Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2015, Article ID 172867, 5 pages
http://dx.doi.org/10.1155/2015/172867

Research Article

Sharp Power Mean Bounds for Sándor Mean

Yu-Ming Chu, Zhen-Hang Yang, and Li-Min Wu

Department of Mathematics, Huzhou Teachers College, Huzhou 313000, China

Correspondence should be addressed to Yu-Ming Chu; chuyuming2005@126.com

Received 12 June 2014; Accepted 14 September 2014

Academic Editor: Agacik Zafer

We prove that the double inequality $M_p(a,b) < X(a,b) < M_q(a,b)$ holds for all $a,b > 0$ with $a \neq b$ if and only if $p \leq 1/3$ and $q \geq \log 2/(1+\log 2) = 0.4093\ldots$, where $X(a,b)$ and $M_r(a,b)$ are the Sándor and rth power means of a and b, respectively.

1. Introduction

Let $p \in \mathbb{R}$ and $a,b > 0$ with $a \neq b$. Then the pth power mean $M_p(a,b)$ of a and b is given by

$$M_p(a,b) = \left(\frac{a^p + b^p}{2}\right)^{1/p} \quad (p \neq 0), \quad M_0(a,b) = \sqrt{ab}. \tag{1}$$

The main properties for the power mean are given in [1]. It is well known that $M_p(a,b)$ is strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a,b > 0$ with $a \neq b$. Many classical means are the special cases of the power mean; for example, $M_{-1}(a,b) = 2ab/(a+b) = H(a,b)$ is the harmonic mean, $M_0(a,b) = \sqrt{ab} = G(a,b)$ is the geometric mean, $M_1(a,b) = (a+b)/2 = A(a,b)$ is the arithmetic mean, and $M_2(a,b) = \sqrt{(a^2+b^2)/2} = Q(a,b)$ is the quadratic mean.

Let $L(a,b) = (a-b)/(\log a - \log b)$, $P(a,b) = (a-b)/[2\arcsin((a-b)/(a+b))]$, $I(a,b) = (a^a/b^b)^{1/(a-b)}/e$, $M(a,b) = (a-b)/[2\sinh^{-1}((a-b)/(a+b))]$, and $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$ be the logarithmic, first Seiffert, identric, Neuman-Sándor, and second Seiffert means of two distinct positive real numbers a and b, respectively. Then it is well known that the inequalities

$$\begin{aligned} H(a,b) &< G(a,b) < L(a,b) < P(a,b) \\ &< I(a,b) < A(a,b) < M(a,b) \\ &< T(a,b) < Q(a,b) \end{aligned} \tag{2}$$

hold for all $a,b > 0$ with $a \neq b$.

Recently, the bounds for certain bivariate means in terms of the power mean have been the subject of intensive research. Seiffert [2] proved that the inequalities

$$\begin{aligned} \frac{2}{\pi}M_1(a,b) &< P(a,b) < M_1(a,b) \\ &< T(a,b) < M_2(a,b) \end{aligned} \tag{3}$$

hold for all $a,b > 0$ with $a \neq b$.

Jagers [3] proved that the double inequality

$$M_{1/2}(a,b) < P(a,b) < M_{2/3}(a,b) \tag{4}$$

holds for all $a,b > 0$ with $a \neq b$.

In [4, 5], Hästö established that

$$\begin{aligned} P(a,b) &> M_{\log 2/\log \pi}(a,b), \\ P(a,b) &> \frac{2\sqrt{2}}{\pi}M_{2/3}(a,b) \end{aligned} \tag{5}$$

for all $a,b > 0$ with $a \neq b$.

Witkowski [6] proved that the double inequality

$$\frac{2\sqrt{2}}{\pi}M_2(a,b) < T(a,b) < \frac{4}{\pi}M_1(a,b) \tag{6}$$

holds for all $a,b > 0$ with $a \neq b$.

In [7], Costin and Toader presented that

$$\begin{aligned} M_{\log 2/(\log \pi - \log 2)}(a,b) &< T(a,b) \\ &< M_{5/3}(a,b) \end{aligned} \tag{7}$$

for all $a,b > 0$ with $a \neq b$.

Chu and Long [8] proved that the double inequality

$$M_p(a,b) < M(a,b) < M_q(a,b) \tag{8}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq \log 2/\log[2\log(1+\sqrt{2})] = 1.224\ldots$ and $q \geq 4/3$.

The following sharp bounds for the logarithmic and identric means in terms of the power means can be found in the literature [9–16]:

$$\begin{aligned}
&M_0(a,b) < L(a,b) < M_{1/3}(a,b),\\
&M_{2/3}(a,b) < I(a,b) < M_{\log 2}(a,b),\\
&M_0(a,b) < L^{1/2}(a,b)\, I^{1/2}(a,b) < M_{1/2}(a,b),\\
&M_{\log 2/(1+\log 2)}(a,b) < \frac{L(a,b)+I(a,b)}{2} < M_{1/2}(a,b)
\end{aligned} \tag{9}$$

for all $a, b > 0$ with $a \neq b$.

Recently, Sándor [17] introduced the Sándor mean $X(a,b)$ of two positive real numbers a and b, which is given by

$$X(a,b) = A(a,b)\, e^{(G(a,b)/P(a,b))-1}. \tag{10}$$

In [18], Sándor proved that

$$\begin{aligned}
&X(a,b) < \frac{P^2(a,b)}{A(a,b)},\\
&\frac{A(a,b)G(a,b)}{P(a,b)} < X(a,b) < \frac{A(a,b)P(a,b)}{2P(a,b)-G(a,b)},\\
&X(a,b) > \frac{A(a,b)L(a,b)}{P(a,b)} e^{(G(a,b)/L(a,b))-1},\\
&X(a,b) > \frac{A(a,b)[P(a,b)+G(a,b)]}{3P(a,b)-G(a,b)},\\
&\frac{A^2(a,b)G(a,b)}{P(a,b)L(a,b)} e^{(L(a,b)/A(a,b))-1} < X(a,b)\\
&\quad < A(a,b)\left[\frac{1}{e} + \left(1-\frac{1}{e}\right)\frac{G(a,b)}{P(a,b)}\right],\\
&A(a,b)+G(a,b)-P(a,b) < X(a,b)\\
&\quad < A^{-1/3}(a,b)\left[\frac{A(a,b)+G(a,b)}{2}\right]^{4/3},\\
&P^{1/(\log\pi-\log 2)}(a,b)\, A^{1-1/(\log\pi-\log 2)}(a,b)\\
&\quad < X(a,b) < P^{-1}(a,b)\left[\frac{A(a,b)+G(a,b)}{2}\right]^2
\end{aligned} \tag{11}$$

for all $a, b > 0$ with $a \neq b$.

In the Introduction we cite only a minor part of the existing literature on the considered means. For example, an important paper on the first Seiffert mean $P(a,b)$ is again due to Sándor [19].

The main purpose of this paper is to present the best possible parameters p and q such that the double inequality $M_p(a,b) < X(a,b) < M_q(a,b)$ holds for all $a, b > 0$ with $a \neq b$.

2. Lemmas

In order to prove our main results we need several lemmas, which we present in this section.

Lemma 1. *Let $g_1 : (0,1)\times\mathbb{R} \to \mathbb{R}$ be defined by*

$$g_1(x,p) = \frac{\sqrt{x}(x-1)\left(x^{p-1}+1\right)}{(x+1)(x^p+1)} - \arcsin\frac{x-1}{x+1}. \tag{12}$$

Then

(1) *$g_1(x,p)$ is strictly decreasing with respect to x on $(0,1)$ if and only if $p \geq 1/2$;*

(2) *$g_1(x,p)$ is strictly increasing with respect to x on $(0,1)$ if and only if $p \leq 1/3$.*

Proof. It follows from (12) that

$$\frac{\partial g_1(x,p)}{\partial x} = \frac{(1-x)\,x^{p-3/2}}{2(x+1)^2(x^p+1)^2} g_2(x,p), \tag{13}$$

where

$$\begin{aligned}
g_2(x,p) = &-3x^{1-p} - x^{2-p} + x^p + 3x^{p+1}\\
&+ (2p-1)x^2 - 2p + 1.
\end{aligned} \tag{14}$$

(1) If $g_1(x,p)$ is strictly decreasing with respect to x on $(0,1)$, then (13) leads to the conclusion that $g_2(x,p) < 0$ for all $x \in (0,1)$. In particular, we have $g_2(0^+,p) \leq 0$. We assert that $p \geq 1/2$. Indeed, from (14) we clearly see that $g_2(0^+,0) = 2$, $g_2(0^+,p) = +\infty$ if $p < 0$, and $g_2(0^+,p) = 1-2p > 0$ if $0 < p < 1/2$.

If $p \geq 1/2$, then it follows from (14) that

$$\begin{aligned}
\frac{\partial g_2(x,p)}{\partial p} = &\left(3x^{p+1} + 3x^{1-p} + x^{2-p} + x^p\right)\log x\\
&- 2\left(1-x^2\right) < 0
\end{aligned} \tag{15}$$

for all $x \in (0,1)$.

Equation (14) and inequality (15) lead to the conclusion that

$$g_2(x,p) \leq g_2\left(x,\frac{1}{2}\right) = -2\sqrt{x}(1-x) < 0 \tag{16}$$

for all $x \in (0,1)$.

Therefore, $g_1(x,p)$ is strictly decreasing with respect to x on $(0,1)$ which follows from (13) and (16).

(2) If $g_1(x,p)$ is strictly increasing with respect to x on $(0,1)$, then (13) leads to the conclusion that $g_2(x,p) > 0$ for all $x \in (0,1)$. In particular, we have

$$\lim_{x\to 1^-} \frac{g_2(x,p)}{1-x} = 4 - 12p \geq 0 \tag{17}$$

and $p \leq 1/3$.

If $p \leq 1/3$, then (14) and (15) lead to the conclusion that

$$g_2(x,p) \geq g_2\left(x,\frac{1}{3}\right) = \frac{1}{3}\left(1+x^{1/3}\right)\left(1+5x^{1/3}+x^{2/3}\right) \times \left(1-x^{1/3}\right)^3 > 0 \tag{18}$$

for all $x \in (0,1)$.

Therefore, $g_1(x,p)$ is strictly increasing with respect to x on $(0,1)$ which follows from (13) and (18). □

Lemma 2. *Let $g_1 : (0,1)\times\mathbb{R} \to \mathbb{R}$ be defined by (12). Then there exists $x_0 \in (0,1)$ such that $g_1(x,p)$ is strictly increasing with respect to x on $(0,x_0]$ and strictly decreasing with respect to x on $[x_0,1)$ if $1/3 < p < 1/2$.*

Proof. Let $p \in (1/3,1/2)$ and $g_2(x,p)$ be defined by (14). Then (14) leads to

$$g_2(0,p) = 1-2p > 0, \qquad g_2(1,p) = 0, \tag{19}$$

$$x^{1-p}\frac{\partial g_2(x,p)}{\partial x} = 3(p-1)x^{1-2p} + (p-2)x^{2-2p} + 2(2p-1)x^{2-p} + 3(p+1)x + p := g_3(x,p), \tag{20}$$

$$g_3(0,p) = p > 0, \qquad g_3(1,p) = 12p-4 > 0, \tag{21}$$

$$x^{2p}\frac{\partial g_3(x,p)}{\partial x} = 3(p+1)x^{2p} - 2(2p-1)(p-2)x^{1+p} - 2(p-1)(p-2)x - 3(2p-1)(p-1) := g_4(x,p), \tag{22}$$

$$g_4(0,p) = -3(1-p)(1-2p) < 0, \quad g_4(1,p) = 4(3p-1)(2-p) > 0, \tag{23}$$

$$\frac{\partial^2 g_4(x,p)}{\partial x^2} = -2p(1-2p)(p+1) \times \left[3+(2-p)x^{1-p}\right]x^{2p-2} < 0 \tag{24}$$

for $x \in (0,1)$.

Inequality (24) implies that $g_4(x,p)$ is strictly convex with respect to x on $(0,1)$. From (22) and (23) together with the strict convexity of $g_4(x,p)$ with respect to x on $(0,1)$ we clearly see that there exists $x_1 \in (0,1)$ such that $g_3(x,p)$ is strictly decreasing with respect to x on $(0,x_1]$ and strictly increasing with respect to x on $[x_1,1)$. We assert that

$$g_3(x_1,p) < 0. \tag{25}$$

Indeed, if $g_3(x_1,p) \geq 0$, then it follows from (20) and the piecewise monotonicity of $g_3(x,p)$ with respect to x on $(0,1)$ that $g_2(x,p)$ is strictly increasing with respect to x on $(0,1)$. Hence, we get $g_2(x,p) < g_2(1,p) = 0$ for all $x \in (0,1)$. This conjunction with Lemma 1 and (13) leads to the conclusion that $p \geq 1/2$, which contradicts with $1/3 < p < 1/2$.

From (20) and (21) together with (25) and the piecewise monotonicity of $g_3(x,p)$ with respect to x on $(0,1)$ we clearly see that there exist $x_{11} \in (0,x_1)$ and $x_{12} \in (x_1,1)$ such that $g_2(x,p)$ is strictly increasing with respect to x on $(0,x_{11}] \cup [x_{12},1)$ and strictly decreasing with respect to x on $[x_{11},x_{12}]$.

Therefore, Lemma 2 follows easily from (13) and (19) together with the piecewise monotonicity of $g_2(x,p)$ with respect to x on $(0,1)$. □

Lemma 3. *Let $g_1 : (0,1)\times\mathbb{R} \to \mathbb{R}$ be defined by (12). Then the following statements are true:*

(1) *$g_1(x,p) > 0$ for all $x \in (0,1)$ if and only if $p \geq 1/2$;*

(2) *$g_1(x,p) < 0$ for all $x \in (0,1)$ if and only if $p \leq 1/3$;*

(3) *if $1/3 < p < 1/2$, then there exists $\mu_0 \in (0,1)$ such that $g_1(\mu_0,p) = 0$, $g_1(x,p) < 0$ for $x \in (0,\mu_0)$, and $g_1(x,p) > 0$ for $x \in (\mu_0,1)$.*

Proof. (1) If $g_1(x,p) > 0$ for all $x \in (0,1)$, then $g_1(0^+,p) \geq 0$. Therefore, $p \geq 1/2$ follows from $g_1(0^+,p) = -\infty$ for $p < 1/2$.

If $p \geq 1/2$, then Lemma 1 (1) leads to the conclusion that $g_1(x,p) > g_1(1,p) = 0$ for all $x \in (0,1)$.

(2) If $g_1(x,p) < 0$ for all $x \in (0,1)$, then by making use of L'Höspital's rules and (12) we get

$$\lim_{x\to 1^-}\frac{g_1(x,p)}{(1-x)^3} = \frac{1}{8}\left(p-\frac{1}{3}\right) \leq 0 \tag{26}$$

and $p \leq 1/3$.

If $p \leq 1/3$, then Lemma 1 (2) leads to the conclusion that $g_1(x,p) < g_1(1,p) = 0$ for all $x \in (0,1)$.

(3) If $1/3 < p < 1/2$, then it follows from (12) that

$$g_1(0^+,p) = -\infty, \qquad g_1(1,p) = 0. \tag{27}$$

Therefore, Lemma 3 (3) follows from Lemma 2 and (27). □

Lemma 4. *Let $g : (0,1)\times(0,\infty) \to \mathbb{R}$ be defined by*

$$g(x,p) = \log\frac{X(1,x)}{M_p(1,x)} = \log\frac{x+1}{2} + \frac{2\sqrt{x}}{1-x}\arcsin\frac{1-x}{1+x} - \frac{1}{p}\log\frac{x^p+1}{2} - 1. \tag{28}$$

Then

(1) *$g(x,p)$ is strictly increasing with respect to x on $(0,1)$ if and only if $p \geq 1/2$;*

(2) *$g(x,p)$ is strictly decreasing with respect to x on $(0,1)$ if and only if $p \leq 1/3$;*

(3) *if $1/3 < p < 1/2$, there exists $\mu_0 \in (0,1)$ such that $g(x,p)$ is strictly decreasing with respect to x on $(0,\mu_0]$ and strictly increasing with respect to x on $[\mu_0,1)$.*

Proof. It follows from (28) that

$$\frac{\partial g(x,p)}{\partial x} = \frac{1+x}{(1-x)^2\sqrt{x}} g_1(x,p), \tag{29}$$

where $g_1(x,p)$ is defined by (12).

Therefore, Lemma 4 follows from Lemma 3 and (29). □

3. Main Results

Theorem 5. *The double inequality*

$$M_p(a,b) < X(a,b) < M_q(a,b) \tag{30}$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $p \leq 1/3$ and $q \geq \log 2/(1+\log 2) = 0.4093\ldots$.

Proof. Since both the Sándor mean $X(a,b)$ and rth power mean $M_r(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a = 1$ and $b = x \in (0,1)$.

We first prove that the inequality $X(1,x) > M_p(1,x)$ holds for all $x \in (0,1)$ if and only if $p \leq 1/3$.

If $p = 1/3$, then from (28) and Lemma 4 (2) we get

$$\log\frac{X(1,x)}{M_{1/3}(1,x)} = g\left(x,\frac{1}{3}\right) > g\left(1^-,\frac{1}{3}\right) = 0 \tag{31}$$

for all $x \in (0,1)$.

Therefore, $X(1,x) > M_p(1,x)$ for all $x \in (0,1)$ and $p \leq 1/3$ follows from (31) and the monotonicity of the function $p \to M_p(1,x)$.

If $X(1,x) > M_p(1,x)$, then (28) leads to $g(x,p) > 0$ for all $x \in (0,1)$. In particular, we have

$$\lim_{x\to 1^-}\frac{g(x,p)}{(1-x)^2} = \frac{1}{8}\left(\frac{1}{3}-p\right) \geq 0 \tag{32}$$

and $p \leq 1/3$.

Next, we prove that the inequality $X(1,x) < M_q(1,x)$ holds for all $x \in (0,1)$ if and only if $q \geq \log 2/(1+\log 2)$.

If $X(1,x) < M_q(1,x)$ holds for all $x \in (0,1)$, then (28) leads to $g(x,q) < 0$ for all $x \in (0,1)$. In particular, we have

$$g(0,q) = \left(\frac{1}{q}-1\right)\log 2 - 1 \leq 0 \tag{33}$$

and $q \geq \log 2/(1+\log 2)$.

If $q = \log 2/(1+\log 2) \in (1/3,1/2)$, then (28) leads to

$$g\left(0,\frac{\log 2}{1+\log 2}\right) = g\left(1,\frac{\log 2}{1+\log 2}\right) = 0. \tag{34}$$

It follows from (28) and (34) together with Lemma 4 (3) that

$$\log\frac{X(1,x)}{M_{\log 2/(1+\log 2)}(1,x)} = g\left(x,\frac{\log 2}{1+\log 2}\right) < 0 \tag{35}$$

for all $x \in (0,1)$.

Therefore, $X(1,x) < M_q(1,x)$ for all $x \in (0,1)$ and $q \geq \log 2/(1+\log 2)$ follows from (35) and the monotonicity of the function $q \to M_q(1,x)$. □

Theorem 6. *Let $a,b > 0$ with $a \neq b$. Then the double inequality*

$$\frac{2}{e}M_{1/2}(a,b) < X(a,b) < \frac{4}{e}M_{1/3}(a,b) \tag{36}$$

holds with the best possible constants $2/e$ and $4/e$.

Proof. Since both the Sándor mean $X(a,b)$ and rth power mean $M_r(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a = 1$ and $b = x \in (0,1)$. It follows from Lemma 4 (1) and (2) together with (28) that

$$\begin{aligned}\log\frac{X(1,x)}{M_{1/2}(1,x)} &= g\left(x,\frac{1}{2}\right) > g\left(0,\frac{1}{2}\right) = \log\frac{2}{e},\\ \log\frac{X(1,x)}{M_{1/3}(1,x)} &= g\left(x,\frac{1}{3}\right) < g\left(0,\frac{1}{3}\right) = \log\frac{4}{e}\end{aligned} \tag{37}$$

for all $x \in (0,1)$.

Therefore, $2/eM_{1/2}(1,x) < X(1,x) < 4/eM_{1/3}(1,x)$ for all $x \in (0,1)$ follows from (37), and the optimality of the parameters $2/e$ and $4/e$ follows from the monotonicity of the functions $g(x,1/2)$ and $g(x,1/3)$. □

Remark 7. For all $a_1,a_2,b_1,b_2 > 0$ with $a_1/b_1 < a_2/b_2 < 1$. Then from Lemma 4 (1) and (2) together with (28) we clearly see that the Ky Fan type inequalities

$$\frac{M_p(a_2,b_2)}{M_p(a_1,b_1)} < \frac{X(a_2,b_2)}{X(a_1,b_1)} < \frac{M_q(a_2,b_2)}{M_q(a_1,b_1)} \tag{38}$$

hold if and only if $p \geq 1/2$ and $q \leq 1/3$.

Let $p \in \mathbb{R}$ and $L_p(a,b) = (a^{p+1}+b^{p+1})/(a^p+b^p)$ be the pth Lehmer mean of two positive real numbers a and b. Then the function $g_1(x,p)$ defined by (12) can be rewritten as

$$g_1(x,p) = \frac{1}{2}(1-x)\left[\frac{1}{P(1,x)} - \frac{G(1,x)}{A(1,x)L_{p-1}(1,x)}\right]. \tag{39}$$

From Lemma 3 and (39) we get Remark 8 as follows.

Remark 8. The double inequality

$$\frac{A(a,b)}{G(a,b)}L_{p-1}(a,b) < P(a,b) < \frac{A(a,b)}{G(a,b)}L_{q-1}(a,b) \tag{40}$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $p \leq 1/3$ and $q \geq 1/2$.

From (5) and (9) together with Theorem 5 one has the following.

Remark 9. The inequalities

$$\begin{aligned}L(a,b) &< M_{1/3}(a,b) < X(a,b) < M_{\log 2/(1+\log 2)}(a,b)\\ &< M_{\log 2/\log\pi}(a,b) < P(a,b)\end{aligned} \tag{41}$$

hold for all $a,b > 0$ with $a \neq b$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This research was supported by the Natural Science Foundation of China under Grants 11171307 and 61374086 and the Natural Science Foundation of Zhejiang Province under Grant LY13A010004.

References

[1] P. S. Bullen, D. S. Mitrinovic, and P. M. Vasić, *Means and Their Inequalities*, D. Reidel, Dordrecht, The Netherlands, 1988.

[2] H.-J. Seiffert, "Aufgabe β16," *Die Wurzel*, vol. 29, pp. 221–222, 1995.

[3] A. A. Jagers, "Solution of problem 887," *Nieuw Archief voor Wiskunde*, vol. 12, no. 4, pp. 230–231, 1994.

[4] P. A. Hästö, "A monotonicity property of ratios of symmetric homogeneous means," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 3, no. 5, article 71, 23 pages, 2002.

[5] P. A. Hästö, "Optimal inequalities between Seiffert's mean and power means," *Mathematical Inequalities & Applications*, vol. 7, no. 1, pp. 47–53, 2004.

[6] A. Witkowski, "Interpolations of Schwab-Borchardt mean," *Mathematical Inequalities & Applications*, vol. 16, no. 1, pp. 193–206, 2013.

[7] I. Costin and G. Toader, "Optimal evaluations of some Seiffert-type means by power means," *Applied Mathematics and Computation*, vol. 219, no. 9, pp. 4745–4754, 2013.

[8] Y.-M. Chu and B.-Y. Long, "Bounds of the Neuman-Sándor mean using power and identric means," *Abstract and Applied Analysis*, vol. 2013, Article ID 832591, 6 pages, 2013.

[9] H. Alzer, "Ungleichungen für mittelwerte," *Archiv der Mathematik*, vol. 47, no. 5, pp. 422–426, 1986.

[10] H. Alzer, "Ungleichungen für $(e/a)^a(b/e)^b$," *Elementary Mathematics*, vol. 40, pp. 120–123, 1985.

[11] F. Burk, "The geometric, logarithmic, and arithmetic mean inequality," *The American Mathematical Monthly*, vol. 94, no. 6, pp. 527–528, 1987.

[12] T. P. Lin, "The power mean and the logarithmic mean," *The American Mathematical Monthly*, vol. 81, pp. 879–883, 1974.

[13] A. O. Pittenger, "Inequalities between arithmetic and logarithmic means," *Publikacije Elektrotehničkog Fakulteta. Serija Matematika i Fizika*, no. 678–715, pp. 15–18, 1980.

[14] A. O. Pittenger, "The symmetric, logarithmic and power means," *Publikacije Elektrotehničkog Fakulteta. Serija Matematika i Fizika*, no. 678–715, pp. 19–23, 1980.

[15] K. B. Stolarsky, "The power and generalized logarithmic means," *The American Mathematical Monthly*, vol. 87, no. 7, pp. 545–548, 1980.

[16] H. Alzer and S.-L. Qiu, "Inequalities for means in two variables," *Archiv der Mathematik*, vol. 80, no. 2, pp. 201–215, 2003.

[17] J. Sándor, "Two sharp inequalities for trigonometric and hyperbolic functions," *Mathematical Inequalities &Applications*, vol. 15, no. 2, pp. 409–413, 2012.

[18] J. Sándor, "On two new means of two variables," *Notes on Number Theory and Discrete Mathematics*, vol. 20, no. 1, pp. 1–9, 2014.

[19] J. Sándor, "On certain inequalities for means III," *Archiv der Mathematik*, vol. 76, no. 1, pp. 34–40, 2001.