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Shepard's Method of "Metric Interpolation" to Bivariate and Multivariate Interpolation

By William J. Gordon and James A. Wixom

Abstract. Shepard developed a scheme for interpolation to arbitrarily spaced discrete bivariate data. This scheme provides an explicit global representation for an interpolant which satisfies a maximum principle and which reproduces constant functions. The interpolation method is basically an inverse distance formula which is generalized to any Euclidean metric. These techniques extend to include interpolation to partial derivative data at the interpolation points.

1. Introduction. In 1968, D. Shepard [4] developed a technique for interpolating irregularly spaced discrete bivariate data and applied this scheme in the context of geographic and demographic data fitting. The techniques developed by Shepard form the basis of a generally applicable class of univariate and multivariate interpolation schemes which we have termed *metric interpolation.* The canonical metric interpolation method is essentially an inverse distance formula, and thus has certain of the properties (e.g., a Maximum Principle) possessed by the harmonic functions of classical potential theory.

In spite of our failure to uncover any references other than [4], the basic simplicity and applicability of these metric interpolation schemes leads one to suspect more antique origins. Be that as it may, this class of methods certainly is not "well known," and does have many interesting mathematical properties and potential applications to practical problems of multivariate data fitting.

The purpose of this paper is to further develop some of the notions introduced by Shepard and to establish certain results relating to the characterization of metric interpolation techniques. In addition, graphical examples are provided which illustrate some of these properties.

2. Shepard's Interpolation Scheme. Since the bivariate case illustrates most of the basic properties of Shepard's interpolation scheme, it will be discussed in detail in this section. Extensions to functions of more than two independent variables can be readily inferred from the bivariate setting. It should also be noted that these same schemes are applicable to univariate functions.

Let $F(P)$ be a function of the point $P = (x, y)$ defined for all P in the real plane R^2, and let $\{P_i\}_{i=1}^N$ be any finite collection of distinct points in R^2. Denote the value of F at P_i by F_i, and let $r_i = |P - P_i|$ be the Euclidean distance between P_i and the generic point P in R^2:

$$r_i = [(x - x_i)^2 + (y - y_i)^2]^{1/2}.$$

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A slight variation of the following result was established by Shepard [4]:

THEOREM 2.1. *The function $U(P)$ given by*

$$U(P) = \left[\sum_{i=1}^{N} F_i\left(\prod_{j\neq i} r_j\right)\right] \Big/ \left[\sum_{i=1}^{N} \prod_{j\neq i} r_j\right], \tag{2.1}$$

is continuous and interpolates F at the N points $\{P_i\}$, i.e.,

$$U(P_k) = F_k, \qquad k = 1, 2, \ldots, N. \tag{2.2}$$

Proof. The continuity of U follows directly from the fact that each r_i is continuous and the denominator of (2.1) never vanishes. The evaluation of (2.1) at the point P_k yields:

$$U(P_k) = \left[F_k \prod_{j\neq k} r_j\right] \Big/ \left[\prod_{j\neq k} r_j\right] = F_k,$$

since all other terms are zero.

The function U of (2.1) can be written in the equivalent form:

$$U(P) = \left[\sum_{i=1}^{N} F_i/r_i\right] \Big/ \left[\sum_{i=1}^{N} 1/r_i\right], \tag{2.3}$$

which is similar to the formulation by Shepard in [4].

The distribution of the points of interpolation P_i is totally arbitrary. This is in sharp contrast to familiar bivariate tensor product interpolation methods which require that the data points P_i be located at the mesh points (x_i, y_i) of a Cartesian product partitioning of a rectangle. This lack of necessary data structure may at first seem to violate the well-known negative result due to Haar [1], which states that there cannot exist a set of N functions $\varphi_i(P)$ such that the interpolation problem (2.2) is soluble as a linear combination of the φ_i for *every* distribution of N distinct points $P_1, P_2, \ldots, P_N$. This apparent dilemma is resolved, however, if we use (2.1) to rewrite $U(P)$ in the form

$$U(P) = \sum_{i=1}^{N} F_i \varphi_i(P; P_1, P_2, \ldots, P_N), \tag{2.4}$$

where

$$\varphi_i(P; P_1, P_2, \ldots, P_N) = \prod_{j\neq i} r_j \Big/ \left[\sum_{k=1}^{N} \prod_{j\neq k} r_j\right], \tag{2.5}$$

and we note that the functions φ_i depend upon the distribution of the point collection $\{P_i\}$—a possibility not covered by the Haar hypothesis. In other words, for a *fixed* distribution of points $\{P_i\}$, one is able to define an N-dimensional linear function space Φ as being the space spanned by the N basis functions $\varphi_i(P; P_1, P_2, \ldots, P_N)$; and, as Theorem 2.1 shows, the interpolation problem is uniquely soluble in this particular linear space. The Haar Theorem says that it is not possible to *a priori* select the linear space and then choose the point set $\{P_i\}$—the topology of the point set must determine the linear space.

The functions φ_i defined by (2.5) have the following *cardinality property*:

$$\varphi_i(P; P_1, P_2, \ldots, P_N)|_{P=P_k} = \delta_{ik} \quad \text{for } i, k = 1, 2, \ldots, N, \tag{2.6}$$

where δ_{ik} is the Kronecker delta. To be more explicit in designating the linear space spanned by the φ_i, we shall sometimes use the notation $\Phi(P_1, P_2, \ldots, P_N)$.

The cardinal basis functions $\varphi_i(P; P_1, P_2, \ldots, P_N)$ are *rational bivariate splines* in the sense that they are analytic (infinitely differentiable) in any region which does not include a point of interpolation, but they are merely continuous (i.e., not even once differentiable) at the points P_i. Thus, one can envision each of the basis functions φ_i as being a smooth two-dimensional transition surface between the points $\{P_i\}$ but with *cusps* at each of the P_i. From (2.6) we note that the value of the φ_i at each of these cusps is either zero or one. It is, therefore, obvious that any linear combination of the φ_i—in particular, the function $U(P)$—will also be analytic except at the points P_i.

For functions of a single independent variable, the r_j in the above formula are just $|x - x_j|$ so that (2.1) becomes

$$U(x) = \sum_{i=1}^{N} F_i\left(\prod_{j\neq i} |x - x_j|\right) \Bigg/ \sum_{i=1}^{N} \prod_{j\neq i} |x - x_j|, \tag{2.1'}$$

which is readily seen to be a rational spline of degree $N - 1$, i.e., both the numerator and denominator of (2.1′) are splines of degree $N - 1$.

An important property of formula (2.1) is the fact that, for all P, the values of $U(P)$ are bounded above by $\max_i F_i$ and below by $\min_i F_i$. In greater detail, we have the following Maximum Principle:

THEOREM 2.2. *Let* $M = \max_i F_i$ *and* $m = \min_i F_i$, *then*

$$m \leqslant U(P) \leqslant M \quad \textit{for all } P \textit{ in } R^2. \tag{2.7}$$

Proof. Let $C = \max(|M|, |m|)$; then

$$\frac{U(P) + C}{M + C} = \left[\sum_{i=1}^{N} \left(\frac{F_i + C}{M + C}\right) \prod_{j\neq i} r_j\right] \Bigg/ \left[\sum_{i=1}^{N} \prod_{j\neq i} r_j\right]. \tag{2.8}$$

Since $M \geqslant F_i$ for all i, we have

$$\frac{F_i + C}{M + C} \leqslant 1 \quad \text{for all } i = 1, 2, \ldots, N, \tag{2.9}$$

from which it follows that

$$\frac{U(P) + C}{M + C} \leqslant 1 \tag{2.10}$$

or equivalently,

$$U(P) \leqslant M. \tag{2.11}$$

By replacing M by m in (2.8) and using a similar argument, one can readily deduce that

$$m \leqslant U(P) \quad \text{for all } P \text{ in } R^2. \tag{2.12}$$

The Maximum Principle described by Theorem 2.2 is, of course, reminiscent of

the same familiar property of harmonic functions (e.g., elastic membranes) and, indeed, the surfaces generated by formula (2.1) look much like a thin elastic membrane supported by point loads of altitude F_i at the interpolation points P_i. This property and others discussed in this section are geometrically illustrated by the examples in the following section.

Examination of formula (2.3) reveals that it is basically an inverse distance formula. That is, for a fixed point P in R^2, the denominator in (2.3) can be considered to be a normalization constant so that the magnitude of $U(P)$ is directly proportional to the value F_i and inversely proportional to the distance from P to P_i. In this sense, the formula is analogous to a "$1/r$ gravitational law". In part, this explains the similarity between the surfaces obtained from this formula and the class of harmonic functions which are rooted in classical potential theory.

In light of Theorem 2.2, we can further characterize the behavior of the functions φ_i given by (2.5). Specifically, since φ_i itself satisfies the cardinal interpolation conditions of (2.6), we have:

COROLLARY 2.1.

$$0 \leqslant \varphi_i(P; P_1, P_2, \ldots, P_N) \leqslant 1 \quad \textit{for all } P \textit{ in } R^2. \tag{2.13}$$

A desirable property for any interpolation scheme is that it approximate constant functions exactly. Formula (2.1) has this property since, from (2.5), the basis functions φ_i can be readily seen to satisfy the relation

$$\sum_{i=1}^{N} \varphi_i(P; P_1, P_2, \ldots, P_N) = 1 \quad \text{for all } P \text{ in } R^2. \tag{2.14}$$

The interpolation of the primitive function F by U as in (2.1) can be viewed as a *projection* of F onto the finite-dimensional linear space $\Phi(P_1, P_2, \ldots, P_N)$. Let $\mathcal{P}$ be the projection operator (projector) so defined, i.e.,

$$\mathcal{P}[F] = U. \tag{2.15}$$

It is easy to check that $\mathcal{P}$ actually is a projector; that is, it has the properties of linearity and idempotency

$$\mathcal{P}[\alpha F + \beta G] = \alpha \mathcal{P}[F] + \beta \mathcal{P}[G], \tag{2.16a}$$

$$\mathcal{P}[\mathcal{P}[F]] = \mathcal{P}^2[F] = \mathcal{P}[F], \tag{2.16b}$$

where F and G are any two continuous bivariate functions and α and β are scalars. Moreover, from Theorem 2.1, it is easy to verify the following theorem; namely, that $\mathcal{P}$ is a *positive operator*–an uncommon property for an interpolation scheme.

THEOREM 2.3. *If $F_i \geqslant 0$ for all $i = 1, 2, \ldots, N$, then $U(P) \geqslant 0$ for all P in R^2.*

If all of the N points P_i are contained within a closed finite subdomain D of the plane, it is interesting to inquire into the asymptotic behavior of the φ_i as P recedes indefinitely outward from D. This query is answered by the following.

THEOREM 2.4. *Let $P_i \in D \subset R^2$ for all $i = 1, 2, \ldots, N$; and let d be the shortest distance from P to the boundary of D. Then,*

$$\lim_{d \to \infty} \varphi_i(P; P_1, P_2, \ldots, P_N) = 1/N. \tag{2.17}$$

Proof. The proof of this theorem is a special case of Theorem 3.2 of Section 3.

From Theorem 2.4, it follows immediately that as P recedes indefinitely from the cluster of interpolation points P_i, $U(P)$ approaches the average of the values of F_i; i.e.,

$$\lim_{\min_i |P-P_i| \to \infty} U(P) = (1/N) \sum_{i=1}^{N} F_i. \tag{2.18}$$

3. Generalized Metric Interpolation. The term *metric interpolation* derives from the fact that the Euclidean distance functions r_i in Eq. (2.1) can be generalized to any metric on the real plane without destroying the interpolatory properties of the function $U(P)$. In fact, if $\{\rho_i(P, Q), i = 1, \ldots, N\}$ is any set of metrics on m-dimensional real space R^m, then the function $U(P)$ defined by

$$U(P) = \left[\sum_{i=1}^{N} F_i \prod_{j \neq i} \rho_j(P_j, P)\right] \Bigg/ \left[\sum_{i=1}^{N} \prod_{j \neq i} \rho_j(P_j, P)\right] \tag{3.1}$$

interpolates to the given F_i at the points P_i in R^m. The corresponding set of cardinal basis functions φ_i are of the form

$$\varphi_i(P; P_1, P_2, \ldots, P_N) = \prod_{j \neq i} \rho_j(P_j, P) \Bigg/ \sum_{k=1}^{N} \prod_{j \neq k} \rho_j(P_j, P). \tag{3.2}$$

The set of φ_i defined by Eq. (3.2) satisfies many of the properties of their counterpart in Eq. (2.5). Specifically,

$$0 \leqslant \varphi_i \leqslant 1, \tag{3.3}$$

and

$$\sum_{i=1}^{N} \varphi_i = 1. \tag{3.4}$$

As a result, the metric interpolant in Eq. (3.1) defines a positive projector. Furthermore, $U(P)$ reproduces constant functions and satisfies the Maximum Principle of Theorem 2.2:

$$m \leqslant U(P) \leqslant M, \tag{3.5}$$

where $m = \min_i F_i$ and $M = \max_i F_i$.

Although the function $U(P)$ in Eq. (3.1) is bounded, the asymptotic behavior as given by Eq. (2.18) of the previous section may fail to hold depending upon the particular choice of the metric functions ρ_i. If we select each ρ_j in (3.1) to be a positive power α_j of the Euclidean distance r_j, then the interpolant is of the form

$$U(P) = \sum_{i=1}^{N} F_i \prod_{j \neq i} r^{\alpha_j}(P, P_j) \Bigg/ \sum_{i=1}^{N} \prod_{j \neq i} r^{\alpha_j}(P, P_j), \tag{3.6}$$

where

$$r(P, P_j) = [(x - x_j)^2 + (y - y_j)^2]^{1/2}. \tag{3.7}$$

When all the $\alpha_j = 1$, we have the interpolant of Eq. (2.1). As indicated in Section 2, this interpolant has cusps at each of the interpolation points P_i. The behavior of $U(P)$ at the P_i for the more general case in Eq. (3.6) is characterized by the following theorem. For simplicity we will discuss only the bivariate case, but the results remain

equally valid for any number of independent variables.

THEOREM 3.1. *The interpolant $U(P)$ in Eq.* (3.6) *has the following properties*:

(i) *If* $\alpha_i > 1$, *then*

$$\lim_{P\to P_i} \frac{\partial U}{\partial x} = \lim_{P\to P_i} \frac{\partial U}{\partial y} = 0,$$

(ii) *If* $0 < \alpha_i \leqslant 1$, *then, in general, the first partial derivatives fail to exist at* P_i.

Proof. For $\alpha_i > 1$,

$$\frac{\partial [r(P, P_i)]^{\alpha_i}}{\partial x} = \alpha_i [r(P, P_i)]^{\alpha_i - 2}(x - x_i),$$

hence

$$\lim_{P\to P_i} \frac{\partial [r(P, P_i)]^{\alpha_i}}{\partial x} = 0. \tag{3.8}$$

Define

$$B_k(P) = \prod_{j\neq k} r(P, P_j)^{\alpha_j},$$

then

$$B_k(P_i) = 0 \quad \text{if } k \neq i,$$

and

$$\frac{\partial B_k}{\partial x} = \sum_{m\neq k} \frac{\partial r(P, P_m)^{\alpha_m}}{\partial x} \prod_{l\neq m;\, l\neq k} r(P, P_l)^{\alpha_l}.$$

Therefore,

$$\lim_{P\to P_i} \frac{\partial B_k}{\partial x} = 0 \quad \text{if } k \neq i. \tag{3.9}$$

From (3.6) we have

$$\frac{\partial U}{\partial x} = \left\{ \left[\sum_{k=1}^{N} B_k(P) \right] \left[\sum_{m=1}^{N} F_m \frac{\partial B_m}{\partial x} \right] \right. \tag{3.10}$$
$$\left. - \left[\sum_{m=1}^{N} F_m B_m(P) \right] \left[\sum_{k=1}^{N} \frac{\partial B_m}{\partial x} \right] \right\} \Bigg/ \left[\sum_{k=1}^{N} B_k(P) \right]^2.$$

Combining (3.8) and (3.9) with (3.10), we have

$$\lim_{P\to P_i} \frac{\partial U}{\partial x} = \left[B_i(P_i) F_i \frac{\partial B_i}{\partial x}\bigg|_{P=P_i} - F_i B_i(P_i) \frac{\partial B_i}{\partial x}\bigg|_{P=P_i} \right] \Big/ B_i(P_i)^2$$
$$= 0.$$

A similar argument establishes

$$\lim_{P\to P_i} \frac{\partial U}{\partial y} = 0,$$

which completes the proof of part (i).

If $0 < \alpha_i \leqslant 1$, then the limits in (3.8) and (3.9) fail to exist; thus, in general, from (3.10), the first partial derivatives do not exist at P_i.

Figures 1–3 illustrate Theorem 3.1. In these figures we have taken $F(P)$ to be a *univariate function* which is interpolated at five equally spaced points. The exponents are all taken to be equal: $\alpha_1 = \alpha_2 = \cdots = \alpha_5 = \alpha$ in each example. In Figure 1, $\alpha = 1$ so that (3.6) reduces to formula (2.1). The various properties established in the previous section are very much in evidence in this figure. Figures 2 and 3 illustrate the vanishing partial derivatives predicted by Theorem 3.1 for values of $\alpha > 1$. In Figure 2 we have taken $\alpha = 2$; and $\alpha = 10$ in Figure 3. Note that as $\alpha \longrightarrow \infty$, the function $U(P)$ behaves like a *step function.* Figure 4 is another graphical representation of the metric interpolant of Eq. (3.6) using five interpolation points. The values of the α_j were selected to be

$$\alpha_1 = 2, \quad \alpha_2 = 1, \quad \alpha_3 = 1/2, \quad \alpha_4 = 15, \quad \alpha_5 = 2.$$

Cusps are produced at the points P_2 and P_3 as a result of α_2 and α_3 being 1 or less. The vanishing of the first derivative of $U(x)$ at the remaining three points is as predicted by Theorem 3.1. The "flatness" of the curve in the neighborhood of P_4 is a result of the large exponent $\alpha_4 = 15$. The maximum of $U(x)$ occurs at both P_1 and P_5 and the minimum at the point P_3.

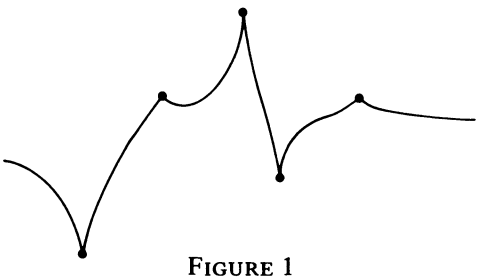

FIGURE 1

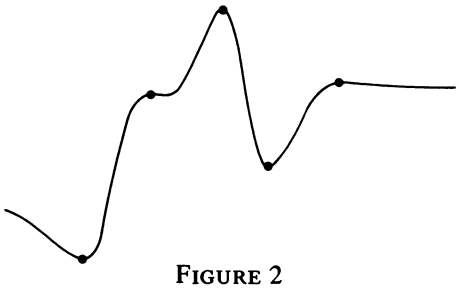

FIGURE 2

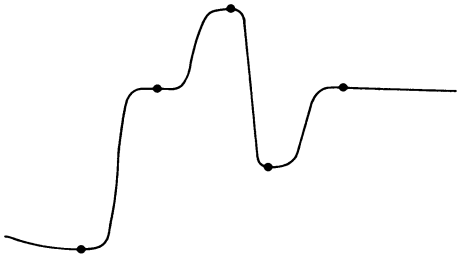

FIGURE 3

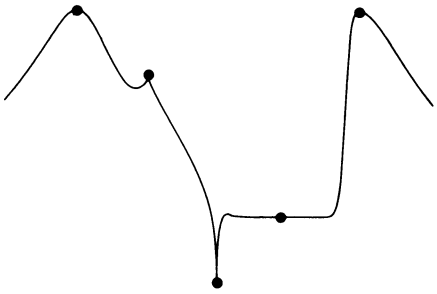

FIGURE 4

Simple illustrations of the behavior of *bivariate* metric interpolants for various values of the α_j are presented in Figures 5–7. The values of α_j used in these figures are given in the following table.

Figure #	α_1	α_2	α_3	α_4	α_5
5	2	2	2	2	2
6	10	1	5	2	3
7	20	20	20	20	20

The above-mentioned properties of metric interpolants are apparent from the figures. Note particularly that for large exponents α_j, the graph of the function $U(P)$ is nearly flat in a large neighborhood of the point P_j, so that as $\alpha_j \longrightarrow \infty$ for all $j = 1, 2, \ldots, N$, $U(P)$ approaches a piecewise constant function.

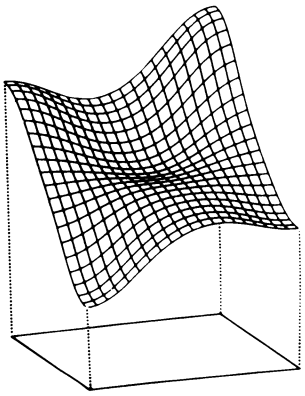

FIGURE 5

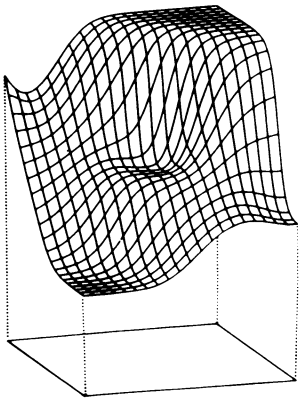

FIGURE 6

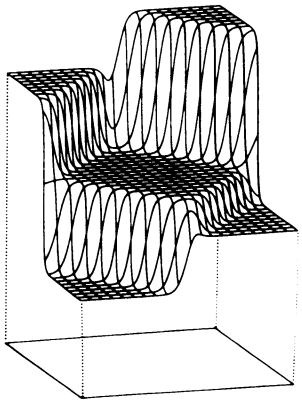

FIGURE 7

This last observation suggests an interesting and novel application of metric interpolation in such fields as demography, ecology and market analysis. To explain: For large values of $\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_N$ one may conclude that the ith basis function $\varphi_i(P; P_1, P_2, \ldots, P_N)$ is essentially nonzero (i.e., $\varphi_i > \epsilon$ for any $\epsilon > 0$) only in the "region of influence" of the point P_i. Thus, if the P_i represent the geographic locations of N competing forces of equal strength, then the region over which the ith one of

these will be expected to dominate is the subdomain over which φ_i is essentially equal to unity (i.e., $\varphi_i > 1 - \epsilon$) since this implies that $\Sigma_{j \neq i} \varphi_j < \epsilon$. From Figure 7 we can see that, within the convex hull of the points P_i, the regions of influence of the points are clearly bounded by straight line segments.

It is interesting to note from Theorem 3.2 below that with $\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_N$, the limiting value of φ_i as P recedes from the cluster of points $\{P_i\}$ is $1/N$ for all values of i. This means that all competitors are equally influential when the minimum distance from P to any of the P_j is much larger than the maximum distance between competitors.

A model such as this may be useful, for instance, in determining the approximate territorial boundaries which would be established by competing individuals of certain animal species or by merchants competing within the same geographic vicinity.

As discussed in Section 2 for the interpolant $U(P)$ in Eq. (2.1), as P recedes indefinitely from the cluster of interpolation points, $U(P)$ approaches the average of the interpolated values F_i, cf. (2.18). For the interpolant in Eq. (3.6), a somewhat different, but analogous result also holds. Let D be a bounded region which contains the interpolation points P_i and assume that the points are indexed such that in Eq. (3.6),

$$\alpha_1 = \alpha_2 = \cdots = \alpha_M < \alpha_{M+1}, \tag{3.12}$$

and

$$\alpha_{M+1} \leqslant \alpha_{M+2} \leqslant \cdots \leqslant \alpha_N, \tag{3.13}$$

i.e., the first M exponents are equal and minimal. The behavior of the cardinal basis functions is determined in the following theorem.

THEOREM 3.2. *If* $d = \min_i r(P; P_i)$, *then the cardinal basis functions*

$$\varphi_i(P; P_1, P_2, \ldots, P_N) = \prod_{j \neq i} r(P, P_j)^{\alpha_j} \Big/ \sum_{k=1}^{N} \prod_{j \neq k} r(P, P_j)^{\alpha_j} \tag{3.14}$$

satisfy

$$\lim_{d \to \infty} \varphi_i(P; P_1, \ldots, P_N) = 1/M \quad \text{if } i \leqslant M, \text{ and} \tag{3.15}$$

$$\lim_{d \to \infty} \varphi_i(P; P_1, \ldots, P_N) = 0 \quad \text{if } i > M, \tag{3.16}$$

where the α_j *satisfy* (3.12) *and* (3.13) *above.*

Proof. If we define

$$\rho = \max_{i,j} r(P_i, P_j) \quad \text{and} \quad \beta = \sum_{j=1}^{N} \alpha_j,$$

then

$$d^{\beta - \alpha_i} \leqslant \prod_{j \neq i} r(P, P_j)^{\alpha_j} \leqslant (d + \rho)^{\beta - \alpha_i}, \quad i = 1, \ldots, M. \tag{3.17}$$

Using (3.17), we obtain the following bounds for each φ_i in (3.14)

$$\frac{d^{\beta-\alpha_i}}{M(d+\rho)^{\beta-\alpha_1} + \Sigma_{k=M+1}^{N}(d+\rho)^{\beta-\alpha_k}}$$

(3.18)

$$\leqslant \varphi_i(P; P_1, \ldots, P_N) \leqslant \frac{(d+\rho)^{\beta-\alpha_i}}{Md^{\beta-\alpha_1} + \Sigma_{k=M+1}^{N} d^{\beta-\alpha_k}}.$$

Equations (3.15) and (3.16) follow directly from (3.18) by computing the appropriate limits. From Theorem 3.2, the behavior of the interpolant $U(P)$ in Eq. (3.6) is readily seen to be

$$\lim_{d\to\infty} U(P) = [1/M] \sum_{i=1}^{M} F_i. \tag{3.19}$$

The surface represented by Figure 8 is generated from the interpolation scheme (3.6) with the α_j as follows:

i	1	2	3	4	5
P_i	(0,0)	(1,1)	(1.2,0.2)	(0,0.5)	(1,0.5)
F_i	4	0	3	1	1
α_i	2.5	2.5	3	4	4

The Maximum Principle (2.7) and the asymptotic properties of Theorem 3.2 are more clearly illustrated in this example since we have plotted the surface over the rectangle $[-2, 3] \times [-2, 3]$.

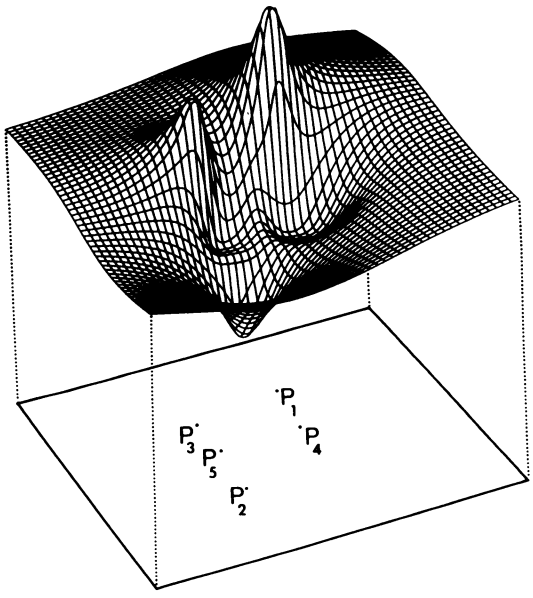

FIGURE 8

For the special case of all $\alpha_i = 2$, Shepard [4] has proposed a technique for interpolating to given first partial derivatives utilizing a formula of the form

$$(3.20)\quad U(P) = \left[\sum_{i=1}^{N} [F_i + F_{x_i}(x - x_i) + F_{y_i}(y - y_i)]/r_i^2\right] \Big/ \left[\sum_{i=1}^{N} 1/r_i^2\right],$$

where

$$F_i = F(P_i), \qquad F_{x_i} = \left.\frac{\partial F}{\partial x}\right|_{P=P_i},$$

and

$$F_{y_i} = \left.\frac{\partial F}{\partial y}\right|_{P=P_i}.$$

Equation (3.20) does indeed interpolate to F, F_x, and F_y at each of the points P_i. Extensions to higher order derivative interpolation can easily be made, but we will not explore such schemes further in this paper. However, as a practical matter, if only function values F_i are known, then some *ad hoc* technique must be employed to estimate such partial derivatives.

One such *ad hoc* method for estimating the requisite first partial derivatives in (3.20) is to employ *local* least squares approximation to data in the vicinity of the point P_i and to extract derivative information from this approximation. This has been tested on several examples using local planar approximations: $a + bx + cy$; and the results have been satisfactory for the engineering applications to which the method has been applied. In [4], Shepard has proposed another *ad hoc* method for determining slopes, but we have not tested his technique.

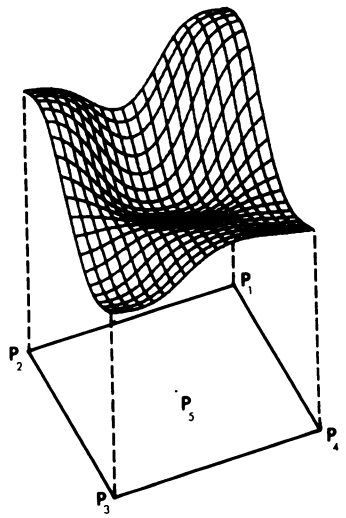

FIGURE 9

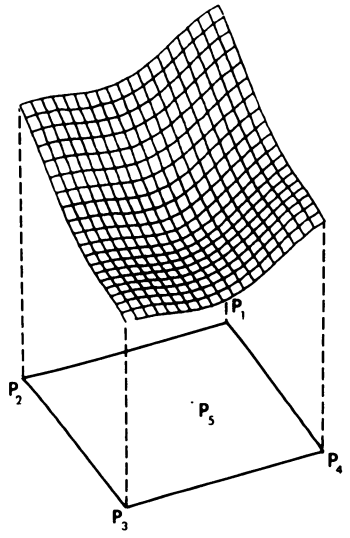

FIGURE 10

To illustrate the effect of imposing nonzero partial derivatives estimated from local least squares approximation by planes, consider Figures 9 and 10. The first of these shows the graph of the function U obtained from metric interpolation to data at the five points $P_1, \ldots, P_5$ with $\alpha_1 = \alpha_2 = \cdots = \alpha_5 = 2$ in (3.6). The flat spots caused by vanishing first partial derivatives are quite evident. Figure 10 shows the result of estimating partial derivatives via a least squares planar approximation to P_i and its three nearest neighbors (four for P_5). It is apparent from the figure that this technique does serve to reduce the extraneous undulations inherent in the interpolant of Figure 9 and therefore would probably be preferred for most practical applications.

Mathematics Department
General Motors Research Laboratories
Warren, Michigan 48090

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