

(2) If  $g_1(x, p) < 0$  for all  $x \in (0, 1)$ , then by making use of L'Höspital's rules and (12) we get

$$\lim_{x \to 1^{-}} \frac{g_1(x, p)}{(1 - x)^3} = \frac{1}{8} \left( p - \frac{1}{3} \right) \le 0 \tag{26}$$

and  $p \le 1/3$ .

If  $p \le 1/3$ , then Lemma 1 (2) leads to the conclusion that  $g_1(x, p) < g_1(1, p) = 0$  for all  $x \in (0, 1)$ .

(3) If 1/3 , then it follows from (12) that

$$g_1(0^+, p) = -\infty, \qquad g_1(1, p) = 0.$$
 (27)

Therefore, Lemma 3 (3) follows from Lemma 2 and (27).

**Lemma 4.** Let  $g:(0,1)\times(0,\infty)\to\mathbb{R}$  be defined by

$$g(x,p) = \log \frac{X(1,x)}{M_p(1,x)}$$

$$= \log \frac{x+1}{2} + \frac{2\sqrt{x}}{1-x} \arcsin \frac{1-x}{1+x}$$

$$-\frac{1}{p} \log \frac{x^p+1}{2} - 1.$$
(28)

Then

- (1) g(x, p) is strictly increasing with respect to x on (0, 1) if and only if  $p \ge 1/2$ ;
- (2) g(x, p) is strictly decreasing with respect to x on (0, 1) if and only if  $p \le 1/3$ ;
- (3) if  $1/3 , there exists <math>\mu_0 \in (0,1)$  such that g(x,p) is strictly decreasing with respect to x on  $(0,\mu_0]$  and strictly increasing with respect to x on  $[\mu_0, 1)$ .

Proof. It follows from (28) that

$$\frac{\partial g\left(x,p\right)}{\partial x} = \frac{1+x}{\left(1-x\right)^{2}\sqrt{x}}g_{1}\left(x,p\right),\tag{29}$$

where  $g_1(x, p)$  is defined by (12).

Therefore, Lemma 4 follows from Lemma 3 and (29).

#### 3. Main Results

**Theorem 5.** *The double inequality* 

$$M_p(a,b) < X(a,b) < M_a(a,b)$$
 (30)

holds for all a, b > 0 with  $a \neq b$  if and only if  $p \leq 1/3$  and  $q \geq \log 2/(1 + \log 2) = 0.4093...$ 

*Proof.* Since both the Sándor mean X(a,b) and rth power mean  $M_r(a,b)$  are symmetric and homogeneous of degree 1, without loss of generality, we assume that a=1 and  $b=x \in (0,1)$ .

We first prove that the inequality  $X(1, x) > M_p(1, x)$  holds for all  $x \in (0, 1)$  if and only if  $p \le 1/3$ .

If p = 1/3, then from (28) and Lemma 4 (2) we get

$$\log \frac{X(1,x)}{M_{1/3}(1,x)} = g\left(x, \frac{1}{3}\right) > g\left(1^{-}, \frac{1}{3}\right) = 0$$
 (31)

for all  $x \in (0, 1)$ .

Therefore,  $X(1,x) > M_p(1,x)$  for all  $x \in (0,1)$  and  $p \le 1/3$  follows from (31) and the monotonicity of the function  $p \to M_p(1,x)$ .

If  $X(1, x) > M_p(1, x)$ , then (28) leads to g(x, p) > 0 for all  $x \in (0, 1)$ . In particular, we have

$$\lim_{x \to 1^{-}} \frac{g(x, p)}{(1 - x)^{2}} = \frac{1}{8} \left( \frac{1}{3} - p \right) \ge 0$$
 (32)

and  $p \le 1/3$ .

Next, we prove that the inequality  $X(1, x) < M_q(1, x)$  holds for all  $x \in (0, 1)$  if and only if  $q \ge \log 2/(1 + \log 2)$ .

If  $X(1,x) < M_q(1,x)$  holds for all  $x \in (0,1)$ , then (28) leads to g(x,q) < 0 for all  $x \in (0,1)$ . In particular, we have

$$g(0,q) = \left(\frac{1}{q} - 1\right)\log 2 - 1 \le 0 \tag{33}$$

and  $q \ge \log 2/(1 + \log 2)$ .

If  $q = \log 2/(1 + \log 2) \in (1/3, 1/2)$ , then (28) leads to

$$g\left(0, \frac{\log 2}{1 + \log 2}\right) = g\left(1, \frac{\log 2}{1 + \log 2}\right) = 0.$$
 (34)

It follows from (28) and (34) together with Lemma 4 (3) that

$$\log \frac{X(1,x)}{M_{\log 2/(1+\log 2)}(1,x)} = g\left(x, \frac{\log 2}{1+\log 2}\right) < 0$$
 (35)

for all  $x \in (0, 1)$ .

Therefore,  $X(1,x) < M_q(1,x)$  for all  $x \in (0,1)$  and  $q \ge \log 2/(1 + \log 2)$  follows from (35) and the monotonicity of the function  $q \to M_q(1,x)$ .

**Theorem 6.** Let a, b > 0 with  $a \neq b$ . Then the double inequality

$$\frac{2}{e}M_{1/2}(a,b) < X(a,b) < \frac{4}{e}M_{1/3}(a,b)$$
 (36)

holds with the best possible constants 2/e and 4/e.

*Proof.* Since both the Sándor mean X(a,b) and rth power mean  $M_r(a,b)$  are symmetric and homogeneous of degree 1, without loss of generality, we assume that a=1 and  $b=x\in(0,1)$ . It follows from Lemma 4 (1) and (2) together with (28) that

$$\log \frac{X(1,x)}{M_{1/2}(1,x)} = g\left(x, \frac{1}{2}\right) > g\left(0, \frac{1}{2}\right) = \log \frac{2}{e},$$

$$\log \frac{X(1,x)}{M_{1/2}(1,x)} = g\left(x, \frac{1}{3}\right) < g\left(0, \frac{1}{3}\right) = \log \frac{4}{e}$$
(37)

for all  $x \in (0, 1)$ .

Therefore,  $2/eM_{1/2}(1,x) < X(1,x) < 4/eM_{1/3}(1,x)$  for all  $x \in (0,1)$  follows from (37), and the optimality of the parameters 2/e and 4/e follows from the monotonicity of the functions g(x,1/2) and g(x,1/3).

Remark 7. For all  $a_1, a_2, b_1, b_2 > 0$  with  $a_1/b_1 < a_2/b_2 < 1$ . Then from Lemma 4 (1) and (2) together with (28) we clearly see that the Ky Fan type inequalities

$$\frac{M_{p}(a_{2},b_{2})}{M_{p}(a_{1},b_{1})} < \frac{X(a_{2},b_{2})}{X(a_{1},b_{1})} < \frac{M_{q}(a_{2},b_{2})}{M_{a}(a_{1},b_{1})}$$
(38)

hold if and only if  $p \ge 1/2$  and  $q \le 1/3$ .

Let  $p \in \mathbb{R}$  and  $L_p(a,b) = (a^{p+1} + b^{p+1})/(a^p + b^p)$  be the pth Lehmer mean of two positive real numbers a and b. Then the function  $g_1(x, p)$  defined by (12) can be rewritten as

$$g_1(x,p) = \frac{1}{2} (1-x) \left[ \frac{1}{P(1,x)} - \frac{G(1,x)}{A(1,x) L_{p-1}(1,x)} \right].$$
(39)

From Lemma 3 and (39) we get Remark 8 as follows.

Remark 8. The double inequality

$$\frac{A(a,b)}{G(a,b)}L_{p-1}(a,b) < P(a,b) < \frac{A(a,b)}{G(a,b)}L_{q-1}(a,b)$$
 (40)

holds for all a, b > 0 with  $a \neq b$  if and only if  $p \leq 1/3$  and  $q \geq 1/2$ .

From (5) and (9) together with Theorem 5 one has the following.

Remark 9. The inequalities

$$L(a,b) < M_{1/3}(a,b) < X(a,b) < M_{\log 2/(1+\log 2)}(a,b)$$
  
 $< M_{\log 2/\log \pi}(a,b) < P(a,b)$  (41)

hold for all a, b > 0 with  $a \neq b$ .

#### **Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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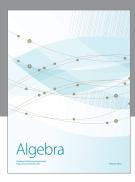
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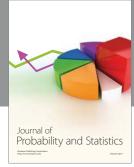
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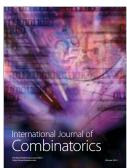






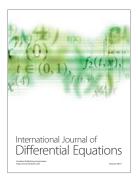


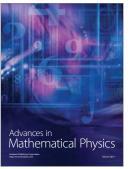


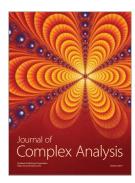


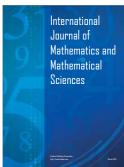


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