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SHIFT-INVARIANT SPACES ON THE REAL LINE

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Abstract. We investigate the structure of shift-invariant spaces generated by a finite number of *compactly supported* functions in $L_p(\mathbb{R})$ $(1 \le p \le \infty)$. Based on a study of linear independence of the shifts of the generators, we characterize such shift-invariant spaces in terms of the semi-convolutions of the generators with sequences on $\mathbb{Z}$. Moreover, we show that such a shift-invariant space provides L_p-approximation order k if and only if it contains all polynomials of degree less than k.

1. Introduction

The purpose of this paper is to investigate the structure of shift-invariant spaces on the real line. In particular, we are interested in those properties of shift-invariant spaces on the real line which are not shared by shift-invariant spaces on higher dimensional spaces $\mathbb{R}^s$, $s > 1$.

Finitely generated shift-invariant subspaces of $L_2(\mathbb{R}^s)$ were studied in [4] by de Boor, DeVore, and Ron, who gave a simple characterization for such spaces in terms of the Fourier transforms of their generators. However, when $p \neq 2$, few results have been known for shift-invariant subspaces of $L_p(\mathbb{R}^s)$.

In this paper, we are mainly concerned with shift-invariant spaces generated by a finite number of *compactly supported* functions in $L_p(\mathbb{R})$ $(1 \le p \le \infty)$. We will give a characterization for such spaces in terms of the semi-convolutions of their generators with sequences on $\mathbb{Z}$. The result is then applied to give a characterization of the approximation order provided by such shift-invariant spaces.

Let S be a linear space of distributions on $\mathbb{R}$. We say that S is *shift-invariant* if

$$f \in S \Rightarrow f(\cdot - j) \in S \quad \forall j \in \mathbb{Z}.$$

A mapping from $\mathbb{Z}$ to $\mathbb{C}$ is called a *sequence*. The linear space of all sequences on $\mathbb{Z}$ is denoted by $\ell(\mathbb{Z})$. Let ϕ be a compactly supported distribution on $\mathbb{R}$, and let $a : \mathbb{Z} \to \mathbb{C}$ be a sequence. The *semi-convolution* of ϕ with a, denoted $\phi *' a$, is defined by

$$\phi *' a := \sum_{j \in \mathbb{Z}} \phi(\cdot - j) a(j).$$

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Given a finite collection Φ of compactly supported distributions on $\mathbb{R}$, we denote by $S_0(\Phi)$ the linear span of $\{\phi(\cdot - j) : \phi \in \Phi, j \in \mathbb{Z}\}$, and by $S(\Phi)$ the linear space of all distributions of the form $\sum_{\phi\in\Phi} \phi *' a_\phi$ with a_ϕ being a sequence on $\mathbb{Z}$ for each $\phi \in \Phi$. The elements in Φ are called the *generators* for $S(\Phi)$.

Now suppose that Φ is a finite subset of $L_p(\mathbb{R})$ for some p with $1 \le p \le \infty$. We denote by $S_p(\Phi)$ the closure of $S_0(\Phi)$ in $L_p(\mathbb{R})$. One of the main results of this paper is a characterization of $S_p(\Phi)$ in terms of semi-convolution. In Section 3, we shall prove that for $1 < p < \infty$, a function $f \in L_p(\mathbb{R})$ lies in $S_p(\Phi)$ if and only if

$$f = \sum_{\phi\in\Phi} \phi *' a_\phi \tag{1.1}$$

for some sequences $a_\phi \in \ell(\mathbb{Z})$. When $p = \infty$, a modified result will also be established.

We observe that this result is not valid for the case $p = 1$. To see this, let χ be the characteristic function of the interval $[0, 1)$, and let $\phi := \chi - \chi(\cdot - 1)$. Then for any $f \in S_1(\phi)$ we have $\int f = 0$; hence $\chi \notin S_1(\phi)$. But $\chi = \sum_{j=0}^{\infty} \phi(\cdot - j)$.

Next, we consider approximation in $L_p(\mathbb{R})$ spaces ($1 \le p \le \infty$). For $f, g \in L_p(\mathbb{R})$, we write $\mathrm{dist}_p(f, g)$ for $\|f - g\|_p$. Moreover, for a subset G of $L_p(\mathbb{R})$, the distance from f to G, denoted $\mathrm{dist}_p(f, G)$, is defined by

$$\mathrm{dist}_p(f, G) := \inf_{g\in G} \|f - g\|_p.$$

Let Φ be a finite collection of compactly supported functions in $L_p(\mathbb{R})$. The preceding result tells us that $S_p(\Phi) = S(\Phi) \cap L_p(\mathbb{R})$ for $1 < p < \infty$. Suppose $1 \le p \le \infty$. Let $S := S(\Phi) \cap L_p(\mathbb{R})$, and let $S^h := \{g(\cdot/h) : g \in S\}$ for $h > 0$. Given a real number $r \ge 0$, we say that $S(\Phi)$ provides L_p-*approximation order* r if, for each sufficiently smooth function $f \in L_p(\mathbb{R})$,

$$\mathrm{dist}_p(f, S^h) \le Ch^r,$$

where C is a positive constant independent of h (C may depend on f). We say that $S(\Phi)$ provides L_p-*density order* r (see [3]) if, for each sufficiently smooth function $f \in L_p(\mathbb{R})$,

$$\lim_{h\to 0^+} \mathrm{dist}_p(f, S^h)/h^r = 0.$$

In [7] Jia characterized the L_∞-approximation order of $S(\Phi)$ in terms of the Strang-Fix conditions (see [16]). When Φ consists of a single generator ϕ, Ron [13] proved that, for a positive integer k, $S(\phi)$ provides L_∞-approximation order k if and only if $S(\phi)$ contains Π_{k-1}, the set of all polynomials of degree $\le k - 1$. Zhao [18] also gave a characterization for the L_p-approximation order ($1 < p < \infty$) provided by $S(\phi)$.

In Section 4, we shall prove that $S(\Phi)$ provides L_p-approximation order ($1 \le p \le \infty$) if and only if $S(\Phi)$ contains Π_{k-1}. This result is no longer true for shift-invariance spaces on $\mathbb{R}^s$, $s > 1$. See the counterexamples given in [5] and [6].

In our study of shift-invariant spaces linear independence plays a crucial role. Let Φ be a finite collection of compactly supported distributions on $\mathbb{R}$. The shifts of the elements in Φ are said to be *linearly independent* if

$$\sum_{\phi\in\Phi} \phi *' a_\phi = 0 \Rightarrow a_\phi = 0 \quad \forall \phi \in \Phi.$$

When the shifts of the elements in Φ are linearly independent, we say that $S(\Phi)$ has linearly independent generators.

In Section 2 we shall show that a finitely generated shift-invariant space always has linearly independent generators. More precisely, if Φ is a finite collection of compactly supported distributions on $\mathbb{R}$, then there exists a finite collection Ψ of compactly supported distributions on $\mathbb{R}$ such that $S(\Psi) = S(\Phi)$ and the shifts of the elements in Ψ are linearly independent. When Φ consists of compactly supported continuous functions, this result was essentially known to de Boor and DeVore (see [2]). When Φ consists of a single generator ϕ, Ron [12] showed that $S(\phi)$ contains a linearly independent generator. Our contribution is to give a concrete construction for Ψ so that Ψ inherits most properties possessed by Φ. For instance, if $\Phi \subset L_p(\mathbb{R})$ for some p with $1 \le p \le \infty$, then Ψ can be chosen to be a subset of $L_p(\mathbb{R})$. Furthermore, for $1 < p \le \infty$, Ψ can be chosen to be a subset of $S_p(\Phi)$. These properties enable us to characterize shift-invariant subspaces of $L_p(\mathbb{R})$ and the approximation order provided by them.

2. Linear independence

This section is devoted to a study of linear independence. Linear independence can be characterized in terms of the Fourier transforms of the generators. For a compactly supported integrable function f on $\mathbb{R}$, the Fourier-Laplace transform of f is given by

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-ix\xi}\,dx, \qquad \xi \in \mathbb{C}.$$

The domain of the Fourier-Laplace transform can be extended to all compactly supported distributions. If f is a compactly supported distribution, then $\hat{f} : \xi \mapsto \hat{f}(\xi)$ is an entire function on $\mathbb{C}$. It is known (see [10] and the references cited there) that the shifts of the elements in Φ are linearly independent if and only if for every $\zeta \in \mathbb{C}$, the sequences $(\hat{\phi}(\zeta + 2\pi k))_{k\in\mathbb{Z}}$, $\phi \in \Phi$, are linearly independent.

For later use we introduce some concepts related to compactly supported distributions. Let ϕ be a compactly supported distribution on $\mathbb{R}$. Suppose $\phi \ne 0$. The support of ϕ, denoted supp ϕ, is a compact subset of $\mathbb{R}$. Let $[r_\phi, s_\phi]$ be the smallest integer-bounded interval containing supp ϕ. The length of the interval $[r_\phi, s_\phi]$ is

$$l(\phi) := s_\phi - r_\phi.$$

We call $l(\phi)$ the *length* of ϕ.

Let Φ be a finite collection of compactly supported distributions on $\mathbb{R}$. The *length* of Φ, denoted $l(\Phi)$, is defined by

$$l(\Phi) := \sum_{\phi\in\Phi} l(\phi).$$

Also, we denote by $\#\Phi$ the number of elements in Φ.

Theorem 1. *Let Φ be a finite collection of nontrivial distributions on $\mathbb{R}$ with compact support. Then there exists a finite collection Ψ of compactly supported distributions on $\mathbb{R}$ with the following properties*:

(a) *The shifts of the elements in Ψ are linearly independent*;
(b) $\#\Psi \le \#\Phi$;
(c) $\Phi \subset S_0(\Psi)$;
(d) $S(\Psi) = S(\Phi)$.

If, in addition, $\Phi \subset L_p(\mathbb{R})$ *for some* p, $1 \le p \le \infty$, *then* Ψ *can be chosen to be a subset of* $L_p(\mathbb{R})$. *Furthermore, for* $1 < p \le \infty$, Ψ *can be chosen to be a subset of* $S_p(\Phi)$.

Proof. It is sufficient to prove that, if the shifts of the elements in Φ are linearly dependent, then there exists Ψ with $l(\Psi) \le l(\Phi) - 1$ satisfying all the conclusions of the theorem, except perhaps (a). Suppose $\Phi = \{\phi_1, \ldots, \phi_m\}$. Let

$$K(\Phi) := \left\{ (b_1, \ldots, b_m) \in (\ell(\mathbb{Z}))^m : \sum_{j=1}^{m} \phi_j *' b_j = 0 \right\}.$$

Then the shifts of the elements in Φ are linearly independent if and only if $K(\Phi) = \{0\}$. If $K(\Phi) = \{0\}$, then we may take $\Psi = \Phi$. Suppose $K(\Phi) \neq \{0\}$. By [10, Theorem 3.3], $K(\Phi) \neq \{0\}$ implies that there exists some $\theta \in \mathbb{C}\backslash\{0\}$ and $(a_1, \ldots, a_m) \in \mathbb{C}^m\backslash\{0\}$ such that

$$(a_1\theta^{()}, \ldots, a_m\theta^{()}) \in K(\Phi), \tag{2.1}$$

where $\theta^{()}$ denotes the sequence $k \mapsto \theta^k$, $k \in \mathbb{Z}$. It follows from (2.1) that

$$\sum_{j=1}^{m} \sum_{k=-\infty}^{\infty} a_j \theta^k \phi_j(\cdot - k) = 0. \tag{2.2}$$

For each ϕ_j, let $r_j := r_{\phi_j}$ and $s_j := s_{\phi_j}$. After shifting the ϕ_j appropriately, we may assume that all $r_j = 0$. Then $s_j = l(\phi_j)$, the length of ϕ_j. Let

$$l := \max\{l(\phi_j) : a_j \neq 0\}.$$

For simplicity, we assume that $a_1 \neq 0$ and $l(\phi_1) = l$. Let

$$\rho := \sum_{j=1}^{m} a_j \phi_j$$

and

$$\psi := \sum_{k=0}^{\infty} \theta^k \rho(\cdot - k). \tag{2.3}$$

By our choice of ρ, we deduce from (2.2) that

$$\sum_{k=-\infty}^{\infty} \theta^k \rho(\cdot - k) = 0. \tag{2.4}$$

Let $\Psi := \{\psi, \phi_2, \ldots, \phi_m\}$. We have

$$\psi - \theta\psi(\cdot - 1) = \sum_{k=0}^{\infty} \theta^k \rho(\cdot - k) - \sum_{k=0}^{\infty} \theta^{k+1} \rho(\cdot - k - 1) = \rho = a_1\phi_1 + \cdots + a_m\phi_m.$$

Since $a_1 \neq 0$, we obtain $\phi_1 \in S_0(\psi, \phi_2, \ldots, \phi_m)$, and hence $\Phi \subset S_0(\Psi)$. It follows that $S(\Phi) \subseteq S(\Psi)$.

Evidently, $\psi \in S(\Phi)$. If $f = \psi *' b$ for some sequence b on $\mathbb{Z}$, then for any bounded open interval E of $\mathbb{R}$, there exists an element $g \in S(\Phi)$ such that g agrees with f on E. Thus, by [8, Theorem 4], f belongs to $S(\Phi)$. This shows $S(\Psi) \subseteq S(\Phi)$. Therefore $S(\Psi) = S(\Phi)$.

Let us show $l(\Psi) < l(\Phi)$. For this purpose we only have to prove $\operatorname{supp} \psi \subseteq [0, l-1]$. Clearly, $\operatorname{supp} \psi \subseteq [0, \infty)$. Hence, it suffices to show that $\langle \psi, u \rangle = 0$ for every $u \in C_c^\infty(\mathbb{R})$ with supp $u \subset (l-1,\infty)$. Let u be such a test function. Note that for each j, $\phi_j(\cdot - k)$ is supported on $[k, l+k]$. Hence $\langle \phi_j(\cdot - k), u\rangle = 0$ for $k \le -1$. It follows that $\langle \rho(\cdot - k), u\rangle = 0$ for $k \le -1$. This in connection with (2.4) gives

$$\langle \psi, u\rangle = \left\langle \sum_{k=0}^{\infty} \theta^k \rho(\cdot - k), u \right\rangle = \left\langle \sum_{k=-\infty}^{\infty} \theta^k \rho(\cdot - k), u \right\rangle = 0.$$

Consequently, supp $\psi \subseteq [0, l-1]$.

Now suppose $\Phi \subset L_p(\mathbb{R})$ for some p, $1 \le p \le \infty$. Then $\rho \in L_p(\mathbb{R})$, and (2.3) tells us that for each integer k, ψ is pth power integrable on the interval $[k, k+1]$. But ψ is compactly supported; hence $\psi \in L_p(\mathbb{R})$.

It remains to prove that $\psi \in S_p(\Phi)$ if $\Phi \subset L_p(\mathbb{R})$ for $1 < p \le \infty$. If $|\theta| < 1$, then (2.3) implies $\psi \in S_p(\Phi)$. If $|\theta| > 1$, then $\psi - \theta\psi(\cdot - 1) = \rho$ implies

$$\psi = \sum_{k=1}^{\infty} -\theta^{-k} \rho(\cdot + k) \in S_p(\Phi).$$

When $|\theta| = 1$, we set

$$f_n := \sum_{k=0}^{n-1} (1 - k/n)\theta^k \rho(\cdot - k),$$

where n is an integer greater than l. Then $f_n \in S_0(\Phi)$. The desired result $\psi \in S_p(\Phi)$ will be established if we can show

$$\|f_n - \psi\|_p \to 0 \quad \text{as } n \to \infty. \tag{2.5}$$

To prove (2.5) we observe that ρ is supported on $[0, l]$, ψ is supported on $[0, l-1]$, and f_n is supported on $[0, n+l-1]$. For $x \in [0, l-1]$ we have

$$\psi(x) - f_n(x) = \sum_{k=0}^{l-1} (k/n)\theta^k \rho(x-k).$$

Hence

$$\|\psi - f_n\|_{L_p([0,l-1])} \to 0 \quad \text{as } n \to \infty. \tag{2.6}$$

For $x \in [n-1, n+l-1]$, we have $\psi(x) = 0$ and

$$\psi(x) - f_n(x) = \sum_{k=n-l}^{n-1} -(1 - k/n)\theta^k \rho(x-k).$$

But $|1 - k/n| \le l/n$ for $n - l \le k \le n-1$; hence

$$\|\psi - f_n\|_{L_p([n-1,n+l-1])} \to 0 \quad \text{as } n \to \infty. \tag{2.7}$$

It remains to prove

$$\|\psi - f_n\|_{L_p([l-1,n-1])} \to 0 \quad \text{as } n \to \infty. \tag{2.8}$$

For this purpose let j be an integer in $[l-1, n-2]$. We observe that for almost every $x \in [j, j+1]$, $\rho(x-k) = 0$ for $k \notin (j-l, j+1)$, and hence by (2.4) we have

$$\sum_{k=j-l+1}^{j} \theta^k \rho(x-k) = \sum_{k=-\infty}^{\infty} \theta^k \rho(x-k) = 0.$$

Therefore, for almost every $x \in [j, j+1]$, we have

$$\psi(x) - f_n(x) = (1 - j/n) \sum_{k=j-l+1}^{j} \theta^k \rho(x-k) - \sum_{k=j-l+1}^{j} (1 - k/n)\theta^k \rho(x-k)$$
$$= \sum_{k=j-l+1}^{j} \frac{k-j}{n} \theta^k \rho(x-k).$$

But $|k-j| \le l$ for $j - l + 1 \le k \le j$. Consequently, (2.8) holds true for $p = \infty$. If $1 < p < \infty$, then there exists a positive constant C independent of n such that

$$\int_{[j,j+1]} |\psi(x) - f_n(x)|^p \, dx \le C^p/n^p, \qquad l - 1 \le j \le n - 2.$$

It follows that

$$\int_{[l-1,n-1]} |\psi(x) - f_n(x)|^p \, dx \le nC^p/n^p = C^p/n^{p-1}.$$

This verifies (2.8) for $1 < p < \infty$. Finally, (2.6), (2.7), and (2.8) together imply (2.5). We conclude that $\psi \in S_p(\Phi)$ for $1 < p \le \infty$.

The results obtained so far can be summarized as follows: If the shifts of the elements in Φ are linearly dependent, then we can find a collection Ψ of distributions such that $\#\Psi \le \#\Phi$, $l(\Psi) < l(\Phi)$, $\Phi \subset S_0(\Psi)$, and $S(\Psi) = S(\Phi)$. Furthermore, if $\Phi \subset L_p(\mathbb{R})$ $(1 \le p \le \infty)$, then Ψ possesses the additional properties stated in the theorem. Repeat the preceding process until $l(\Psi)$ achieves its minimum. The resulting set Ψ has the property that the shifts of the elements in Ψ are linearly independent. Moreover, Ψ meets the requirement of the theorem. □

3. Characterization of shift-invariant spaces

In this section we investigate the structure of shift-invariant spaces generated by a finite number of compactly supported functions in $L_p(\mathbb{R})$ $(1 \le p \le \infty)$.

We use $\ell_0(\mathbb{Z})$ to denote the linear space of all finitely supported sequences on $\mathbb{Z}$. Then, for $1 \le p < \infty$, $\ell_0(\mathbb{Z})$ is dense in $\ell_p(\mathbb{Z})$. For $p = \infty$, the closure of $\ell_0(\mathbb{Z})$ in $\ell_\infty(\mathbb{Z})$ is $c_0(\mathbb{Z})$, the linear space of all sequences a on $\mathbb{Z}$ such that $\lim_{|k|\to\infty} a(k) = 0$. For a measurable subset E of $\mathbb{R}$ and a measurable function f on $\mathbb{R}$, we denote by $\|f\|_\infty(E)$ the essential supremum of f on E. Let $L_{\infty,0}(\mathbb{R})$ be the linear space of all functions $f \in L_\infty(\mathbb{R})$ for which $\lim_{r\to\infty} \|f\|_\infty(\mathbb{R}\backslash[-r, r]) = 0$.

Let $\Phi = \{\phi_1, \ldots, \phi_m\}$ be a finite collection of compactly supported functions in $L_p(\mathbb{R})$. We say that the shifts of the functions of Φ are *stable*, if there exist two positive constants C_1 and C_2 such that for any choice of sequences $a_1, \ldots, a_m \in \ell_p(\mathbb{Z})$,

$$C_1 \sum_{j=1}^{m} \|a_j\|_{\ell_p(\mathbb{Z})} \le \left\| \sum_{j=1}^{m} \phi_j \;*'\; a_j \right\|_{L_p(\mathbb{R})} \le C_2 \sum_{j=1}^{m} \|a_j\|_{\ell_p(\mathbb{Z})}.$$

It was proved by Jia and Micchelli in [10] and [11] that the shifts of the functions in Φ are stable if and only if for every $\xi \in \mathbb{R}$, the sequences $(\hat{\phi}_j(\xi + 2\pi k))_{k\in\mathbb{Z}}$, $j = 1, \ldots, m$, are linearly independent. Thus, if the shifts of the functions in Φ are linearly independent, then they are stable.

Consider the linear mapping T_Φ from$(\ell_p(\mathbb{Z}))^m$ to $L_p(\mathbb{R})$ given by

$$T_\Phi(a_1,\dots,a_m)=\sum_{j=1}^m \phi_j *' a_j, \qquad a_1,\dots,a_m\in\ell_p(\mathbb{Z}).$$

If the shifts of the functions in Φ are stable, then T_Φ is a continuous mapping and the range of T_Φ is closed (see [14, p. 70]). Therefore, for $1\le p<\infty$, $S_p(\Phi)$ is the range of T_Φ. In other words, for $1\le p<\infty$, f lies in $S_p(\Phi)$ if and only if $f=\sum_{\phi\in\Phi}\phi *' a_\phi$ for some sequences $a_\phi\in\ell_p(\mathbb{Z})$, $\phi\in\Phi$. In the case $p=\infty$, $f\in S_\infty(\Phi)$ if and only if $f=\sum_{\phi\in\Phi}\phi *' a_\phi$ for some sequences $a_\phi\in c_0(\mathbb{Z})$, $\phi\in\Phi$.

In general, we have the following characterization for $S_p(\Phi)$ $(1<p\le\infty)$, where the stability condition is not assumed.

Theorem 2. *Let Φ be a finite collection of compactly supported functions in $L_p(\mathbb{R})$. Then for $1\le p\le\infty$, $S(\Phi)\cap L_p(\mathbb{R})$ is closed in $L_p(\mathbb{R})$. Moreover, for $1<p<\infty$,*

$$S(\Phi)\cap L_p(\mathbb{R})=S_p(\Phi). \tag{3.1}$$

In other words, for $1<p<\infty$, a function f lies in $S_p(\Phi)$ if and only if $f\in L_p(\mathbb{R})$ and

$$f=\sum_{\phi\in\Phi}\phi *' a_\phi \tag{3.2}$$

for some sequences $a_\phi\in\ell(\mathbb{Z})$. In the case $p=\infty$, $f\in S_\infty(\Phi)$ if and only if $f\in L_{\infty,0}(\mathbb{R})$ and (3.2) holds true for some sequences $a_\phi\in\ell(\mathbb{Z})$.

Proof. By Theorem 1, there exists a finite collection $\Psi\subset L_p(\mathbb{R})$ such that $S(\Psi)=S(\Phi)$ and the shifts of the functions in Ψ are linearly independent. Moreover, for $1<p\le\infty$, Ψ can be so chosen that $S_p(\Psi)=S_p(\Phi)$.

We first show that $S(\Phi)\cap L_p(\mathbb{R})$ is closed in $L_p(\mathbb{R})$ $(1\le p\le\infty)$. This can be derived from [8, Theorem 4]. Here we establish this result by using the dual functionals discussed in [1] and [17]. Suppose $\Psi=\{\psi_1,\dots,\psi_m\}$. Let $f\in S(\Psi)\cap L_p(\mathbb{R})$. Then

$$f=\sum_{j=1}^m \psi_j *' a_j, \tag{3.3}$$

where $a_j\in\ell(\mathbb{Z})$, $j=1,\dots,m$. From [1] and [17] we see that there are functions $u_1,\dots,u_m\in C_c^\infty(\mathbb{R})$ such that for $j,\,k=1,\dots,m$ and $\alpha\in\mathbb{Z}$,

$$\langle\psi_j,u_k(\cdot-\alpha)\rangle=\delta_{jk}\delta_{\alpha 0},$$

where δ_{jk} stands for the Kronecker sign: $\delta_{jk}=1$ for $j=k$ and $\delta_{jk}=0$ for $j\ne k$. It follows that

$$a_j(\alpha)=\langle f,u_j(\cdot-\alpha)\rangle, \qquad \alpha\in\mathbb{Z}. \tag{3.4}$$

Since $f\in L_p(\mathbb{R})$, we obtain $a_j\in\ell_p(\mathbb{Z})$ for $j=1,\dots,m$ (see [11, Theorem 3.1]). Thus, by the discussion at the beginning of this section, $S(\Psi)\cap L_p(\mathbb{R})$ is closed in $L_p(\mathbb{R})$. But $S(\Phi)=S(\Psi)$. Hence $S(\Phi)\cap L_p(\mathbb{R})$ is closed in $\ell_p(\mathbb{R})$.

Furthermore, for $1\le p<\infty$, $S(\Psi)\cap L_p(\mathbb{R})=S_p(\Psi)$. But, for $1<p\le\infty$, we have $S_p(\Psi)=S_p(\Phi)$. Therefore, (3.1) is true for $1<p<\infty$.

Finally, it is easily seen that $S_\infty(\Psi)\subseteq S(\Psi)\cap L_{\infty,0}(\mathbb{R})$. If $f\in S(\Psi)\cap L_{\infty,0}(\mathbb{R})$ has the expression as in (3.3), then it follows from (3.4) that $a_j\in c_0(\mathbb{Z})$ for $j=1,\dots,m$. Hence $f\in S_\infty(\Psi)$. This shows that $S_\infty(\Psi)=S(\Psi)\cap L_{\infty,0}(\mathbb{R})$. But $S(\Phi)=S(\Psi)$

and $S_\infty(\Phi) = S_\infty(\Psi)$. We therefore conclude that $S_\infty(\Phi) = S(\Phi) \cap L_{\infty,0}(\mathbb{R})$. This verifies the last statement of the theorem. □

4. Approximation order

We are now in a position to consider approximation in $L_p(\mathbb{R})$ spaces ($1 \le p \le \infty$).

Theorem 3. *Let Φ be a finite collection of compactly supported functions in $L_p(\mathbb{R})$, $1 \le p \le \infty$. Let k be a positive integer. Then the following statements are equivalent.*

(a) *$S(\Phi)$ provides L_p-approximation order k.*
(b) *$S(\Phi)$ provides L_p-density order $k-1$.*
(c) *$S(\Phi)$ contains Π_{k-1}, the set of all polynomials of degree $\le k-1$.*
(d) *$S(\Phi)$ contains a compactly supported function ψ such that*

$$\sum_{\beta\in\mathbb{Z}} q(\beta)\psi(\cdot-\beta) = q \quad \forall q \in \Pi_{k-1}. \tag{4.1}$$

Proof. It is obvious that (a) implies (b). It was proved in [8] that (b) implies (c). The implication (d) ⇒ (a) is well known. See [9] for an explicit L_p-approximation scheme. It remains to prove (c) ⇒ (d). By Theorem 1, we may assume that the shifts of the functions in Φ are linearly independent. Suppose $\Phi = \{\phi_1, \dots, \phi_m\}$.

Since the shifts of the functions in Φ are linearly independent, there exist test functions $u_1, \dots, u_m \in C_c^\infty(\mathbb{R})$ such that

$$\langle \phi_r(\cdot-\alpha), u_s(\cdot-\beta)\rangle = \delta_{rs}\delta_{\alpha\beta}, \qquad r,s \in \{1,\dots,m\},\ \alpha,\beta\in\mathbb{Z}. \tag{4.2}$$

By condition (c), $q \in S(\Phi)$ for $q \in \Pi_{k-1}$. Hence by (4.2) we have

$$q = \sum_{j=1}^m \sum_{\alpha\in\mathbb{Z}} \phi_j(\cdot-\alpha)\langle q(\cdot+\alpha), u_j\rangle.$$

Let $(\ell_r : r = 1, \dots, k)$ be the Lagrange polynomials of degree $k-1$ for the points $1, \dots, k$. Then, for any $q \in \Pi_{k-1}$,

$$q = \sum_{\alpha\in\mathbb{Z}} \sum_{j=1}^m \phi_j(\cdot-\alpha) \left\langle \sum_{r=1}^k q(r+\alpha)\ell_r, u_j \right\rangle = \sum_{\beta\in\mathbb{Z}} \psi(\cdot-\beta) q(\beta),$$

with

$$\psi := \sum_{j=1}^m \sum_{r=1}^k \phi_j(r+\cdot)\langle \ell_r, u_j\rangle$$

certainly a compactly supported element of $S(\Phi)$. Therefore, (c) implies (d). □

It was proved by Schoenberg [15] that (4.1) is equivalent to the following conditions: $D^\alpha\hat{\psi}(0) = \delta_{\alpha 0}$ and $D^\alpha\hat{\psi}(2\pi j) = 0$ for $0 \le \alpha < k$ and $j \in \mathbb{Z}\backslash\{0\}$. Now these conditions are referred to as the Strang-Fix conditions (see [16]).

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