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Field line coordinates are used to obtain a concise set of three coupled partial differential equations for the modes in ideal magnetohydrodynamic (MHD) plasmas of arbitrary geometry. Zero plasma pressure is assumed throughout. The shear Alfvén continuum and the pressureless remnant of the ballooning continuum are readily obtained. The equations are suitable, in particular, for the study of the spatial dependence of continuum modes close to singular magnetic surfaces or field lines.

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1. Introduction

The shear Alfvén continuum of ideal magnetohydrodynamics (MHD) has received considerable interest and attention in the past because of its potential importance for plasma heating [1] and current drive [2], its possible effects on plasma stability in the presence of fast particles [3], its relevance to magnetospheric observations [4], and its inherent interest as a basic plasma physics phenomenon [5]. The references cited here are a few among the immense body of literature in this imported field. The shear Alfvén wave continuum is connected with a spatial singularity of the mode amplitudes. This singularity implies local energy accumulation, which is relevant for the practical applications already mentioned. In the present investigation, which is Part I of a two part study of the MHD continua in asymmetric plasma configurations, the linearized equations are cast in a form that is useful to study the continuum modes and their singularities in equilibria with arbitrary geometry. The derivation and analysis of the continuum singularities is carried out in Part II [6]. The essence of Part I is that, with a proper choice of coordinates and in the low-β limit, the seven scalar MHD mode equations can be reduced to a compact system of three coupled scalar equations, which are Eqs. (36), (37) and (38) below. They clearly suggest resonance phenomena and spatial mode singularities corresponding to continuous spectra. This prepares the way for the investigation presented in Part II. An important advantage of our reduced system of three equations is a new access and a better understanding of the common origin of the shear Alfvén continuum and the ballooning continuum, a fact that has been previously pointed out [5, 7]. This could be of pedagogical value, since in particular the term "Alfvén mode" and other related designations are non-standardized and are commonly used by different authors with rather different meanings. Our designations are discussed in Section 4.

Since the focus of this investigation is the shear Alfvén continuum, which is stable regardless of the plasma pressure [8, 9, 10], we feel that it is justifiable to consider only the pressureless limit, which avoids the complications from pressure-related modes. For ballooning modes, which may become unstable for large enough pressure, this is a much more restrictive but still potentially useful assumption, e.g. for the study of global modes [11].

An important aspect in the present work is the choice of the coordinate system. Among three arbitrary curvilinear coordinates (r^1, r^2, r^3), we select r^1 and r^2 as field line coordinates. The surfaces $r^1(\mathbf{r}) = \text{const}$ and $r^2(\mathbf{r}) = \text{const}$ are then magnetic surfaces, while their intersection defines magnetic field lines. The corresponding contravariant components of the magnetic field $\mathbf{B}$, namely B^1 and B^2, vanish,

$$\mathbf{B} \cdot \nabla r^1(\mathbf{r}) = 0, \qquad \mathbf{B} \cdot \nabla r^2(\mathbf{r}) = 0, \tag{1}$$

where $\mathbf{r}$ is the position vector. The coordinate function $r^3(\mathbf{r})$ is arbitrary but linearly independent of $r^1(\mathbf{r})$ and $r^2(\mathbf{r})$. At fixed r^1 and r^2, the coordinate r^3 labels points on a field line. Field line coordinates facilitate the derivation of our final triplet of equations and make it rather concise. The construction of field line coordinates is always locally possible [12]. A disadvantage of them is that their *global* use on irrational toroidal magnetic surfaces is not straightforward. It is easily possible, however, to transform our *final* system of equations to other coordinates which are better suited for toroidal geometry. In Part II, for example, a transformation to Boozer coordinates [13] is performed. In field line coordinates the magnetic field has the Clebsch representation

$$\mathbf{B} = f(r^1, r^2)\, [\nabla r^1 \times \nabla r^2], \tag{2}$$

where f is an arbitrary function of its arguments, and $\operatorname{div}\mathbf{B} = 0$ implies that f is independent of r^3. The function f can always be transformed into unity [12]. Although $f \neq 1$ is employed occasionally, e.g. in Ref. [14], we select $f \equiv 1$ throughout the present study.

Part I of our two part study is organized as follows. In Section 2 the basic linearized mode equations are presented in general curvilinear coordinates and thereafter restricted to field line coordinates. In Section 3 the mode equations are reduced to a coupled system of three partial differential equations for three variables, which describe the MHD discrete and continuous spectra and modes of a pressureless plasma of arbitrary geometry. This system of equations is discussed in Section 4. Intermediate steps in the derivation are provided in Appendix A. Simplified forms of the reduced system that are often used e.g. in magnetospheric physics are given in Appendix B. In Appendix C, an equation which can clarify the relationship of the ballooning continuum to the shear Alfvén continuum is derived. This relationship is also discussed in Section 4.

2. Basic Equations and Choice of Coordinates

The linearized MHD equations [15] can be written in the form

$$i\omega\mu_0\rho\mathbf{v} = [\operatorname{curl}\mathbf{B} \times \mathbf{b}] + [\operatorname{curl}\mathbf{b} \times \mathbf{B}] - \mu_0\nabla p, \tag{3}$$

$$i\omega\mathbf{b} = \operatorname{curl}[\mathbf{v} \times \mathbf{B}], \tag{4}$$

$$i\omega p = -\mathbf{v}\cdot\nabla P - \gamma P \operatorname{div}\mathbf{v}, \tag{5}$$

where $\mathbf{b}$, $\mathbf{v}$, and p represent, respectively, the magnetic field, the velocity field and the pressure of the wave, and the parameters μ_0 and γ designate, respectively, the vacuum permeability and the ratio of specific heats. Capital letters denote equilibrium values. In addition, a harmonic time dependence of the form $\exp(i\omega t)$ with frequency ω has been assumed. As mentioned in the introduction we neglect the effects of finite pressure, and assume $P = 0$. Consequently, the MHD equilibrium equation $[\text{curl}\,\mathbf{B} \times \mathbf{B}] = \mu_0 \nabla P$ reduces to

$$[\text{curl}\,\mathbf{B} \times \mathbf{B}] = 0, \tag{6}$$

and Eq. (5) goes over into

$$p = 0. \tag{7}$$

It is assumed that the equilibrium equation (6) has asymmetric solutions. This assumption is of course non-trivial, but as in most treatments of asymmetric configurations, it has to be made in order to proceed further. From Eqs. (3) and (6), it follows that

$$\mathbf{v} \cdot \mathbf{B} = 0. \tag{8}$$

In the pressureless case the displacement vector of the mode contains no component in the direction of the magnetic field. From Eqs. (4), (6) and (8), and the vector identity $\mathbf{a} \cdot \text{curl}\,\mathbf{c} = \mathbf{c} \cdot \text{curl}\,\mathbf{a} - \text{div}\,[\mathbf{a} \times \mathbf{c}]$ it follows that

$$i\omega \mathbf{b} \cdot \mathbf{B} = -\,\text{div}\,(|\mathbf{B}|^2\, \mathbf{v}). \tag{9}$$

The perturbed magnetic pressure is designated $\mu_0 p^*$, where

$$p^* = \mathbf{b} \cdot \mathbf{B}. \tag{10}$$

Let $r^1(\mathbf{r})$, $r^2(\mathbf{r})$ and $r^3(\mathbf{r})$ be an arbitrary, nonsingular triple of coordinates. Both contra- and covariant vector components with respect to r^i, $i = 1, 2, 3$, are important in the present analysis. Their definitions, for an arbitrary vector $\mathbf{a}$, are standard, $a^i = \mathbf{a} \cdot \nabla r^i$ for contravariant components and $a_i = \mathbf{a} \cdot \mathbf{e}_i$ for covariant components, where $\mathbf{e}_i$ designates the basis vector, $\mathbf{e}_i = \partial \mathbf{r} / \partial r^i$. The metric tensors g^{ik} and g_{ik} are defined by $g^{ik} = \nabla r^i \cdot \nabla r^k$ and $g_{ik} = \mathbf{e}_i \cdot \mathbf{e}_k$. Furthermore,

$$a^i = g^{ik} a_k, \quad a_i = g_{ik} a^k, \quad g_{ij} g^{jk} = \delta_i^k, \tag{11}$$

where the summation convention over repeated indices is understood and δ_i^k is the Kronecker delta with the values 0 and 1. The determinant of the matrix $\{g_{ik}\}$ is denoted by g.

In covariant notation the equilibrium condition (6) becomes

$$B^j \left(\frac{\partial B_i}{\partial r^j} - \frac{\partial B_j}{\partial r^i} \right) = 0, \tag{12}$$

while Eq. (8) acquires the form

$$B^i v_i = 0, \tag{13}$$

or, alternatively,

$$B_i v^i = 0. \tag{14}$$

Equation (9), written in covariant notation, is

$$i\omega p^* = -\frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}\,|\mathbf{B}|^2 v^i)}{\partial r^i}, \tag{15}$$

with

$$B^i b_i = p^*. \tag{16}$$

In order to obtain the covariant representation of Eq. (3), use is made of the well-known relations [12] $[\mathbf{a}\times\mathbf{b}]_i = \sqrt{g}\,\epsilon_{ijk}a^j b^k$ and $\sqrt{g}\,(\mathrm{curl}\ \mathbf{a})^l = \epsilon^{lmn}\partial a_n/\partial r^m$ for arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$. Here, $\epsilon_{ijk} = \epsilon^{ijk} = 0$ if any two indices coincide, $\epsilon_{ijk} = \epsilon^{ijk} = 1$ if i, j and k are cyclic permutations of 1, 2 and 3, and $\epsilon_{ijk} = \epsilon^{ijk} = -1$ otherwise. With the relation $\epsilon_{ijk}\,\epsilon^{kmn} = \delta_i^m\,\delta_j^n - \delta_i^n\,\delta_j^m$, where δ_i^k is the Kronecker delta, and with Eq. (16), there results after some algebra

$$\mu_0 i\omega\rho v_k = -\frac{\partial p^*}{\partial r^k} + Db_k + b^j\left(\frac{\partial B_k}{\partial r^j} - \frac{\partial B_j}{\partial r^k}\right) + b_j\frac{\partial B^j}{\partial r^k}. \tag{17}$$

Here, the operator D is defined by

$$D \equiv \mathbf{B}\cdot\nabla = B^j\frac{\partial}{\partial r^j}. \tag{18}$$

Similarly, from Eqs. (4) and (15) there results

$$i\omega b^k = \frac{i\omega p^*}{|\mathbf{B}|^2}B^k + Dv^k + v^m\left(-\frac{\partial B^k}{\partial r^m} + \frac{B^k}{|\mathbf{B}|^2}\frac{\partial|\mathbf{B}|^2}{\partial r^m}\right). \tag{19}$$

For later purposes it is advantageous to express the covariant components v_k and b_k of $\mathbf{v}$ and $\mathbf{b}$ not by their contravariant counterparts but in a mixed representation. In Appendix B, v_1 and v_2 are expressed in terms of v^1, v^2 and v_3, and analogously with b_1 and b_2, see Eqs. (A.1) – (A.3).

We now select the coordinates $r^1(\mathbf{r})$ and $r^2(\mathbf{r})$ such that they are field line coordinates, i.e. such that they satisfy Eq. (1), $B^1 = B^2 = 0$. With this choice there results from Eqs. (11) – (12)

$$|\mathbf{B}|^2 = B_3 B^3, \quad B_i = g_{i3}B^3 \quad \text{for } i = 1,2,3. \tag{20}$$

$$\frac{\partial B_1}{\partial r^3} - \frac{\partial B_3}{\partial r^1} = 0, \qquad \frac{\partial B_2}{\partial r^3} - \frac{\partial B_3}{\partial r^2} = 0, \tag{21}$$

and from Eq. (13),

$$v_3 = 0. \tag{22}$$

Equations (14) and (15) remain unchanged. Equations (16) – (19) now read

$$B^3 b_3 = p^*, \tag{23}$$

$$\mu_0 i\omega\rho v_1 = -\frac{\partial p^*}{\partial r^1} + Db_1 + b^2\left(\frac{\partial B_1}{\partial r^2} - \frac{\partial B_2}{\partial r^1}\right) + b_3\frac{\partial B^3}{\partial r^1}, \tag{24}$$

$$\mu_0 i\omega\rho v_2 = -\frac{\partial p^*}{\partial r^2} + Db_2 + b^1\left(\frac{\partial B_2}{\partial r^1} - \frac{\partial B_1}{\partial r^2}\right) + b_3\frac{\partial B^3}{\partial r^2}, \tag{25}$$

where the operator D is now

$$D = B^3\partial/\partial r^3, \tag{26}$$

and

$$i\omega b^1 = Dv^1, \qquad i\omega b^2 = Dv^2. \tag{27}$$

In field line coordinates, (v_1, v_2) and (b_1, b_2, b_3) are given by Eqs. (A.4) – (A.7) in Appendix B. The bracketed terms in Eqs. (24) and (25) can be expressed in coordinate invariant form with the parallel component of the equilibrium current density $\mathbf{J} = \mathrm{curl}\,\mathbf{B}/\mu_0$. From

$$J^3 = \frac{1}{\mu_0\sqrt{g}}\left(\frac{\partial B_2}{\partial r^1} - \frac{\partial B_1}{\partial r^2}\right) \tag{28}$$

and $J^1 = J^2 = 0$, see Eq. (21), the coordinate invariant form reads

$$\frac{\partial B_2}{\partial r^1} - \frac{\partial B_1}{\partial r^2} = \mu_0\frac{\mathbf{J}\cdot\mathbf{B}}{|\mathbf{B}|^2}. \tag{29}$$

In the derivation of Eq. (29) use was made of $\sqrt{g}B^3 = 1$, which follows from $1/\sqrt{g} = [\nabla r^1 \times \nabla r^2]\cdot\nabla r^3$ and Eq. (2) with $f = 1$. In the pressureless case, with $\mathbf{J}\times\mathbf{B} = 0$, it holds that

$$D\frac{\mathbf{J}\cdot\mathbf{B}}{|\mathbf{B}|^2} = 0. \tag{30}$$

This follows from $\mathbf{J} = \widehat{\sigma}\mathbf{B}$ and $\mathrm{div}\,\mathbf{J} = \widehat{\sigma}\,\mathrm{div}\,\mathbf{B} + \mathbf{B}\cdot\nabla\widehat{\sigma} = 0$, where $\widehat{\sigma} = \mathbf{J}\cdot\mathbf{B}/|\mathbf{B}|^2$.

3. Set of Three Coupled Equations

A set of three coupled partial differential equations for the three variables v^1, v^2 and p^* alone can be derived from Eqs. (15), (24) and (25). For this purpose, v^3 is eliminated with the help of Eq. (14), while v_1, v_2 and b^1, b^2, b_3 are inserted from Eqs. (A.4) – (A.7). Furthermore, the equilibrium relations (21) are applied to Eq. (15). The result is

$$i\omega p^* = \frac{1}{\sqrt{g}}\left\{-B_3\left[\frac{\partial v^1}{\partial r^1} + \frac{\partial v^2}{\partial r^2}\right] + B_1\frac{\partial v^1}{\partial r^3} + B_2\frac{\partial v^2}{\partial r^3}\right\}. \tag{31}$$

$$\frac{\partial\, i\omega p^*}{\partial r^1} - \frac{1}{B^3}\frac{\partial B^3}{\partial r^1} i\omega p^* - D\left(\frac{B_1}{|\mathbf{B}|^2}\, i\omega p^*\right)$$
$$= \mathcal{L}^{22}\, v^1 - \mathcal{L}^{21}\, v^2 - \mu_0 \frac{\mathbf{J}\cdot\mathbf{B}}{|\mathbf{B}|^2} D v^2, \qquad (32)$$

$$\frac{\partial\, i\omega p^*}{\partial r^2} - \frac{1}{B^3}\frac{\partial B^3}{\partial r^2} i\omega p^* - D\left(\frac{B_2}{|\mathbf{B}|^2}\, i\omega p^*\right)$$
$$= \mathcal{L}^{11}\, v^2 - \mathcal{L}^{12}\, v^1 + \mu_0 \frac{\mathbf{J}\cdot\mathbf{B}}{|\mathbf{B}|^2} D v^1. \qquad (33)$$

Here, the operators $\mathcal{L}^{ik} = \mathcal{L}^{ki}$, $i, k = 1, 2$, are defined by

$$\mathcal{L}^{ik} \equiv D\frac{g^{ik}}{|\mathbf{B}|^2} D + \mu_0 \rho \omega^2 \frac{g^{ik}}{|\mathbf{B}|^2}. \qquad (34)$$

A slightly more compact form of Eqs. (31) – (33) is obtained if the covariant components E_1 and E_2 of the electric field perturbation $\mathbf{E}$ are used instead of v^1 and v^2. From Ohm's law, $\mathbf{E} + [\mathbf{v} \times \mathbf{B}] = 0$, which has been used already in the induction equation (4) above, it follows that

$$E_1 = -v^2, \quad E_2 = v^1. \qquad (35)$$

Also, p^* can be expressed in terms of b_3. The final result is

$$i\omega b_3 = B_3\left(\frac{\partial E_1}{\partial r^2} - \frac{\partial E_2}{\partial r^1}\right) + \frac{B_1}{B^3} D E_2 - \frac{B_2}{B^3} D E_1, \qquad (36)$$

$$B^3\,\frac{\partial\, i\omega b_3}{\partial r^1} - D\left(\frac{B_1}{B_3}\, i\omega b_3\right) = + \left(\mathcal{L}^{21}\, E_1 + \mathcal{L}^{22}\, E_2\right) + \mu_0 \frac{\mathbf{J}\cdot\mathbf{B}}{|\mathbf{B}|^2} D E_1, \qquad (37)$$

$$B^3\,\frac{\partial\, i\omega b_3}{\partial r^2} - D\left(\frac{B_2}{B_3}\, i\omega b_3\right) = - \left(\mathcal{L}^{11}\, E_1 + \mathcal{L}^{12}\, E_2\right) + \mu_0 \frac{\mathbf{J}\cdot\mathbf{B}}{|\mathbf{B}|^2} D E_2. \qquad (38)$$

The analysis of the spatial dependence of modes in the shear Alfvén continuum, which is performed in Part II of the present investigation, is based on Eqs. (36) – (38).

Some past investigations of continuum resonances, in particular in magnetospheric physics [16], are based on simplified versions of Eqs. (36) – (38). A derivation of theses forms is presented in Appendix B.

4. Discussion

The compact form of equations (36) – (38) for E_1, E_2 and b_3 permits a straightforward discussion of some relevant features.

First, plausible conditions for the occurrence of continuous spectra are easily identified, namely spatial singularities of modes. Conditions for them to occur are readily obtained. Assume that on a magnetic surface $r^1 = \text{const}$ the components E_2 and b_3 of the modes are known. In that case their derivatives $\partial E_2/\partial r^2$ and $\partial b_3/\partial r^2$ *within* that surface are also known. The same is true for the action of the operators D and $\mathcal{L}^{ik}$ since they only contain derivatives along the field lines within the surface. The derivatives $\partial E_2/\partial r^1$ and $\partial b_3/\partial r^1$ *out of* the surface are then bounded and are determined by Eqs. (36) and (37) provided E_1 on that surface can be expressed in terms of E_2 and b_3. According to Eq. (38) this is possible if the equation

$$\mathcal{L}^{11} E_1 = \widehat{s} \tag{39}$$

can be solved unambiguously for E_1 in terms of the remaining parts of Eq. (38), collectively denoted by $\widehat{s}$. This is guaranteed if the homogeneous equation for a function u

$$\mathcal{L}^{11}\, u = 0, \tag{40}$$

together with the same boundary conditions that E_1 satisfies in Eq. (39), only has the trivial solution $u = 0$. The equations with the r^1 derivative can then be integrated out of the initial surface and a smooth spatial dependence ensues. However, if Eq. (40) has a nontrivial solution, Eq. (39) may not have a spatially smooth solution for E_1. Ultimately, singular surfaces $r^1 = \text{const}$ indeed result, i.e. surfaces about which the modes become singular, see e.g. Refs. [9, 17, 18] and Part II of the present investigation. Hence, Eq. (40), and its analogue with respect to the surfaces $r^2 = \text{const}$,

$$\mathcal{L}^{22}\, v = 0, \tag{41}$$

according to this heuristic derivation are the conditions for continua in pressureless MHD plasmas.

A rigorous proof that Eqs. (40) and (41), generalized to include finite pressure, indeed are the correct equations for the spectral continua of the MHD equations was given by Hameiri in Ref. [5] for the general toroidal case and in Ref. [7] for the case of field lines with finite length. For axisymmetric plasma configurations pertinent exact results were obtained also in Refs. [19] and [20].

Equations (40) and (41) are eigenvalue equations with eigenvalue parameter ω^2 and with ordinary derivatives along field lines only. This indicates that the resonances occur primarily on field *lines* and that resonant *surfaces* are in a certain sense secondary phenomena. This point is stressed in Refs. [5] and [7]. The distinction between these two different forms of the continua becomes relevant in particular for configurations with toroidally closed magnetic surfaces $\psi(\mathbf{r}) = \text{const}$, say, where it occurs in a twofold manner. In these geometries the coordinate r^1, for example, can be identified with the label ψ of the magnetic flux surfaces. Equation (40) then agrees with the well known eigenvalue equation for modes in the shear Alfvén continuum of low-pressure plasmas [21]. It makes sense to look for solutions of Eq. (40) which are bounded and smooth on the closed magnetic surfaces, in other words to consider Eq. (40) as a boundary value problem with periodic boundary conditions in the poloidal and toroidal directions. A first indication of the above mentioned dichotomy is found in numerical model calculations [22] which suggest that increasing nonaxisymmetry can change the character of shear Alfvén continuum modes from the surface filling type into a type which is localized exponentially along a field line. A second, and more fundamental indication of the dichotomy between the surface and the field line aspects of the resonance phenomena is related to the magnetic and coordinate surfaces $r^2 = \text{const}$ which determine $g^{22} = \nabla r^2 \cdot \nabla r^2$ in the operator $\mathcal{L}^{22}$ in Eq. (41). These surfaces, denoted by R_2, say, may be chosen with a considerable amount of freedom [12]. Consider an arbitrary surface S transverse to a field line in a point P_0, say. On S, chose an arbitrary curve $\mathcal{C}$, say, passing through P_0. Those field lines which pass through $\mathcal{C}$ form a surface which, by construction, represents a magnetic surface R_2. Whatever the form of S and $\mathcal{C}$, the surface R_2 does not have the same toroidal topology as the surfaces $\psi = \text{const}$ and, for a field line on an irrational magnetic surface, it is a rather pathological object in which the surface comes arbitrarily close to itself. This makes it impossible to consider Eq. (41) as a boundary value problem with doubly periodic boundary conditions, as is possible with Eq. (40). It shows that the field line aspect for Eq. (41) is more natural. The formal symmetry of Eqs. (40) and (41) with respect to the coordinates r^1 and r^2, see Eq. (34), therefore is deceptive when applied to toroidal configurations.

The differences between Eqs. (40) and (41) in toroidal geometry can be detailed even further. If a family of coordinate surfaces $\psi = \text{const}$ is known and if the magnetic shear s along field lines is also known, g^{22} is almost completely fixed. In Appendix D, the relation

$$g^{22} = \frac{|\mathbf{B}|^2}{g^{11}} \left[1 + \left(\frac{g^{11}}{|\mathbf{B}|} \int_{\ell_0}^{\ell} s \, \frac{d\ell}{|\mathbf{B}|} \right)^2 \right] \tag{42}$$

is derived where the integration is along a field line and starts at an arbitrary position ℓ_0 along it. If the averaged value of the weighted shear along the field line does not vanish, the integral in Eq. (42) leads to a term which is secular in the distance from the starting point along the field line. Under the above mentioned conditions such a secular behavior in $\mathcal{L}^{22}$ is absent in the operator $\mathcal{L}^{11}$. Owing to the poloidal and toroidal periodicity of the ψ surfaces, $g^{11} = \nabla\psi \cdot \nabla\psi$ and hence the coefficients in the operator $\mathcal{L}^{11}$, are doubly periodic and bounded along field lines. This is a fundamental difference between the two operators. In axisymmetric or helically symmetric configurations, for example, it implies the absence of gap modes in Eq. (40) but not in Eq. (41).

Equation (41), with g^{22} from Eq. (42), coincides with the eigenvalue equation for ballooning modes as derived for example in Ref. [23] for the axisymmetric case and in Ref. [24] for the general toroidal case, if the pressure terms are neglected there. Of course, pressure effects are vital in causing ballooning instability so that Eq. (41) without pressure terms would be pointless to use for this particular purpose. For the study of resonance phenomena in ballooning stable low-pressure plasmas, however, Eq. (41) is well suited, see e.g. Ref. [11], and contributes to the continuous spectrum no less than Eq. (40). As already mentioned above, the discrete gap modes, such as the TAE [25] and HAE [11] modes ("toroidicity and helicity induced Alfvén eigenmodes"), are contained in Eq. (41).

The relationship of the shear Alfvén continuum and ballooning modes has been pointed out previously by Hameiri [5, 7] and by Spies [26]. In the magnetospheric context the two resonances from Eqs. (40) and (41) are also well known and are the focus of continued studies, e.g. in Refs. [4, 16, 19, 27, 28]. Field lines in a large part of the magnetosphere have dipolar instead of toroidal topology, with fixed ends. In such cases the distinction between Eqs. (40) and (41) tends to be a more quantitative than a qualitative one.

In the present context it may be worth recalling that no agreed-upon terminology exists in the literature as regards the modes or the continua connected with the operators $\mathcal{L}^{11}$ and $\mathcal{L}^{22}$, with or without the effects of finite pressure included. The term "continuous spectrum" or "shear Alfvén continuum" is often used for the spectrum of Eq. (40), see e.g. Refs. [21, 29, 30]. In Ref. [5] the designation "Alfvén" is reserved for phenomena connected with the operator $\mathcal{L}^{11}$ while phenomena associated with the operator $\mathcal{L}^{22}$ are referred to as "ballooning" effects. From the early days of ballooning stability theory this name was quite commonly used in connection with Eq. (41), with the addition of pressure effects. However, in treatments of the gap modes, which are also derived from Eq. (41), the designation "shear Alfvén" is also quite common in the literature, see e.g. Ref. [25] and the many studies that follow from it. In other studies of the gap modes the term "ballooning" is still employed [11]. In the magnetospheric literature the two operators $\mathcal{L}^{11}$ and $\mathcal{L}^{22}$ are on occasion referred to as "toroidal" and "poloidal", respectively [4]. The combined spectra and/or solutions of Eqs. (40) and (41) have several designations in the literature. For example one finds "essential spectrum" [5], "Alfvén" [7], "quasi-Alfvén" [19] and "continuous (ballooning and Alfvén)" [31]. In our study we use "shear Alfvén" in connection with Eq. (40) and "ballooning" in connection with Eq. (41).

Part II of our investigation is devoted to the study of shear Alfvén continuum modes in the vicinity of singular toroidal *surfaces*. The expansion method employed is equivalent to that used for axisymmetric toroidal configurations [17]. Singularities around field *lines* are not discussed since they require a different approach, for example a modified WKB method as employed in Ref. [32], in the study of the spatial properties of ballooning modes.

Appendix A: Transformations for v_k and b_k

From Eq. (11) one obtains

$$\begin{aligned} v_1 &= g_{11}v^1 + g_{12}v^2 + g_{13}v^3 = g_{11}v^1 + g_{12}v^2 + \frac{g_{13}}{g_{33}}\left(v_3 - g_{31}v^1 - g_{32}v^2\right) \\ &= \frac{g_{13}}{g_{33}}v_3 + \frac{1}{g_{33}}\left[\left(g_{11}g_{33} - g_{13}^2\right)v^1 + \left(g_{12}g_{33} - g_{13}g_{23}\right)v^2\right] \\ &= \frac{g_{13}}{g_{33}}v_3 + \frac{g}{g_{33}}\left(g^{22}v^1 - g^{12}v^2\right) \end{aligned} \tag{A.1}$$

and similarly

$$v_2 = \frac{g_{23}}{g_{33}}v_3 - \frac{g}{g_{33}}\left(g^{12}v^1 - g^{11}v^2\right). \tag{A.2}$$

The analogous equations for b_k are

$$b_1 = \frac{g_{13}}{g_{33}}b_3 + \frac{g}{g_{33}}\left(g^{22}b^1 - g^{12}b^2\right), \quad b_2 = \frac{g_{23}}{g_{33}}b_3 - \frac{g}{g_{33}}\left(g^{12}b^1 - g^{11}b^2\right). \tag{A.3}$$

In field line coordinates these expressions become

$$v_1 = \frac{1}{|\mathbf{B}|^2}\left(g^{22}v^1 - g^{12}v^2\right), \quad v_2 = \frac{1}{|\mathbf{B}|^2}\left(-g^{21}v^1 + g^{11}v^2\right), \tag{A.4}$$

where Eqs. (20) and (22) have been used, and, with Eq. (27),

$$i\omega b_1 = \frac{B_1}{|\mathbf{B}|^2}i\omega p^* + \frac{1}{|\mathbf{B}|^2}\left(g^{22}Dv^1 - g^{12}Dv^2\right), \tag{A.5}$$

$$i\omega b_2 = \frac{B_2}{|\mathbf{B}|^2}i\omega p^* - \frac{1}{|\mathbf{B}|^2}\left(g^{12}Dv^1 - g^{11}Dv^2\right), \tag{A.6}$$

$$i\omega b_3 = \frac{B_3}{|\mathbf{B}|^2}i\omega p^*. \tag{A.7}$$

Appendix B: Simplified Triplets of Equations

In many investigations of shear Alfvén or ballooning resonances in the magnetosphere it is assumed that field line coordinates r^1, r^2 and a third coordinate r^3 can be chosen such that they are mutually orthogonal. While this assumption significantly limits the magnetic configurations that can be considered [33], it does drastically simplify the triplet of equations (36) – (38) on which the analysis can be based. Since $B^i = g^{ii} B_i$ (without summation over repeated indices), it follows that $B_1 = B_2 = 0$. Also, from $g^{12} = 0$, there results

$$\mathcal{L}^{12} = 0. \tag{B.1}$$

If, in addition, the parallel-current density is set equal to zero, the remaining equations are [16]

$$\frac{\partial E_1}{\partial r^2} - \frac{\partial E_2}{\partial r^1} = \frac{1}{B_3} i\omega b_3, \tag{B.2}$$

$$L^{22}\, E_2 = \frac{\partial\, i\omega b_3}{\partial r^1}, \tag{B.3}$$

$$L^{11}\, E_1 = -\frac{\partial\, i\omega b_3}{\partial r^2}, \tag{B.4}$$

with $L^{ii} \equiv \sqrt{g}\,\mathcal{L}^{ii}$. In orthogonal coordinates it holds that $g = g_{11} g_{22} g_{33}$ and $g_{ii} = 1/g^{ii}$. With these relations it is easily seen that the operators L^{ii} assume the form [16, 19, 34]

$$L^{11} = \frac{\partial}{\partial r^3} \frac{1}{\alpha_2} \frac{\partial}{\partial r^3} + \frac{\mu_0 \rho \omega^2}{|\mathbf{B}|^2} \alpha_1, \tag{B.5}$$

$$L^{22} = \frac{\partial}{\partial r^3} \frac{1}{\alpha_1} \frac{\partial}{\partial r^3} + \frac{\mu_0 \rho \omega^2}{|\mathbf{B}|^2} \alpha_2, \tag{B.6}$$

where $\alpha_i \equiv \sqrt{g}/g_{ii}$. The apparent simplicity of Eqs. (B.2) – (B.4) is deceptive. They still retain the essence of shear Alfvén and ballooning resonances, and they may serve e.g. as model equations for the study of 3-D effects in nontrivial geometries.

Appendix C: Expression for g^{22}

In field line coordinates it follows from Eq. (2) with $f = 1$ and from the definition of the g^{ik} that $|\mathbf{B}|^2 = g^{11} g^{22} - (g^{12})^2$. Hence

$$g^{22} = \frac{|\mathbf{B}|^2}{g^{11}} \left[1 + \left(\frac{g^{11}}{|\mathbf{B}|} \frac{g^{12}}{g^{11}} \right)^2 \right]. \tag{C.1}$$

The local shear is defined by [35]

$$s \equiv \frac{1}{|\nabla r^1|^4} [\nabla r^1 \times \mathbf{B}] \cdot \mathrm{curl}\, [\nabla r^1 \times \mathbf{B}]. \tag{C.2}$$

With $[\nabla r^1 \times \mathbf{B}] = g^{12}\nabla r^1 - g^{11}\nabla r^2$, the evaluation of $\mathrm{curl}\,[\nabla r^1 \times \mathbf{B}]$ and the relation $[\nabla r^1 \times \nabla r^2] \cdot \nabla r^3 = 1/\sqrt{g}$, there results after some algebra

$$D\frac{g^{12}}{g^{11}} = -\, s. \tag{C.3}$$

This relation can be integrated along a field line. With Eq. (C.1) and $D = |\mathbf{B}| \mathrm{d}/\mathrm{d}\ell$ where l is the length along the field line, it gives rise to the expression (42).

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