# Shifted $1 / N$ expansion for the Klein-Gordon equation with vector and scalar potentials 

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#### Abstract

The shifted $1 / N$ expansion method has been extended to solve the Klein-Gordon equation with both scalar and vector potentials. The calculations are carried out to the third-order correction in the energy series. The analytical results are applied to a linear scalar potential to obtain the relativistic energy eigenvalues. Our numerical results are compared with those obtained by Gunion and Li [Phys. Rev. D 12, 3583 (1975)].


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## I. INTRODUCTION

Recently the shifted $1 / N$ expansion technique [1] has received much attention in solving the Schrödinger equation. It has been applied to a large number of physically interesting potentials yielding highly accurate results [1-7]. Very recently the Klein-Gordon (KG) and Dirac equations have been studied by some authors [8-11] to determine the energy eigenvalues of these equations.

In the present paper we have extended the method to deal with the $K G$ equation for any radially symmetric vector and scalar potentials. While the formalism applies for both vector- and scalar-type potentials, numerical results are obtained only for a scalar potential of the form $S(r)=A r$, which is a Lorentz scalar used in the study of quarkonium systems $[8,12-14]$. Accurate numerical results are used to compare with our results.

In Sec. II we extend the formalism of Ref. [1] and apply a different approach [11] to deal with both vector and scalar potentials in the KG equation. In Sec. III we present our numerical results and compare with those given in Ref. [13]. Section IV is left for concluding remarks.

## II. THE METHOD

The radial part of the $N$-dimensional KG equation (in units $\hbar=c=1$ ) for radially symmetric vector and scalar potentials [15] can be written as

$$
\begin{align*}
& {\left[-\frac{d^{2}}{d r^{2}}+\frac{(k-1)(k-3)}{4 r^{2}}\right.} \\
& \left.\quad+\left\{[m+S(r)]^{2}-[E-V(r)]^{2}\right\}\right] \phi(r)=0 \tag{1}
\end{align*}
$$

where $S(r)$ is a scalar potential and $V(r)$ the fourth component of a vector potential, $k=N+2 l$, and $\phi(r)$ is the radial wave function.

Following Ref. [1], we use $\bar{k}$, which is defined as

$$
\begin{equation*}
\bar{k}=k-a \tag{2}
\end{equation*}
$$

and shift the origin of coordinate by

$$
\begin{equation*}
x=\bar{k}^{1 / 2}\left(r-r_{0}\right) / r_{0} \tag{3}
\end{equation*}
$$

and also accordingly expand $V(r), S(r)$, and $E$ as

$$
\begin{align*}
& V(r)=\left(\bar{k}^{2} / Q\right)\left[V\left(r_{0}\right)+V^{\prime}\left(r_{0}\right) r_{0} x / \bar{k}^{1 / 2}\right. \\
& +V^{\prime \prime}\left(r_{0}\right) r_{0}^{2} x^{2} /(2 \bar{k}+\cdots],  \tag{4a}\\
& S(r)=\left(\bar{k}^{2} / Q\right)\left[S\left(r_{0}\right)+S^{\prime}\left(r_{0}\right) r_{0} x / \bar{k}^{1 / 2}\right. \\
& \left.+S^{\prime \prime}\left(r_{0}\right) r_{0}^{2} x^{2} /(2 \bar{k})+\cdots\right],  \tag{4b}\\
& E=E_{0}+E_{1} / \bar{k}+E_{2} / \bar{k}^{2}+E_{3} / \bar{k}^{3}+\cdots, \tag{4c}
\end{align*}
$$

where $Q$ is a scale to be determined later. After substituting Eqs. (4a)-(4c) in Eq. (1), we obtain a Schrödinger-like equation which has been solved by Imbo, Pagnamenta, and Sukhatme [1]. We therefore just quote the results and give the final expression for the energy eigenvalue.

$$
\begin{equation*}
E=E_{0}+[\beta(1)+\beta(2) / \bar{k}] / 2 E_{0} r_{0}^{2} \tag{5}
\end{equation*}
$$

where $\beta(1)$ and $\beta(2)$ are defined in Ref. [8] and

$$
\begin{equation*}
E_{0}=V\left(r_{0}\right)+\left\{\left[S\left(r_{0}\right)+m\right]^{2}+Q / 4 r_{0}^{2}\right\}^{1 / 2} \tag{6}
\end{equation*}
$$

where $r_{0}$ is chosen to be the minimum of $E_{0}$. Hence $r_{0}$ satisfies the relation

$$
\begin{equation*}
Q=b\left(r_{0}\right)+\left[b^{2}\left(r_{0}\right)+c\left(r_{0}\right)\right] \tag{7}
\end{equation*}
$$

in which

$$
\begin{align*}
& b\left(r_{0}\right)=4 r_{0}^{3} S^{\prime}\left(r_{0}\right)\left[S\left(r_{0}\right)+m\right]+2 r_{0}^{4} V^{\prime}\left(r_{0}\right)^{2}  \tag{8}\\
& c\left(r_{0}\right)=16 r_{0}^{6}\left[S\left(r_{0}\right)+m\right]^{2}\left[V^{\prime}\left(r_{0}\right)^{2}-S^{\prime}\left(r_{0}\right)^{2}\right] \tag{9}
\end{align*}
$$

The shifting parameter $a$ is chosen so as to make the first-order correction $E_{1} / \bar{k}$ vanish. Consequently
$a=2-\left(1+2 n_{r}\right) w$,
where

$$
\begin{equation*}
w=\left\{3+\left(4 r_{0}^{4} / Q\right)\left[\left[S\left(r_{0}\right)+m\right]\left(S^{\prime \prime}\left(r_{0}\right)+\left\{1+\left(Q / 4 r_{0}^{2}\right)\left[S\left(r_{0}\right)+m\right]^{2}\right\}^{1 / 2} V^{\prime \prime}\left(r_{0}\right)\right)+S^{\prime}\left(r_{0}\right)^{2}-V^{\prime}\left(r_{0}\right)^{2}\right]\right\} \tag{11}
\end{equation*}
$$

TABLE I. Klein-Gordon results for part of the energy levels (in GeV ) of $\psi, \psi^{\prime}$ system with $A=0.137 \mathrm{GeV}^{2}$ and $m=1.12$ GeV . The values in parentheses are those given by Gunion and Li [13].

|  | $L$ |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| $n_{r}$ | 0 | 1 | 2 | 3 |
| 0 | 3.12 | 3.46 | 3.74 | 3.99 |
|  | $(3.1)$ | $(3.47)$ | $(3.73)$ | $(3.98)$ |
| 1 | 3.70 | 3.96 | 4.18 | 4.387 |
|  | $(3.7)$ | $(3.95)$ | $(4.17)$ | $(4.39)$ |
| 2 | 4.16 | 4.36 | 4.556 | 4.738 |
|  | $(4.17)$ | $(4.38)$ | $(4.56)$ | $(4.73)$ |
| 3 | 4.537 | 4.719 | 4.89 | 5.06 |
|  | $(4.54)$ | $(4.72)$ | $(4.90)$ | $(5.05)$ |
| 4 | 4.9 | 5.039 | 5.196 | 5.346 |
|  | $(4.8)$ | $(5.04)$ | $(5.2)$ | $(5.35)$ |

Equations (7)-(9) along with $Q=\bar{k}^{2}$ and Eqs. (10) and (11) read
$\left\{b\left(r_{0}\right)+\left[b\left(r_{0}\right)^{2}+c\left(r_{0}\right)\right]^{1 / 2}\right\}^{1 / 2}=1+2 l+\left(1-2 n_{r}\right) w$.

## III. APPLICATION TO A PURE SCALAR POTENTIAL $[V(r)=0, S(r)=A r]$

By this case we are precisely referring to the quarkconfining linear potential regarded as a Lorentz scalar.

From Eq. (5) the eigenvalue $E$ is calculated and accordingly the Klein-Gordon results for the energy levels are listed in Tables I-III. Our results are compared with the numerical results obtained by Gunion and Li [13].

## IV. CONCLUDING REMARKS

We have developed a general formalism for the shifted $1 / N$ expansion of the Klein-Gordon equation with both vector and scalar potentials. The case where $V(r)=0$ and $S(r)=A r$ only has been treated in this paper, leaving the other cases for later investigations. The comparison of our results with those of Gunion and Li [13] gives no doubt about the good agreement between them. In Table I the accuracy ranges from $97.96 \%$ to $100.00 \%$. In Table II the accuracy is noted to range between $99.21 \%$ and $\mathbf{9 9 . 8 8 \%}$. In Table III the accuracy is between $\mathbf{9 8 . 8 7 \%}$ and $99.62 \%$. It has also been noted that the term contributing most to the energy levels is the leading term $E_{0}$ of Eq. (5) in the sense that the ratio of the leading term contribution to the contribution of $E$, Eq. (5), ranges between 0.9962 and 0.9998 for Table I, 0.9987 and 0.9997 for Table II, and 0.9992 and 0.9998 for Table III. All in all we can say that the shifted $1 / N$ expansion works well for the KG equation with a scalar potential of the form $S(r)=A r$.

## APPENDIX

We list below the definitions of $\epsilon_{j}$ and $\delta_{j}$ :

$$
\begin{align*}
& \epsilon_{1}=(2-a), \quad \epsilon_{2}=-3(2-a) / 2, \\
& \epsilon_{3}=-1+\left(r_{0}^{5} / 3 Q\right)\left[m S^{\prime \prime \prime}\left(r_{0}\right)+E_{0} V^{\prime \prime \prime}\left(r_{0}\right)+S\left(r_{0}\right) S^{\prime \prime \prime}\left(r_{0}\right)-V\left(r_{0}\right) V^{\prime \prime \prime}\left(r_{0}\right)+3 S^{\prime}\left(r_{0}\right) S^{\prime \prime}\left(r_{0}\right)-3 V^{\prime}\left(r_{0}\right) V^{\prime \prime}\left(r_{0}\right)\right], \\
& \begin{array}{c}
\epsilon_{4}=\frac{5}{4}+\left(r_{0}^{6} / 12 Q\right)\left[m S^{\prime \prime \prime \prime}\left(r_{0}\right) S^{\prime \prime \prime \prime}\left(r_{0}\right)+E_{0} V^{\prime \prime \prime \prime}\left(r_{0}\right)+S\left(r_{0}\right) S^{\prime \prime \prime \prime}\left(r_{0}\right)+4 S^{\prime}\left(r_{0}\right) S^{\prime \prime \prime}\left(r_{0}\right)+3 S^{\prime \prime \prime}\left(r_{0}\right)^{2}\right. \\
\\
\left.\quad-V\left(r_{0}\right) V^{\prime \prime \prime \prime}\left(r_{0}\right)-4 V^{\prime}\left(r_{0}\right) V^{\prime \prime \prime}\left(r_{0}\right)-3 V^{\prime \prime}\left(r_{0}\right)^{2}\right]
\end{array} \tag{A1}
\end{align*}
$$

and

TABLE II. Klein-Gordon results for part of the energy levels (in GeV ) of the $\rho$ system, using $\rho^{\prime}(1.25)$ as a first excitation, with $A=0.07 \mathrm{GeV}^{2}$ and $m=0.15 \mathrm{GeV}$. The values in parentheses are those given by Gunion and Li [13].

|  | $L$ |  |
| :---: | :---: | :---: |
| $n_{r}$ | 0 | 1 |
| 0 | 1.127 | 1.396 |
|  | $(1.13)$ | $(1.4)$ |
| 1 | 1.60 | 1.792 |
|  | $(1.61)$ | $(1.79)$ |
| 2 | 1.955 | 2.11 |
|  | $(1.96)$ | $(2.12)$ |
| 3 | 2.247 | 2.38 |
|  | $(2.25)$ | $(2.39)$ |
| 4 | 2.50 | 2.62 |
|  | $(2.52)$ | $(2.63)$ |

TABLE III. Klein-Gordon results for part of the energy levels (in GeV ) of the $\rho$ system, using $\rho^{\prime}(1.6)$ as a first excitation, with $A=0.21 \mathrm{GeV}^{2}$ and $m=0.15 \mathrm{GeV}$. The values in parentheses are those given by Gunion and Li [13].

|  | $L$ |  |
| :---: | :---: | :---: |
| $n_{r}$ | 0 |  |
|  |  |  |
| 0 | 1.79 | 2.26 |
|  | $(1.8)$ | $(2.28)$ |
| 1 | 2.63 | 2.95 |
|  | $(2.64)$ | $(2.98)$ |
| 2 | 3.24 | 3.50 |
|  | $(3.27)$ | $(3.54)$ |
| 3 | 3.74 | 3.98 |
|  | $(3.78)$ | $(4.0)$ |

$$
\begin{align*}
& \delta_{1}=-(1-a)(3-a) / 2, \quad \delta_{2}=3(1-a)(3-a) / 4 \text {, } \\
& \delta_{3}=2(2-a), \quad \delta_{4}=-5(2-a) / 2, \\
& \delta_{5}=-\frac{3}{2}+\left(r_{0}^{7} / 60 Q\right)\left[m S^{\prime \prime \prime \prime \prime}\left(r_{0}\right)+E_{0} V^{\prime \prime \prime \prime \prime}\left(r_{0}\right)+S\left(r_{0}\right) S^{\prime \prime \prime \prime \prime}\left(r_{0}\right)+5 S^{\prime}\left(r_{0}\right) S^{\prime \prime \prime \prime}\left(r_{0}\right)+15 S^{\prime \prime \prime}\left(r_{0}\right) S^{\prime \prime \prime \prime}\left(r_{0}\right)\right. \\
& \left.-V\left(r_{0}\right) V^{\prime \prime \prime \prime \prime}\left(r_{0}\right)-5 V^{\prime}\left(r_{0}\right) V^{\prime \prime \prime \prime}\left(r_{0}\right)-15 V^{\prime \prime}\left(r_{0}\right) V^{\prime \prime \prime}\left(r_{0}\right)\right],  \tag{A2}\\
& \delta_{6}=\frac{7}{4}+\left(r_{0}^{8} / 360 Q\right)\left[m S^{\prime \prime \prime \prime \prime \prime}\left(r_{0}\right)+E_{0} V^{\prime \prime \prime \prime \prime \prime}\left(r_{0}\right)+S\left(r_{0}\right) S^{\prime \prime \prime \prime \prime \prime}\left(r_{0}\right)+6 S^{\prime}\left(r_{0}\right) S^{\prime \prime \prime \prime \prime}\left(r_{0}\right)+15 S^{\prime \prime}\left(r_{0}\right) S^{\prime \prime \prime \prime}\left(r_{0}\right)\right. \\
& \left.+10 S^{\prime \prime \prime}\left(r_{0}\right)^{2}-V\left(r_{0}\right) V^{\prime \prime \prime \prime \prime \prime}\left(r_{0}\right)-6 V^{\prime}\left(r_{0}\right) V^{\prime \prime \prime \prime}\left(r_{0}\right)-15 V^{\prime \prime}\left(r_{0}\right) V^{\prime \prime \prime \prime}\left(r_{0}\right)-10 V^{\prime \prime \prime}\left(r_{0}\right)^{2}\right] .
\end{align*}
$$

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