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Sharpening and generalizations of Shafer-Fink's double inequality for the arc sine function

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Abstract. In the paper, the authors elementarily sharpen and generalize Shafer-Fink's double inequality for the arc sine function.

1. Introduction and main results

In [4, p. 247, 3.4.31], it was listed that the inequality

$$\arcsin x > \frac{6\left(\sqrt{1+x}-\sqrt{1-x}\right)}{4+\sqrt{1+x}+\sqrt{1-x}} > \frac{3x}{2+\sqrt{1-x^2}} \tag{1.1}$$

holds for $0 < x < 1$. It was also pointed out in [4, p. 247, 3.4.31] that these inequalities are due to R. E. Shafer, but no other related reference is cited. By now we do not know the very original source of inequalities in (1.1).

In the first part of the short paper [2], the inequality between the very ends of (1.1) was recovered and an upper bound for the arc sine function was also established as follows:

$$\frac{3x}{2+\sqrt{1-x^2}} \le \arcsin x \le \frac{\pi x}{2+\sqrt{1-x^2}}, \quad 0 \le x \le 1. \tag{1.2}$$

Therefore, we call (1.2) Shafer-Fink's double inequality for the arc sine function.

In [3], the right hand side inequality in (1.2) was improved to

$$\arcsin x \le \frac{\pi x/(\pi-2)}{2/(\pi-2)+\sqrt{1-x^2}}, \quad 0 \le x \le 1. \tag{1.3}$$

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In [13], the inequality (1.3) was recovered and the following Shafer-Fink type inequalities were derived:

$$\frac{\pi(4-\pi)x}{2/(\pi-2)+\sqrt{1-x^2}} \le \arcsin x \quad\text{and}\quad \frac{(\pi/2)x}{1+\sqrt{1-x^2}} \le \arcsin x, \quad 0 \le x \le 1. \tag{1.4}$$

Note that the lower bounds in (1.2) and (1.4) are not included in each other.

The main aim of this paper is to elementarily sharpen and generalize the above Shafer-Fink type double inequalities.

Our main results can be stated as follows.

Theorem 1.1. *For $\alpha \in \mathbb{R}$ and $x \in (0,1]$, the function $f_\alpha(x) = \left(\alpha + \sqrt{1-x^2}\right)\frac{\arcsin x}{x}$ is strictly*

1. *increasing if and only if $\alpha \ge 2$;*
2. *decreasing if and only if $\alpha \le \frac{\pi}{2}$.*

When $\frac{\pi}{2} < \alpha < 2$, the function $f_\alpha(x)$ has a unique minimum on $(0,1)$.

As straightforward consequences of Theorem 1.1, the following double inequalities may be readily derived.

Theorem 1.2. *If $\alpha \ge 2$, the double inequality*

$$\frac{(\alpha+1)x}{\alpha+\sqrt{1-x^2}} \le \arcsin x \le \frac{(\pi\alpha/2)x}{\alpha+\sqrt{1-x^2}} \tag{1.5}$$

holds on $[0,1]$. If $0 < \alpha \le \frac{\pi}{2}$, the double inequality (1.5) reverses. If $\frac{\pi}{2} < \alpha < 2$, then the double inequality

$$\frac{4\left(1-1/\alpha^2\right)x}{\alpha+\sqrt{1-x^2}} \le \arcsin x \le \frac{\max\{\pi\alpha/2, \alpha+1\}x}{\alpha+\sqrt{1-x^2}} \tag{1.6}$$

holds on $[0,1]$.

2. Remarks

Before proving our theorems, we would like to give several remarks on them.

Remark 2.1. Letting $x = \sin t$ for $t \in \left[0, \frac{\pi}{2}\right]$ yields the restatement of Theorem 1.2 as follows.

1. If $\alpha \ge 2$, then
$$\frac{(\alpha+1)\sin t}{\alpha+\cos t} \le t \le \frac{(\pi\alpha/2)\sin t}{\alpha+\cos t}, \quad 0 \le t \le \frac{\pi}{2}. \tag{2.1}$$
2. If $0 < \alpha \le \frac{\pi}{2}$, the double inequality (1.5) reverses.
3. If $\frac{\pi}{2} < \alpha < 2$, then
$$\frac{4\left(1-1/\alpha^2\right)\sin t}{\alpha+\cos t} \le t \le \frac{\max\{\pi\alpha/2, \alpha+1\}\sin t}{\alpha+\cos t}, \quad 0 \le t \le \frac{\pi}{2}. \tag{2.2}$$

For more information on the inequalities in (2.1) and (2.2), please refer to [8] and closely related references therein.

Remark 2.2. The Shafer-Fink's double inequality (1.2) is the special case $\alpha = 2$ in (1.5).

Remark 2.3. Taking $\alpha = \frac{\pi}{2}$ in (1.5) gives

$$\frac{\left(\pi^2/4\right)x}{\pi/2+\sqrt{1-x^2}} \le \arcsin x \le \frac{(\pi/2+1)x}{\pi/2+\sqrt{1-x^2}}, \quad 0 \le x \le 1. \tag{2.3}$$

This improves the first inequality in (1.4) and recovers the right hand side inequality in [13, p. 61, Theorem 8].

The left hand side inequalities in (1.4) and (2.3) are not included in each other.

The lower bound in (2.3) and those in (1.1) are not included in each other.

All the above comparisons can be carried out by straightforward arguments or by, with the help of the famous software Mathematica, plotting the graphs of differences between corresponding bounds for $\arcsin x$.

Remark 2.4. Since $\frac{\pi\alpha}{2} = \alpha + 1$ has a unique root $\alpha = \frac{2}{\pi-2} \in \left(\frac{\pi}{2}, 2\right)$, the inequality (1.3) follows from taking $\alpha = \frac{2}{\pi-2}$ in (1.6).

Remark 2.5. Let $G_x(\alpha) = \frac{1-1/\alpha^2}{\alpha+\sqrt{1-x^2}}$ for $\frac{\pi}{2} < \alpha < 2$ and $x \in (0,1)$. Then

$$\alpha\left(3-\alpha^2\right) < \alpha^3\left(\alpha+\sqrt{1-x^2}\right)^2 G_x'(\alpha) = 3\alpha - \alpha^3 + 2\sqrt{1-x^2} < 2 + 3\alpha - \alpha^3.$$

This means that

1. when $\frac{\pi}{2} < \alpha \le \sqrt{3}$ the function $\alpha \mapsto G_x(\alpha)$ is increasing;
2. when $\sqrt{3} < \alpha < 2$ the function $\alpha \mapsto G_x(\alpha)$ attains its maximum

   $$\frac{4\cos^2\varphi(x)-1}{4\left\{2\cos\varphi(x)+\sqrt{1-x^2}\right\}\cos^2\varphi(x)}$$

   at the point

   $$2\cos\left[\frac{1}{3}\arctan\left(\frac{x}{\sqrt{1-x^2}}\right)\right], \quad x \in (0,1),$$

   where

   $$\varphi(x) = \frac{1}{3}\arctan\left(\frac{x}{\sqrt{1-x^2}}\right).$$

Therefore, the following two sharp inequalities may be derived from the left hand side inequality in (1.6) for $x \in (0,1)$:

$$\arcsin x > \frac{(8/3)x}{\sqrt{3}+\sqrt{1-x^2}} \quad \text{and} \quad \arcsin x > \frac{x\left\{4\cos^2\varphi(x)-1\right\}}{\left\{2\cos\varphi(x)+\sqrt{1-x^2}\right\}\cos^2\varphi(x)}. \tag{2.4}$$

By the famous software Mathematica, we reveal that the inequality (2.4) is better than the left hand side inequality in (1.2), the inequality (1.4), and the first one in (2.4), and does not include the first inequality in (1.4) and the left hand side inequality in (2.3).

Remark 2.6. Letting $\arcsin x = t$ in (2.4) yields

$$\frac{\sin t}{t} > \frac{[2\cos(t/3)+\cos t]\cos^2(t/3)}{4\cos^2(t/3)-1}, \quad 0 < t < \frac{\pi}{2}. \tag{2.5}$$

This inequality has something to do with the first open problem posed in [9, Section 7.6].

Remark 2.7. The method to prove Theorems 1.1 and 1.2 in next section have been used in [5, 7, 8, 10–12] and closely related references therein.

Remark 2.8. The method used in next section to prove Theorems 1.1 and 1.2 is more elementary than the ones utilized in [2, 3, 13–15].

Remark 2.9. We note that Shafer type inequalities from [3] were applied recently in [1] for obtaining upper and lower bounds on the Gaussian Q-function.

3. Proofs of theorems

Now we are in a position to prove our theorems.

3.1. Proof of Theorem 1.1

Direct differentiation yields

$$f_\alpha'(x)=\frac{\alpha+1/\sqrt{1-x^2}}{x^2}\left[\frac{x\left(\alpha+\sqrt{1-x^2}\right)}{1+\alpha\sqrt{1-x^2}}-\arcsin x\right]\triangleq\frac{\alpha+1/\sqrt{1-x^2}}{x^2}h_\alpha(x)$$

and

$$h_\alpha'(x)=\frac{x^2\left(\alpha^2-2-\alpha\sqrt{1-x^2}\right)}{\left(1+\alpha\sqrt{1-x^2}\right)^2\sqrt{1-x^2}}.$$

Because $\alpha^2-\alpha-2\le\alpha^2-2-\alpha\sqrt{1-x^2}\le\alpha^2-2$ on $[0,1]$, the derivative $h_\alpha'(x)$ is negative (or positive respectively) when $0<\alpha\le\sqrt{2}$ (or $\alpha\ge 2$ respectively). Moreover, if $\sqrt{2}<\alpha<2$, the derivative $h_\alpha'(x)$ has a unique zero on $(0,1)$, at which the value of $h_\alpha'(x)$ becomes from negative to positive. As a result, the function $h_\alpha(x)$ is increasing (or decreasing respectively) when $\alpha\ge 2$ (or $0<\alpha\le\sqrt{2}$ respectively) and has a unique minimum on $(0,1)$ when $\sqrt{2}<\alpha<2$. It is easy to obtain that $h_\alpha(0)=0$ and $h_\alpha(1)=\alpha-\frac{\pi}{2}$. Hence,

1. when $\alpha\ge 2$, the functions $h_\alpha(x)$ and $f_\alpha'(x)$ are positive, and so $f_\alpha(x)$ is strictly increasing on $(0,1)$;
2. when $0<\alpha\le\sqrt{2}$, the functions $h_\alpha(x)$ and $f_\alpha'(x)$ are negative, and so $f_\alpha(x)$ is strictly decreasing on $(0,1)$;
3. when $\sqrt{2}<\alpha<2$ and $\alpha\le\frac{\pi}{2}$, the functions $h_\alpha(x)$ and $f_\alpha'(x)$ are also negative, and so $f_\alpha(x)$ is also strictly decreasing on $(0,1)$;
4. when $\sqrt{2}<\alpha<2$ and $\alpha>\frac{\pi}{2}$, the functions $h_\alpha(x)$ and $f_\alpha'(x)$ have the same unique zero on $(0,1)$, and so the function $f_\alpha(x)$ has a unique minimum on $(0,1)$.

On other hand, the derivative $f_\alpha'(x)$ may be rearranged as

$$f_\alpha'(x)=\frac{1}{x^2}\left[x\left(1+\frac{\alpha}{\sqrt{1-x^2}}\right)-\left(\alpha+\frac{1}{\sqrt{1-x^2}}\right)\arcsin x\right]\triangleq\frac{1}{x^2}H_\alpha(x)$$

and

$$H_\alpha'(x)=-\frac{x\left[x\left(\sqrt{1-x^2}-\alpha\right)+\arcsin x\right]}{(1-x^2)^{3/2}}.$$

When $\alpha\le 0$, the derivative $H_\alpha'(x)$ is negative, and so the function $H_\alpha(x)$ is strictly decreasing on $(0,1)$. From $\lim_{x\to0^+}H_\alpha(x)=0$, it follows that $H_\alpha(x)<0$ on $(0,1)$. Therefore, when $\alpha\le 0$, the derivative $f_\alpha'(x)$ is negative and the function $f_\alpha(x)$ is strictly decreasing on $(0,1)$. The proof of Theorem 1.1 is complete.

3.2. Proof of Theorem 1.2

It is easy to obtain that $\lim_{x\to0^+}f_\alpha(x)=\alpha+1$ and $f_\alpha(1)=\frac{\pi}{2}\alpha$. From the monotonicity obtained in Theorem 1.1, it follows that

1. when $\alpha\ge 2$, we have

$$\alpha+1<\left(\alpha+\sqrt{1-x^2}\right)\frac{\arcsin x}{x}\le\frac{\pi}{2}\alpha \tag{3.1}$$

on $(0,1]$, which can be rewritten as the inequality (1.5);

2. when $0 < \alpha \le \frac{\pi}{2}$, the inequality (3.1) is reversed;
3. when $\frac{\pi}{2} < \alpha < 2$, we have

$$\left(\alpha + \sqrt{1-x^2}\right)\frac{\arcsin x}{x} \le \max\left\{\frac{\pi}{2}\alpha, \alpha+1\right\}$$

   which can be rearranged as the right hand side inequality in (1.6).

On the other hand, when $\frac{\pi}{2} < \alpha < 2$, the minimum point $x_0 \in [0,1]$ of $f_\alpha(x)$ satisfies

$$\frac{\arcsin x_0}{x_0} = \frac{\alpha + \sqrt{1-x_0^2}}{1+\alpha\sqrt{1-x_0^2}}.$$

Hence, the minimum equals

$$f_\alpha(x_0) = \frac{\left(\alpha + \sqrt{1-x_0^2}\right)^2}{1+\alpha\sqrt{1-x_0^2}} = \frac{(\alpha+u_0)^2}{1+\alpha u_0} \ge 4\left(1-\frac{1}{\alpha^2}\right), \quad u_0 \in [0,1].$$

The left hand side inequality in (1.6) follows. The proof of Theorem 1.2 is complete.

Remark 3.1. This paper is a slightly modified version of the preprint [6].

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