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SHARP WEIGHTED ESTIMATES FOR VECTOR-VALUED SINGULAR INTEGRAL OPERATORS AND COMMUTATORS

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Abstract. We prove sharp weighted norm inequalities for vector-valued singular integral operators and commutators. We first consider the strong (p, p) case with $p > 1$ and then the weak-type $(1, 1)$ estimate. Our results do not assume any condition on the weight function and involve iterations of the classical Hardy-Littlewood maximal function.

1. Introduction and Main Results. The purpose of this paper is to sharpen the results obtained in [5] for vector valued singular integral operators. Indeed, the method considered in that paper is based on extrapolation ideas. This method is very general and the results hold for any kind of operators. Furthermore, it is not possible to derive better results with such a generality. However, we are going to show that for vector valued singular integral operators we can improve those results.

To be more precise, we let T be a classical Calderón-Zygmund operator with kernel K (see Section 2.2), and let T_q, $q > 0$, be the vector-valued singular integral operator associated to T by

$$T_q f(x) = |Tf(x)|_q = \left(\sum_{j=1}^{\infty} |Tf_j(x)|^q \right)^{1/q},$$

where, by abuse of notation, we also denote by T the vector valued extension of the scalar operator T. For any Calderón-Zygmund singular integral operator T the first author proved in [11] that whenever $p > 1$ and $\varepsilon > 0$,

$$\int_{\boldsymbol{R}^n} |Tf(y)|^p w(y)dy \le C \int_{\boldsymbol{R}^n} |f(y)|^p M_{L(\log L)^{p-1+\varepsilon}}(w)(y)dy .$$

See Section 2.3 for the definition and main properties of the maximal operator of $M_{L(\log L)^\alpha}$, $\alpha > 0$. On the other hand, it is not hard to see that if we apply the extrapolation method from [5, Theorem 1.4] to the vector-valued singular operator T_q, we obtain, for $p > q > 1$ and $\varepsilon > 0$,

$$\int_{\boldsymbol{R}^n} |T_q f(y)|^p \, w(y)dy \le C \int_{\boldsymbol{R}^n} |f(y)|_q^p M_{L(\log L)^{p^2/q-1+\varepsilon}}(w)(y)dy , \tag{1.1}$$

where $|f(x)|_q = (\sum_{j=1}^{\infty} |f_j(x)|^q)^{1/q}$.

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We show in this paper that this estimate can be improved by using different techniques. Our result on (p, p)-type weighted estimates is the following.

THEOREM 1.1. *Let $1 < p, q < \infty$, $w(x)$ be a weight and T_q be a vector-valued singular integral operator. Suppose that $A(t)$ is a Young function satisfying the condition*

$$\int_c^\infty \left(\frac{t}{A(t)}\right)^{p'-1} \frac{dt}{t} < \infty \tag{1.2}$$

for some $c > 0$. Then there exists a constant $C > 0$ such that

$$\int_{\boldsymbol{R}^n} (T_q f(x))^p w(x) dx \le C \int_{\boldsymbol{R}^n} |f(x)|_q^p M_A(w)(x) dx\,. \tag{1.3}$$

Observe that (1.2) is independent of q and this is not the case of (1.1).

As a corollary we have the following result.

COROLLARY 1.2. *Let $1 < p, q < \infty$, $w(x)$ be a weight and T_q be a vector-valued singular integral operator.*

a) *Let $\varepsilon > 0$. Then there exists a constant $C > 0$ such that*

$$\int_{\boldsymbol{R}^n} (T_q f(x))^p w(x) dx \le C \int_{\boldsymbol{R}^n} |f(x)|_q^p M_{L(\log L)^{p-1+\varepsilon}}(w)(x) dx\,. \tag{1.4}$$

b) *As a consequence, we have that there exists a constant $C > 0$ such that*

$$\int_{\boldsymbol{R}^n} (T_q f(x))^p w(x) dx \le C \int_{\boldsymbol{R}^n} |f(x)|_q^p \, M^{[p]+1}(w)(x) dx\,. \tag{1.5}$$

Estimates (1.5) and (1.4) are sharp because they coincide with the corresponding scalar results, where the results are already optimal [11].

We remark that the first author obtained in [15] an estimate similar to (1.5) (also to (1.4)) for the vector-valued maximal operator

$$M_q f(x) = \left(\sum_{j=1}^{\infty} (M f_j(x))^q\right)^{1/q}.$$

The main result from [15] is

$$\int_{\boldsymbol{R}^n} (M_q f(x))^p w(x) dx \le C \int_{\boldsymbol{R}^n} |f(x)|_q^p M^{[p/q]+1}(w)(x) dx\,.$$

Observe that the operator $M^{[p]+1}$ is replaced by the pointwise smaller operator $M^{[p/q]+1}$. This result is also sharp, but is different from the corresponding scalar result, namely the celebrated Fefferman and Stein weighted estimate

$$\int_{\boldsymbol{R}^n} (Mf(x))^p w(x) dx \le c \int_{\boldsymbol{R}^n} |f(x)|^p Mw(x) dx\,.$$

As in the scalar situation, the proof of Theorem 1.1 is based on the Calderón-Zygmund classical principle which establishes the control of the singular integral operator by the Hardy-Littlewood maximal operator (see Theorem 1.3 below). Also our approach makes use of a pointwise estimate between the maximal operators $M_\delta^\#$ and M:

$$M_\delta^\#(T_q f)(x) \le C M(|f|_q)(x)\,, \tag{1.6}$$

where $0 < \delta < 1$ (see Lemma 3.1 for details). When $\delta = 1$, estimate (1.6) is false and the right hand side should be replaced by $M(|f|_q^r)(x)^{1/r}$, $r > 1$. In this case we believe that this estimate was known, but it is not sharp enough to derive our results.

As a consequence of (1.6), we deduce the following vector-valued version of the classical estimate of Coifman [2] which is used in the proof of Theorem 1.1.

THEOREM 1.3. *Let $1 < q < \infty$ and $0 < p < \infty$. Let $w(x)$ be a weight satisfying the A_∞ condition. Then the following a priori estimate holds: there exists a constant $C > 0$ such that*

$$\int_{\mathbf{R}^n} (T_q f(x))^p w(x)dx \le C[w]_{A_\infty}^p \int_{\mathbf{R}^n} (M(|f|_q)(x))^p w(x)dx \tag{1.7}$$

for any smooth function f for which the left hand side is finite. Similarly, we have that there exists a constant $C > 0$ such that

$$\|T_q f\|_{L^{p,\infty}(w)} \le C\|M(|f|_q)\|_{L^{p,\infty}(w)} \tag{1.8}$$

for any smooth vector function f for which the left hand side is finite.

We remark that it is not clear how to prove (1.7) adapting the good-λ inequality derived in [2].

The weighted weak-type (1, 1) estimate version of (1.3) is the following.

THEOREM 1.4. *Let $1 < q < \infty$ and $\varepsilon > 0$. Then there exists a constant $C > 0$ such that for any weight w and $\lambda > 0$*

$$w(\{y \in \mathbf{R}^n;\, |T_q f(y)| > \lambda\}) \le \frac{C}{\lambda} \int_{\mathbf{R}^n} |f(x)|_q M_{L(\log L)^\varepsilon}(w)(x)dx\,,$$

where f is an arbitrary smooth vector function.

REMARK 1.5. As above this result is a vector valued extension of the corresponding scalar result [11]. When T is replaced by M, the weight on the right hand side is the best possible, namely Mw ([15]). The result can be sharpened by replacing the maximal operator $M_{L(\log L)^\varepsilon}$ by the maximal operator M_A, where A is any Young function satisfying (1.2) for all $p > 1$.

We also consider in this paper vector-valued extensions of the by now classical commutator of Coifman-Rochberg-Weiss $[h, T]$ defined by the formula

$$[h, T]f(x) = h(x)Tf(x) - T(hf)(x) = \int_{\mathbf{R}^n} (h(x) - h(y))K(x, y)f(y)dy\,.$$

Here h is a locally integrable function and is usually called the symbol of the operator. T is any Calderón-Zygmund operator with kernel K. The main result from [3] establishes that, whenever the symbol h is a B.M.O. function, the commutator is bounded on $L^p(\mathbf{R}^n)$, $p > 1$. Later on this result was extended to the case $L^p(w)$, $w \in A_p$. The first author has shown in

[14] that there is a version of Coifman's estimate [2] where the role played by the maximal function M is replaced by $M^2 = M \circ M$. This is also the point of view of [12], where it is shown that commutators with B.M.O. functions are not of weak type $(1,1)$ but that they satisfy a $L(\log L)$ type estimate. These results show that somehow commutators with B.M.O. functions carry a higher degree of singularity. We will extend these estimates to the vector-valued context.

As above, for a sequence $f(x) = \{f_j(x)\}_{j=1}^{\infty}$ of functions, the vector–valued version of the commutator $[h, T]$ is given by the expression

$$[h, T]_q f(x) = |[h, T] f(x)|_q = \left(\sum_{j=1}^{\infty} |[h, T] f_j(x)|^q \right)^{1/q} .$$

As in the case of singular integrals, we first need an appropriate version of the Calderón-Zygmund principle. The precise estimate is given as follows.

THEOREM 1.6. *Let $1 < q < \infty$, $0 < p < \infty$, $w \in A_\infty$ and $h \in BMO$. Then there exists a constant $C > 0$ such that*

$$\int_{\mathbf{R}^n} ([h, T]_q f(x))^p w(x) dx \leq C [w]_{A_\infty}^{2p} \|h\|_{BMO}^p \int_{\mathbf{R}^n} (M_{L \log L}(|f|_q)(x))^p w(x) dx \tag{1.9}$$

for any smooth vector function f such that the left hand side is finite.

The proof of this theorem is also based on a pointwise estimate very much in the spirit of the pointwise estimate (1.6) required for the proof of Theorem 1.3, namely

$$M_\delta^{\#}([h, T]_q f)(x) \leq C \|h\|_{BMO} (M_\varepsilon(T_q f)(x) + M_{L \log L}(|f|_q)(x)) , \tag{1.10}$$

where $0 < \delta < \varepsilon$ (see Lemma 3.2). This time there is an extra term involving the maximal operator $M_{L \log L}$, which is pointwise comparable to M^2. This is optimal and explains why commutators have a higher degree of singularity.

By arguing as in [14, Theorem 2], we obtain the sharp two-weight estimates where no assumption is assumed on the weight w.

THEOREMM 1.7. *Let $1 < p, q < \infty$, $\delta > 0$ and $h \in BMO$. Then there exists a constant $C > 0$ such that for each weight w*

$$\int_{\mathbf{R}^n} ([h, T]_q f(x))^p w(x) dx \leq C \|h\|_{BMO}^p \int_{\mathbf{R}^n} |f(x)|_q^p M_{L(\log L)^{2p-1+\delta}}(w)(x) dx , \tag{1.11}$$

where $f = \{f_i\}_{i=1}^{\infty}$ is any sequence of bounded functions with compact support.

As in the singular integral operator case, (1.11) improves the result obtained for $[h, T]_q$ as an application of the general extrapolation theorem from [5] derived from the scalar estimate given in [14, Theorem 2].

The weighted weak-type $(1, 1)$ estimate version of (1.11) is the following.

THEOREM 1.8. *Let $1 < q < \infty$, $\varepsilon > 0$ and $h \in BMO$. Then there exists a constant $C > 0$ such that for any weight w and $\lambda > 0$*

$$w(\{x \in \boldsymbol{R}^n; |[h, T]_q f(x)| > \lambda\}) \le C \int_{\boldsymbol{R}^n} \Phi\left(\|h\|_{BMO} \frac{|f(x)|_q}{\lambda}\right) M_{L(\log L)^{1+\varepsilon}}(w)(x)dx\,,$$

where $\Phi(t) = t\log(e + t)$. The constant C is independent of the weight w, f and $\lambda > 0$.

This result is a vector valued version of the main result proved in [16].

2. Preliminaries. In this section we introduce the basic tools needed for the proof of the main results.

2.1. A_p weights and maximal operators. By a weight we mean a positive and locally integrable function. We say that a weight w belongs to the class A_p, $1 < p < \infty$, if there is a constant C such that

$$\left(\frac{1}{|Q|}\int_Q w(y)dy\right)\left(\frac{1}{|Q|}\int_Q w(y)^{1-p'}dy\right)^{p-1} \le C$$

for each cube Q and where as usual $1/p + 1/p' = 1$. A weight w belongs to the class A_1 if there is a constant C such that

$$\frac{1}{|Q|}\int_Q w(y)dy \le C \inf_Q w\,.$$

We will denote the infimum of the constants C by $[w]_{A_p}$. Observe that $[w]_{A_p} \ge 1$ by Hölder's inequality.

Since the A_p classes are increasing with respect to p, the A_∞ class of weights is defined in a natural way by $A_\infty = \bigcup_{p>1} A_p$. However, the following characterization is more interesting in applications: there are positive constants c and ρ such that for any cube Q and any measurable set E contained in Q

$$\frac{w(E)}{w(Q)} \le c\left(\frac{|E|}{|Q|}\right)^\rho.$$

We recall now the definitions of classical maximal operators. If, as usual, M denotes the Hardy-Littlewood maximal operator, we consider for $\delta > 0$

$$M_\delta f(x) = M(|f|^\delta)(x)^{1/\delta} = \left(\sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(y)|^\delta dy\right)^{1/\delta},$$

$$M^\#(f)(x) = \sup_{Q\ni x}\inf_c \frac{1}{|Q|}\int_Q |f(y) - c|dy \approx \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y) - (f)_Q|dy\,,$$

where as usual $(f)_Q$ denotes the average of f on Q, and a variant of this sharp maximal operator, which will become the main tool in our scheme, $M^\#_\delta f(x) = M^\#(|f|^\delta)(x)^{1/\delta}$.

The main inequality between these operators to be used is a version of the classical one due to Fefferman and Stein (see [6], [[9]).

THEOREM 2.1. *Let $0 < p, \delta < \infty$ and $w \in A_\infty$. There exists a positive constant C such that*

$$\int_{\boldsymbol{R}^n} M_\delta f(x)^p w(x)dx \le C[w]_{A_\infty}^p \int_{\boldsymbol{R}^n} M_\delta^\# f(x)^p w(x)dx \tag{2.1}$$

for every function f such that the left hand side is finite.

2.2. Calderón-Zygmund operators. By a kernel K in $\boldsymbol{R}^n \times \boldsymbol{R}^n$ we mean a locally integrable function defined away from the diagonal. We say that K satisfies the standard estimates if there exist positive and finite constants γ and C such that, for all distinct $x, y \in \boldsymbol{R}^n$ and all z with $2|x-z| < |x-y|$, it verifies

i) $|K(x,y)| \le C|x-y|^{-n}$,

ii) $|K(x,y) - K(z,y)| \le C\left|\dfrac{x-z}{x-y}\right|^\gamma |x-y|^{-n}$,

iii) $|K(y,x) - K(y,z)| \le C\left|\dfrac{x-z}{x-y}\right|^\gamma |x-y|^{-n}$.

We define a linear and continuous operator $T : C_0^\infty(\boldsymbol{R}^n) \to \mathcal{D}'(\boldsymbol{R}^n)$ associated to the kernel K by

$$Tf(x) = \int_{\boldsymbol{R}^n} K(x,y)f(y)dy\,,$$

where $f \in C_0^\infty(\boldsymbol{R}^n)$ and x is not in the support of f. T is called a Calderón-Zygmund operator if K satisfies the standard estimates and if it extends to a bounded linear operator on $L^2(\boldsymbol{R}^n)$. It is well known that under these conditions T can be extended to a bounded operator on $L^p(\boldsymbol{R}^n)$, $1 < p < \infty$ and is of weak type-(1, 1). For more information on this subject see [1], [4], [6] or [9].

We next define the vector-valued singular operator T_q associated to the operator T by

$$T_q f(x) = |Tf(x)|_q = \left(\sum_{j=1}^\infty |Tf_j(x)|^q\right)^{1/q}.$$

It is well-known that, for $1 < q < \infty$, T_q is of type-(p, p), $1 < p < \infty$, and weak type-(1, 1). Moreover, the A_p condition also implies the corresponding weighted estimate. For a complete study on these results see [8, Chapter V].

2.3. Orlicz maximal functions. By a Young function $A(t)$ we shall mean a continuous, nonnegative, strictly increasing and convex function on $[0,\infty)$ with $A(0) = 0$ and $A(t) \to \infty$ as $t \to \infty$. In this paper any Young function A will be doubling, namely $A(2t) \le CA(t)$ for $t > 0$.

We define the A-averages of a function f over a cube Q by

$$\|f\|_{A,Q} = \inf\left\{\lambda > 0\,; \frac{1}{|Q|}\int_Q A\left(\frac{|f(x)|}{\lambda}\right)dx \le 1\right\}.$$

An equivalent norm, which is often useful in calculations, is (see [10, p. 92] or [17, p. 69]):

$$\|f\|_{A,Q} \leq \inf_{\mu>0}\left\{\mu + \frac{\mu}{|Q|}\int_Q A\left(\frac{|f|}{\mu}\right)dx\right\} \leq 2\|f\|_{A,Q}\,. \tag{2.2}$$

If A, B and C are Young functions such that

$$A^{-1}(t)B^{-1}(t) \leq C^{-1}(t)\,,$$

then

$$\|fg\|_{C,R} \leq 2\|f\|_{A,R}\|g\|_{B,R}\,.$$

The examples to be considered in our study will be $A^{-1}(t) = \log(1+t)$, $B^{-1}(t) = t/\log(e+t)$ and $C^{-1}(t) = t$. Then $A(t) \approx e^t$ and $B(t) \approx t\log(e+t)$, which gives the Hölder inequality

$$\frac{1}{|Q|}\int_Q |fg|dx \leq C\|f\|_{\exp L,Q}\|g\|_{L\log L,Q}\,. \tag{2.3}$$

For these examples we recall that, if $h \in BMO$ and $(h)_Q$ denotes its average on the cube Q, then

$$\|h - (h)_Q\|_{\exp L,Q} \leq C\|h\|_{BMO} \tag{2.4}$$

by the classical John-Nirenberg inequality.

Associate to this average, for any Young function $A(t)$ we can define a maximal operator M_A given by

$$M_A f(x) = \sup_{Q\ni x}\|f\|_{A,Q}\,,$$

where the supremum is taken over all the cubes containing x.

The following result from [13] will be very useful.

THEOREM 2.2. *Let $1 < p < \infty$. Suppose that A is a Young function. Then the following are equivalent*:

i) *There exists a positive constant c such that*

$$\int_c^\infty \left(\frac{t}{A(t)}\right)^{p-1}\frac{dt}{t} < \infty\,. \tag{2.5}$$

ii) *There exists a constant C such that*

$$\int_{\mathbf{R}^n} Mf(x)^p\frac{w(x)}{[M_A(u)(x)]^{p-1}}dx \leq C\int_{\mathbf{R}^n} f(x)^p\frac{Mw(x)}{u(x)^{p-1}}dx \tag{2.6}$$

for all non-negative, locally integrable functions f and all weights w and u.

3. Pointwise estimates. In this section we prove the basic pointwise estimates for the vector-valued singular integral operator and commutator.

LEMMA 3.1. *Let $1 < q < \infty$ and $0 < \delta < 1$. Then there exists a constant $C > 0$ such that*

$$M^{\#}_\delta(T_q f)(x) \leq CM(|f|_q)(x) \tag{3.7}$$

for any smooth vector function $f = \{f_j\}_{j=1}^{\infty}$ *and for every* $x \in \boldsymbol{R}^n$.

PROOF. Let $f = \{f_j\}$ be any smooth vector function. Fix $x \in \boldsymbol{R}^n$ and let B be a ball centered at x of radius r. Decompose $f = f^1 + f^2$, where $f^1 = f\chi_{2B} = \{f_j\chi_{2B}\}$. As usual, $2B$ denotes the ball concentric with B and radius two times the radius of B. Set

$$c = |(Tf^2)_B|_q = \left(\sum_{j=1}^{\infty} |(Tf_j^2)_B|^q\right)^{1/q}.$$

Since for any $0 < r < \infty$ it follows

$$|\alpha^r - \beta^r| \le C_r|\alpha - \beta|^r \tag{3.8}$$

for any $\alpha, \beta \in \boldsymbol{C}$ with $C_r = \max\{1, 2^{r-1}\}$, we can estimate

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B ||T_q f(y)|^\delta - c^\delta| dy\right)^{1/\delta} \\
&\quad \le \left(\frac{1}{|B|}\int_B ||Tf(y)|_q - |(Tf^2)_B|_q|^\delta dy\right)^{1/\delta} \le \left(\frac{1}{|B|}\int_B |Tf(y) - (Tf^2)_B|_q^\delta dy\right)^{1/\delta} \\
&\quad \le C\left[\left(\frac{1}{|B|}\int_B |Tf^1(y)|_q^\delta dy\right)^{1/\delta} + \left(\frac{1}{|B|}\int_B |Tf^2(y) - (Tf^2)_B|_q^\delta dy\right)^{1/\delta}\right] \\
&\quad = I + II\,.
\end{aligned}$$

For I we recall that T_q is weak type-(1, 1). Then by Kolmogorov's inequality ([18, p. 104]),

$$I \le C\frac{1}{|B|}\|T_q f^1\|_{L^{1,\infty}} \le \frac{C}{|2B|}\int_{2B} |f(y)|_q dy \le CM(|f|_q)(x)\,. \tag{3.9}$$

To estimate II we will use Jensen's inequality, the definition of T, the basic estimates of the kernel K and Minkowski's inequality to obtain the following:

$$\begin{aligned}
II &\le \frac{C}{|B|}\int_B |Tf^2(y) - (Tf^2)_B|_q dy \\
&= \frac{C}{|B|}\int_B \left(\sum_{j=1}^{\infty} |Tf_j^2(y) - (Tf_j^2)_B|^q\right)^{1/q} dy \\
&= \frac{C}{|B|}\int_B \left(\sum_{j=1}^{\infty} \left|\frac{1}{|B|}\int_B (Tf_j^2(y) - Tf_j^2(z))dz\right|^q\right)^{1/q} dy \\
&= \frac{C}{|B|}\int_B \left(\sum_{j=1}^{\infty} \left|\frac{1}{|B|}\int_B\int_{\boldsymbol{R}^n\setminus 2B} (K(y,w) - K(z,w))f_j(w)dwdz\right|^q\right)^{1/q} dy \\
&\le \frac{C}{|B|}\frac{1}{|B|}\int_B\int_B\int_{\boldsymbol{R}^n\setminus 2B} \left(\sum_{j=1}^{\infty} |K(y,w) - K(z,w)|^q |f_j(w)|^q\right)^{1/q} dwdzdy \qquad (3.10) \\
&\le \frac{C}{|B|}\frac{1}{|B|}\int_B\int_B\int_{\boldsymbol{R}^n\setminus 2B} \left(\sum_{j=1}^{\infty} \left|\frac{y-z}{y-w}\right|^{\gamma q} \frac{1}{|y-w|^{nq}} |f_j(w)|^q\right)^{1/q} dwdzdy
\end{aligned}$$

$$\leq C \sum_{k=1}^{\infty} \int_{2^k r \leq |w-x| < 2^{k+1} r} \left(\sum_{j=1}^{\infty} \left(\frac{2r}{2^k r} \right)^{\gamma q} \frac{1}{(2^k r)^{nq}} |f_j(w)|^q \right)^{1/q} dw$$
$$\leq C \sum_{k=1}^{\infty} \frac{1}{2^{k\gamma}} \frac{1}{(2^{k+1} r)^n} \int_{2^{k+1} B} \left(\sum_{j=1}^{\infty} |f_j(w)|^q \right)^{1/q} dw$$
$$\leq C M(|f|_q)(x) .$$

Finally, (3.7) follows from (3.9) and (3.10) and the proof of the lemma is concluded. □

As mentioned in the introduction, we need a similar estimate for the commutator.

LEMMA 3.2. *Let $h \in BMO$ and let $0 < \delta < \varepsilon$. Then there exists a constant $C > 0$ such that*

$$M_\delta^{\#}([h, T]_q f)(x) \leq C \|h\|_{BMO} (M_\varepsilon(T_q f)(x) + M_{L \log L}(|f|_q)(x)) \tag{3.11}$$

for any smooth vector function $f = \{f_j\}_{j=1}^{\infty}$ and for every $x \in \boldsymbol{R}^n$.

PROOF. Observe that for any constant λ

$$[h, T] f(x) = (h(x) - \lambda) T f(x) - T((h - \lambda) f)(x) .$$

As above we fix $x \in \boldsymbol{R}^n$ and let B be a ball centered at x of radius $r > 0$. We split $f = f^1 + f^2$, where $f^1 = f \chi_{2B} = \{f_j \chi_{2B}\}$. Let λ be a constant and $c = \{c_j\}_{j=1}^{\infty}$ a sequence of constants to be fixed along the proof.

By (3.8) we have

$$\left(\frac{1}{|B|} \int_B ||[h, T]_q f(y)|^\delta - |c|_q^\delta| dy \right)^{1/\delta}$$
$$\leq C_\delta \left(\frac{1}{|B|} \int_B ||[h, T] f(y)|_q - |c|_q|^\delta dy \right)^{1/\delta} \leq C_\delta \left(\frac{1}{|B|} \int_B |[h, T] f(y) - c|_q^\delta \, dy \right)^{1/\delta}$$
$$= C_\delta \left(\frac{1}{|B|} \int_B |(h(y) - \lambda) T f(y) - T((h - \lambda) f)(y) - c|_q^\delta \, dy \right)^{1/\delta}$$
$$\leq C_\delta \Bigg[\left(\frac{1}{|B|} \int_B |(h(y) - \lambda) T f(y)|_q^\delta \, dy \right)^{1/\delta} + \left(\frac{1}{|B|} \int_B |T((h - \lambda) f^1)(y)|_q^\delta dy \right)^{1/\delta}$$
$$+ \left(\frac{1}{|B|} \int_B |T((h - \lambda) f^2)(y) - c|_q^\delta dy \right)^{1/\delta} \Bigg]$$
$$= I + II + III .$$

To deal with I, we first fix $\lambda = (h)_{2B}$, the average of h on $2B$. Then, for any $1 < p < \varepsilon/\delta$, we have

$$
\begin{aligned}
I &= C_\delta \left(\frac{1}{|B|} \int_B |h(y) - (h)_{2B}|^\delta |Tf(y)|_q^\delta dy \right)^{1/\delta} \\
&\le C_\delta \left(\frac{1}{|2B|} \int_{2B} |h(y) - (h)_{2B}|^{p'\delta} dy \right)^{1/p'\delta} \left(\frac{1}{|B|} \int_B (T_q f(y))^{p\delta} dy \right)^{1/p\delta} \\
&\le C\|h\|_{BMO} M_{\delta p}(T_q f)(x) \\
&\le C\|h\|_{BMO} M_\varepsilon (T_q f)(x)\,.
\end{aligned} \tag{3.12}
$$

For II we make use of Kolmogorov's inequality again. Then

$$
\begin{aligned}
II &\le C \frac{1}{|B|} \int_B |h(y) - (h)_{2B}||f^1(y)|_q dy \le C \frac{1}{|2B|} \int_{2B} |h(y) - (h)_{2B}||f(y)|_q dy \\
&\le C\|h - (h)_{2B}\|_{\exp L, 2B} \||f|_q\|_{L \log L, 2B} \le C\|h\|_{BMO} M_{L \log L}(|f|_q)(x)\,,
\end{aligned} \tag{3.13}
$$

where we have used (2.3) and (2.4).

Finally, for III we first fix the value of c by taking $c = \{(T((h - (h)_{2B})f_j^2))_B\}_{j=1}^\infty$, the average of each $T((h - (h)_{2B})f_j^2)$ on B. Then, by Jensen and Minkowski's inequalities, respectively, and the basic estimates of the kernel K, we have

$$
\begin{aligned}
III &\le C_\delta \frac{1}{|B|} \int_B |T((h - \lambda) f^2)(y) - c|_q dy \\
&= C_\delta \frac{1}{|B|} \int_B \left(\sum_{j=1}^\infty |T((h - (h)_{2B}) f_j^2)(y) - T((h - (h)_{2B}) f_j^2)_B|^q \right)^{1/q} dy \\
&= C_\delta \frac{1}{|B|} \int_B \left(\sum_{j=1}^\infty \left| \frac{1}{|B|} \int_B \{ T((h - (h)_{2B}) f_j^2)(y) \right.\right. \\
&\qquad\qquad \left.\left. - T((h - (h)_{2B}) f_j^2)(z) \} dz \right|^q \right)^{1/q} dy \\
&= \frac{C}{|B|} \int_B \left(\sum_{j=1}^\infty \left| \frac{1}{|B|} \int_B \int_{\mathbf{R}^n \setminus 2B} (K(y, w) - K(z, w)) \right.\right. \\
&\qquad\qquad \left.\left. \times (h(w) - (h)_{2B}) f_j(w) dw dz \right|^q \right)^{1/q} dy \\
&\le \frac{C}{|B|} \frac{1}{|B|} \int_B \int_B \int_{\mathbf{R}^n \setminus 2B} \left(\sum_{j=1}^\infty \left| \frac{y - z}{y - w} \right|^{\gamma q} \frac{1}{|y - w|^{nq}} \right. \\
&\qquad\qquad \left. \times |(h(w) - (h)_{2B}) f_j(w)|^q \right)^{1/q} dw dz dy
\end{aligned} \tag{3.14}
$$

$$
\begin{aligned}
&= C\sum_{k=1}^{\infty}\int_{2^k r\le |w-x|<2^{k+1}r}\left(\frac{2r}{2^k r}\right)^{\gamma}\frac{1}{(2^k r)^n}|h(w)-(h)_{2B}||f(w)|_q\,dw\\
&\le C\sum_{k=1}^{\infty}\frac{1}{2^{k\gamma}}\frac{1}{(2^{k+1}r)^n}\int_{2^{k+1}B}|h(w)-(h)_{2B}||f(w)|_q\,dw\\
&\le C\sum_{k=1}^{\infty}\frac{1}{2^{k\gamma}}\frac{1}{(2^{k+1}r)^n}\int_{2^{k+1}B}|h(w)-(h)_{2^{k+1}B}||f(w)|_q\,dw\\
&\quad + C\sum_{k=1}^{\infty}\frac{1}{2^{k\gamma}}|(h)_{2^{k+1}B}-(h)_{2B}|\frac{1}{(2^{k+1}r)^n}\int_{2^{k+1}B}|f(w)|_q\,dw\\
&\le C\sum_{k=1}^{\infty}\frac{1}{2^{k\gamma}}\|h-(h)_{2^{k+1}B}\|_{\exp L,2^{k+1}B}\||f|_q\|_{L\log L,2^{k+1}B}\\
&\quad + C\|h\|_{BMO}M(|f|_q)(x)\left(\sum_{k=1}^{\infty}\frac{k}{2^{k\gamma}}\right)\\
&\le C\|h\|_{BMO}M_{L\log L}(|f|_q)(x)\,,
\end{aligned}
$$

where in the last inequality we have used that $|(h)_{2^{k+1}B}-(h)_{2B}|\le 2k\|h\|_{BMO}$.

From (3.12), (3.13) and (3.14) we get (3.11) and the proof is finished. □

4. Proof of the theorems.

4.1. Proof of Theorem1.3. In order to prove

$$
\int_{\mathbf{R}^n}(T_q f(x))^p w(x)dx \le C\int_{\mathbf{R}^n}(M(|f|_q)(x))^p w(x)dx\,, \tag{4.15}
$$

we make some reductions. First, we assume that the right hand side of (4.15) is finite, since otherwise there is nothing to prove. Next we restrict to a finite number of elements $f_m = (f_1, f_2, \dots, f_m, 0, \dots)$ and prove (4.15) with a constant independent of m. Then we let m go to ∞. To apply Theorem 2.1, take it for granted that

$$
\int_{\mathbf{R}^n}(M_\delta(T_q(f_m))(x))^p w(x)\,dx < \infty\,. \tag{4.16}
$$

Then, since $w\in A_\infty$, we can combine Theorem 2.1 together with Lemma 3.1 with $0<\delta<1$ to get

$$
\begin{aligned}
\int_{\mathbf{R}^n}(T_q f_m(x))^p w(x)dx &\le \int_{\mathbf{R}^n}(M_\delta(T_q f_m)(x))^p w(x)dx\\
&\le C\int_{\mathbf{R}^n}(M^{\#}_\delta(T_q f_m)(x))^p w(x)dx\\
&\le C\int_{\mathbf{R}^n}(M(|f_m|_q)(x))^p w(x)dx\,.
\end{aligned}
$$

It only remains to show (4.16). Indeed, since $w \in A_\infty$, there exists $r > 1$ such that $w \in A_r$ and we can choose δ small enough so that $p/\delta > r$. Then, by Muckenhoupt's theorem, all is reduced to checking that $\|T_q f_m\|_{L^p(w)} < \infty$. Now, by the classical Coifman [2] estimate we have

$$\begin{aligned}\int_{\boldsymbol{R}^n}\left(\sum_{j=1}^m |Tf_j(x)|^q\right)^{p/q} w(x)dx &\le C_m \sum_{j=1}^m \int_{\boldsymbol{R}^n} |Tf_j(x)|^p w(x)dx \\ &\le C_m \sum_{j=1}^m \int_{\boldsymbol{R}^n} (M(f_j)(x))^p w(x)dx \\ &\le C_m \int_{\boldsymbol{R}^n} (M(|f|_q)(x))^p w(x)dx\,.\end{aligned}$$

The proof of the theorem is complete. □

4.2. Proof of Theorem 1.1. We want to show that the vector valued extension of T is a bounded operator from $L^p_{l^q}(M_A w(x))$ into $L^p_{l^q}(w)$ (the definition of $L^p_{l^q}(\mu)$ is standard, see [8, Chapter V]). A simple duality argument shows that this is equivalent to see that the adjoint operator T^* is bounded from $L^{p'}_{l^{q'}}(w^{1-p'})$ into $L^{p'}_{l^{q'}}((M_A w(x))^{1-p'})$.

So, the estimate to be established is

$$\int_{\boldsymbol{R}^n} (T^*_{q'} f(x))^{p'} (M_A w(x))^{1-p'} dx \le C \int_{\boldsymbol{R}^n} |f(x)|^{p'}_{q'}\, w(x)^{1-p'} dx\,. \tag{4.17}$$

As above we may restrict to a finite number of elements $f_m = (f_1, f_2, \dots, f_m, 0, \dots)$ and show the estimate with a constant independent of m. First, we note that $(M_A w(x))^{1-p'} \in A_\infty$ (see [11, p. 300]). Thus, since T^* is also a Calderón-Zygmund operator, we can apply Theorem 1.3 combined with Theorem 2.2 to deduce

$$\begin{aligned}\int_{\boldsymbol{R}^n} (T^*_{q'} f(x))^{p'} (M_A w(x))^{1-p'} dx &\le C \int_{\boldsymbol{R}^n} (M(|f|_{q'})(x))^{p'} (M_A w(x))^{1-p'} dx \\ &\le C \int_{\boldsymbol{R}^n} |f(x)|^{p'}_{q'}\, w(x)^{1-p'} dx\,,\end{aligned}$$

whenever

$$\int_{\boldsymbol{R}^n} (T^*_{q'} f(x))^{p'} (M_A w(x))^{1-p'} dx < \infty\,.$$

To show this we use an argument similar to the proof of Theorem 1.3, where now we make use of the scalar version of (1.3) derived in [11], since we are assuming that

$$\int_c^\infty \left(\frac{t}{A(t)}\right)^{p'-1} \frac{dt}{t} < \infty\,.$$

□

4.3. Proof of Theorem 1.4. Fix $\lambda > 0$ and let $\{Q_j\}$ be the standard family of nonoverlapping dyadic cubes satisfying

$$\lambda < \frac{1}{|Q_j|}\int_{Q_j} |f(x)|_q dx \le 2^n \lambda\,, \tag{4.18}$$

maximal with respect to left hand side inequality. Denote by z_j and r_j the center and side-length of each Q_j, respectively. As usual, if we denote $\Omega = \bigcup_j Q_j$, then $|f(x)|_q \leq \lambda$ a. e. $x \in \boldsymbol{R}^n \setminus \Omega$.

Now we proceed to construct an slightly different version of the classical Calderón-Zygmund decomposition. Split f as $f = g + b$, where $g = \{g_i\}_{i=1}^{\infty}$ is given by

$$g_i(x) = \begin{cases} f_i(x) & \text{for } x \in \boldsymbol{R}^n \setminus \Omega\,, \\ (f_i)_{Q_j} & \text{for } x \in Q_j\,, \end{cases}$$

$(f_i)_{Q_j}$ being, as usual, the average of f_i on the cube Q_j, and

$$b(x) = \{b_i(x)\}_{i=1}^{\infty} = \left\{ \sum_{Q_j} b_{ij}(x) \right\}_{i=1}^{\infty}$$

with $b_{ij}(x) = (f_i(x) - (f_i)_{Q_j})\chi_{Q_j}(x)$. Let $\tilde{\Omega} = \bigcup_j 2Q_j$. We then have

$$\begin{aligned} w(\{y \in \boldsymbol{R}^n; |T_q f(y)| > \lambda\}) &\leq w(\{y \in \boldsymbol{R}^n \setminus \tilde{\Omega}; |T_q g(y)| > \lambda/2\}) + w(\tilde{\Omega}) \\ &\quad + w(\{y \in \boldsymbol{R}^n \setminus \tilde{\Omega}; |T_q b(y)| > \lambda/2\})\,. \end{aligned} \tag{4.19}$$

For the first term we invoke Theorem 1.1. Let $\varepsilon > 0$. By choosing $1 < p < 1 + \varepsilon$, we have that $A_\varepsilon(t) = t\log^\varepsilon(1+t)$ satisfies (1.2). Thus,

$$\begin{aligned} &w(\{y \in \boldsymbol{R}^n \setminus \tilde{\Omega}; |T_q g(y)| > \lambda/2\}) \\ &\quad \leq \frac{C}{\lambda^p} \int_{\boldsymbol{R}^n \setminus \tilde{\Omega}} (T_q g(y))^p w(y)dy \leq \frac{C}{\lambda^p} \int_{\boldsymbol{R}^n} |g(y)|_q^p\, M_{L(\log L)^\varepsilon}(\chi_{\boldsymbol{R}^n \setminus \tilde{\Omega}} w)(y)dy \\ &\quad \leq \frac{C}{\lambda^p} \int_{\boldsymbol{R}^n \setminus \Omega} |f(y)|_q^p\, M_{L(\log L)^\varepsilon}(w)(y)dy + \frac{C}{\lambda^p} \int_{\Omega} |g(y)|_q^p\, M_{L(\log L)^\varepsilon}(\chi_{\boldsymbol{R}^n \setminus \tilde{\Omega}} w)(y)dy \\ &\quad = I + II\,. \end{aligned}$$

The estimate of I is immediate; since $|f(x)|_q \leq \lambda$ a. e. $x \in \boldsymbol{R}^n \setminus \Omega$,

$$I \leq \frac{C}{\lambda} \int_{\boldsymbol{R}^n} |f(y)|_q\, M_{L(\log L)^\varepsilon}(w)(y)dy\,.$$

For II, taking into account that for any j

$$M_{L(\log L)^\varepsilon}(\chi_{\boldsymbol{R}^n \setminus 2Q_j} w)(y) \approx M_{L(\log L)^\varepsilon}(\chi_{\boldsymbol{R}^n \setminus 2Q_j} w)(z)$$

for all $y, z \in Q_j$ (see [11, p. 303]), we have by Minkowski's inequality

$$
\begin{aligned}
II &= \frac{C}{\lambda^p}\sum_{Q_j}\int_{Q_j}|g(y)|_q^p\, M_{L(\log L)^\varepsilon}(\chi_{\boldsymbol{R}^n\setminus\tilde{\Omega}}w)(y)dy\\
&= \frac{C}{\lambda^p}\sum_{Q_j}\int_{Q_j}\left(\sum_{i=1}^{\infty}\left|(f_i)_{Q_j}\right|^q\right)^{p/q} M_{L(\log L)^\varepsilon}(\chi_{\boldsymbol{R}^n\setminus\tilde{\Omega}}w)(y)dy\\
&\le \frac{C}{\lambda^p}\sum_{Q_j}\left(\sum_{i=1}^{\infty}\left|\frac{1}{|Q_j|}\int_{Q_j}f_i(z)dz\right|^q\right)^{p/q}|Q_j|\inf_{y\in Q_j}M_{L(\log L)^\varepsilon}(\chi_{\boldsymbol{R}^n\setminus 2Q_j}w)(y)\\
&\le \frac{C}{\lambda^p}\sum_{Q_j}\left(\frac{1}{|Q_j|}\int_{Q_j}|f(z)|_q dz\right)^p|Q_j|\inf_{y\in Q_j}M_{L(\log L)^\varepsilon}(w)(y) \qquad (4.20)\\
&\le \frac{C}{\lambda}\sum_{Q_j}\left(\frac{1}{|Q_j|}\int_{Q_j}|f(z)|_q dz\right)|Q_j|\inf_{y\in Q_j}M_{L(\log L)^\varepsilon}(w)(y)\\
&\le \frac{C}{\lambda}\sum_{Q_j}\int_{Q_j}|f(z)|_q M_{L(\log L)^\varepsilon}(w)(z)dz\\
&\le \frac{C}{\lambda}\int_{\boldsymbol{R}^n}|f(z)|_q M_{L(\log L)^\varepsilon}(w)(z)dz\,.
\end{aligned}
$$

where the fifth inequality follows by (4.18).

For the second term of (4.19) we proceed as follows. Again by (4.18)

$$
\begin{aligned}
w(\tilde{\Omega}) &\le C\sum_{Q_j}\frac{w(2Q_j)}{|2Q_j|}|2Q_j| \le \frac{C}{\lambda}\sum_{Q_j}\frac{w(2Q_j)}{|2Q_j|}\int_{Q_j}|f(y)|_q dy\\
&\le \frac{C}{\lambda}\sum_{Q_j}\int_{Q_j}|f(y)|_q Mw(y)dy \le \frac{C}{\lambda}\int_{\boldsymbol{R}^n}|f(y)|_q M_{L(\log L)^\varepsilon}w(y)dy
\end{aligned} \qquad (4.21)
$$

since $Mw(y) \le M_{L(\log L)^\varepsilon}w(y)$.

Finally, for the third term of (4.19) we recall that each b_{ij} has zero average on Q_j. Hence, if z_j denotes the center of Q_j, we have

$$
\begin{aligned}
&w(\{y\in\boldsymbol{R}^n\setminus\tilde{\Omega};\ |T_q b(y)|>\lambda/2\})\\
&\le \frac{C}{\lambda}\int_{\boldsymbol{R}^n\setminus\tilde{\Omega}}T_q b(y)w(y)dy = \frac{C}{\lambda}\int_{\boldsymbol{R}^n\setminus\tilde{\Omega}}\left[\sum_{i=1}^{\infty}|Tb_i(y)|^q\right]^{1/q}w(y)dy\\
&= \frac{C}{\lambda}\int_{\boldsymbol{R}^n\setminus\tilde{\Omega}}\left[\sum_{i=1}^{\infty}\left|\sum_{Q_j}\int_{Q_j}K(y,z)b_{ij}(z)dz\right|^q\right]^{1/q}w(y)dy\\
&= \frac{C}{\lambda}\int_{\boldsymbol{R}^n\setminus\tilde{\Omega}}\left[\sum_{i=1}^{\infty}\left|\sum_{Q_j}\int_{Q_j}(K(y,z)-K(y,z_j))b_{ij}(z)dz\right|^q\right]^{1/q}w(y)dy
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{C}{\lambda}\int_{\boldsymbol{R}^n\setminus\tilde{\Omega}}\Bigg(\sum_{Q_j}\int_{Q_j}\Bigg[\sum_{i=1}^{\infty}|K(y,z)-K(y,z_j)|^q|b_{ij}(z)|^q\Bigg]^{1/q}dz\Bigg)w(y)dy\\
&\le \frac{C}{\lambda}\sum_{Q_j}\int_{Q_j}\Bigg(\int_{\boldsymbol{R}^n\setminus 2Q_j}\Bigg[\sum_{i=1}^{\infty}|K(y,z)-K(y,z_j)|^q|b_{ij}(z)|^q\Bigg]^{1/q}w(y)dy\Bigg)dz \qquad (4.22)\\
&\le \frac{C}{\lambda}\sum_{Q_j}\int_{Q_j}\Bigg(\int_{\boldsymbol{R}^n\setminus 2Q_j}\left|\frac{z-z_j}{z-y}\right|^{\gamma}\frac{1}{|x-y|^n}w(y)dy\Bigg)\Bigg[\sum_{i=1}^{\infty}|b_{ij}(z)|^q\Bigg]^{1/q}dz\\
&\le \frac{C}{\lambda}\sum_{Q_j}\int_{Q_j}\Bigg(\sum_{k=1}^{\infty}\int_{2^k r_j\le|y-z_j|<2^{k+1}r_j}\left(\frac{2r_j}{2^k r_j}\right)^{\gamma}\frac{1}{(2^k r_j)^n}w(y)dy\Bigg)\\
&\qquad\qquad \times\Bigg[\sum_{i=1}^{\infty}|b_{ij}(z)|^q\Bigg]^{1/q}dz\\
&\le \frac{C}{\lambda}\sum_{Q_j}\int_{Q_j}\Bigg[\sum_{i=1}^{\infty}|b_{ij}(z)|^q\Bigg]^{1/q}M(\chi_{\boldsymbol{R}^n\setminus 2Q_j}w)(z)dz\\
&\le \frac{C}{\lambda}\sum_{Q_j}\Bigg[\int_{Q_j}|f(z)|_q M(\chi_{\boldsymbol{R}^n\setminus 2Q_j}w)(z)dz+\int_{Q_j}|g(z)|_q M(\chi_{\boldsymbol{R}^n\setminus 2Q_j}w)(z)dz\Bigg]\\
&= III+IV\,.
\end{aligned}
$$

Trivially,

$$
III \le \frac{C}{\lambda}\int_{\boldsymbol{R}^n}|f(z)|_q M_{L(\log L)^{\varepsilon}}w(z)dz\,.
$$

On the other hand,

$$
\begin{aligned}
IV &\le \frac{C}{\lambda}\sum_{Q_j}\Bigg[\sum_{i=1}^{\infty}\left|\frac{1}{|Q_j|}\int_{Q_j}f_i(z)dz\right|^q\Bigg]^{1/q}\int_{Q_j}M(\chi_{\boldsymbol{R}^n\setminus 2Q_j}w)(y)dy\\
&\le \frac{C}{\lambda}\sum_{Q_j}\frac{1}{|Q_j|}\int_{Q_j}|f(z)|_q dz|Q_j|\inf_{y\in Q_j}M(\chi_{\boldsymbol{R}^n\setminus 2Q_j}w)(y)\\
&\le \frac{C}{\lambda}\sum_{Q_j}\frac{1}{|Q_j|}\int_{Q_j}|f(z)|_q dz|Q_j|\inf_{y\in Q_j}M(w)(y) \qquad (4.23)\\
&\le \frac{C}{\lambda}\sum_{Q_j}\int_{Q_j}|f(z)|_q M(w)(z)dz\\
&\le \frac{C}{\lambda}\int_{\boldsymbol{R}^n}|f(z)|_q M(w)(z)dz\\
&\le \frac{C}{\lambda}\int_{\boldsymbol{R}^n}|f(z)|_q M_{L(\log L)^{\varepsilon}}(w)(z)dz\,,
\end{aligned}
$$

concluding the proof of the theorem. □

4.4. Proof of Theorem 1.6. As in the proof of Theorem 1.3, we fix $f = f_m = (f_1, f_2, \dots, f_m, 0, \dots)$ with a finite amount of smooth functions components with compact support. We also assume that the right hand side of (1.9) is finite, since otherwise there is nothing to prove.

We will prove the estimates with constant independent of the number of elements of f. By (2.1), since $w \in A_\infty$, it follows from Lemma 3.2 and Lemma 3.1, for $0 < \delta < \varepsilon < 1$, that

$$\begin{aligned}
\|[h,T]_q f\|_{L^p(w)} &\le \|M_\delta([h,T]_q f)\|_{L^p(w)} \\
&\le C[w]_{A_\infty}\|M^\#_\delta([h,T]_q f)\|_{L^p(w)} \\
&\le C[w]_{A_\infty}\|h\|_{BMO}(\|M_\varepsilon(T_q f)\|_{L^p(w)} + \|M_{L\log L}(|f|_q)\|_{L^p(w)}) \\
&\le C[w]^2_{A_\infty}\|h\|_{BMO}(\|M^\#_\varepsilon(T_q f)\|_{L^p(w)} + \|M_{L\log L}(|f|_q)\|_{L^p(w)}) \\
&\le C[w]^2_{A_\infty}\|h\|_{BMO}(\|M(|f|_q)\|_{L^p(w)} + \|M_{L\log L}(|f|_q)\|_{L^p(w)}) \\
&\le C[w]^2_{A_\infty}\|h\|_{BMO}\|M_{L\log L}(|f|_q)\|_{L^p(w)}\,,
\end{aligned}$$

whenever we are able to prove that $\|M_\delta([h,T]_q f)\|_{L^p(w)}$ and $\|M_\varepsilon(T_q f)\|_{L^p(w)}$ are both finite as Theorem 2.1 requires.

Since $w \in A_\infty$, there exists $r > 1$ such that $w \in A_r$ and we can choose δ and ε small enough so that $p/\delta, p/\varepsilon > r$. Then, by Muckenhoupt's theorem, all is reduced to check that $\|[h,T]_q f\|_{L^p(w)} < \infty$ and $\|T_q f\|_{L^p(w)} < \infty$. But this is a consequence of the scalar situation: $\|[h,T]g\|_{L^p(w)} \le \|M_{L\log L}(g)\|_{L^p(w)} < \infty$ [14] and $\|Tg\|_{L^p(w)} \le \|Mg\|_{L^p(w)} < \infty$ [2], when g is a smooth function with compact support, since the amount of elements on f is finite. Recall that the right hand side of (1.9) is finite. The theorem is proved. □

4.5. Proof of Theorem 1.8. A simple homogeneity argument shows that we may assume that $\|h\|_{BMO} = 1$, and with this assumption it suffices to show that

$$w(\{x \in \boldsymbol{R}^n;\, |[h,T]_q f(x)| > \lambda\}) \le C\int_{\boldsymbol{R}^n} \Phi\left(\frac{|f(x)|_q}{\lambda}\right) M_{L(\log L)^{1+\varepsilon}}(w)(x)\,dx\,,$$

where $\Phi(t) = t\log(e+t)$.

We proceed as in the proof of Theorem 1.4, using essentially the same notation. Let $\{Q_j\}$ be the family of non-overlapping dyadic cubes which are maximal with respect to the condition

$$\lambda < \frac{1}{|Q_j|}\int_{Q_j} |f(x)|_q\,dx \le 2^n\lambda\,. \tag{4.24}$$

For each j we let z_j and r_j be the center and side-length of Q_j. If we denote $\Omega = \bigcup_j Q_j$, then $|f(x)|_q \le \lambda$ a. e. $x \in \boldsymbol{R}^n \setminus \Omega$.

Split f as $f = g + b$, where $g = \{g_i\}_{i=1}^\infty$ is given by

$$g_i(x) = \begin{cases} f_i(x) & \text{for } x \in \boldsymbol{R}^n \setminus \Omega\,, \\ (f_i)_{Q_j} & \text{for } x \in Q_j\,, \end{cases}$$

$(f_i)_{Q_j}$ being, as usual, the average of f_i on the cube Q_j, and

$$b(x) = \{b_i(x)\}_{i=1}^{\infty} = \left\{ \sum_{Q_j} b_{ij}(x) \right\}_{i=1}^{\infty}$$

with $b_{ij}(x) = (f_i(x) - (f_i)_{Q_j})\chi_{Q_j}(x)$. Let $\tilde{\Omega} = \bigcup_j 2Q_j$. We then have

$$\begin{aligned} &w(\{y \in \boldsymbol{R}^n ; |[h,T]_q f(y)| > \lambda\}) \\ &\qquad \le w(\{y \in \boldsymbol{R}^n \setminus \tilde{\Omega} ; |[h,T]_q g(y)| > \lambda/2\}) + w(\tilde{\Omega}) \\ &\qquad\quad + w(\{y \in \boldsymbol{R}^n \setminus \tilde{\Omega} ; |[h,T]_q b(y)| > \lambda/2\}) . \end{aligned} \tag{4.25}$$

Let $\varepsilon > 0$. We use Theorem 1.7, with p and δ such that $1 < p < 1 + \varepsilon/2$ and $\delta = \varepsilon - 2(p-1) > 0$. Then

$$\begin{aligned} &w(\{y \in \boldsymbol{R}^n \setminus \tilde{\Omega} ; |[h,T]_q g(y)| > \lambda/2\}) \\ &\quad \le \frac{C}{\lambda^p} \int_{\boldsymbol{R}^n \setminus \tilde{\Omega}} ([h,T]_q g(y))^p w(y) dy \le \frac{C}{\lambda^p} \int_{\boldsymbol{R}^n} |g(y)|_q^p M_{L(\log L)^{1+\varepsilon}}(\chi_{\boldsymbol{R}^n \setminus \tilde{\Omega}} w)(y) dy \\ &\quad \le \frac{C}{\lambda^p} \int_{\boldsymbol{R}^n \setminus \Omega} |f(y)|_q^p M_{L(\log L)^{1+\varepsilon}}(w)(y) dy + \frac{C}{\lambda^p} \int_{\Omega} |g(y)|_q^p M_{L(\log L)^{1+\varepsilon}}(\chi_{\boldsymbol{R}^n \setminus \tilde{\Omega}} w)(y) dy \\ &\quad = I + II . \end{aligned}$$

The estimate of I is immediate; since $|f(x)|_q \le \lambda$ a. e. $x \in \boldsymbol{R}^n \setminus \Omega$,

$$I \le \frac{C}{\lambda} \int_{\boldsymbol{R}^n} |f(y)|_q \, M_{L(\log L)^{1+\varepsilon}}(w)(y) dy \le C \int_{\boldsymbol{R}^n} \Phi\left(\frac{|f(y)|_q}{\lambda} \right) M_{L(\log L)^{1+\varepsilon}}(w)(y) dy .$$

For II we proceed as in the proof of (4.20) obtaining

$$\begin{aligned} II &\le \frac{C}{\lambda} \int_{\boldsymbol{R}^n} |f(y)|_q M_{L(\log L)^{1+\varepsilon}}(w)(y) dy \\ &\le C \int_{\boldsymbol{R}^n} \Phi\left(\frac{|f(y)|_q}{\lambda} \right) M_{L(\log L)^{1+\varepsilon}}(w)(y) dy . \end{aligned}$$

For the second term of (4.25) we proceed as in the proof of (4.21). Then

$$\begin{aligned} w(\tilde{\Omega}) &\le \frac{C}{\lambda} \int_{\boldsymbol{R}^n} |f(y)|_q M_{L(\log L)^{1+\varepsilon}}(w)(y) dy \\ &\le C \int_{\boldsymbol{R}^n} \Phi\left(\frac{|f(y)|_q}{\lambda} \right) M_{L(\log L)^{1+\varepsilon}}(w)(y) dy . \end{aligned}$$

Finally, taking into account the following decomposition

$$\begin{aligned} [h,T]_q b(x) &= \left(\sum_i \left| \sum_j (h(x) - (h)_{Q_j}) T(b_{ij})(x) - T((h - (h)_{Q_j}) b_{ij})(x) \right|^q \right)^{1/q} \\ &\le \sum_j |h(x) - (h)_{Q_j}| \left(\sum_i |T(b_{ij})(x)|^q \right)^{1/q} \end{aligned}$$

$$+\left[\sum_i \left|T\Big(\sum_j (h-(h)_{Q_j})b_{ij}\Big)(x)\right|^q\right]^{1/q}$$
$$= A(x)+B(x)\,,$$

the third term of (4.25) is estimated by

$$w\left(\left\{x\in \boldsymbol{R}^n\setminus\tilde{\Omega};\, A(x)>\frac{\lambda}{4}\right\}\right)+w\left(\left\{x\in \boldsymbol{R}^n\setminus\tilde{\Omega};\, B(x)>\frac{\lambda}{4}\right\}\right)=IV+V\,.$$

Using the standard estimates of the kernel K and the cancellation of b_{ij} over Q_j we have

$$\begin{aligned}
IV &\le \sum_j \frac{C}{\lambda}\int_{\boldsymbol{R}^n\setminus\tilde{\Omega}} |h(x)-(h)_{Q_j}|\Big(\sum_i |T(b_{ij})(x)|^q\Big)^{1/q} w(x)dx\\
&\le \frac{C}{\lambda}\sum_j \int_{\boldsymbol{R}^n\setminus 2Q_j} |h(x)-(h)_{Q_j}|\\
&\quad\times \int_{Q_j} |K(x,y)-K(x,z_j)|\Big(\sum_i |b_{ij}(y)|^q\Big)^{1/q} w(x)dydx\\
&\le \frac{C}{\lambda}\sum_j \int_{Q_j}\Big(\sum_i |b_{ij}(y)|^q\Big)^{1/q}\\
&\quad\times\Big(\int_{\boldsymbol{R}^n\setminus 2Q_j} |K(x,y)-K(x,z_j)||h(x)-(h)_{Q_j}|w(x)dx\Big)dy\\
&\le \frac{C}{\lambda}\sum_j \int_{Q_j}\Big(\sum_i |b_{ij}(y)|^q\Big)^{1/q}\\
&\quad\times\Big(\sum_{k=1}^{\infty}\int_{2^k r_j\le|x-z_j|<2^{k+1}r_j}\left|\frac{y-z_j}{x-y}\right|^{\gamma}\frac{1}{|x-y|^n}|h(x)-(h)_{Q_j}|w(x)dx\Big)dy\\
&\le \frac{C}{\lambda}\sum_j \Big(\int_{Q_j}\Big(\sum_i |b_{ij}(y)|^q\Big)^{1/q}dy\Big)\\
&\quad\times\sum_{k=1}^{\infty}\frac{2^{-k\gamma}}{(2^{k+1}r_j)^n}\int_{|x-z_j|<2^{k+1}r_j}|h(x)-(h)_{Q_j}|\chi_{\boldsymbol{R}^n\setminus 2Q_j}(x)w(x)dx\,.
\end{aligned}$$

To control the sum on k we use again standard estimates together with the generalized Hölder inequality and John-Nirenberg's theorem. Indeed, if $y\in Q_j$, we have

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{2^{-k\gamma}}{(2^{k+1}r_j)^n}\int_{|x-z_j|<2^{k+1}r_j}|h(x)-(h)_{Q_j}|\chi_{\boldsymbol{R}^n\setminus 2Q_j}(x)w(x)dx\\
&\qquad\le C\sum_{k=1}^{\infty}\frac{2^{-k\gamma}}{(2^{k+1}r_j)^n}\int_{2^{k+1}Q_j}|h(x)-(h)_{2^{k+1}Q_j}|\chi_{\boldsymbol{R}^n\setminus 2Q_j}(x)w(x)dx
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{\infty}|(h)_{2^{k+1}Q_j}-(h)_{Q_j}|\frac{2^{-k\gamma}}{(2^{k+1}r_j)^n}\int_{2^{k+1}Q_j}\chi_{\boldsymbol{R}^n\setminus 2Q_j}(x)w(x)dx\\
&\leq C\sum_{k=1}^{\infty}2^{-k\gamma}\|h-(h)_{2^{k+1}Q_j}\|_{\exp L,2^{k+1}Q_j}\|\chi_{\boldsymbol{R}^n\setminus 2Q_j}w\|_{L\log L,2^{k+1}Q_j}\\
&+\sum_{k=1}^{\infty}2^{-k\gamma}(k+1)M(\chi_{\boldsymbol{R}^n\setminus 2Q_j}w)(y)\\
&\leq C\left(M_{L(\log L)}(\chi_{\boldsymbol{R}^n\setminus 2Q_j}w)(y)\sum_{k=1}^{\infty}2^{-k\gamma}+M(\chi_{\boldsymbol{R}^n\setminus 2Q_j}w)(y)\sum_{k=1}^{\infty}2^{-k\gamma}k\right)\\
&\leq CM_{L(\log L)^{1+\varepsilon}}(\chi_{\boldsymbol{R}^n\setminus 2Q_j}w)(y)\,.
\end{aligned}
$$

Thus we have

$$
IV\leq\frac{C}{\lambda}\sum_j\int_{Q_j}\left(\sum_i|b_{ij}(y)|^q\right)^{1/q}M_{L(\log L)^{1+\varepsilon}}(\chi_{\boldsymbol{R}^n\setminus 2Q_j}w)(y)dy\,,
$$

and we can continue the estimate of IV in the same way as in the proof of (4.22) with M replaced by $M_{L(\log L)^{1+\varepsilon}}$. We conclude that

$$
\begin{aligned}
IV&\leq\frac{C}{\lambda}\int_{\boldsymbol{R}^n}|f(y)|_qM_{L(\log L)^{1+\varepsilon}}(w)(y)dy\\
&\leq C\int_{\boldsymbol{R}^n}\Phi\left(\frac{|f(x)|_q}{\lambda}\right)M_{L(\log L)^{1+\varepsilon}}(w)(x)dx\,.
\end{aligned}
$$

To estimate V we will use Theorem 1.4 for singular integrals:

$$
\begin{aligned}
V&=w\left(\left\{x\in\boldsymbol{R}^n\setminus\tilde{\Omega};\,B(x)>\frac{\lambda}{4}\right\}\right)\\
&\leq\frac{C}{\lambda}\int_{\boldsymbol{R}^n}\left[\sum_i\left|\sum_j(h(x)-(h)_{Q_j})b_{ij}(x)\right|^q\right]^{1/q}M_{L(\log L)^{\varepsilon}}(\chi_{\boldsymbol{R}^n\setminus\tilde{\Omega}}w)(x)dx\\
&\leq\frac{C}{\lambda}\sum_j\int_{Q_j}|h(x)-(h)_{Q_j}|\left(\sum_i|b_{ij}(x)|^q\right)^{1/q}M_{L(\log L)^{\varepsilon}}(\chi_{\boldsymbol{R}^n\setminus 2Q_j}w)(x)dx\\
&\leq\frac{C}{\lambda}\sum_j\inf_{Q_j}M_{L(\log L)^{\varepsilon}}(\chi_{\boldsymbol{R}^n\setminus 2Q_j}w)(x)\\
&\quad\times\left(\int_{Q_j}|h(x)-(h)_{Q_j}||f(x)|_qdx+\int_{Q_j}|h(x)-(h)_{Q_j}||g(x)|_qdx\right)\\
&=V_1+V_2\,.
\end{aligned}
$$

To estimate V_2 we combine the argument to prove (4.23), replacing M by $M_{L\log L}$, together with the definition of BMO:

$$V_2 \le C \int_{\mathbf{R}^n} |f(x)|_q M_{L(\log L)^\varepsilon}(w)(x)dx\,.$$

For V_1 we have by the generalized Hölder inequality (2.3)

$$\begin{aligned} V_1 &= \frac{C}{\lambda} \sum_j \inf_{Q_j} M_{L(\log L)^\varepsilon}(\chi_{\mathbf{R}^n\setminus 2Q_j} w)(x) \int_{Q_j} |h(x) - (h)_{Q_j}||f(x)|_q dx \\ &\le \frac{C}{\lambda} \sum_j \inf_{Q_j} M_{L(\log L)^\varepsilon}(\chi_{\mathbf{R}^n\setminus 2Q_j} w)(x)|Q_j|\||f|_q\|_{L\log L,Q_j}. \end{aligned}$$

Now, combining formula (2.2) with (4.24) and recalling that $\Phi(t) = t\log(e+t)$, we have

$$\begin{aligned} \frac{1}{\lambda}|Q_j|\|f_q\|_{L\log L,Q_j} &\le \frac{1}{\lambda}|Q_j| \inf_{\mu>0} \left\{\mu + \frac{\mu}{|Q_j|}\int_{Q_j} \Phi\left(\frac{|f(x)|_q}{\mu}\right)dx\right\} \\ &\le |Q_j| + \int_{Q_j} \Phi\left(\frac{|f(x)|_q}{\lambda}\right)dx \\ &\le \frac{1}{\lambda}\int_{Q_j} |f(x)|_q dx + \int_{Q_j} \Phi\left(\frac{|f(x)|_q}{\lambda}\right)dx \\ &\le 2\int_{Q_j} \Phi\left(\frac{|f(x)|_q}{\lambda}\right)dx\,. \end{aligned}$$

Then

$$\begin{aligned} V_1 &\le C\int_{Q_j} \Phi\left(\frac{|f(x)|_q}{\lambda}\right) M_{L(\log L)^\varepsilon}(\chi_{\mathbf{R}^n\setminus 2Q_j} w)(x)dx \\ &\le C\int_{\mathbf{R}^n} \Phi\left(\frac{|f(x)|_q}{\lambda}\right) M_{L(\log L)^{1+\varepsilon}}(w)(x)dx\,. \end{aligned}$$

The proof of the theorem is finished. □

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