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SHIFTABLE VERSUS NON-SHIFTABLE CAPITAL: A SYNTHESIS

BY MARTIN L. WEITZMAN[1]

Non-transferable capital is an essential feature of the Fel'dman two-sector growth model. This paper is primarily an answer to the following question. Given that capital is really partially shiftable between its two sectors, when can the Fel'dman model be meaningfully used?

1. INTRODUCTION

TO AN ECONOMIST the study of economic development is in large part an investigation into the mechanics of capital formation. At least in theory, the output options open to a developing economy are more restricted in the case where possibilities for obtaining foreign exchange via trade or aid are relatively limited. Society's menu of choices is even easier to enumerate if it is further assumed that labor is surplus in the sense that labor supply is a non-binding constraint on economic development now and for some time to come. These conditions are roughly descriptive of the historical situation confronting some large underdeveloped nations wishing to industrialize rapidly; the U.S.S.R. in the thirties is a classic example.

In such situations the key to economic growth is the capacity of the domestic capital goods sector. Increasing that capacity by ploughing back a high proportion of investment goods for purposes of self-reproduction will permit high consumption levels eventually, but not just in the near future. The reverse is true if, by bolting down a substantial percentage of investment goods there, the consumer goods sector is presently expanded.

These thoughts underlie a very interesting model of economic development first propounded by the Soviet engineering economist G. A. Fel'dman in 1928 [7].[2] We are indebted to Professor Domar [6] for pointing out the significance of this model and for relating it to current growth theory as well as to the Soviet industrialization debate of the twenties. The same model has been independently formulated by the Indian statistician P. C. Mahalanobis [9] who places somewhat greater emphasis on making it operational enough to serve as a rough guide of sorts for Indian long term planning.[3]

In its simplest form this model splits an economy into two departments, investment and consumption. Investment goods are general ex ante and can be used to increase the capacity of either sector. But ex post, capital is specific to the

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that category to which more of the (embodied) coal is ultimately destined. The same is true of many other intermediates—chemicals, petroleum, electricity, transportation, forest products, etc. Ideally one wants to place all capacity increasing activities in one category and all those merely sustaining output at current levels in the other. But this does not tell us how to split up meaningfully the troublesome double purpose intermediate sectors.

A related problem lurks behind the whole notion of a capital stock which cannot be transferred between the consumption and investment departments of an economy. Undoubtedly houses or wheat-growing land are not well suited for making tractor engines. Likewise machine tools are relatively useless for baking bread. Nevertheless, because most intermediate commodities are used, directly or indirectly, for *both* consumption and investment, plant and equipment engaged in producing these raw materials is easily transferable in the sense that a change in the intermediate product's final destination is tantamount to shifting capital from one department to the other. Railroads can transport construction materials as easily as they can sacks of flour, and it is irrelevant to the operation of a steam turbine whether its electricity goes to power factory machinery engaged in cutting metals or sewing shoes.

Clearly the Fel'dman model exaggerates the significance of capital ossification. In order to examine the consequences of greater realism it is necessary to go over to an economy with more than two categories. A classification scheme for a model with three sectors is described in Table I.[6]

Introducing an extra sector has hardly banished the arbitrariness which must be involved in assigning certain industries to one of three categories, although the present arrangement along more usable functional guidelines is at least an improvement over the two sector classification. The assumption (to be made) that such gigantic sectors as these are subject to aggregate production relations (of the

TABLE I

Symbol	Sector	Definition	Examples
I	Investment	All final and intermediate goods and services used directly or indirectly to produce *investment only*	construction, cement, machine tools, metal working
C	Consumption	All final and intermediate goods and services used directly or indirectly to produce *consumption only*	bread, flour, clothing textiles
R	Raw materials	All intermediate goods and services used indirectly in producing *both investment and consumption*	freight transportation, fuels, chemicals, electricity.

[6] Raj and Sen [13] split up an economy according to similar criteria, but for a different purpose.

simple fixed-coefficients surplus-labor type, no less) should identify this model as yet another species of the non-operational "suggestive" variety.

3. A THREE SECTOR MODEL

For evaluation purposes we take as an appropriate social objective the infinite integral of a discounted instantaneous utility function which is defined over current consumption, C. For ease of manipulation we choose the instantaneous utility function to be of the constant elasticity of marginal utility type:

$$U'(C) \equiv \frac{dU}{dC} = \frac{1}{C^{\eta}} \tag{1}$$

where the parameter η is minus the (constant) elasticity of marginal utility.

Likewise for convenience, the discount factor is chosen to be of the exponential form $e^{-\rho t}$, where ρ is the social rate of pure time discount parameter.

From (1), the instantaneous (undiscounted) social utility of consumption at time t is given by[7]

$$U(C(t)) = \begin{cases} \dfrac{1}{1-\eta}(C(t))^{1-\eta} & \text{for} \quad \eta \neq 1, \\ \log(C(t)) & \text{for} \quad \eta = 1. \end{cases} \tag{2}$$

Because the primary goal of this paper is to capture sharply the issue of shiftable versus non-shiftable capital, a high premium is placed on the use of analytically convenient functional forms. Many basic features of the present model would remain under more general representations of tastes and technology. But since the basic message would tend to get diluted, this approach is not taken.[8]

The three sector problem, henceforth called problem (iii), is to maximize

$$\int_0^{\infty} U(C)e^{-\rho t}\, dt \tag{3}$$

[7] Any time invariant utility function obtained by a linear transformation of U $(a + bU,\ b > 0)$ could serve equally well as an instantaneous utility index and would yield an identical solution for the problem under consideration. In (2) utility is expressed as a function of total consumption. Only a minor adjustment is necessary to deal with utility as a function of per-capita consumption and with exponential population growth. Subtract the term μK_j $(j = I, C, R)$ from the right hand side of equations (7)–(9). Now interpret all variables as if they are expressed per capita. Interpret μ as equalling the rate of true physical depreciation plus the rate of population growth. The new value of ρ is either the same as the old, if total welfare of each *generation* counts equally (except for the pure time preference factor), or is less by the rate of population growth if total welfare of each *person* counts equally (except for the pure time preference factor). In between situations are handled by in between values of ρ.

[8] For example, there is no difficulty in handling a more general utility function along the lines of Bose [**1**, Appendix 1]. Exponential capital deterioration at a common rate could be easily handled; Weitzman [**15**] contains the details. Later on we discuss what happens if more sectors are added to the present model.

subject to

(4) $\frac{I}{\gamma_I} \leqslant K_I: \quad \pi_I$

(5) $\frac{C}{\gamma_C} \leqslant K_C: \quad \pi_C$

(6) $\frac{a_I I}{\gamma_R} + \frac{a_C C}{\gamma_R} \leqslant K_R: \quad \pi_R$

(7) $\dot{K}_I = \lambda_I I: \quad p_I$

(8) $\dot{K}_C = \lambda_C I: \quad p_C$

(9) $\dot{K}_R = \lambda_R I: \quad p_R$

(10) $\lambda_I + \lambda_C + \lambda_R = 1$

(11) $I, C, \lambda_C, \lambda_I, \lambda_R \geqslant 0$

(12) $[K_I(0), K_C(0), K_R(0)] = [K_I^0, K_C^0, K_R^0]$, given.

We follow the convention of writing (undiscounted) price or co-state variables to the right of a double colon following the equation to which they are dual. Variables are not explicitly specified as a function of time where this interpretation is otherwise clear.

Letting $j = I$, C, or R, $K_j(t)$ is capital stock in sector j at time t, a state variable; γ_j is output per unit of capital in sector j, a parameter; $\lambda_j(t)$ represents the fraction of investment allocated to sector j at time t, a control variable; a_I is the amount of input R required per unit production of I, a parameter; and a_C is the amount of input R required per unit production of C, a parameter.

Because each sector of economy (iii) is viewed as an enclave, extensive netting out of intermediate stages internal to a sector must be assumed to have taken place. This interpretation has to be considered in translating the sectoral output-capital ratios; γ_j represents the final net-of-intermediate-stages output of sector j per unit of capital stock which is spread out in the appropriate proportions over all the stages of production internal to sector j leading up to and including the production of the sector's final product.

Let β_j represent the final output of j per unit of direct and indirect capital in *all* sectors; β_j differs from γ_j in that account is taken of the fact that raw materials used up in the production of I and C require the use of (direct and indirect) capital in sector R. It is easily seen that

(13) $$\beta_I = \frac{1}{1/\gamma_I + a_I/\gamma_R},$$

(14) $$\beta_C = \frac{1}{1/\gamma_C + a_C/\gamma_R},$$

(15) $$\beta_R = \gamma_R.$$

The units of I are naturally fixed in terms of K (or vice versa). Thus γ_I and (because a_I/γ_R is independent of the units of R) β_I are scale free. It is convenient to choose the units of C and R so that $\beta_I = \beta_C = \beta_R$; this common value is called β without ambiguity. In the context of a capital theory of value this is a natural way to define units of C and R. Note that the values of a_I, a_C, γ_C, and γ_R are automatically fixed by a choice of units for C and R.

We assume that β and η are positive and that ρ is non-negative. To be able to analyze the more realistic and interesting case where genuine growth occurs, it is postulated that

$$(16) \qquad \beta > \rho.$$

Otherwise the net productivity of capital is exceeded by the discount rate imposed on the system; it never pays to plough back any investment into the investment goods sector because society is too impatient to exploit the productivity of capital.

A final assumption is that

$$(17) \qquad \rho > (1 - \eta)\beta.$$

This requirement is necessary to ensure the existence of a meaningful solution to problem (iii). Otherwise a feasible solution will exist which results in an unbounded objective.[9]

Since no-one expects a condition of labor redundancy to persist for ever, an infinite horizon formulation of a model incorporating a labor surplus technology might appear pointless. Such a formulation can be rationalized as an acceptable approximation as long as the day when every member of the labor force will be absorbed is in the distant future.[10]

[9] For some M, any $\theta > 0$, $\int_0^\infty U(C^*(t))e^{-\rho t}\,dt \geqslant \int_0^\infty U((1-\theta)Me^{\beta\theta t})e^{-\rho t}\,dt$. If $\rho < (1-\eta)\beta$, the latter integral goes to infinity as $\theta \to 1$. The case $\rho = (1-\eta)\beta$ is essentially uninteresting because any efficient (with respect to consumption) policy is also optimal.

[10] This justification can easily be made rigorous. Let Φ_e represent the optimal value of the social objective integral (3) in problem (iii) which, it turns out, is relatively easy to solve. Now consider a *harder* problem (problem (h)) with the same objective function but where the potential labor force at time t is fixed, say at $\bar{L}e^{nt}$, and α_j ($j = I, C, R$) represents the labor-output ratio for sector j under the surplus labor regime. In problem (h) the labor surplus technology described by (4), (5), and (6) is appropriate only so long as $\alpha_I I + \alpha_C C + \alpha_R R < \bar{L}e^{nt}$. The moment this constraint becomes binding we must move on to other sets of techniques which economize on labor at the expense of capital. Problem (h) is thus a fully general, much more realistic three sector, non-shiftable, putty-clay model with labor a primary input and multiple production techniques available. Let the optimal value of the social objective (3) for problem (h) be denoted Φ_h.

Now consider the following *feasible* solution to problem (h). Follow exactly the optimal solution to problem (iii) (not yet enumerated) until time T when the equation $\alpha_I I(T) + \alpha_C C(T) + \alpha_R(T) = \bar{L}e^{nT}$ holds for the first time. From then on follow an optimal solution with respect to the given capital stocks at time T and the full production possibilities of problem (h). Let the social objective (3) for this feasible but clearly non-optimal solution to problem (h) be denoted Φ_f. Obviously $\Phi_f \leqslant \Phi_h \leqslant \Phi_e$. It is relatively straightforward to show that $\lim_{\bar{L}\to\infty} \Phi_f = \Phi_e$, implying $\lim_{\bar{L}\to\infty} \Phi_h = \Phi_e$.

In the presence of a large initial reserve of unemployed labor the planners cannot go very far wrong in starting off by implementing the solution to problem (iii) in the early years even though they know (h) in fact to be the real situation. It is this result which can be interpreted as justifying our interest in the infinite horizon surplus labor problem.

4. ONE AND TWO SECTOR MODELS

The behavior of model (iii) is easiest to understand in terms of simpler one and two sector models of the same family. With this in mind, it is convenient to pretend that the "real world" is being portrayed by the model (iii) and then to consider how a macro economist would build a one and two sector model out of the same situation using the same data.

The analogous one sector model, called model (i), is to maximize

$$\int_0^\infty U(C)e^{-\rho t}\,dt \tag{18}$$

subject to

$$Y = \beta K, \tag{19}$$

$$\dot{K} = sY, \tag{20}$$

$$C = (1 - s)Y, \tag{21}$$

$$0 \leqslant s \leqslant 1, \tag{22}$$

$$K(0) = K^0, \quad \text{given.} \tag{23}$$

Using the calculus of variations, it is easy to show[11] that the complete solution calls for Y, K, C, and I to each grow at the same steady rate g, where

$$g \equiv \frac{\beta - \rho}{\eta}. \tag{24}$$

This information, plus the conditions (19)–(23), specify the complete time paths of all relevant variables. In particular, it can be shown that $I(t)/Y(t) = s^*$ for all t, where

$$s^* \equiv \frac{\beta - \rho}{\eta\beta}. \tag{25}$$

Note that $0 < s^* < 1$ by (16) and (17).

The appropriate two sector non-shiftable model, called model (ii), is to maximize[12]

$$\int_0^\infty U(C)e^{-\rho t}\,dt \tag{26}$$

subject to

$$C = \beta K_2, \tag{27}$$

$$I = \beta K_1, \tag{28}$$

[11] For details see Chakravarty [**2**].

[12] The production structure is that of the basic Fel'dman model. Bose [**1**] has obtained a complete characterization of the optimal path using Pontryagin's principle; see [**1**, Section 2, 466–70]. With slight modification we rely on Bose's results, omitting details of the proof.

$$\dot{K}_1 = \lambda I: \qquad q_1, \tag{29}$$

$$\dot{K}_2 = (1 - \lambda)I: \qquad q_2, \tag{30}$$

$$0 \leqslant \lambda \leqslant 1, \tag{31}$$

$$[K_1(0), K_2(0)] = [K_1^0, K_2^0], \quad \text{given.} \tag{32}$$

Define $s(t)$ to be the gross savings rate at time t for this economy:

$$s(t) \equiv \frac{I(t)}{I(t) + C(t)}. \tag{33}$$

(We will also use this definition for model (iii).) In contrast to the one sector situation, in model (ii) the authorities are not free to choose the savings rate. At any time it is fixed in terms of the prevailing capital stock structure, although it can be changed over time by manipulating λ.

Roughly speaking, the two sector economy (ii) wants to grow in the same way as the optimal solution to the one sector model (i). But only infrequently is it to be expected that $s(0) = s^*$. In all but a razor's edge case, therefore, a specialization phase comes first. All investment in this initial specialization phase is devoted to the relatively underdeveloped sector. In this way capital stock proportions are restructured to achieve the optimal gross savings rate s^* as quickly as possible. Thereafter balanced growth at rate g occurs which maintains the optimal savings rate s^*.[13]

Let x^* be the capital structure ratio K_1/K_2 (or (I/C)) which corresponds to the optimal savings rate s^*. Obviously $x^* = s^*/(1 - s^*)$.

The solution is easily portrayed diagrammatically. In Figure 1, the line N of slope x^* passing through the origin divides the quadrant $\{K_1, K_2 \colon K_1 \geqslant 0,$

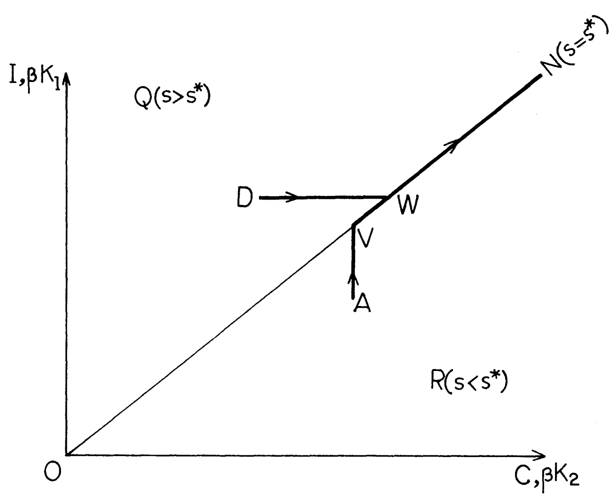

FIGURE 1.—Optimal growth in the two-sector model.

[13] Cf. Bose [1, Theorem 1, p. 469].

$K_2 \geqslant 0\}$ into two regions. Any point belonging to Q is one such that $s > s^*$, whereas a point is in R if and only if $s < s^*$. If $D = [K_1(0), K_2(0)] \in Q$, all I goes into increasing K_2 ($\lambda = 0$) until line N is reached at point W. In the historically more interesting case of $A = [K_1(0), K_2(0)] \in R$, all initial investment is ploughed back into sector 1 ($\lambda = 1$) until the resulting trajectory intersects line N at point V. The case $s(0) < s^*$ is considered to be historically more relevant in the context of an underdeveloped economy under new leadership because after a revolution tastes are imposed which typically *raise* the desired growth rate. In both cases, once line N (the non-specialization phase) is reached, the optimal program remains on it forever. Thus, if $s(0) = s^*$, the solution is identical with that of the one sector model from the very beginning.[14]

Applying the Pontryagin "maximum principle," the relevant optimality conditions[15] yield

$$\lambda(t)\begin{cases} = 1 & \text{if} \quad q_1(t) > q_2(t), \\ \in [0, 1] & \text{if} \quad q_1(t) = q_2(t), \\ = 0 & \text{if} \quad q_1(t) < q_2(t). \end{cases} \tag{34}$$

The undiscounted non-negative prices $[q_1, q_2]$ are at all times continuous and satisfy

$$\dot{q}_1 = \rho q_1 - \beta q, \tag{35}$$

$$\dot{q}_2 = \rho q_2 - \beta U'(C), \tag{36}$$

$$\lim_{t \to \infty} q_j(t) e^{-\rho t} K_j(t) = 0 \qquad (j = 1, 2),$$

where

$$q(t) \equiv \max\{q_1(t), q_2(t)\}. \tag{37}$$

Define $q_0 \equiv q(0)$.

We now derive a relation between q_0 and $U'(C(0))$ for the case $s(0) < s^*$ which will help explain the solution to model (iii). If initial conditions start economy (ii) in region R of Fig. 1, all investment goes first into building up K_1. Let τ be the time spent in the initial specialization phase. $\dot{K}_1 = \beta K_1$ and $\dot{K}_2 = 0$ for $0 \leqslant t < \tau$ implies

$$\frac{I(0)e^{\beta\tau}}{C(0)} = \frac{K_1(0)e^{\beta\tau}}{K_2(0)} = x^*,$$

or that

$$\tau = \frac{1}{\beta} \log\left(\frac{x^*}{I(0)/C(0)}\right). \tag{38}$$

[14] With a general concave utility function the "optimal savings line" N becomes a curve. But the qualitative properties of the optimal solution remain the same, provided sufficiently strong regularity conditions are imposed—cf. Bose [**1**, Appendix II].

[15] Bose [**1**, Equations (9)–(13), p. 467], which is based on Pontryagin [**10**, Theorem 1, p. 19, and the discussion of pp. 189–191].

During the specialization phase ($t < \tau$), $\lambda = 1$. Hence, from (34) and (37), $q_1(t) = q(t)$ for $0 \leqslant t < \tau$. From (35),

(39) $$q(\tau) = q_0 e^{(\rho - \beta)\tau}.$$

Throughout the non-specialization phase ($t \geqslant \tau$), investment is balanced between I and C. With $0 < \lambda = s^* < 1$, (34) and (37) yield $q_1(t) = q_2(t) = q(t)$ for $t \geqslant \tau$. From (35) and (36), $U'(C(t)) = q(t)$ for $t \geqslant \tau$. In particular, letting $t = \tau$,

(40) $$U'(C(\tau)) = q(\tau).$$

But

(41) $$U'(C(\tau)) = U'(C(0)),$$

since $\dot{C} = 0$ for $0 \leqslant t < \tau$.

Putting together (39), (40), and (41) yields

(42) $$U'(C(0)) = q_0 e^{(\rho - \beta)\tau}.$$

Combining (42) with (38),

(43) $$U'(C(0)) = q_0 \left(\frac{I(0)/C(0)}{x^*} \right)^{(\beta - \rho)/\beta}.$$

The expression (43) will prove useful in the sequel.

5. SOLUTION OF THE THREE SECTOR PROBLEM

In the full three sector model (iii), introduction of the common intermediate sector R offers the possibility of a richer and more realistic kind of substitution between I and C. At any point of time the constraints (4), (5), and (6) define a

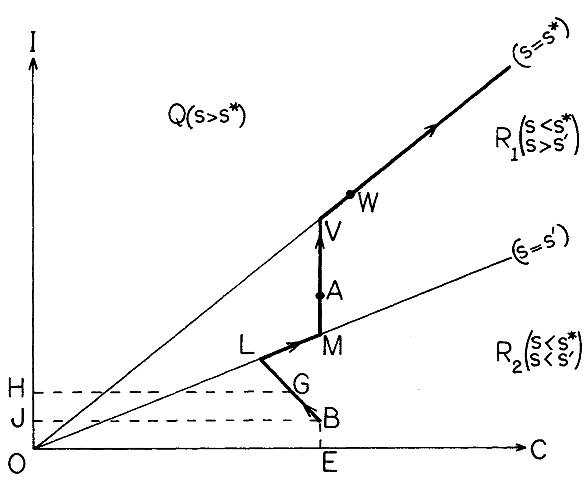

FIGURE 2.—Optimal growth in the three sector model.

production possibilities surface (PPS) with I and C as final products. In general the PPS can have any one of several shapes, depending upon which configuration of effective constraints holds at the given moment. For reasons that will be made clear, two configurations are of special interest: (1) the degenerate case (in the linear programming sense) where the three constraints hold simultaneously at a single point, forming a rectangular PPS represented by *OEBJO* in Figure 2; (2) the case where the raw materials constraint (6) cuts across the rectangle given by the two specific capacity constraints (4) and (5) so that the PPS is a pentagon represented by *OEBGHO* in Figure 2. The efficient operating regions are point B in case (1) and line GB in case (2).

While there is often a temptation to analyze patterns of optimal growth parametrically, as a function of all possible initial endowments, it is unlikely that the given historical capital stocks of any real economy would take on completely arbitrary values. We assume that the economy starts off in configuration (1). The rationale behind a full capacity endowment is the notion that previous to the historical discontinuity of time zero an internally consistent *ancien régime* was moving along in some kind of non-specialization phase without excess capacity.[16] *Après la révolution* the new planning board inherits an historically determined savings rate $s(0)$ which is likely to be very different from the desired rate s^* best suited to its own newly enforceable social values.

The two sector approach suggests starting off by splitting $K_R(0)$ between departments 1 and 2. Defining $K_1(0) \equiv (1 + (a_I\gamma_I/\gamma_R))K_I(0)$ and $K_2(0) \equiv (1 + (a_C\gamma_C/\gamma_R))K_C(0)$, we could pretend that the resulting model obeyed (27)–(32) instead of (4)–(12). Translating back from model (ii) to model (iii) would be accomplished with the aid of the relations $K_I(t) = (1 - a_I)K_1(t)$, $K_C(t) = (1 - a_C)K_2(t)$, and $K_R(t) = a_IK_1(t) + a_CK_2(t)$. We could then proceed by optimizing model (ii) with the given initial conditions. Suppose for concreteness the historically more relevant situation $s(0) < s^*$. The solution would be to specialize initially all investment to the investment goods department (which now includes K_R capital) increasing s to s^* and thereafter maintaining it at that rate by a policy of balanced investment. Starting from an initial PPS of *OEBJO* in Figure 2, such a policy would proceed from B to V and then move out along the ray $s = s^*$. At any time the PPS would be a full capacity rectangle.

This full capital employment program is certainly feasible in the context of model (iii), but is it optimal? If more investment is strongly enough desired in the beginning, it could be rapidly built up at the expense of consumption via a program of reinvesting only in sector I and re-routing to that sector some R previously destined for sector C. In Figure 2 this means moving from B to G, where the line BG has slope $-a_C/a_I$. Excess capacity would be created in sector C, changing the PPS from a full employment rectangle (*OEBJO*) to a pentagonal PPS (*OEBGHO*) in which the raw materials constraint (6) is operative as the line BG.

[16] In theory the general case of any initial configuration of capital endowments could be handled without difficulty by using the same methods we employ for the initial full capacity situation. The rub is that the number of possible cases becomes unwieldy. For this reason it seems better to sacrifice full generality in favor of focusing on a particular historically interesting case.

In this situation the pseudo departments 1 and 2 would be meaningless because they would lack stability. The basic aim of this paper is to determine conditions under which stability is non-existent because capital is in effect shifted from one department to another by transferring the destination of R. The following theorem is the main result.

THEOREM: *Assume that all capital is initially fully employed. Let q_0 be the undiscounted social price of investment in model (ii) at time zero. The full capacity model (ii) solution is also optimal for model (iii) if and only if the following conditions are fulfilled*:

$$(44) \qquad s(0) < s^*: \qquad \frac{U'(C(0))}{a_C} \geqslant q_0,$$

$$(45) \qquad s(0) > s^*: \qquad U'(C(0)) \leqslant \frac{q_0}{a_I},$$

$$(46) \qquad s(0) = s^*: \qquad \text{unconditional.}$$

In (46), the solution to model (iii) is just like the solution to model (i): grow at rate g and save at rate s^*. Conditions (44) and (45) are also easy to interpret. Suppose, in model (ii) with $s(0) < s^*$, that after recalling the true three sector nature of the model it is decided to accelerate growth of the relatively underdeveloped I sector by transferring a unit of R from C to I at time zero. When all necessary sub-optimal adjustments have been made, the gain would be q_0, the value of an extra unit of investment at time zero measured in terms of the utility index. Consumption would be diminished by $1/a_C$ units, resulting in a utility decrease of $U'(C(0))/a_C$. The relevant question is whether the loss of utility is matched by a sufficiently high value of investment to justify the transfer of R. A similar interpretation can be made for case (45).

Conditions (44) and (45) can be translated into a form where they are expressed only in terms of the original parameters and initial capital stocks of problem (iii). We concentrate on the case of historical interest. If $s(0) < s^*$ it follows from (43) that condition (44) is equivalent to

$$(47) \qquad \frac{I(0)}{C(0)} \geqslant x^* a_C^\alpha$$

with $\alpha \equiv \beta/(\beta - \rho)$.

Let s' be the savings rate which corresponds to the capital structure $I/C = x^* a_C^\alpha$. Obviously

$$(48) \qquad s' = \frac{x^* a_C^\alpha}{1 + x^* a_C^\alpha} = \frac{s^* a_C^\alpha}{1 - s^*(1 - a_C^\alpha)}.$$

It follows from (47) that with $s(0) < s^*$ the full capacity model (ii) solution is also optimal for model (iii) if and only if

$$(49) \qquad s(0) \geqslant s'$$

with s' defined by (48).

Condition (49) is very easy to interpret. The ray $s = s'$ of slope $x^* a_C^\alpha$ is drawn in Figure 2 lying below and to the right of the ray $s = s^*$ of slope x^* (because $a_C < 1$). Any initial full capacity endowment, like A, lying in the region R_1 between the rays $s = s^*$ and $s = s'$ will lead to a full capacity solution AVW. An initial full endowment capacity like B, lying in the region R_2 between the rays $s = s'$ and $s = 0$ will result in capital shifting.

As values of a_C are made smaller, the ray $s = s'$ rotates clockwise and the region R_1 expands. Other things being equal, condition (49) would be more likely to hold the lower is a_C. If little R is used per unit of C, it would be foolish to sacrifice a great deal of consumption to free a small amount of intermediate commodities for investment purposes.

Treating s^* as fixed, condition (49) is also more likely to prevail, ceterus paribus, the closer is $s(0)$ to s^*. An economy starting off with a savings rate near that which it desires to attain is less likely to be in such a hurry to speed up the growth of investment as to tolerate excess capacity in the consumption goods sector.

The most meaningful consumption goods in the context of an economically underdeveloped country are primarily food products, soft goods, and housing services. It seems reasonable to suppose that the transportation, fuel, electricity, and selected industrial materials necessary to maintain consumption levels at existing capacity are probably negligible compared with the loss of consumption entailed by transferring these intermediate materials to the investment sector.[17] Even though investment goods were top priority in post-1928 Soviet development strategy, it might have been foolish to have conveyed capital goods in vehicles which formerly carried consumption goods simply in order to avoid having to invest in the transportation system.

These kinds of arguments suggest that a_C may be low enough so that in practice capital might not be shifted even though in theory it could be. If this is so, the relevant historical case is $s' \leqslant s(0) < s^*$. The initial savings rate is lower than the desired rate, but not so low as to encourage capital shifting. The Fel'dman story about a two-department specific capital economy may not be literally true, but it probably makes a good parable. The effectiveness of the parable depends upon a certain stickiness in model (iii) around the initial capital stocks. Although it costs a rouble to increase consumption by a rouble, salvaging a rouble's worth of intermediate commodities requires sacrificing *more* than a rouble's worth of

[17] Casual playing with numbers supports the feeling that (49) holds. Look at $\alpha = \beta/(\beta - \rho)$ as $1/\eta s^*$. The biggest unknown is η. Suppose a logarithmic utility function ($\eta = 1$). Even with $s(0)$ as low as .05, s^* as high as .3, and a_C as high as .5, condition (49) would be fulfilled.

consumption. The higher this asymmetric adjustment cost, the less profitable it becomes to shift capital.[18]

The case $s(0) > s^*$ could be treated analogously. One could transform (45) into a condition that depends only on parameters and initial values. This condition could be given a roughly similar interpretation to that which was placed on (49).

6. PROOF OF THE THEOREM

We prove only (44). The proof of (45) is completely analogous and (46) is trivial. Let τ be specified by (38) and q_0 by (43). Define g and s^* by (24) and (25).

For $t < \tau$, set

(50) $$p_I(t) = p_R(t) = q_0 e^{-(\beta-\rho)t},$$

(51) $$p_C(t) = \frac{q_0 e^{-(\beta-\rho)\tau}}{\rho(1-a_C)}[\beta - (\beta-\rho)e^{\rho(t-\tau)} - \rho a_C e^{-(\beta-\rho)(t-\tau)}],$$

(52) $$\lambda_I(t) = 1 - a_I,$$

(53) $$\lambda_C(t) = 0,$$

(54) $$\lambda_R(t) = a_I,$$

(55) $$I(t) = I(0)e^{\beta t},$$

(56) $$C(t) = C(0),$$

(57) $$R(t) = a_I I(t) + a_C C(t),$$

(58) $$K_I(t) = K_I(0)e^{\beta t},$$

(59) $$K_C(t) = K_C(0),$$

(60) $$K_R(t) = \frac{a_I}{1-a_I}K_I(t) + \frac{a_C}{1-a_C}K_C(t).$$

For $t \geqslant \tau$, set

(61) $$p_I(t) = p_C(t) = p_R(t) = q_0 e^{-(\beta-\rho)t},$$

(62) $$\lambda_I(t) = s^*(1 - a_I),$$

(63) $$\lambda_C(t) = (1 - s^*)(1 - a_C),$$

(64) $$\lambda_R(t) = s^* a_I + (1 - s^*)a_C,$$

(65) $$I(t) = I(\tau)e^{g(t-\tau)},$$

[18] The behavior of model (iii) vis a vis model (ii) should be contrasted with the behavior of (ii) vis a vis (i). In all but a razor's edge case the two sector model yields an optimal growth path different from that which would prevail in the one sector case. As we have seen, however, the behavior of (iii) may well duplicate that of (ii). The mathematical reason is that although in (iii) an extra sector has been added, an extra initial condition has also been included with the stipulation that economy (iii) starts off without any excess capacity.

$$(66)\qquad C(t) = C(\tau)e^{g(t-\tau)},$$

$$(67)\qquad R(t) = R(\tau)e^{g(t-\tau)},$$

$$(68)\qquad K_I(t) = K_I(\tau)e^{g(t-\tau)},$$

$$(69)\qquad K_C(t) = K_C(\tau)e^{g(t-\tau)},$$

$$(70)\qquad K_R(t) = K_R(\tau)e^{g(t-\tau)}.$$

In (65)–(70) $I(\tau)$, $C(\tau)$, $R(\tau)$, $K_I(\tau)$, $K_C(\tau)$, and $K_R(\tau)$ are fixed at time τ by (55)–(60), whereas $I(0)$ and $C(0)$ in (55) and (56) are simply given initial full-capacity output values at time zero.

We must show that in the model (iii) solution with (44) holding no shifting or excess capacity occurs. The optimal policy is to devote all investment initially to building up the I and R sectors together until time τ when $s = s^*$. Thereafter all sectors grow at rate g. The method of proof for the *if* part of the theorem is to directly verify that the system (50)–(70) satisfies all the relevant Pontryagin conditions.[19]

Using the dual variables to equation (7)–(9), the following undiscounted Hamiltonian form is introduced:

$$(71)\qquad He^{\rho t} \equiv U(C) + I(p_I\lambda_I + p_C\lambda_C + p_R\lambda_R).$$

The quantity part of the proposed solution (50)–(70) is feasible because it satisfies (4)–(12). In addition, the following conditions (A) and (B) must be fulfilled.

CONDITION (A): *At each t, control variables (I, C, λ_I, λ_C, λ_R) are set at feasible values which maximize $He^{\rho t}$.*

Let

$$p(t) \equiv \max\{p_I(t), p_C(t), p_R(t)\}.$$

Maximizing (71) with respect to non-negative $(\lambda_I, \lambda_C, \lambda_R)$ subject to (10) yields

$$(72)\qquad He^{\rho t} = U(C) + pI.$$

For $t < \tau$, $p_C(t) < \min\{p_I(t), p_R(t)\}$ implies $\lambda_C(t) = 0$; no new information of this sort is revealed for $t \geqslant \tau$ since $p_I(t) = p_C(t) = p_R(t)$ during that period.

Maximizing (72) over non-negative I and C satisfying (4)–(6) and (58)–(60), (68)–(70) will always call forth the full capacity solutions (55)–(57), (65)–(67). In this sub-problem π_I, π_C, and π_R are dual, respectively, to equations (4), (5), and (6).

[19] See Pontryagin [**10**, Theorem 1, p. 19, and the discussion of pp. 189–191]. Due to convexity in production and strict concavity of utility, we are assured that the Pontryagin necessary conditions are also sufficient for the proposed solution to be optimal. Strictly speaking, the "transversality condition" (78) is generally a sufficient rather than a necessary condition for optimality and optimal paths may exist which do not satisfy it. The present case, however, is different. As we show constructively, there exists a solution satisfying (78) which obeys all the other Pontryagin conditions. Since from strict concavity of utility an optimal solution is unique, the sufficient conditions must also be necessary.

From duality theory, we have the following equations[20] for all $t \geqslant 0$:

(73) $$p(t) = \frac{\pi_I(t)}{\gamma_I} + \frac{a_I \pi_R(t)}{\gamma_R},$$

(74) $$U'(C(t)) = \frac{\pi_C(t)}{\gamma_C} + \frac{a_C \pi_R(t)}{\gamma_R},$$

(75) $$\pi_I(t), \pi_C(t), \pi_R(t) \geqslant 0.$$

CONDITION (B): *$p_I(t)$, $p_C(t)$, and $p_R(t)$ are non-negative, continuous, and must satisfy*

(76) $$\dot{p}_I = \rho p_I - \pi_I,$$

(77) $$\dot{p}_C = \rho p_C - \pi_C,$$

(78) $$\dot{p}_R = \rho p_R - \pi_R,$$

(79) $$\lim_{t \to \infty} p_j(t) e^{-\rho t} K_j(t) = 0 \qquad (j = I, C, R).$$

The proposed prices (p_I, p_C, p_R) defined by (50), (51), and (61) are clearly continuous and non-negative. Using the conditions (16) and (17) and the equations (68)–(70), it is easy to verify that the transversality conditions (79) hold.

In order for (76)–(78) to be consistent with (61) during $t \geqslant \tau$, we must have $\pi_I(t) = \pi_C(t) = \pi_R(t)$. This common value of $\pi_j(t)$ must equal $\beta p(t)$ from (13) and (73), i.e.,

(80) $$\pi_I(t) = \pi_C(t) = \pi_R(t) = \beta p(t).$$

Using (14), it is easily seen that conditions (73)–(75) are fulfilled for values (80) so long as $U'(C(t)) = p(t)$, which holds for $t \geqslant \tau$ by (1), (42), (61), and (66).

As for $t < \tau$, (76) and (78) will be consistent with (50) if and only if $\pi_I(t) = \pi_R(t)$. From (13) and (73),

(81) $$\pi_I(t) = \pi_R(t) = \beta p(t).$$

Obviously (73) is satisfied by values (81). Substitution of (81) into (74) and using (15) yields

$$\pi_C(t) = \gamma_C(U'(C(t)) - a_C p(t)).$$

It follows that for $t < \tau$, $\pi_C(t) \geqslant 0$ if and only if $U'(C(t)) \geqslant a_C p(t)$. Substituting from (50) and (56), $\pi_C(t) \geqslant 0$ if and only if $U'(C(0)) \geqslant a_C q_0 e^{-(\beta - \rho)t}$ which will hold for *all* $t < \tau$ if and only if $U'(C(0))/a_C \geqslant q_0$.

It remains only to verify that the price solutions (50), (51), and (61) satisfy the differential equations (76)–(78). This is straightforward, if tedious, and finishes the necessity part of the proof.

For the *only if* part of the theorem, it is not difficult to show that so long as $s(0) < s^*$, and whether or not (44) holds, the initial phase must consist of $\lambda_C = 0$.

[20] Equations (73) and (74) must hold with full equality because in our proposed solution C and I are strictly positive.

A proof by contradiction then proceeds as follows. In order for no excess capacity to be created in the first phase, we must have λ_I and λ_R both positive. This will be possible only if $p_I(t) = p_R(t) \geqslant p_C(t)$ during the initial phase which in turn is *not* allowable (as we have just seen in the proof of the sufficiency part of the theorem) if

$$(82) \qquad U'(C(0))/a_C < p(0).$$

If capital is allowed to be shifted, the initial value of an extra unit of investment must be at least as high as if no shifting is permitted implying $p(0) \geqslant q_0$. Thus, if (44) does *not* hold, (82) holds and the optimality of non-shiftability must be violated. This demonstrates the sufficiency part of the theorem and finishes the proof.

7. CONCLUDING REMARKS

For completeness we include a brief description of optimal three sector growth in the case where capital is shifted and excess capacity is created. Since it is of little intrinsic interest, the proof is omitted—it is mostly a tedious verification of optimality conditions, somewhat in the unhappy spirit of the proof of the previous theorem.

Suppose the historically more interesting case $s(0) < s^*$ (the case $s(0) > s^*$ is analogous). If capital is to be shifted, we must be given that $U'(C(0))/a_C < q_0$, or that $s(0) < s'$. In Figure 2, the initial full capacity endowment B (lying in R_2) is the starting point of the geometric discussion.

In the first phase all investment goes into I ($\lambda_C = \lambda_R = 0$) and R is transferred from C to I, creating excess capacity in the C sector ($\pi_C = 0$). This moves output from B toward L, creating a pentagonal PPS. The line BL has slope $(-a_C/a_I)$. The purpose of phase 1 is to rapidly build up I, which grows at the rate γ_I, from (13) faster than the full capacity maximal rate β. In this stage $p = p_I > p_R > p_C$, $U'(C) < a_C p$, and $U'(C)/p$ increases over time.

Phase 2 begins as soon as $U'(C) = a_C p$ or $s = s'$ (at point L). This is a stage when the full capacity C output is recouped by investing more in R than is needed to increase I alone. Both I and C grow at the same rate, so that s is constant at s'. Throughout this phase $U'(C) = a_C p$, $\lambda_C = 0$, $\pi_C = 0$, and $p = p_I = p_R > p_C$. In Figure 2 the second phase is represented by movement from L to M.[21]

Phase 3 begins at point M when no excess capacity exists anywhere in the economy, as in the beginning of phase 1. But this time I and R are built up together just like in the opening phase of the optimal path described in the previous theorem. Throughout this stage $U'(C)/a_C p$ increases from unity, $\lambda_C = 0$, $p = p_I = p_R > p_C$, and no excess capacity exists ($\pi_I, \pi_C, \pi_R > 0$). The economy moves from M to V.

Phase 4 is the balanced growth phase which begins at point V when for the first time $s = s^*$. All stocks and flows grow at rate g, and $p = p_I = p_C = p_R$.

[21] Figure 2 may be somewhat misleading because it looks as if it might be quicker to go directly from B to M than to proceed via L. In terms of time, the route BLM is faster because high levels of I are initially built up quickly at the expense of C.

If $s' \leqslant s(0) < s^*$ the optimal program starts off in phase 3 of the present situation and only the last two growth phases are relevant. It is only in this kind of circumstance that the story about non-shiftable capital in a two-department economy can be defended as a meaningful parable.

We conclude with a few words about what happens if the number of sectors is made greater than three.

Increasing the number of consumption goods changes very little. Complete non-shiftability now holds if and only if conditions like (44) or (45) are true for *each* consumption good. With $s(0) < s^*$, this puts the greatest pressure for shifting, other things being equal, on those consumer goods having the highest content of salvageable raw materials.

An enlarged number of jointly shared intermediate raw materials sectors can also be treated by an easy modification. Now interpreting a_C and a_I as the *total* jointly used raw materials per unit output of, repectively, C and I, the necessary and sufficient conditions for complete non-shiftability in the present case are still (44) or (45). However, the description of exactly what happens if capital *is* shifted can become very complicated.

As usual, working with more than one investment goods sector opens a Pandora's box of practical difficulties. There does not seem to be an easy way of cataloging results in this case.

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