Shifted Poisson Structures and deformation quantization

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Abstract

This paper is a sequel to [PTVV]. We develop a general and flexible context for differential calculus in derived geometry, including the de Rham algebra and the mixed algebra of polyvector fields. We then introduce the formalism of formal derived stacks and prove formal localization and gluing results. These allow us to define shifted Poisson structures on general derived Artin stacks, and to prove that the non-degenerate Poisson structures correspond exactly to shifted symplectic forms. Shifted deformation quantization for a derived Artin stack endowed with a shifted Poisson structure is discussed in the last section. This paves the road for shifted deformation quantization of many interesting derived moduli spaces, like those studied in [PTVV] and many others.

Contents

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Introduction

This work is a sequel of [PTVV]. We introduce the notion of a *shifted Poisson structure* on a general derived Artin stack, study its relation to the shifted symplectic structures from [PTVV], and construct a deformation quantization of it. As a consequence, we construct a deformation quantization of any derived Artin stack endowed with an n-shifted symplectic structure, as soon as $n \neq 0$. In particular we quantize many derived moduli spaces studied in [PTVV]. In a nutshell the results of this work are summarized as follows.

Main results A

1. *There exists a meaningful notion of* n-shifted Poisson structures *on derived Artin stacks locally of finite presentation, which recovers the usual notion of Poisson structures on smooth schemes when* $n = 0$.

2. *For a given derived Artin stack* X*, the space of* n*-shifted symplectic structures on* X *is naturally equivalent to the space of non-degenerate* n*-shifted Poisson structures on* X*.*

3. *Let* X *be any derived Artin stack locally of finite presentation endowed with an* n*-shifted Poisson structure* π*. For* $n \neq 0$ *there exists a canonical deformation quantization of* X *along* π*, realized as an* $E_{|n|}$*-monoidal* ∞*-category* $\mathsf{Perf}(X, \pi)$*, which is a deformation of the symmetric monoidal* ∞*-category* $\mathsf{Perf}(X)$ *of perfect complexes on* X*.*

As a corollary of these, we obtain the existence of deformation quantization of most derived moduli stacks studied in [PTVV], e.g. of the derived moduli of G-bundles on smooth and proper Calabi-Yau schemes, or the derived moduli of G-local systems on compact oriented topological manifolds. The existence of these deformation quantizations is a completely new result, which is a far reaching generalization of the construction of deformation quantization of character varieties heavily studied in topology, and provides a new world of quantized moduli spaces to explore in the future.

The above items are not easy to achieve. Some ideas of what *n-shifted Poisson structures* should be have been available in the literature for a while (see [Me, To2, To3]), but up until now no general definition of n-shifted Poisson structures on derived Artin stacks existed outside of the rather restrictive case of Deligne-Mumford stacks. The fact that Artin stacks have affine covers only up to smooth submersions is an important technical obstacle which we have to deal with already when we define shifted Poisson structures in this general setting. Indeed, in contrast to differential forms, polyvectors do not pull-back along smooth morphisms, so the well understood definition in the affine setting (see [Me, To2]) can not be transplanted to an Artin stack without additional effort, and such a transplant requires a new idea. A different complication lies in the fact that the comparison between non-degenerate shifted Poisson structures and their symplectic counterparts requires keeping track of non-trivial homotopy coherences even in the case of an affine derived scheme. One reason for this is that non-degeneracy is only defined up to quasi-isomorphism, and so converting a symplectic structure into a Poisson structure by dualization can not be performed easily. Finally, the existence of deformation quantization requires the construction of a deformation of the globally defined ∞-category of perfect complexes on a derived Artin stack. These ∞-categories are of global nature, and their deformations are not easily understood in terms of local data.

In order to overcome the above mentioned technical challenges we introduce a new point of view on derived Artin stacks by developing tools and ideas from formal geometry in the derived setting. This new approach is one of the technical hearts of the paper, and we believe it will be an important general tool in derived geometry, even outside the applications to shifted Poisson and symplectic structures discussed in this work. The key new idea here is to understand a given derived Artin stack X by means of its various formal completions $\widehat{X}_x$, at all of its points x in a coherent fashion. For a smooth algebraic variety, this idea has been already used with great success in the setting of deformation quantization (see for instance [Fe, Ko1, Bez-Ka]), but the extension we propose here in the setting of derived Artin stacks is new. By [Lu2], the geometry of a given formal completion $\widehat{X}_x$ is controlled by a dg-Lie algebra, and our approach, in a way, rephrases many problems concerning derived Artin stacks in terms of dg-Lie algebras. In this work we explain how shifted symplectic and Poisson structures, as well as ∞-categories of perfect complexes, can be expressed in those terms. Having this formalism at our disposal is what makes our Main statement A accessible. The formalism essentially allows us to reduce the problem to statements concerning dg-Lie algebras over general base rings and their Chevalley complexes. The general formal geometry results we prove on the way are of independent interest and will be useful for many other questions related to derived Artin stacks.

Let us now discuss the mathematical content of the paper in more detail. To start with, let us explain the general strategy and the general philosophy developed all along this manuscript. For a given derived Artin stack X, locally of finite presentation over a base commutative ring k of characteristic 0, we consider the corresponding de Rham stack X_{DR} of [Si1, Si2]. As an ∞-functor on commutative dg-algebras, X_{DR} sends A to $X(A_{red})$, the A_{red}-points of X (where A_{red} is defined to be the reduced ordinary commutative ring $\pi_0(A)_{red}$). The natural projection $\pi : X \longrightarrow X_{DR}$ realizes X as a family of formal stacks over X_{DR}: the fiber of π at a given point $x \in X_{DR}$, is the formal completion $\widehat{X}_x$ of X at x. By [Lu2] this formal completion is determined by a dg-Lie algebra l_x. However, the dg-Lie algebra l_x itself does not exist globally as a sheaf of dg-Lie algebras over X_{DR}, simply because its underlying complex is $\mathbb{T}_X[-1]$, the shifted tangent complex of X, which in general does not have a flat connection and thus does not descend to X_{DR}. However, the Chevalley complex of l_x, viewed as a graded mixed commutative dg-algebra (*cdga* for short) can be constructed as a global object $\mathcal{B}_X$ over X_{DR}. To be more precise we construct $\mathcal{B}_X$ as the derived de Rham complex of the natural inclusion $X_{red} \longrightarrow X$, suitably sheafified over X_{DR}. One of the key insights of this work is the following result, expressing global geometric objects on X as sheafified notions on X_{DR} related to $\mathcal{B}_X$.

Main results B *With the notation above:*

1. *The ∞-category $\mathsf{Perf}(X)$, of perfect complexes on X, is naturally equivalent, as a symmetric monoidal ∞-category, to the ∞-category of perfect sheaves of graded mixed $\mathcal{B}_X$-dg-modules on X_{DR}:*
$$\mathsf{Perf}(X) \simeq \mathcal{B}_X - Mod^{\mathsf{Perf}}_{\epsilon-\mathbf{dg}^{gr}}.$$
2. *There is an equivalence between the space of n-shifted symplectic structures on X, and the space of closed and non-degenerate 2-forms on the sheaf of graded mixed cdgas $\mathcal{B}_X$.*

The results above state that the geometry of X is largely recovered from X_{DR} together with the sheaf of graded mixed cdgas $\mathcal{B}_X$, and that the assignment $X \mapsto (X_{DR}, \mathcal{B}_X)$ behaves in a faithful manner from the perspective of derived algebraic geometry. In the last part of the paper, we take advantage of this in order to define the deformation quantization problem for objects belonging to general categories over k. In particular, we study and quantize shifted Poisson structures on X, by considering compatible brackets on the sheaf $\mathcal{B}_X$. Finally, we give details for three relevant quantizations and compare them to the existing literature. The procedure of replacing X with $(X_{DR}, \mathcal{B}_X)$ is crucial for derived Artins stacks because it essentially reduces statements and notions to the case of a sheaf of graded mixed cdgas. As graded mixed cdgas can also be understood as cdgas endowed with an action of a derived group stack, this further reduces statements to the case of (possibly unbounded) cdgas, and thus to an affine situation.

Description of the paper.

In the first section, we start with a very general and flexible context for (relative) differential calculus. We introduce the *internal cotangent complex* $\mathbb{L}^{int}_A$ and *internal de Rham complex* $\mathbf{DR}^{int}(A)$ associated with a commutative algebra A in a good enough symmetric monoidal stable k-linear ∞-category $\mathcal{M}$ (see Section 1.1 and Section 1.2 for the exact assumptions we put on $\mathcal{M}$). The internal de Rham complex $\mathbf{DR}^{int}(A)$ is defined as a graded mixed commutative algebra in $\mathcal{M}$. Next we recall from [PTVV] and extend to our general context the spaces $\mathcal{A}^{p,cl}(A,n)$ of (closed) p-forms of degree n on A, as well as of the space $\mathsf{Symp}(A,n)$ of n-shifted symplectic forms on A. We finally introduce (see also [PTVV, Me, To2, To3]) the *object* $\mathbf{Pol}^{int}(A,n)$ of *internal n-shifted polyvectors on A*, which is a graded n-shifted Poisson algebra in $\mathcal{M}$. In particular, $\mathbf{Pol}^{int}(A,n)[n]$ is a graded Lie algebra object in $\mathcal{M}$. We recall from [Me] that the space $\mathsf{Pois}(A,n)$ of graded n-shifted Poisson structures on A is equivalent to the mapping space from $\mathbf{1}(2)[-1]$ to $\mathbf{Pol}^{int}(A,n+1)[n+1]$ in the ∞-category of graded Lie algebras in $\mathcal{M}$, and we thus obtain a reasonable definition of non-degeneracy for graded n-shifted Poisson structures. Here $\mathbf{1}(2)[-1]$ denotes the looping of the monoidal unit of $\mathcal{M}$ sitting in pure weight 2 (for the grading). We finally show that

Corollary 1.5.5 *If $\mathbb{L}_A^{int}$ is a dualizable A-module in $\mathcal{M}$, then there is natural morphism*

$$\mathsf{Pois}^{nd}(A,n) \longrightarrow \mathsf{Symp}(A,n)$$

from the space $\mathsf{Pois}^{nd}(A,n)$ of non-degenerate n-shifted Poisson structures on A to the space $\mathsf{Symp}(A,n)$ of n-shifted symplectic structures on A.

We end the first part of the paper by a discussion of what happens when $\mathcal{M}$ is chosen to be the ∞-category $\epsilon-(k-mod)^{gr}$ of graded mixed complexes, which will be our main case of study in order to deal with the sheaf $\mathcal{B}_X$ on X_{DR} mentioned above. We then describe two lax symmetric monoidal functors $|-|, |-|^t : \epsilon-\mathcal{M}^{gr} \to \mathcal{M}$, called standard realization and Tate realization. We can apply the Tate realization to all of the previous internal constructions and get in particular the notions of Tate n-shifted symplectic form and non-degenerate Tate n-shifted Poisson structure. We prove that, as before, these are equivalent as soon as $\mathbb{L}_A^{int}$ is a dualizable A-module in $\mathcal{M}$.

One of the main difficulties in dealing with n-shifted polyvectors (and thus with n-shifted Poisson structures) is that they do not have sufficiently good functoriality properties. Therefore, in contrast with the situation with forms and closed forms, there is no straightforward and easy global definition of n-shifted polyvectors and n-shifted Poisson structures. Our strategy is to use ideas from formal geometry and define an n-shifted Poisson structure on a derived stack X as a flat family of n-shifted Poisson structures on the family of all formal neighborhoods of points in X. The main goal of the second part of the paper is to make sense of the previous sentence for general enough derived stacks, i.e. for locally almost finitely presented derived Artin stacks over k. In order to achieve this, we develop a very general theory of *derived formal localization* that will be certainly very useful in other applications of derived geometry, as well.

We therefore start the second part by introducing various notions of formal derived stacks: formal derived stack, affine formal derived stack, good formal derived stack over A, and perfect formal derived stack over A. It is important to note that if X is a derived Artin stack, then

- the formal completion $\widehat{X}_f : X \times_{X_{DR}} F_{DR}$ along any map $f : F \to X$ is a formal derived stack.
- the formal completion $\widehat{X}_x$ along a point $x : \mathbf{Spec}(A) \to X$ is an affine formal derived stack.
- each fiber $X \times_{X_{DR}} \mathbf{Spec}(A)$ of $X \to X_{DR}$ is a good formal derived stack over A, which is moreover perfect if X is locally of finite presentation.

Our main result here is the following

Theorem 2.2.2 *There exists an ∞-functor $\mathbb{D}$ from affine formal derived stacks to commutative algebras in $\mathcal{M} = \epsilon - (k - mod)^{gr}$, together with a conservative ∞-functor*

$$\phi_X : \mathsf{QCoh}(X) \to \mathbb{D}(X) - mod_{\mathcal{M}},$$

which becomes fully faithful when restricted to perfect modules.

Therefore, $\mathsf{Perf}(X)$ is identified with a full sub-∞-category $\mathbb{D}(X) - mod_{\mathcal{M}}^{perf}$ of $\mathbb{D}(X) - mod_{\mathcal{M}}$ that we explicitly determine. We then prove that the de Rham theories of X and of $\mathbb{D}(X)$ are equivalent for a perfect algebraisable formal derived stack over A. Namely we show that:

$$\mathbf{DR}\big(\mathbb{D}(X)/\mathbb{D}(\mathbf{Spec}\, A)\big) \simeq \mathbf{DR}^t\big(\mathbb{D}(X)/\mathbb{D}(\mathbf{Spec}\, A)\big) \simeq \mathbf{DR}(X/\mathbf{Spec}\, A)$$

as commutative algebras in $\epsilon - (A - mod)^{gr}$. We finally extend the above to the case of families $X \to Y$ of algebraisable perfect formal derived stacks, i.e. families for which every fiber $X_A := X \times_Y \mathbf{Spec}\, A \to \mathbf{Spec}\, A$ is an algebraisable perfect formal derived stack. We get an equivalence of symmetric monoidal ∞-categories $\phi_X : \mathsf{Perf}(X) \simeq \mathbb{D}_{X/Y} - mod_{\mathcal{M}}^{perf}$ as well as equivalences:

$$\Gamma\Big(Y, \mathbf{DR}\big(\mathbb{D}_{X/Y}/\mathbb{D}(Y)\big)\Big) \simeq \Gamma\Big(Y, \mathbf{DR}^t\big(\mathbb{D}_{X/Y}/\mathbb{D}_Y\big)\Big) \simeq \mathbf{DR}(X/Y)$$

of commutative algebras in $\mathcal{M}$

In particular, whenever $Y = X_{DR}$, we get a description of the de Rham (graded mixed) algebra $\mathbf{DR}(X) \simeq \mathbf{DR}(X/X_{DR})$ by means of the global sections of the relative Tate de Rham (graded mixed) algebra $\mathcal{B}_X := \mathbb{D}_{X/X_{DR}}$ over $\mathbb{D}_{X_{DR}}$. Informally speaking, we prove that a (closed) form on X is a float family of (closed) forms on the family of all formal completions of X at various points.

The above justifies the definitions of shifted polyvector fields and shifted Poisson structures that we introduce in the third part of the paper. Namely, the n-shifted Poisson algebra $\mathbf{Pol}(X/Y, n)$ of n-shifted polyvector fields on a family of algebraisable perfect formal derived stacks $X \to Y$ is defined to be

$$\Gamma\big(Y, \mathbf{Pol}^t(\mathbb{D}_{X/Y}/\mathbb{D}_X, n)\big)$$

The space of n-shifted Poisson structures $\mathsf{Pois}(X/Y, n)$ is then defined as the mapping space from $k(2)[-1]$ to $\mathbf{Pol}(X/Y, n+1)[n+1]$ in the ∞-category of graded Lie algebras in $\mathcal{M}$. Following the affine case treated in the first part (see also [Me]), we again prove that this is equivalent to the space of $\mathbb{D}_Y$-linear n-shifted Poisson algebra structures on $\mathbb{D}_{X/Y}$. We then prove[1] the following

[1]Recently J. Pridham proved this comparison theorem for derived Deligne-Mumford stacks by a different approach [Pri].

Theorem 3.2.4 *The subspace of non-degenerate elements in* $\mathsf{Pois}(X,n) := \mathsf{Pois}(X/X_{DR}, n)$ *is equivalent to* $\mathsf{Symp}(X,n)$ *for any derived Artin stack that is locally of finite presentation.*

We conclude the third Section by defining the deformation quantization problem for n-shifted Poisson structures, whenever $n \geq -1$. For every such n, we consider a $\mathbb{G}_m$-equivariant $\mathbb{A}^1_k$-family of k-dg-operads $\mathbb{BD}_{n+1}$ such that its 0-fiber is the Poisson operad $\mathbb{P}_{n+1}$ and its generic fiber is the k-dg-operad $\mathbb{E}_{n+1}$ of chains of the little $(n+1)$-disk topological operad. The general deformation quantization problem can then be stated as follows:

Deformation Quantization Problem. *Given a* $\mathbb{P}_{n+1}$*-algebra stucture on an object, does it exist a family of* $\mathbb{BD}_{n+1}$*-algebra structures such that its* 0*-fiber is the original* $\mathbb{P}_{n+1}$*-algebra structure?*

Let X be a derived Artin stack locally of finite presentation over k, and equipped with an n-shifted Poisson structure. Using the formality of $\mathbb{E}_{n+1}$ for $n \geq 1$, we can solve the deformation quantization problem for the $\mathbb{D}_{X_{DR}}$-linear $\mathbb{P}_{n+1}$-algebra structure on $\mathcal{B}_X$. This gives us, in particular, a $\mathbb{G}_m$-equivariant 1-parameter family of $\mathbb{D}_{X_{DR}}$-linear $\mathbb{E}_{n+1}$-algebra structures on $\mathcal{B}_X$. Passing to perfect modules we get a 1-parameter deformation of $\mathsf{Perf}(X)$ as an $\mathbb{E}_n$-monoidal ∞-category, which we call the *n-quantization of X*.

We work out three important examples in some details:

- the case of an n-shifted symplectic structure on the formal neighborhood of a k-point in X: we recover Markarian's Weyl n-algebra from [Mar].
- the case of those 1-shifted Poisson structure on BG that are given by elements in $\wedge^3(\mathfrak{g})^{\mathfrak{g}}$: we obtain a deformation, as a monoidal k-linear category, of the category $\mathsf{Rep}^{fd}(\mathfrak{g})$ of finite dimensional representations of $\mathfrak{g}$.
- the case of 2-shifted Poisson structures on BG, given by elements in $Sym^2(\mathfrak{g})^{\mathfrak{g}}$: we obtain a deformation of $\mathsf{Rep}^{fd}(\mathfrak{g})$ as a braided monoidal category.

Finally, Appendices A and B contains some technical results used in Sections 1 and 3, respectively.

Further directions and future works. In order to finish this introduction, let us mention that the present work does not treat some important questions related to quantization, which we hope to address in the future. For instance, we introduce a general notion of coisotropic structures for maps with an n-shifted Poisson target, analogous to the notion of Lagrangian structures from [PTVV]. However, the definition itself requires a certain additivity theorem, whose proof has been announced recently by N. Rozenblyum but is not available yet. Also, we did not address the question of comparing Lagrangian structure and co-isotropic structures that would be a relative version of our comparison between shifted symplectic and non-degenerate Poisson structures. Neither did we address the question of quantization of coisotropic structures. In a different direction, our deformation quantizations are only constructed under the restriction $n \neq 0$. The case $n = 0$ is presently being investigated, but at the moment is still open. In the same spirit, when $n = -1$ and $n = -2$, deformation quantization admits an interpretation different from our construction (see for example [To3, Section 6.2]). We believe that our present formal geometry approach can also be applied to these two specific cases.

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Notation.

- Throughout this paper k will denote a Noetherian commutative $\mathbb{Q}$-algebra.
- We will use $(\infty, 1)$-categories ([Lu1]) as our model for ∞-categories. They will be simply called ∞-categories.
- For a model category N, we will denote by $L(N)$ the ∞-category defined as the homotopy coherent nerve of the Dwyer-Kan localization of N along its weak equivalences.
- The ∞-category $\mathcal{T} := L(\mathbf{sSets})$ will be called the ∞-category of *spaces*.
- All symmetric monoidal categories we use will be symmetric monoidal (bi)closed categories.
- $\mathbf{cdga}_k$ will denote the ∞-category of non-positively graded differential graded k-algebras (with differential increasing the degree). For $A \in \mathbf{cdga}_k$, we will write $\pi_i\, A = H^{-i}(A)$ for any $i \geq 0$.

- For $A \in \mathbf{cdga}_k$, we will write either $\mathrm{L}(A)$ or $\mathrm{L}_{\mathrm{QCoh}}(A)$ for the ∞-category of A-dg-modules
- For $A \in \mathbf{cdga}_k$, we will denote by $\mathrm{L}_{\mathsf{Perf}}(A)$ the full sub-∞-category of $\mathrm{L}(A)$ consisting of perfect A-dg-modules.
- If X is a derived geometric stack, we will denote by either $\mathsf{QCoh}(X)$ or $\mathrm{L}_{\mathsf{QCoh}}(X)$ the k-linear symmetric monoidal dg-category of quasi-coherent complexes on X.
- If X is a derived geometric stack, we will denote by either $\mathsf{Perf}(X)$ or $\mathrm{L}_{\mathsf{Perf}}(X)$ the symmetric monoidal sub-dg-category of $\mathsf{QCoh}(X)$ consisting of dualizable objects.
- If X is a derived geometric stack, we will denote by either $\mathsf{Coh}(X)$ or $\mathrm{L}_{\mathrm{Coh}}(X)$ the full sub-dg category of $\mathsf{QCoh}(X)$ consisting of complexes whose cohomology sheaves are coherent over the truncation $\mathsf{t}_0 X$.
- For a derived stack X, $\Gamma(X, -)$ will always denote the *derived* functor of global sections on X (i.e. hypercohomology).

1 Relative differential calculus

In this section we describe the basics of differential calculus *inside* any reasonable k-linear symmetric monoidal ∞-category. In particular, we introduce cotangent complexes, De Rham mixed dg-algebras, shifted (closed) forms and polyvectors, and two different *realizations* (standard and Tate) of such objects over k.

1.1 Model categories setting

Let k be a Noetherian commutative $\mathbb{Q}$-algebra, and let $C(k) = \mathbf{dg}_k$ be the category of (unbounded, cochain) k-dg-modules. We endow $C(k)$ with its standard model category structure whose equivalences are quasi-isomorphisms and whose fibrations are epimorphisms ([Hov, Theorem 2.3.11]). The natural tensor product $-\otimes_k -$ of dg-modules endows $C(k)$ with the structure of a symmetric monoidal model category ([Hov, Proposition 4.2.13]). As a monoidal model category $C(k)$ satisfies the monoid axiom of [SS, Definition 3.3], and moreover, since k is a $\mathbb{Q}$-algebra, $C(k)$ is freely-powered in the sense of [Lu6, Definition 4.5.4.2].

Suppose next that M is a symmetric monoidal model category that is combinatorial as a model category ([Lu1, Definition A.2.6.1]). Assume furthermore that M admits a $C(k)$-enrichment (with tensor and cotensor) compatible with both the model and the monoidal structures, i.e. M is a symmetric monoidal $C(k)$-model algebra as in [Hov, Definition 4.2.20]. As a consequence (see our Proposition A.1.1) such an M is a *stable* model category, i.e. it is pointed and the suspension functor is a self equivalence of its homotopy category.
All along this first section, and as a reference for the rest of the paper, we make the following standing assumptions on M

1. The unit $\mathbf{1}$ is a cofibrant object in M.

2. For any cofibration $j : X \to Y$ in M, any object $A \in M$, and for any morphism $u : A \otimes X \to C$ in M the push-out square in M
$$\begin{tikzcd} C \arrow[r] & D \\ A \otimes X \arrow[u, "u"] \arrow[r, "id\otimes j"'] & A \otimes Y, \arrow[u] \end{tikzcd}$$
is a homotopy push-out square.

3. For a cofibrant object $X \in M$, the functor $X \otimes - : M \longrightarrow M$ preserves equivalences (i.e. *cofibrant objects in M are $\otimes$-flat*).

4. M is a *tractable* model category, i.e. there are generating sets of cofibrations I, and trivial cofibrations J in M with cofibrant domains.

5. Equivalences are stable under filtered colimits and finite products in M.

We note that conditions $(2)-(5)$ together imply that M satisfies the monoid axiom of [SS, Definition 3.3] In particular ([SS, Theorem 4.1 (2)]), for any commutative monoid $A \in Comm(M)$, the category of A-modules in M, denoted by $A-Mod_M$, is endowed with the structure of a symmetric monoidal combinatorial model category, for which the equivalences and fibrations are defined in M, and it again satisfies the monoid axiom. Moreover, $A-Mod_M$ comes with an induced compatible $C(k)$-enrichment (with tensor and cotensor). Moreover, as shown in Proposition A.1.3, the conditions $(2)-(5)$ on M imply that if $A \longrightarrow B$ is an equivalence in $Comm(M)$, then the induced restriction-extension of scalars Quillen adjunction

$$A - Mod_M \longleftrightarrow B - Mod_M$$

is a Quillen equivalence.

As k is a $\mathbb{Q}$-algebra, M is itself a $\mathbb{Q}$-linear category. This implies that M is freely-powered in the sense of [Lu6, Definition 4.5.4.2], since quotients by finite group actions are split epimorphisms in characteristic 0. As a consequence, the category $Comm(M)$ of commutative and unital monoids in M, is again a combinatorial model category for which the equivalences and fibrations are defined via the forgetful functor to M, and whose generating (trivial) cofibrations are given by $\mathrm{Sym}(I)$ (respectively, $\mathrm{Sym}(J)$), where I (respectively J) are generating (trivial) cofibrations in M ([Lu6, Proposition 4.5.4.6]).

Let B be a k-linear commutative and cocommutative Hopf dg-algebra. We let $B-\mathbf{codg}_M$ be the category of B-comodules in M, i.e. the category whose

- objects are objects P in M equipped with a morphism $\rho_P : P \to P \otimes_k B$ in M ($\otimes_k : M \times C(k) \to M$ being the tensor product given by the $C(k)$-enrichment[2]) satisfying the usual identities

$$(\rho_P \otimes_k \mathrm{id}_B) \circ \rho_P = (\mathrm{id}_P \otimes_k \Delta_B) \circ \rho_P$$
$$(\mathrm{id}_P \otimes_k \varepsilon_B) \circ \rho_P = \mathrm{id}_P$$

  where Δ_B (respectively ε_B) denotes the comultiplication (respectively the counit) of B, and we have implicitly identified P with $P \otimes k$ via the obvious M-isomorphism $P \otimes_k k \to P$;

- morphisms are given by M-morphisms commuting with the structure maps ρ.

The category $B-\mathbf{codg}_M$ comes equipped with a left adjoint forgetful functor $B-\mathbf{codg}_M \longrightarrow M$, whose right adjoint sends an object $X \in M$ to $X \otimes B$ endowed with its natural B-

[2]Note that this slightly abusive notation for the tensor enrichment $\otimes_k := \otimes_{C(k)}$ is justified by the fact that the properties of the enrichment give a canonical isomorphism $P \otimes_{C(k)} (B \otimes_k B) \simeq (P \otimes_{C(k)} B) \otimes_{C(k)} B$.

comodule structure. The multiplication in B endows $B-\mathbf{codg}_M$ with a natural symmetric monoidal structure for which the forgetful functor $B-\mathbf{codg}_M \longrightarrow M$ becomes a symmetric monoidal functor.

We will be especially interested in the case where $B = k[t,t^{-1}] \otimes_k k[\epsilon]$ defined as follows. Here $k[\epsilon] := Sym_k(k[1])$ is the free commutative k-dg-algebra generated by one generator ϵ in cohomological degree -1, and $k[t,t^{-1}]$ is the usual commutative algebra of functions on $\mathbb{G}_m$ (so that t sits in degree 0). The comultiplication on B is defined by the dg-algebra map

$$\begin{aligned}
\Delta_B : \qquad & B \longrightarrow B \otimes_k B \\
& t \equiv t \otimes 1 \longmapsto (t \otimes 1) \otimes (t \otimes 1) \equiv t \otimes t \\
& \epsilon \equiv 1 \otimes \epsilon \longmapsto (1 \otimes \epsilon) \otimes (1 \otimes 1) + (t \otimes 1) \otimes (1 \otimes \epsilon) \equiv \epsilon \otimes 1 + t \otimes \epsilon
\end{aligned}$$

where $\equiv$ is used as a concise, hopefully clear notation for canonical identifications. Together with the counit dg-algebra map

$$\varepsilon_B : B \longrightarrow k \ , \ t \mapsto 1 \, , \, \epsilon \mapsto 0 \, ,$$

B becomes a commutative and cocommutative k-linear Hopf dg-algebra.

Remark 1.1.1 Note that B can be identified geometrically with the dg-algebra of functions on the affine group stack $\mathbb{G}_m \ltimes \Omega_0\mathbb{G}_a$, semi-direct product of $\mathbb{G}_m$ with $\Omega_0\mathbb{G}_a = K(\mathbb{G}_a, -1) = \mathbb{G}_a[-1]$ induced by the natural action of the multiplicative group on the additive group. This is similar to [PTVV, Remark 1.1] where we used the algebra of functions on $\mathbb{G}_m \ltimes B\mathbb{G}_a = \mathbb{G}_m \ltimes \mathbb{G}_a[1]$ instead. In fact these two Hopf dg algebras have equivalent module theories and can be used interchangeably (see Remark 1.1.2).

The category of B-comodules $B-\mathbf{codg}_{C(k)}$ identifies naturally with the category of *graded mixed complexes of k-dg-modules*. Its objects consist of families of k-dg-modules $\{E(p)\}_{p\in\mathbb{Z}}$, together with families of morphisms

$$\epsilon : E(p) \longrightarrow E(p+1)[1],$$

such that $\epsilon^2 = 0$. The identification, actually an isomorphism of categories, is made by observing that the co-restriction functor

$$p_* : B-\mathbf{codg}_{C(k)} \to k[t,t^{-1}]-\mathbf{codg}_{C(k)}$$

along the coalgebra map $p : B \to k[t,t^{-1}]$ (sending ϵ to 0), yields the usual $C(k)$-isomorphism $\oplus_{p\in\mathbb{Z}} E(p) \to E$, where

$$E(p) := \rho_{p_*E}^{-1}(E \otimes_k k \cdot t^p) \, , \, p \in \mathbb{Z}$$

or, equivalently,

$$E(p) := \rho_E^{-1}(E \otimes_k (k \cdot t^p \oplus k[t,t^{-1}]\epsilon) \, , \, p \in \mathbb{Z}.$$

Note that the morphism $\epsilon : E(p) \longrightarrow E(p+1)[1]$ is then defined by sending $x_i \in E(p)^i$ to the image of x_i under the composite map

$$E \xrightarrow{\rho} E \otimes_k B \xrightarrow{\mathrm{pr}} E \otimes_k k \cdot t^{p+1}\epsilon \ .$$

Therefore, objects in $B-\mathbf{codg}_{C(k)}$ will be often simply denoted by $E = \oplus_p E(p)$, and the corresponding mixed differential by ϵ.

In order to avoid confusion, we will refer to the decomposition $E = \oplus_p E(p)$ as the *weight decomposition*, and refer to p as the *weight degree* in order to distinguish it from the *cohomological* or *internal degree.*

Remark 1.1.2 Note that here we have adopted a convention opposite to the one in [PTVV, 1.1]: the category $B - \mathbf{codg}_{C(k)}$ of graded mixed complexes introduced above, is naturally equivalent to the category of graded mixed complexes used in [PTVV, 1.1] where the mixed structures decrease the cohomological degrees by one. An explicit equivalence is given by sending an object $\oplus_p E(p)$ in $B-\mathbf{codg}_{C(k)}$ to $\oplus_p(E(p)[2p])$ together with its natural induced mixed structure (which now decreases the cohomological degree by 1).

More generally, the category of *graded mixed objects in* M is defined to be $B-\mathbf{codg}_M$, the category of B-comodules in M, with $B = k[t, t^{-1}] \otimes_k k[\epsilon]$, and will be denoted by $\epsilon - M^{gr}$. Its objects consist of

- $\mathbb{Z}$-families $\{E(p)\}_{p\in\mathbb{Z}}$ of objects of M,
- together with morphisms in M

$$\epsilon \equiv \{\epsilon_p : E(p) \longrightarrow E(p+1)[1]\}_{p\in\mathbb{Z}} \, ,$$

  where for $P \in M$ and $n \in \mathbb{Z}$ we define $P[n] := P^{k[-n]}$ using the (cotensored) $C(k)$-enrichment, and we require that $\epsilon^2 = 0$, i.e. that the composition

$$E(p) \xrightarrow{\epsilon_p} E(p+1)[1] \xrightarrow{\epsilon_{p+1}[1]} E(p+2)[2]$$

  is zero for any $p \in \mathbb{Z}$.

Note that, by adjunction, ϵ_p can also be specified by giving a map $E(p) \otimes_k k[-1] \to E(p+1)$ in M or, equivalently, a map $k[1] \to \underline{\mathrm{Hom}}(E(p), E(p+1))$ in $C(k)$, (where $\underline{\mathrm{Hom}}$ denotes the $C(k)$-enriched hom in M). The morphisms ϵ will sometimes be called *mixed maps* or *mixed differentials*, following the analogy with the case $M = C(k)$.

The category $M^{gr} := \prod_{p\in\mathbb{Z}} M$ is naturally a symmetric monoidal model category with weak equivalences (respectively cofibrations, respectively fibrations) defined component-wise, and a monoidal structure defined by

$$(E \otimes E')(p) := \bigoplus_{i+j=p} E(i) \otimes E'(j)$$

where $\oplus$ denotes the coproduct in M, and the symmetry constraint *does not* involve signs, and simply consists in exchanging the two factors in $E(i) \otimes E'(j)$. It is easy to check, using our standing assumptions $(1) - (5)$ on M, that $\epsilon - M^{gr}$ comes equipped with a combinatorial symmetric monoidal model category structure for which the equivalences and cofibrations are defined through the forgetful functor

$$\epsilon - M^{gr} \longrightarrow M^{gr}.$$

Again the symmetric monoidal structure on $\epsilon - M^{gr}$ can be described on the level of graded objects by the formula $(E \otimes E')(p) := \oplus_{i+j=p} E(i) \otimes E'(j)$ where $\oplus$ denotes the coproduct in M, and again the symmetry constraint *does not* involve signs, and simply consist of the exchange of the two factors in $E(i) \otimes E'(j)$. The mixed differentials on $E \otimes E'$ are then defined by the usual formula, taking the sums (i.e. coproducts) of all maps

$$\begin{array}{rl} \epsilon \otimes 1 + 1 \otimes \epsilon' : \quad E(i) \otimes E'(j) \longrightarrow & (E(i+1)[1] \otimes E'(j)) \bigoplus (E(i) \otimes E'(j+1)[1]) \\ & \qquad\qquad\qquad \| \\ & ((E(i+1) \otimes E'(j)) \bigoplus (E(i) \otimes E'(j+1)))\,[1] \end{array}$$

As a symmetric monoidal model category $\epsilon - M^{gr}$ again satisfies all of our standing assumptions $(1) - (5)$, and the forgetful functor $\epsilon - M^{gr} \longrightarrow M^{gr}$ comes equipped with a natural symmetric monoidal structure.

Note that $\epsilon - M^{gr}$ is also an $\epsilon - C(k)^{gr}$-enriched symmetric monoidal model category. Let us just briefly define the graded mixed complex $\underline{\mathrm{Hom}}_{\epsilon}^{gr}(E, F)$, for $E, F \in \epsilon - M^{gr}$, leaving the other details and properties of this enrichment to the reader. We define

- $\underline{\mathrm{Hom}}_{\epsilon}^{gr}(E, F)(p) := \prod_{q\in\mathbb{Z}} \underline{\mathrm{Hom}}_k(E(q), F(q+p))$, for any $p \in \mathbb{Z}$
- the mixed differential $\epsilon_p : \underline{\mathrm{Hom}}_{\epsilon}^{gr}(E, F)(p) \to \underline{\mathrm{Hom}}_{\epsilon}^{gr}(E, F)(p+1)[1]$ as the map whose q-component

$$\prod_{q'\in\mathbb{Z}} \underline{\mathrm{Hom}}_k(E(q'), F(q'+p)) \longrightarrow \underline{\mathrm{Hom}}_k(E(q), F(p+q+1))[1] \simeq \underline{\mathrm{Hom}}_k(E(q), F(p+q+1)^{k[-1]}$$

is given by the sum $\alpha + \beta$ where

$$\begin{tikzcd} \prod_{q' \in \mathbb{Z}} \underline{\mathrm{Hom}}_k(E(q'), F(q'+p)) \arrow[r, "\mathrm{pr}"] \arrow[dr, "\alpha"'] & \underline{\mathrm{Hom}}_k(E(q), F(q+p)) \arrow[d, "\alpha'"] \\ & \underline{\mathrm{Hom}}_k(E(q), F(p+q+1))[1] \end{tikzcd}$$

α' being adjoint to the composite

$$\underline{\mathrm{Hom}}_k(E(q), F(q+p)) \otimes E(q) \xrightarrow{\mathrm{can}} F(q+p) \xrightarrow{\epsilon_F} F(q+p+1)^{k[-1]},$$

and

$$\begin{tikzcd} \prod_{q' \in \mathbb{Z}} \underline{\mathrm{Hom}}_k(E(q'), F(q'+p)) \arrow[r, "\mathrm{pr}"] \arrow[dr, "\beta"'] & \underline{\mathrm{Hom}}_k(E(q+1), F(q+1+p)) \arrow[d, "\beta'"] \\ & \underline{\mathrm{Hom}}_k(E(q), F(p+q+1))[1] \end{tikzcd}$$

β' being adjoint to the composite

$$\begin{tikzcd} \underline{\mathrm{Hom}}_k(E(q+1), F(q+1+p)) \otimes_k (E(q) \otimes_k k[-1]) \arrow[d, "id \otimes \epsilon_E"'] & \\ \underline{\mathrm{Hom}}_k(E(q+1), F(q+p+1)) \otimes_k E(q+1) \arrow[r, "\mathrm{can}"] & F(q+p+1). \end{tikzcd}$$

Therefore, as already observed for M, the category $Comm(\epsilon - M^{gr})$, of commutative and unital monoids in graded mixed objects in M, is a combinatorial model category whose weak equivalences and fibrations are defined through the forgetful functor $Comm(\epsilon - M^{gr}) \longrightarrow \epsilon - M^{gr}$ ([Lu6, Proposition 4.5.4.6]).

1.2 ∞-Categories setting

We will denote by $\mathcal{M} := L(M)$ the ∞-category obtained from M by inverting the equivalences (see [To2, §2.1]). Since M is a stable model category (Proposition A.1.1), $\mathcal{M}$ is automatically a stable ∞-category. Moreover, as explained in [To-Ve-1, §2.1], $\mathcal{M}$ possesses a natural induced symmetric monoidal structure. An explicit model for $\mathcal{M}$ is the simplicial category of fibrant and cofibrant objects in M, where the simplicial sets of morphisms are defined by applying the Dold-Kan construction to the truncation in non-negative degrees of the complexes of morphisms coming from the $C(k)$-enrichment (see [Tab]). The symmetric monoidal structure on $\mathcal{M}$ is harder to describe explicitly, and we will not discuss it here since it will not be used in an essential way in what follows. Parallel results hold for $\mathcal{M}^{gr} := L(M^{gr})$. We refer to [To-Ve-1, §2.1] for more about localization of symmetric monoidal model categories.

We recall from Section 1.1 that $Comm(M)$ is the model category of commutative monoids in M, and we let

$$\mathbf{cdga}_{\mathcal{M}} := L(Comm(M)),$$

to be the ∞-category obtained by localizing $Comm(M)$ along weak equivalences. Note that our notation suggests that $\mathbf{cdga}_{\mathcal{M}}$ is the ∞-category of *commutative dg-algebras internal to* $\mathcal{M}$ in the sense of [Lu6]. This is justified by the existence of a natural equivalence of ∞-categories

$$L(Comm(M)) \simeq Comm(LM).$$

This equivalence is a consequence of [Lu6, Theorem 4.5.4.7], since by Proposition A.1.4 the forgetful functor $Comm(M) \to M$ preserves fibrant-cofibrant objects.

The Quillen adjunction $\epsilon - M^{gr} \longleftrightarrow M^{gr}$ (see Section 1.1) induces an adjunction of ∞-categories $\epsilon - \mathcal{M}^{gr} = L(\epsilon - M^{gr}) \longleftrightarrow \mathcal{M}^{gr} := L(M^{gr})$.

Definition 1.2.1 *The symmetric monoidal ∞-category $\epsilon - \mathcal{M}^{gr}$ of* graded mixed objects in $\mathcal{M}$ *is defined as $\epsilon - \mathcal{M}^{gr} := L(\epsilon - M^{gr})$. The ∞-category $\epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}}$ of* graded mixed commutative dg-algebras in $\mathcal{M}$ *is defined as $\epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}} := L(Comm(\epsilon - M^{gr}))$.*

Note that, again, [Lu6, Theorem 4.5.4.7] and Proposition A.1.4 imply that we have a natural equivalence of ∞-categories

$$Comm(\epsilon - \mathcal{M}^{gr}) \simeq L(Comm(\epsilon - M^{gr})),$$

and so $\epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}}$ can also be considered as the ∞-category of commutative monoid objects in the symmetric monoidal ∞-category $\epsilon - \mathcal{M}^{gr}$. We have an adjunction of ∞-categories

$$\epsilon - \mathcal{M}^{gr} \longleftrightarrow \epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}},$$

where the right adjoint forgets the algebra structure.

At a more concrete level, objects in $\epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}}$ can be described as commutative monoids in $\epsilon - M^{gr}$, i.e. as the following collections of data

1. a family of objects $\{A(p) \in M\}_{p\in\mathbb{Z}}$.

2. a family of morphisms $\epsilon \equiv \{\epsilon_p : A(p) \longrightarrow A(p+1)[1]\}_{p\in\mathbb{Z}}$, satisfying $\epsilon^2 = 0$.

3. a family of multiplication maps

$$\{A(p) \otimes A(q) \longrightarrow A(p+q)\}_{(p,q)\in\mathbb{Z}\times\mathbb{Z}},$$

which are associative, unital, graded commutative, and compatible with the maps ϵ above.

Remark 1.2.2 Since $\mathcal{M}$ is stable, we have equivalences in $\mathcal{M}$

$$\Sigma X \simeq X \otimes_k k[1] \simeq X[1] = X^{k[-1]} \simeq \Omega^{-1} X$$

where the the tensor and cotensor products are to be understood in the ∞-categorical sense (i.e. in the derived sense when looking at M). These equivalences are natural in $X \in \mathcal{M}$. In particular there is no ambiguity about what $X[n]$ means in $\mathcal{M}$, for any $n \in \mathbb{Z}$: $X[n] \simeq X \otimes_k k[n] \simeq X^{k[-n]}$. Beware that these formulas are not correct, on the nose, in M, unless X is fibrant and cofibrant.

1.3 De Rham theory in a relative setting

Let M be a symmetric monoidal model category satisfying the conditions from Section 1.1. We denote the corresponding ∞-category by $\mathcal{M}$. As above we have the category $\epsilon - M^{gr}$ of graded mixed objects in M and the corresponding ∞-category $\epsilon - \mathcal{M}^{gr}$ of graded mixed objects in $\mathcal{M}$.

Since $\mathbf{1}_M$ is cofibrant in M, there is a natural Quillen adjunction

$$- \otimes \mathbf{1}_M : C(k) \longleftrightarrow M : \underline{Hom}(\mathbf{1}_M, -),$$

where the left adjoint sends an object $x \in C(k)$ to $x \otimes \mathbf{1} \in M$ (tensor enrichment of M over $C(k)$), while the right adjoint is given by the $C(k)$-hom enrichment. The induced adjunction on the corresponding ∞-categories will be denoted by

$$- \otimes \mathbf{1}_M : \mathbf{dg}_k = \mathrm{L}(k) \longleftrightarrow \mathcal{M} : |-| := \mathbb{R}\underline{Hom}(\mathbf{1}_M, -).$$

Since $\mathbf{1}_M$ is a comonoid object in M, the right Quillen functor $\underline{Hom}(\mathbf{1}_M, -)$ is lax symmetric monoidal. Therefore, we get similar adjunctions at the commutative monoids and graded mixed level (simply denoted through the corresponding right adjoints)

$$\mathbf{cdga}_k \longleftrightarrow \mathbf{cdga}_{\mathcal{M}} : |-|$$

$$\epsilon - \mathbf{cdga}_k \longleftrightarrow \epsilon - \mathbf{cdga}_{\mathcal{M}} : |-|$$

$$\epsilon - \mathbf{dg}_k^{gr} \longleftrightarrow \epsilon - \mathcal{M}^{gr} : |-|$$

$$\epsilon - \mathbf{cdga}_k^{gr} \longleftrightarrow \epsilon - \mathbf{cdga}_{\mathcal{M}}^{gr} : |-|$$

Definition 1.3.1 *The right adjoint ∞-functors $|-|$ defined above will be called the* realization ∞-functors*.*

Remark 1.3.2 Note that if $A \in \mathbf{cdga}_{\mathcal{M}}$ and $P \in A - Mod_{\mathcal{M}}$, then $|P| \in |A| - \mathbf{dg}_k$, and we get a refined realization functor

$$|-|: A - Mod_{\mathcal{M}} \longrightarrow |A| - \mathbf{dg}_k.$$

1.3.1 Cotangent complexes.

We start with the notion of a cotangent complex for a commutative dg-algebra inside $\mathcal{M}$. For $A \in \mathbf{cdga}_{\mathcal{M}}$ we have an ∞-category $A - Mod_{\mathcal{M}}$ of A-modules in $\mathcal{M}$. If the object A corresponds to $A \in Comm(M)$, the ∞-category $A - Mod_{\mathcal{M}}$ can be defined as the localization of the category $A - Mod_M$, of A-modules in M, along the equivalences. The model category $A - Mod_M$ is a stable model category and thus $A - Mod_{\mathcal{M}}$ is itself a presentable stable ∞-category. As A is commutative, $A - Mod_M$ is a symmetric monoidal category in a natural way, for the tensor product $- \otimes_A -$ of A-modules. This makes $A - Mod_M$ a symmetric monoidal model category which satisfies again the conditions $(1) - (5)$ (see Proposition A.1.2). The corresponding ∞-category $A - Mod_{\mathcal{M}}$ is thus itself a symmetric monoidal presentable and stable ∞-category.

For an A-module $N \in A - Mod_M$, we endow $A \oplus N$ with the trivial square zero structure, as in [HAG-II, 1.2.1]. We denoted the coproduct in M by $\oplus$; note however that since $A - Mod_M$ is stable, any finite coproduct is identified with the corresponding finite product. The projection $A \oplus N \to A$ defines an object $A \oplus N \in Comm(M)/A$, as well as an object in the comma ∞-category $A \oplus N \in \mathbf{cdga}_{\mathcal{M}}/A$ of commutative monoids in $\mathcal{M}$ augmented to A.

Definition 1.3.3 *In the notations above, the* space of derivations from A to N *is defined by*

$$Der(A, N) := \mathsf{Map}_{\mathbf{cdga}_{\mathcal{M}}/A}(A, A \oplus N) \in \mathcal{T}.$$

For a fixed $A \in \mathbf{cdga}_{\mathcal{M}}$, the construction $N \mapsto Der(A, N)$ can be naturally promoted to an ∞-functor

$$Der(A, -) : A - Mod_{\mathcal{M}} \longrightarrow \mathcal{T}.$$

Lemma 1.3.4 *For any $A \in A - Mod_{\mathcal{M}}$, the ∞-functor $Der(A, -)$ is corepresentable by an object $\mathbb{L}_A^{int} \in A - Mod_{\mathcal{M}}$.*

Proof: This is a direct application of [Lu1, Proposition 5.5.2.7], since $A - Mod_{\mathcal{M}}$ and $\mathcal{T}$ are both presentable ∞-categories, and the ∞-functor $Der(A, -)$ is accessible and commutes with small limits. $\square$

Definition 1.3.5 *Let $A \in \mathbf{cdga}_{\mathcal{M}}$.*

1. *The object $\mathbb{L}_A^{int} \in A - Mod_{\mathcal{M}}$ is called the* cotangent complex of A, internal to $\mathcal{M}$.

2. The absolute cotangent complex *(or simply* the cotangent complex *of A) is*

$$\mathbb{L}_A := |\mathbb{L}_A^{int}| \in \mathbf{dg}_k,$$

where $|-|:\mathcal{M} \longleftrightarrow \mathbf{dg}_k$ *is the realization* ∞*-functor of definition 1.3.1.*

Remark 1.3.6 Both $A - Mod_{\mathcal{M}}$ and $\mathbf{cdga}_{\mathcal{M}}/A$ are presentable ∞-categories, and the ∞-functor $N \mapsto A \oplus N$ is accessible and preserves limits, therefore ([Lu1, Corollary 5.5.2.9]) it admits a left adjoint $\mathbf{L}^{int} : \mathbf{cdga}_{\mathcal{M}}/A \to A - Mod_{\mathcal{M}}$, and we have $\mathbb{L}_A^{int} = \mathbf{L}^{int}(A)$.

The construction $A \mapsto \mathbb{L}_A^{int}$ possesses all standard and expected properties. For a morphism $A \longrightarrow B$ in $\mathbf{cdga}_{\mathcal{M}}$, we have an adjunction of ∞-categories

$$B \otimes_A - : A - Mod_{\mathcal{M}} \longleftrightarrow B - Mod_{\mathcal{M}} : \mathsf{forg}$$

where forg is the forgetful ∞-functor, and we have a natural morphism $B \otimes_A \mathbb{L}_A^{int} \longrightarrow \mathbb{L}_B^{int}$ in $B - Mod_{\mathcal{M}}$. The cofiber of this morphism, in the ∞-category $B - Mod_{\mathcal{M}}$, is denoted by $\mathbb{L}_{B/A}^{int}$, and is called the *relative cotangent complex of* $A \to B$ *internal to* $\mathcal{M}$. We have, by definition, a fibration-cofibration sequence of B-modules

$$B \otimes_A \mathbb{L}_A^{int} \longrightarrow \mathbb{L}_B^{int} \longrightarrow \mathbb{L}_{B/A}^{int}.$$

Moreover, the internal cotangent complex is compatible with push-outs in $\mathbf{cdga}_{\mathcal{M}}$, in the following sense. For a cocartesian square of objects in $\mathbf{cdga}_{\mathcal{M}}$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D, \end{tikzcd}$$

the induced square of objects in $B - Mod_{\mathcal{M}}$

$$\begin{tikzcd} D \otimes_A \mathbb{L}_A^{int} \arrow[r] \arrow[d] & D \otimes_B \mathbb{L}_B^{int} \arrow[d] \\ D \otimes_C \mathbb{L}_C^{int} \arrow[r] & \mathbb{L}_D^{int} \end{tikzcd}$$

is again cocartesian.

Remark 1.3.7 The above definition of an internal cotangent complex gives the usual cotangent complex of commutative dg-algebras A over k when one takes $M = C(k)$. More precisely, for $M = C(k)$, the ∞-functor $|-|$ is isomorphic to the forgetful functor $\mathsf{forg} : A - Mod \to C(k)$, and we have $\mathsf{forg}(\mathbb{L}_A^{int}) \simeq \mathbb{L}_A$ in $C(k)$.

1.3.2 De Rham complexes.

We have defined, for any object $A \in \mathbf{cdga}_{\mathcal{M}}$ a cotangent complex $\mathbb{L}_A^{int} \in A - Mod_{\mathcal{M}}$. We will now show how to associate to any $A \in \mathbf{cdga}_{\mathcal{M}}$ its de Rham complex. As for cotangent complexes we will have two versions, an internal de Rham complex $\mathbf{DR}^{int}(A)$, and an absolute one $\mathbf{DR}(A)$, respectively related to $\mathbb{L}_A^{int}$ and $\mathbb{L}_A$. The first version, $\mathbf{DR}^{int}(A)$ will be a graded mixed cdga in $\mathcal{M}$, whereas $\mathbf{DR}(A)$ will be a graded mixed cgda in $\mathbf{dg}_k$. These of course will be related by the formula

$$\mathbf{DR}(A) = |\mathbf{DR}^{int}(A)|$$

where $|-| : \mathcal{M} \longrightarrow \mathbf{dg}_k$ (or equivalently, $|-| : \epsilon - \mathbf{cdga}_{\mathcal{M}}^{gr} \longrightarrow \epsilon - \mathbf{cdga}_k^{gr}$) is the realization ∞-functor of Definition 1.3.1.

We recall from Section 1.2 that a mixed graded commutative dg-algebra A in $\mathcal{M}$ can be described as the following data

1. a family of objects $\{A(p) \in M\}_{p\in\mathbb{Z}}$.

2. a family of morphisms $\epsilon\ = \{\equiv \epsilon_p : A(p) \longrightarrow A(p+1)[1]\}_{p\in\mathbb{Z}}$, satisfying $\epsilon_{p+1}[1] \circ \epsilon_p = 0$.

3. a family of multiplication maps

$$\{A(p) \otimes A(q) \longrightarrow A(p+q)\}_{(p,q)\in\mathbb{Z}\times\mathbb{Z}},$$

   which are associative, unital, graded commutative, and compatible with the maps ϵ.

The (formal) decomposition $A = \oplus A(p)$ will be called the *weight decomposition*, and $A(p)$ the *weight p part* of A.

By point 3. above, for $A \in \epsilon - \mathbf{cdga}_{\mathcal{M}}^{gr}$, the weight 0 object $A(0) \in \mathcal{M}$ comes equipped with an induced commutative monoid structure and thus defines an object $A(0) \in \mathbf{cdga}_{\mathcal{M}}$. This defines an ∞-functor

$$(-)(0) : \epsilon - \mathbf{cdga}_{\mathcal{M}}^{gr} \longrightarrow \mathbf{cdga}_{\mathcal{M}}$$

which picks out the part of weight degree 0 only. The compatibility of the multiplication with the mixed structure ϵ expresses in particular that the property that the morphism $A(0) \longrightarrow A(1)[1]$ is a derivation of the commutative monoid $A(0)$ with values in $A(1)[1]$. We thus have a natural induced morphism in the stable ∞-category of $A(0)$-modules

$$\varphi_\epsilon : \mathbb{L}_{A(0)}^{int}[-1] \longrightarrow A(1).$$

Proposition 1.3.8 *The ∞-functor*

$$(-)(0) : \epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}} \longrightarrow \mathbf{cdga}_{\mathcal{M}},$$

has a left adjoint

$$\mathbf{DR}^{int} : \mathbf{cdga}_{\mathcal{M}} \longrightarrow \epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}}.$$

Proof. This is an application of the adjoint functor theorem ([Lu1, Corollary 5.5.2.9]). We just need to show that the ∞-functor $A \mapsto A(0)$ is accessible and preserves limits. For this, we use the commutative diagram of ∞-categories

$$\begin{tikzcd} \epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}} \arrow[r] \arrow[d] & \mathbf{cdga}_{\mathcal{M}} \arrow[d] \\ \epsilon - \mathcal{M}^{gr} \arrow[r] & \mathcal{M}, \end{tikzcd}$$

where the vertical ∞-functors forget the commutative monoid structures and the horizontal ∞-functors select the parts of weight 0. These vertical ∞-functors are conservative and commute with all limits. We are thus reduced to checking that the bottom horizontal ∞-functor $\epsilon - \mathcal{M}^{gr} \longrightarrow \mathcal{M}$ preserves limits. This last ∞-functor has in fact an explicit left adjoint, obtained by sending an object $X \in \mathcal{M}$, to the graded mixed object E defined by

$$E(0) = X \qquad E(1) = X[-1] \qquad E(i) = 0 \quad \forall i \neq 0, 1,$$

and with $\epsilon : E(0) \to E(1)[1]$ being the canonical isomorphism $X[-1][1] \simeq X$. $\square$

Definition 1.3.9 *Let $A \in \mathbf{cdga}_{\mathcal{M}}$ be a commutative dg-algebra in $\mathcal{M}$.*

1. *The* internal de Rham object of A *is the graded mixed commutative dg-algebra over $\mathcal{M}$ defined by*

$$\mathbf{DR}^{int}(A) \in \epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}}.$$

2. *The* absolute de Rham object of A *(or simply the* de Rham object*) is the graded mixed commutative dg-algebra over k defined by*

$$\mathbf{DR}(A) := |\mathbf{DR}^{int}(A)| \in \epsilon - \mathbf{cdga}^{gr}_{k}$$

where $|-| : \epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}} \longrightarrow \epsilon - \mathbf{cdga}^{gr}_{k}$ is the realization ∞-functor of Definition 1.3.1.

Remark 1.3.10 Abusing the language we will often refer to the de Rham objects $\mathbf{DR}^{int}(A)$ and $\mathbf{DR}(A)$ as the (internal or absolute) *de Rham complexes of* A, even though they are not just complexes but a rather objects of $\epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}}$ or of $\epsilon - \mathbf{cdga}^{gr}_{k}$.

We will also need the following

Definition 1.3.11 *Let $Comm(M)^{gr}$ be the category with objects $\mathbb{Z}$-indexed families $\{A(n)\}_{n\in\mathbb{Z}}$ of objects in $Comm(M)$, and morphisms $\mathbb{Z}$-indexed families $\{A(n) \to B(n)\}_{n\in\mathbb{Z}}$ of morphisms in $Comm(M)$.*

$Comm(M)^{gr}$ has a model structure with fibrations, weak equivalences (and cofibrations) defined levelwise. Its localization $L(Comm(M)^{gr})$ along weak equivalences will be denoted by $\mathbf{cdga}^{gr}_{\mathcal{M}}$ and called the ∞-category of graded (non-mixed) commutative dg-algebras in $\mathcal{M}$.

By definition, the de Rham object $\mathbf{DR}^{int}(A)$ comes equipped with an adjunction morphism $A \longrightarrow \mathbf{DR}^{int}(A)(0)$ in $\mathbf{cdga}_{\mathcal{M}}$. Moreover, the structure of a mixed graded cdga on $\mathbf{DR}^{int}(A)$ defines a derivation $\mathbf{DR}^{int}(A)(0) \longrightarrow \mathbf{DR}^{int}(A)(1)[1]$, and thus a canonical morphism in the ∞-category of $\mathbf{DR}^{int}(A)(0)$-modules

$$\mathbb{L}^{int}_{A} \otimes_A \mathbf{DR}^{int}(A)(0) \longrightarrow \mathbb{L}^{int}_{\mathbf{DR}^{int}(A)(0)} \longrightarrow \mathbf{DR}^{int}(A)(1)[1].$$

Note that this is the same as a morphism

$$\mathbb{L}^{int}_{A}[-1] \longrightarrow \mathbf{DR}^{int}(A)(1)$$

in the stable ∞-category of A-modules.

This extends to a morphism in $\mathbf{cdga}^{gr}_{\mathcal{M}}$

$$\phi_A : Sym_A(\mathbb{L}^{int}_{A}[-1]) \longrightarrow \mathbf{DR}^{int}(A),$$

where the grading on the left hand side is defined by letting $\mathbb{L}^{int}_{A}[-1]$ be pure of weight 1. Note that, by construction, the morphism ϕ_A is natural in A.

Proposition 1.3.12 *For all $A \in \mathbf{cdga}_{\mathcal{M}}$ the above morphism*

$$\phi_A : Sym_A(\mathbb{L}^{int}_{A}[-1]) \longrightarrow \mathbf{DR}^{int}(A)$$

is an equivalence in $\mathbf{cdga}^{gr}_{\mathcal{M}}$.

Proof. The morphism ϕ_A is functorial in A, and moreover, any commutative dg-algebra in $\mathcal{M}$ is a colimit of free commutative dg-algebras (see, e.g. [Lu6, 3.2.3]). It is therefore enough to prove the following two assertions

1. The morphism $\phi_A : Sym_A(\mathbb{L}_A^{int}[-1]) \longrightarrow \mathbf{DR}^{int}(A)$ is an equivalence when $A = Sym(X)$ is the free commutative dg-algebra over an object $X \in \mathcal{M}$.

2. The two ∞-functors $A \mapsto Sym_A(\mathbb{L}_A^{int}[-1])$ and $A \mapsto \mathbf{DR}^{int}(A)$, from commutative dg-algebras in $\mathcal{M}$ to graded commutative algebras in $\mathcal{M}$, commute with all colimits.

Proof of 1. Let $A = Sym(X) \in \mathbf{cdga}_{\mathcal{M}}$ be a free object. Explicitly its de Rham object $\mathbf{DR}^{int}(A)$ can be described as follows. Let us denote by $Y \in \epsilon - \mathcal{M}^{gr}$ the free graded mixed object over X, the free graded mixed object functor being left adjoint to the forgetful functor $\epsilon - \mathcal{M}^{gr} \longrightarrow \mathcal{M}$. As already observed, we have $Y(0) = X$, $Y(1) = X[-1]$, $Y(i) = 0$ if $i \neq 0, 1$, and with the canonical mixed structure $X \simeq X[-1][1]$. The de Rham object $\mathbf{DR}^{int}(A)$, is then the free commutative monoid object in $\epsilon - \mathbf{cdga}_{\mathcal{M}}^{gr}$ over Y. We simply denote by $X \oplus X[-1]$ the graded object in $\mathcal{M}$ obtained by forgetting the mixed differential in Y. As forgetting the mixed structure is a symmetric monoidal left adjoint, the graded commutative algebra underlying $\mathbf{DR}^{int}(A)$ is thus given by

$$\begin{aligned}\mathbf{DR}^{int}(Sym(X)) &\simeq Sym(X \oplus X[-1]) \simeq Sym(X) \otimes Sym(X[-1]) \simeq Sym_{Sym(X)}(A \otimes X[-1]) \\ &\simeq Sym_A(\mathbb{L}_A^{int}[-1]),\end{aligned}$$

where $Sym : \mathcal{M}^{gr} \to \mathbf{cdga}_{\mathcal{M}}^{gr}$ denotes the left adjoint to the forgetful functor. Note that, for $Y \in \mathcal{M}^{gr}$ sitting entirely in weight 0, $Sym(Y)$ sits entirely in weight 0. On the other hand if $Z \in \mathcal{M}$, and we write $Z(1) \in \mathcal{M}^{gr}$ for Z sitting in degree 1, then $Sym(Z(1))$ coincides with $Sym(Z)$ with its "usual" full $\mathbb{N}$-weight-grading (with Z sitting in weight 1). This proves 1..

Proof of 2. This follows because both ∞-functors are obtained by composition of various left adjoint ∞-functors. Indeed, for the case of $A \mapsto \mathbf{DR}^{int}(A)$ this is the composition of the ∞-functor $\mathbf{DR}^{int}$ from lemma 1.3.8 with the forgetful ∞-functor from $\epsilon - \mathbf{cdga}_{\mathcal{M}}^{gr} \longrightarrow \mathbf{cdga}_{\mathcal{M}}^{gr}$ which are both left adjoints. For the second ∞-functor, we have, for any $B \in \mathbf{cdga}_{\mathcal{M}}^{gr}$, a natural morphism of spaces

$$\mathsf{Map}_{\mathbf{cdga}_{\mathcal{M}}^{gr}}(Sym_A(\mathbb{L}_A^{int}[-1]), B) \longrightarrow \mathsf{Map}_{\mathbf{cdga}_{\mathcal{M}}}(A, B(0)).$$

The fiber of this map at a given morphism $A \to B(0)$, is naturally equivalent to $\mathsf{Map}_{A-Mod_{\mathcal{M}}}(\mathbb{L}_A^{int}[-1], B(1))$. By the definition of the cotangent complex this fiber is also naturally equivalent to $\mathsf{Map}_{\mathbf{cdga}_{\mathcal{M}}/B(0)}(A, B(0) \oplus B(1)[1])$. This implies that, for a fixed $B \in \mathbf{cdga}_{\mathcal{M}}^{gr}$, the ∞-functor $A \mapsto \mathsf{Map}_{\mathbf{cdga}_{\mathcal{M}}^{gr}}(Sym_A(\mathbb{L}_A^{int}[-1]), B)$ transforms colimits into limits, and thus that $A \mapsto Sym_A(\mathbb{L}_A^{int}[-1])$, as an ∞-functor $\mathbf{cdga}_{\mathcal{M}} \to \mathbf{cdga}_{\mathcal{M}}^{gr}$ preserves colimits.

□

Remark 1.3.13 Observe that $\phi_A : Sym_A(\mathbb{L}_A^{int}[-1]) \longrightarrow \mathbf{DR}^{int}(A)$ is actually an equivalence in the under-category $A/\mathbf{cdga}^{gr}_{\mathcal{M}}$ (where A sits in pure weight 0), simply by inducing the map $A \to \mathbf{DR}^{int}(A)$ using $\phi(A)$ and the canonical map $A \to Sym_A(\mathbb{L}^{int}_A[-1])$.

An important corollary of the previous proposition is the existence of a *de Rham differential*, for any object $A \in \mathbf{cdga}_{\mathcal{M}}$.

Corollary 1.3.14 *For any object $A \in \mathbf{cdga}_{\mathcal{M}}$, the graded commutative dg-algebra $Sym_A(\mathbb{L}^{int}_A[-1])$ possesses a canonical mixed structure making it into a mixed graded commutative dg-algebra in $\mathcal{M}$. The corresponding mixed differential is called the* de Rham differential.

Remark 1.3.15 Note that, from the point of view of ∞-categories (which is the point of view adopted in its statement), Corollary 1.3.14 is almost tautological. In fact, from this point of view, for a graded cdga B in M, a mixed structure on B means a *weak* mixed structure, i.e. a pair (B', u), where B' is a graded mixed cdga in M and $u : B' \simeq B$ is an equivalence of graded cdga. This is the exact content of Cororllary 1.3.14.

Relative $\mathbf{DR}^{int}$. We conclude this subsection with the *relative* version of $\mathbf{DR}^{int}$. Let $A \in \mathbf{cdga}_{\mathcal{M}}$, and consider the ∞-functor

$$(-)(0) : A/\epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}} \longrightarrow A/\mathbf{cdga}_{\mathcal{M}} \quad C \longmapsto C(0)$$

where in $A/\epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}}$, A is considered as concentrated in pure weight 0 (hence with trivial mixed differential).

Proposition 1.3.16 *For any $A \in \mathbf{cdga}_{\mathcal{M}}$, the ∞-functor*

$$(-)(0) : A/\epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}} \longrightarrow A/\mathbf{cdga}_{\mathcal{M}}$$

has a left adjoint, denoted as

$$\mathbf{DR}^{int}(-/A) : A/\mathbf{cdga}_{\mathcal{M}} \longrightarrow A/\epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}} \quad (A \to B) \longmapsto \mathbf{DR}^{int}(B/A)$$

Proof. This is an application of the adjoint functor theorem ([Lu1, Corollary 5.5.2.9]), completely analogous to the proof of Proposition 1.3.8. We leave the details to the reader $\square$

Proceeding as in Proposition 1.3.12, we also get

Proposition 1.3.17 *For all $A \in \mathbf{cdga}_{\mathcal{M}}$ there is a canonical morphism*

$$\phi_{/A} : Sym_B(\mathbb{L}^{int}_{B/A}[-1]) \longrightarrow \mathbf{DR}^{int}(B/A)$$

is an equivalence in $A/\mathbf{cdga}^{gr}_{\mathcal{M}}$.

Consider the ∞-functor

$$\mathbf{DR}^{int} : \mathsf{Mor}(\mathbf{cdga}_{\mathcal{M}}) \longrightarrow \epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}},$$

sending a morphism $A \to B$ to $\mathbf{DR}^{int}(B/A)$. This ∞-functor can be explicitly constructed as the localization along equivalences of the functor

$$DR^{str} : \mathsf{Cof}(cdga_{\mathcal{M}}) \longrightarrow \epsilon - cdga^{gr}_{\mathcal{M}},$$

from the category of cofibrations between cofibrant cdga to the category of graded mixed cdga, sending a cofibration $A \to B$ to $DR^{str}(B/A) = Sym_B(\Omega^1_{B/A}[-1])$, with mixed structure given by the de Rham differential. The following result gives a useful description of $\mathbf{DR}^{int}(B/A)$.

Lemma 1.3.18 *For the ∞-functor*

$$\mathbf{DR}^{int} : \mathsf{Mor}(\mathbf{cdga}_{\mathcal{M}}) \longrightarrow \epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}} \quad (A \to B) \longmapsto \mathbf{DR}^{int}(B/A),$$

we have an equivalence in $A/\epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}}$

$$\mathbf{DR}^{int}(B/A) \simeq \mathbf{DR}^{int}(B) \otimes_{\mathbf{DR}^{int}(A)} A$$

where A is concentrated in weight 0 (hence, with trivial mixed differential), and the rhs denotes the obvious pushout in the category $\epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}}$.

Proof. We have to prove that the ∞-functor

$$A/\mathbf{cdga}_{\mathcal{M}} \longrightarrow A/\epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}} \quad (A \to B) \longmapsto \mathbf{DR}^{int}(B) \otimes_{\mathbf{DR}^{int}(A)} A$$

is left adjoint to the functor sending C to $C(0)$. Now,

$$\mathsf{Map}_{A/\epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}}}(X, C) \simeq \mathsf{Map}_{\epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}}}(X, C) \times_{\mathsf{Map}_{\epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}}}(A,C)} \{*\}$$

where the map $\{*\} \to \mathsf{Map}_{\epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}}}(A, C)$ is induced by the structure map $\rho : A \to C$, defining C as an object in $A/\epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}}$. Taking $X = \mathbf{DR}^{int}(B) \otimes_{\mathbf{DR}^{int}(A)} A$, and using the shortcut notation

$\mathsf{Map} := \mathsf{Map}_{\mathbf{cdga}_{\mathcal{M}}}$, we thus get

$$\begin{aligned}
&\mathsf{Map}_{A/\epsilon-\mathbf{cdga}^{gr}_{\mathcal{M}}}(\mathbf{DR}^{int}(B) \otimes_{\mathbf{DR}^{int}(A)} A, C) \\
&\qquad\simeq (\mathsf{Map}(B, C(0)) \times_{\mathsf{Map}(A,C(0))} \mathsf{Map}_{\epsilon-\mathbf{cdga}^{gr}_{\mathcal{M}}}(A, C)) \times_{\mathsf{Map}_{\epsilon-\mathbf{cdga}^{gr}_{\mathcal{M}}}(A,C)} \{*\} \\
&\qquad\simeq \mathsf{Map}(B, C(0)) \times_{\mathsf{Map}(A,C(0))} \{*\}
\end{aligned}$$

where the map $\{*\} \to \mathsf{Map}(A, C(0))$ is induced by the weight 0 component $\rho(0)$ of ρ. Therefore

$$\begin{aligned}
\mathsf{Map}_{A/\epsilon-\mathbf{cdga}^{gr}_{\mathcal{M}}}(\mathbf{DR}^{int}(B) \otimes_{\mathbf{DR}^{int}(A)} A, C) &\simeq \mathsf{Map}(B, C(0)) \times_{\mathsf{Map}(A,C(0))} \{*\} \\
&\simeq \mathsf{Map}_{A/\epsilon-\mathbf{cdga}^{gr}_{\mathcal{M}}}(B, C(0))
\end{aligned}$$

as we wanted. $\square$

1.3.3 Strict models.

For future reference we give here *strict models* for both the cotangent complex $\mathbb{L}^{int}_A$ and the de Rham object $\mathbf{DR}^{int}(A)$. For $A \in \mathbf{cdga}_{\mathcal{M}}$, corresponding to an object $A \in Comm(M)$, we can consider the functor

$$Der^{str}(A, -) : A - Mod_M \longrightarrow Set,$$

sending an A-module M to the set $Hom_{Comm(M)/A}(A, A \oplus M)$. This functor commutes with limits and thus is corepresentable by an A-module $\Omega^1_A \in A - Mod_M$.

Let $Q(A) \longrightarrow A$ be a cofibrant replacement inside $Comm(M)$. As this is an equivalence it induces an equivalence of homotopy categories

$$Ho(A - Mod_{\mathcal{M}}) \simeq Ho(A - Mod_M) \simeq Ho(Q(A) - Mod).$$

Through these identifications, we have a natural isomorphism in $Ho(A - Mod_{\mathcal{M}})$

$$\Omega^1_{Q(A)} \simeq \mathbb{L}^{int}_A.$$

In particular, when A is cofibrant Ω^1_A is a model for the cotangent complex of A.

De Rham complexes also possess similarly defined strict models. We have the functor

$$Comm(\epsilon - M^{gr}) \longrightarrow Comm(M),$$

sending a graded mixed commutative monoid A to its part of weight zero $A(0)$.

This functor commutes with limits and thus possesses a left adjoint

$$DR^{str} : Comm(M) \longrightarrow Comm(\epsilon - M^{gr}).$$

For the same formal reasons, the analogue of the Lemma 1.3.12 remains correct, and for any $A \in Comm(M)$, we have a functorial isomorphism of graded commutative monoids in M

$$Sym_A(\Omega^1_A[-1]) \simeq DR^{str}(A).$$

In particular, $Sym_A(\Omega^1_A[-1])$ has a uniquely defined mixed structure compatible with its natural grading and multiplicative structure. This mixed structure is given by a map in M

$$\epsilon : \Omega^1_A \longrightarrow \wedge^2 \Omega^1_A$$

which is called the *strict de Rham differential.*

If $Q(A)$ is a cofibrant model for A in $Comm(M)$, we have a natural equivalence of mixed graded commutative dg-algebras in $\mathcal{M}$

$$DR^{str}(Q(A)) \simeq \mathbf{DR}^{int}(A).$$

Therefore, the explicit graded mixed commutative monoid $Sym_{Q(A)}(\Omega^1_{Q(A)}[-1])$ is a model for $\mathbf{DR}^{int}(A)$.

Remark 1.3.19 When $M = C(k)$, and A is a commutative dg-algebra over k, $\mathbf{DR}^{int}(A)$ coincides with the de Rham object $\mathbf{DR}(A/k)$ constructed in [To-Ve-2].

1.4 Differential forms and polyvectors

Next we describe the notions of *differential forms*, *closed differential forms* and *symplectic structure*, as well as the notion of $\mathbb{P}_n$*-structure* on commutative dg-algebras over a fixed base ∞-category $\mathcal{M}$. We explain a first relation between Poisson and symplectic structures, by constructing the symplectic structure associated to a *non-degenerate* Poisson structure.

1.4.1 Forms and closed forms.

Let $A \in \mathbf{cdga}_{\mathcal{M}}$ be a commutative dg-algebra over $\mathcal{M}$. As explained in Section 1.3.2 we have the associated de Rham object $\mathbf{DR}^{int}(A) \in \epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}}$. We let $\mathbf{1}$ be the unit object in $\mathcal{M}$, considered as an object in $\epsilon - \mathcal{M}^{gr}$ in a trivial manner (pure of weight zero and with zero mixed structure). We let similarly $\mathbf{1}(p)$ be its twist by $p \in \mathbb{Z}$: it is now pure of weight p again with the zero mixed structure. Finally, we have shifted versions $\mathbf{1}[n](p) \equiv \mathbf{1}(p)[n] \in \epsilon - \mathcal{M}^{gr}$ for any $n \in \mathbb{Z}$.

For $q \in \mathbb{Z}$, we will denote the weight-degree shift by q functor as

$$(-)((q)) : \epsilon - \mathcal{M}^{gr} \longrightarrow \epsilon - \mathcal{M}^{gr} \, E \longmapsto E((q)) \ ;$$

it sends $E = \{E(p), \epsilon\}_{p \in \mathbb{Z}}$ to the graded mixed object in $\mathcal{M}$ having $E(p+q)$ in weight p, and with the obvious induced mixed structure (with no signs involved). Note that $(-)((q))$ is an equivalence for any $q \in \mathbb{Z}$, it commutes with the cohomological-degree shift, and that, in our previous notation, we have $\mathbf{1}(p) = \mathbf{1}((-p))$.

We will also write $\mathrm{Free}^{gr}_{\epsilon,0} : \mathcal{M} \to \epsilon - \mathcal{M}^{gr}$ for the left adjoint to the weight-zero functor $\epsilon - \mathcal{M}^{gr} \to \mathcal{M}$ sending $E = \{E(p), \epsilon\}_{p\in\mathbb{Z}}$ to its weight-zero part $E(0)$. Note that, then, the functor $\epsilon - \mathcal{M}^{gr} \to \mathcal{M}$ sending $E = \{E(p), \epsilon\}_{p\in\mathbb{Z}}$ to its weight-q part $E(q)$ is right adjoint to the functor $X \mapsto (\mathrm{Free}^{gr}_{\epsilon,0}(X))((-q))$.

Below we will not distinguish notationally between $\mathbf{DR}^{int}(A)$ and its image under the forgetful functor $\epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}} \to \epsilon - \mathcal{M}^{gr}$, for $A \in \mathbf{cdga}_{\mathcal{M}}$. The same for $\mathbf{DR}(A)$ and its image under the forgetful functor $\epsilon - \mathbf{cdga}^{gr}_{k} \to \epsilon - \mathbf{dg}^{gr}_{k}$, and for $\wedge^p_A \mathbb{L}^{int}_A$ and its image under the forgetful functor $A - Mod_{\mathcal{M}} \to \mathcal{M}$.

Definition 1.4.1 *For any $A \in \mathbf{cdga}_{\mathcal{M}}$, and any integers $p \geq 0$ and $n \in \mathbb{Z}$, we define the* space of closed p-forms of degree n on A *by*

$$\mathcal{A}^{p,cl}(A, n) := \mathsf{Map}_{\epsilon-\mathcal{M}^{gr}}(\mathbf{1}(p)[-p-n], \mathbf{DR}^{int}(A)) \in \mathcal{T}.$$

The space of p-forms of degree n on A *is defined by*

$$\mathcal{A}^{p}(A, n) := \mathsf{Map}_{\mathcal{M}}(\mathbf{1}[-n], \wedge^p_A \mathbb{L}^{int}_A) \in \mathcal{T}.$$

Remark 1.4.2 Note that by definition of realization functors (Definition 1.3.1), we have natural identifications

$$\mathcal{A}^{p,cl}(A, n) = \mathsf{Map}_{\epsilon-\mathbf{dg}^{gr}_k}(k(p)[-p-n], \mathbf{DR}(A))$$
$$\mathcal{A}^{p}(A, n) = \mathsf{Map}_{\mathbf{dg}_k}(k[-n], \wedge^p_{|A|} \mathbb{L}_A)$$

where $|A| \in \mathbf{cdga}_k$. Note also that $| \wedge^p_A \mathbb{L}^{int}_A | \simeq \wedge^p_{|A|} \mathbb{L}_A$.

By Proposition 1.3.12, we have

$$\mathcal{A}^{p}(A, n) = \mathsf{Map}_{\mathcal{M}}(\mathbf{1}[-n], \wedge^p_A \mathbb{L}^{int}_A) \simeq \mathsf{Map}_{\epsilon-\mathcal{M}^{gr}}(\mathrm{Free}^{gr}_{\epsilon,0}(\mathbf{1})((-p)), \mathbf{DR}^{int}(A)[p+n])$$

and the identity map $\mathbf{1} \to \mathbf{1}$ induces a map $\mathrm{Free}^{gr}_{\epsilon,0}(\mathbf{1})((-p)) \to \mathbf{1}((-p))$ in $\epsilon - \mathcal{M}^{gr}$ (where, in the target we abuse notation and write $\mathbf{1}$ for the object $\mathbf{1}$ in pure weight zero). In particular, we get an induced canonical map

$$\mathcal{A}^{p,cl}(A, n) \longrightarrow \mathcal{A}^{p}(A, n)$$

which should be thought of as the map assigning to a closed p-form its *underlying p-form.*

In order to gain a better understanding of the spaces $\mathcal{A}^{p,cl}(A, n)$, we observe that the object

$\mathbf{1} \in \epsilon - \mathcal{M}^{gr}$ possesses a natural cell decomposition consisting of a sequence of push-outs in $\epsilon - \mathcal{M}^{gr}$

$$\begin{tikzcd} X_m \arrow[r] & X_{m+1} \\ L_{m+1}[-1] \arrow[u] \arrow[r] & 0, \arrow[u] \end{tikzcd}$$

with the following properties

1. $X_{-1} \simeq 0$.

2. $L_m \in \epsilon - \mathcal{M}^{gr}$ is the free graded mixed object in $\mathcal{M}$ generated by $\mathbf{1} \in \mathcal{M}$, and weight-shifted by $(-m)$, i.e. $L_m := (\mathrm{Free}^{gr}_{\epsilon,0}(\mathbf{1}))((-m))$. Note that L_m is **not** concentrated in one single weight.

3. There is a natural equivalence $colim_m\, X_m \simeq \mathbf{1}$.

We can give a completely explicit description of this cell decomposition, by first studying the case of the enriching category $M = C(k)$. In $\epsilon - C(k)^{gr}$ there is a natural cell model for $k = k(0)$, considered as a trivial graded mixed complex pure of weight zero. The underlying k-module is generated by a countable number of variables $\{x_n, y_n\}_{n\geq 0}$, where x_n is of cohomological degree 0 and y_n of cohomological degree 1, and the cohomological differential is defined by $d(x_n) = y_{n-1}$ (with the convention $y_{-1} = 0$). The weight-grading is defined by declaring x_n to be pure of weight n and y_n pure of weight $(n+1)$. Finally, the mixed structure is defined by $\epsilon(x_n) = y_n$. This graded mixed complex will be denoted by $\widetilde{k}$ and is easily seen to be equivalent to k via the natural augmentation $\widetilde{k} \to k$ sending x_0 to 1 and all other generators to zero. Note that while k is cofibrant in the injective model structure on $\epsilon - C(k)^{gr}$ (where cofibrations and weak equivalences are detected through the forgetful functor $\mathrm{U}_\epsilon : \epsilon - C(k)^{gr} \to C(k)^{gr}$), it is not cofibrant in the *projective* model structure on $\epsilon - C(k)^{gr}$ (where fibrations and weak equivalences are detected through the same forgetful functor U_ϵ). In fact the map $\widetilde{k} \to k$ is a cofibrant replacement of k in the projective model structure on $\epsilon - C(k)^{gr}$. Moreover, the graded mixed complex $\widetilde{k}$ comes naturally endowed with a filtration by sub-objects $\widetilde{k} = \cup_{m\geq -1} Z_m$, where Z_m is the sub-object spanned by the x_n's and y_n's, for all $n \leq m$.

For a general symmetric monoidal model category M, enriched over $C(k)$ as in Section 1.1, we can consider $\widetilde{k} \otimes \mathbf{1}$ as a graded mixed object in M. Since $(-) \otimes_k \mathbf{1}$ is left Quillen, the cell decomposition of $\widetilde{k}$ defined above, induces the required cell decomposition in $\epsilon - \mathcal{M}^{gr}$

$$colim_m\, X_m \simeq \mathbf{1},$$

where $X_m := Z_m \otimes \mathbf{1}$.

In particular, we have, for all $m \geq -1$ ($X_{-1} := 0$), a cofibration sequence in $\epsilon - \mathcal{M}^{gr}$

$$X_m \longrightarrow X_{m+1} \longrightarrow L_{m+1}.$$

Passing to mapping spaces, we obtain, for all graded mixed object $E \in \epsilon-\mathcal{M}^{gr}$, a tower decomposition

$$\mathsf{Map}_{\epsilon-\mathcal{M}^{gr}}(\mathbf{1}, E) \simeq lim_m \, \mathsf{Map}_{\epsilon-\mathcal{M}^{gr}}(X_m, E),$$

together with fibration sequences

$$\mathsf{Map}_{\epsilon-\mathcal{M}^{gr}}(L_{m+1}, E) \simeq \mathsf{Map}_{\mathcal{M}}(\mathbf{1}, E(m+1)) \longrightarrow \mathsf{Map}_{\epsilon-\mathcal{M}^{gr}}(X_{m+1}, E) \longrightarrow \mathsf{Map}_{\epsilon-\mathcal{M}^{gr}}(X_m, E).$$

Note that , for any $(n, q) \in \mathbb{Z}^2$, the degree-shift and weight-shift functors

$$[n]\, ,\, ((q)) : \epsilon - \mathcal{M}^{gr} \to \epsilon - \mathcal{M}^{gr}$$

are equivalences, hence commute with colimits. Therefore by taking E to be the graded mixed object $\mathbf{DR}^{int}(A)[n+p]((p))$, we have the following decomposition of the space of closed p-forms of degree n

$$\mathcal{A}^{p,cl}(A, n) \simeq lim_m \, \mathcal{A}^{p,cl}(A, n)(\leq m),$$

where

$$\mathcal{A}^{p,cl}(A, n)(\leq m) := \mathsf{Map}_{\epsilon-\mathcal{M}^{gr}}(X_m, \mathbf{DR}^{int}(A)[n+p]((p))).$$

These data are all packaged in fibration sequences

$$\mathsf{Map}_{\mathcal{M}}(\mathbf{1}, (\wedge_A^{p+m+1} \mathbb{L}_A^{int})[n-m-1]) \longrightarrow \mathcal{A}^{p,cl}(A, n)(\leq m+1) \longrightarrow \mathcal{A}^{p,cl}(A, n)(\leq m)$$

where we have used Proposition 1.3.12 to identify

$$\mathbf{DR}^{int}(A)[n+p](m+1+p) \simeq Sym_A^{m+p+1}(\mathbb{L}_A^{int}[-1])[n+p] \simeq (\wedge_A^{p+m+1} \mathbb{L}_A^{int})[n-m-1].$$

These successive fibration sequences embody the *Hodge filtration* on the de Rham complex of A. Note that $L_0 \simeq X_0$ so that $\mathcal{A}^{p,cl}(A, n)(\leq -1) \simeq \mathcal{A}^p(A, n)$. In particular, the canonical map $\mathcal{A}^{p,cl}(A, n) \longrightarrow \mathcal{A}^p(A, n)$ from closed p-forms to $p-$forms, defined above, can be re-obtained as the canonical map

$$lim_m \, \mathcal{A}^{p,cl}(A, n)(\leq m) \longrightarrow \mathcal{A}^{p,cl}(A, n)(\leq -1)$$

from the limit to the level (≤ -1) of the tower.

We are now ready to define the notion of a shifted symplectic structure on a commutative dg-algebra in $\mathcal{M}$. Let $A \in \mathbf{cdga}_{\mathcal{M}}$ and $A - Mod_{\mathcal{M}}$ be the symmetric monoidal ∞-category of A-modules in $\mathcal{M}$. The symmetric monoidal ∞-category $A - Mod_{\mathcal{M}}$ is closed, so any object M possesses a dual

$$M^{\vee} := \underline{Hom}_{\mathcal{M}}(M, A) \in A - Mod_{\mathcal{M}}.$$

For an object $M \in A-Mod_{\mathcal{M}}$, and a morphism $w : A \longrightarrow M \wedge_A M[n]$, we have an adjoint morphism

$$\Theta_w : M^\vee \longrightarrow M[n]$$

where $M^\vee$ is the dual object of M.

Definition 1.4.3 *For $A \in \mathbf{cdga}_{\mathcal{M}}$ the* internal tangent complex of A is defined by

$$\mathbb{T}_A^{int} := (\mathbb{L}_A^{int})^\vee \in A - Mod_{\mathcal{M}}.$$

Note that the space of (non-closed) p-forms of degree n on A can be canonically identified as the mapping space

$$\mathcal{A}^p(A,n) \simeq \mathsf{Map}_{A-Mod_{\mathcal{M}}}(A, \wedge^p \mathbb{L}_A^{int}[n]).$$

In particular, when $p=2$ and when $\mathbb{L}_A^{int}$ is a dualizable A-module, any 2-form ω_0 of degree n induces a morphism of A-modules

$$\Theta_{\omega_0} : \mathbb{T}_A^{int} \longrightarrow \mathbb{L}_A^{int}[n].$$

Definition 1.4.4 *Let $A \in \mathbf{cdga}_{\mathcal{M}}$. We assume that $\mathbb{L}_A^{int}$ is a dualizable object in the symmetric monoidal ∞-category of A-modules in $\mathcal{M}$.*

1. *A closed 2-form $\omega \in \pi_0(\mathcal{A}^{2,cl}(A,n))$ of degree n on A is* non-degenerate *if the underlying 2-form $\omega_0 \in \pi_0(\mathcal{A}^2(A,n))$ induces an equivalence of A-modules*

$$\Theta_{\omega_0} : \mathbb{T}_A^{int} \simeq \mathbb{L}_A^{int}[n].$$

2. *The* space $\mathsf{Symp}(A;n)$ of n-shifted symplectic structures on A *is the subspace of $\mathcal{A}^{2,cl}(A,n)$ consisting of the union of connected components corresponding to non-degenerate elements.*

De Rham objects have strict models, as explained in our previous subsection, so the same is true for the space of forms and closed forms. Let $A \in \mathbf{cdga}_{\mathcal{M}}$ be a commutative dg-algebra in $\mathcal{M}$, and choose a cofibrant model $A' \in Comm(M)$ for A. Then, the space of closed p-forms on A can be described as follows. We consider the unit $\mathbf{1} \in M$, and set

$$| - | : M \longrightarrow C(k)$$

the functor defined by sending $x \in M$ to $\underline{Hom}_k(\mathbf{1}, R(x)) \in C(k)$, where $R(x)$ is a (functorial) fibrant replacement of x in M and $\underline{Hom}_k$ is the enriched hom of M with values in $C(k)$. The graded mixed object $\mathbf{DR}^{int}(A)$ can be represented by $DR^{str}(Q(A))$, and $\mathbf{DR}(A)$ by $|DR^{str}(Q(A))|$. We have by construction

$$\mathcal{A}^{p,cl}(A,n) \simeq \mathsf{Map}_{\epsilon-C(k)^{gr}}(k(p)[-p-n], |DR^{str}(Q(A))|).$$

In order to compute this mapping space we observe that the injective model structure on $\epsilon - C(k)^{gr}$ (where cofibrations and weak equivalences are detected through the forgetful functor $\mathrm{U}_\epsilon : \epsilon - C(k)^{gr} \to C(k)^{gr}$) is Quillen equivalent to the projective model structure on $\epsilon - C(k)^{gr}$ (where fibrations and weak equivalences are detected through the same forgetful functor U_ϵ), therefore the corresponding mapping spaces are equivalent objects in $\mathcal{T}$. It is then convenient to compute $\mathsf{Map}_{\epsilon - C(k)^{gr}}(k(p)[-p-n], |DR^{str}(Q(A))|)$ in the projective model structure, since any object is fibrant here, and we have already constructed an explicit (projective) cofibrant resolution $\widetilde{k}$ of k. This way, we get the following explicit strict model for the space of closed forms on A

$$\begin{aligned}\mathcal{A}^{p,cl}(A,n) &\simeq \mathsf{Map}_{C(k)}(k[-n], \prod_{j\geq p} | \wedge^j_{A'} \Omega^1_{A'}|[-j]) \\ &= \mathsf{Map}_{C(k)}(k[-n], \prod_{j\geq p} \mathbf{DR}(A)(j)).\end{aligned}$$

Here $\prod_{j\geq p} | \wedge^j_{A'} \Omega^1_{A'}|[-j]$ is the complex with the total differential, which is sum of the cohomological differential and mixed structure as in [To2, §5].

1.4.2 Shifted polyvectors.

We will now introduce the dual notion to differential forms, namely *polyvector fields*. Here we start with strict models, as the ∞-categorical aspects are not totally straightforward and will be dealt with more conveniently in a second step.

Graded dg shifted Poisson algebras in $\mathcal{M}$. Let us start with the case $M = C(k)$, $n \in \mathbb{Z}$, and consider the *graded n-shifted Poisson operad* $\mathbb{P}_n^{gr} \in Op(C(k)^{gr})$ defined as follows. As an operad in $C(k)$ (i.e. as an ungraded dg-operad), it is freely generated by two operations $\cdot, [-,-]$, of arity 2 and respective cohomological degrees 0 and $(1-n)$

$$\cdot \in \mathbb{P}_n^{gr}(2)^0 \qquad [-,-] \in \mathbb{P}_n^{gr}(2)^{1-n},$$

with the standard relations expressing the conditions that $\cdot$ is a graded commutative product, and that $[-,-]$ is a biderivation of cohomological degree $1-n$ with respect to the product $\cdot$.

A $\mathbb{P}_n^{gr}$-algebra in $C(k)$ is just a commutative dg-algebra A endowed with a compatible Poisson bracket of degree $(1-n)$

$$[-,-] : A \otimes_k A \longrightarrow A[1-n].$$

The weight-grading on $\mathbb{P}_n^{gr}$ is then defined by letting $\cdot$ be of weight 0 and $[-,-]$ be of weight -1. When $n > 1$, the operad $\mathbb{P}_n$ is also the operad $H_\bullet(E_n)$ of homology of the topological little n-disks or E_n-operad, endowed with its natural weight-grading for which H_0 is of weight 0 and H_{n-1} of weight -1 (see [Coh] or [Sin] for a very detailed account).

We consider M^{gr}, the category of $\mathbb{Z}$-graded objects in M, endowed with its natural symmetric monoidal structure. With fibrations and equivalences defined levelwise, M^{gr} is a symmetric monoidal model category satisfying our standing assumptions $(1)-(5)$ of 1.1. We can then consider $\mathsf{Op}(M^{gr})$ the category of (symmetric) operads in M^{gr}. As already observed, the category M^{gr} is naturally enriched over $C(k)^{gr}$, via a symmetric monoidal functor $C(k)^{gr} \to M^{gr}$. This induces a functor $\mathsf{Op}((C(k))^{gr}) \to \mathsf{Op}(M^{gr})$, and we will denote by $\mathbb{P}^{gr}_{M,n} \in \mathsf{Op}(M^{gr})$ the image of $\mathbb{P}^{gr}_n$ under this functor. The category of $\mathbb{P}^{gr}_{M,n}$-algebras will be denoted by $\mathbb{P}_n - \mathbf{cdga}^{gr}_M$, and its objects will be called *graded n-Poisson commutative dg-algebras in M*. Such and algebra consists of the following data.

1. A family of objects $A(p) \in M$, for $p \in \mathbb{Z}$.
2. A family of multiplication maps
$$A(p) \otimes A(q) \longrightarrow A(p+q),$$
which are associative, unital, and graded commutative.
3. A family of morphisms
$$[-,-] : A(p) \otimes A(q) \longrightarrow A(p+q-1)[1-n].$$

These data are furthermore required to satisfy the obvious compatibility conditions for a Poisson algebra (see [Ge-Jo, §1.3] for the ungraded dg-case). We just recall that, in particular, $A(0)$ should be a commutative monoid in M, and that the morphism

$$[-,-] : A(1) \otimes A(1) \longrightarrow A(1)[1-n]$$

has to make $A(1)$ into a n-Lie algebra object in M, or equivalently, $A(1)[n-1]$ has to be a Lie algebra object in M when endowed with the induced pairing

$$A(1)[n-1] \otimes A(1)[n-1] \simeq (A(1) \otimes A(1))[2n-2] \longrightarrow A(1)[n-1].$$

Since the bracket is a derivation with respect to the product, this Lie algebra object acts naturally on $A(0)$ by derivations, making the pair $(A(0), A(1)[n-1])$ into a Lie algebroid object in M (see [Vez]). Moreover, $A[n-1]$ is a Lie algebra object in M^{gr}.

Remark 1.4.5 Note that for any dg-operad $\mathcal{O}$ over k, and for any symmetric monoidal $E : C(k)$-model category M as in §1.1, the symmetric monoidal functor $C(k) \to M$ has a natural extension to a symmetric monoidal functor $E^{gr} : C(k)^{gr} \to M^{gr}$. Moreover $\mathcal{O}$ has a *naive* extension to an operad $\mathcal{O}^{gr,\mathrm{naive}}$ in $C(k)^{gr}$, and via E^{gr} there is an induced operad $\mathcal{O}^{gr,\mathrm{naive}}_M$ on M^{gr}. However, for $\mathcal{O} = \mathbb{P}_n$ the n-Poisson operad, and for $\mathcal{O} = \mathsf{Lie}$ the Lie operad, the graded versions $\mathbb{P}^{gr}_n$ and Lie^{gr} we are considering here are *not* the naive versions, due to the non-zero weight of the bracket operation. The same is true for our operads $\mathbb{P}^{gr}_{M,n}$ and Lie^{gr}_{M}.

Definition 1.4.6 *The* ∞*-category of graded* n*-Poisson commutative dg-algebras in the* ∞*-category* $\mathcal{M}$ *is defined to be*

$$\mathbb{P}_n - \mathbf{cdga}^{gr}_{\mathcal{M}} := L(\mathbb{P}_n - \mathbf{cdga}^{gr}_{M}).$$

Shifted polyvectors. Let $A \in \mathbf{cdga}_M$ be a commutative monoid in M. We define a graded $\mathbb{P}_n$-algebra of n-shifted polyvectors on A as follows. As in the case of forms, we will have an internal and external version of shifted polyvectors on A. We consider the A-module Ω^1_A corepresenting derivations (see 1.3.3), and we write

$$T(A,n) := \underline{Hom}_A(\Omega^1_A, A[n]) \in A - Mod_M$$

for the A-module object of derivations from A to to the A-module $A[n]$ (note that $T(A,n)$ is a model for $\mathbb{T}^{int}_A[n]$ of Definition 1.4.3 only when A is cofibrant and fibrant object in $\mathbf{cdga}_M$).

Note that $T(A,n)$ can also be identified as follows. Consider the canonical map

$$\alpha : \underline{Hom}_M(A, A[n]) \otimes A \longrightarrow A[n]$$

in M, adjoint to the identity of $\underline{Hom}_M(A, A[n])$, and the multiplication map

$$m : A \otimes A \longrightarrow A.$$

Then, we consider the following three maps

- μ'_1 defined as the composition

$$\underline{Hom}_M(A, A[n]) \otimes A \otimes A \xrightarrow{\mathrm{id}_A \otimes m} \underline{Hom}_M(A, A[n]) \otimes A \xrightarrow{\alpha} A[n]$$

- u'_1 defined as the composition

$$\underline{Hom}_M(A, A[n]) \otimes A \otimes A \xrightarrow{\alpha \otimes \mathrm{id}_A} A[n] \otimes A \xrightarrow{r} A[n]$$

  where r is the right A-module structure on $A[n]$;

- v'_1 defined as the composition

$$\underline{Hom}_M(A, A[n]) \otimes A \otimes A \xrightarrow{\sigma \otimes \mathrm{id}_A} A \otimes \underline{Hom}_M(A, A[n]) \otimes A \xrightarrow{\mathrm{id}_A \otimes \alpha} A \otimes A[n] \xrightarrow{l} A[n]$$

  where l is the left A-module structure on $A[n]$, and σ is the symmetry for $\underline{Hom}_M(A, A[n]) \otimes A$;

If we denote by $\mu_1, u_1, v_1 : \underline{Hom}_M(A, A[n]) \longrightarrow \underline{Hom}_M(A \otimes A, A[n])$ the adjoint maps to μ'_1, u'_1, v'_1, then the object $T(A, n)$ is the kernel of the morphism

$$\mu_1 - u_1 - v_1 : \underline{Hom}_M(A, A[n]) \longrightarrow \underline{Hom}_M(A^{\otimes 2}, A[n]).$$

More generally, for any $p \geq 0$, we define $T^{(p)}(A, n)$ the A-module of *p-multiderivations* from $A^{\otimes p}$ to $A[np]$. This is the A-module of morphisms $A^{\otimes p} \longrightarrow A[np]$ which are derivations in each variable separately. More precisely, let us consider the canonical map

$$\alpha_p : \underline{Hom}_M(A^{\otimes p}, A[np]) \otimes A^{\otimes p} \longrightarrow A[np]$$

in M, adjoint to the identity of $\underline{Hom}_M(A^{\otimes p}, A[n])$, the multiplication map $m : A \otimes A \longrightarrow A$, and, for any pair (P, Q) of A-modules, let us denote by $\sigma(P, Q)$ the symmetry map $P \otimes Q \to Q \otimes P$. Then, for any $1 \leq i \leq p$, we can define the following three morphisms

- μ'_i defined as the composition

$$\underline{Hom}_M(A^{\otimes p}, A[np]) \otimes A^{\otimes p+1} \xrightarrow{\mathrm{id} \otimes m \otimes \mathrm{id}} \underline{Hom}_M(A^{\otimes p}, A[np]) \otimes A^{\otimes p} \xrightarrow{\alpha_p} A[n]$$

  where m is the multiplication map $A_{(i)} \otimes A_{(i+1)} \to A$ on the $(i, i+1)$ factors of $A^{\otimes p+1}$;

- u'_i defined as the composition

$$\begin{tikzcd} \underline{Hom}_M(A^{\otimes p}, A[np]) \otimes A^{\otimes p+1} \arrow[r, "\mathrm{id} \otimes \sigma_{(i+1)}"] & \underline{Hom}_M(A^{\otimes p}, A[np]) \otimes A^{\otimes p+1} \arrow[r, "\alpha \otimes \mathrm{id}_A"] & A[n] \otimes A \arrow[d, "r"] \\ & & A[n] \end{tikzcd}$$

  where $\sigma_{(i+1)} := \sigma(A_{(i+1)}, A^{\otimes p-i-1})$, and r is the right A-module structure on $A[n]$;

- v'_i defined as the composition

$$\begin{tikzcd} \underline{Hom}_M(A^{\otimes p}, A[np]) \otimes A^{\otimes p+1} \arrow[r, "\tau_{(i)} \otimes \mathrm{id}"] & A \otimes \underline{Hom}_M(A^{\otimes p}, A[np]) \otimes A^{\otimes p} \arrow[r, "\mathrm{id}_A \otimes \alpha"] & A \otimes A[n] \arrow[d, "l"] \\ & & A[n] \end{tikzcd}$$

  where $\tau_{(i)} := \sigma(\underline{Hom}_M(A^{\otimes p}, A[np]) \otimes A^{\otimes i-1}, A_{(i)})$, and l is the left A-module structure on $A[n]$.

We denote by $\mu_i, u_i, v_i : \underline{Hom}_M(A^{\otimes p}, A[np]) \longrightarrow \underline{Hom}_M(A^{\otimes p+1}, A[np])$ the adjoint maps to μ'_i, u'_i, v'_i.

We have, for each $1 \leq i \leq p$ a sub-object in M

$$Ker(\mu_i - u_i - v_i) \subset \underline{Hom}_M(A^{\otimes p}, A[np]).$$

The intersection of all these sub-objects defines

$$T^{(p)}(A,n) := \cap Ker(\mu_i - u_i - v_i) \subset \underline{Hom}_M(A^{\otimes p}, A[np]).$$

The symmetric group Σ_p acts on $\underline{Hom}_M(A^{\otimes p}, A[np])$, by its standard action on $A^{\otimes p}$, and by $(-1)^n{\cdot}\mathrm{Sign}$ on $A[np]$ which is the natural action when $A[np]$ is identified with $A[n]^{\otimes_A p}$. This action stabilizes the sub-object $T^{(p)}(A,n)$ and thus induces a Σ_p-action on $T^{(p)}(A,n)$. We set[3]

$$\mathsf{Pol}^{int}(A,n) := \bigoplus_{p\geq 0} (T^{(p)}(A,-n))^{\Sigma_p} \in M,$$

and call it the *object of internal n-shifted polyvectors on A.*

The object $\mathsf{Pol}^{int}(A,n)$ is naturally endowed with a structure of a graded $(n+1)$-Poisson commutative dg-algebra in M as follows.

- The weight $\mathbb{Z}$-grading is the usual one, with $(T^{(p)}(A,-n))^{\Sigma_p}$ being of weight p by definition. The multiplication morphisms

$$(T^{(p)}(A,-n))^{\Sigma_p} \otimes (T^{(q)}(A,-n))^{\Sigma_q} \longrightarrow (T^{(p+q)}(A,-n))^{\Sigma_{p+q}}$$

  are induced by composing the natural morphisms

$$\underline{Hom}_M(A^{\otimes p}, A[-np]) \otimes \underline{Hom}_M(A^{\otimes q}, A[-nq]) \longrightarrow \underline{Hom}_M(A^{\otimes p+q}, A[-np] \otimes A[-nq]),$$

  with the multiplication in the monoid A:

$$A[-np] \otimes A[-nq] \simeq (A \otimes A)[-n(p+q]] \longrightarrow A[-n(p+q)],$$

  and then applying the symmetrization with respect to Σ_{p+q}. This endows the object $\mathsf{Pol}^{int}(A,n)$ with the structure of a graded commutative monoid object in M.

- The Lie structure, shifted by $-n$, on $\mathsf{Pol}^{int}(A,n)$ is itself a version of the Schouten-Nijenhuis bracket on polyvector fields. One way to define it categorically is to consider the graded object

[3] Since we work in characteristic 0, we could have used coinvariants instead of invariants.

$\mathsf{Pol}^{int}(A,n)[n]$ as a sub-object of

$$Conv(A,n) := \bigoplus_{p\geq 0} \underline{Hom}_M(A^{\otimes p}, A[-np])^{\Sigma_p}[n].$$

The graded object $Conv(A,n)$ is a graded Lie algebra in M, where the Lie bracket is given by natural explicit formulas given by generalized commutators (the notation $Conv$ here refers to the convolution Lie algebra of the operad $Comm$ with the endomorphism operad of A, see [Lo-Va]). We refer to [Lo-Va, 10.1.7] and [Me, §2] for more details. This Lie bracket restricts to a graded Lie algebra structure on $\mathsf{Pol}^{int}(A,n)[n]$.

The Lie bracket $\mathsf{Pol}^{int}(A,n)$ is easily seen to be compatible with the graded algebra structure, i.e. $\mathsf{Pol}^{int}(A,n)$ is a graded $\mathbb{P}_{n+1}$-algebra object in M.

Definition 1.4.7 *Let $A \in \mathbf{cdga}_M$ be a commutative monoid in M. The* graded $\mathbb{P}_{n+1}$-algebra of n-shifted polyvectors on A *is defined to be*

$$\mathsf{Pol}^{int}(A,n) \in \mathbb{P}_{n+1} - \mathbf{cdga}_M^{gr}$$

described above.

For a commutative monoid $A \in Comm(M^{gr})$, the graded $\mathbb{P}_{n+1}$-algebra $\mathsf{Pol}^{int}(A,n)$ is related to the set of (non graded) $\mathbb{P}_n$-structures on A in the following way. The commutative monoid structure on A is given by a morphism of (symmetric) operads in $C(k)$

$$\phi_A : Comm \longrightarrow \underline{Hom}_k(A^{\otimes \bullet}, A),$$

where the right hand side is the usual endomorphism operad of $A \in M$ (which is an operad in $C(k)$). We have a natural morphism of operads $Comm \longrightarrow \mathbb{P}_n$, inducing the forgetful functor from $\mathbb{P}_n$-algebras to commutative monoids, by forgetting the Lie bracket. The set of $\mathbb{P}_n$-algebra structures on A is by definition the set of lifts of ϕ_A to a morphism $\mathbb{P}_n \longrightarrow \underline{Hom}_{C(k)}(A^{\otimes \bullet}, A)$

$$\mathbb{P}_n^{str}(A) := Hom_{Comm/\mathsf{Op}}(\mathbb{P}_n, \underline{Hom}_k(A^{\otimes \bullet}, A)).$$

The superscript *str* stands for *strict*, and is used to distinguish this operad from its ∞-categorical version that will be introduced below. Recall that $\mathsf{Pol}^{int}(A,n)[n]$ is a Lie algebra object in M^{gr}, and consider another Lie algebra object $\mathbf{1}(2)[-1]$ in M^{gr} given by $\mathbf{1}[-1] \in M$ with zero bracket and pure weight grading equal to 2.

Proposition 1.4.8 *There is a natural bijection*

$$\mathbb{P}_n^{str}(A) \simeq Hom_{Lie_M^{gr}}(\mathbf{1}(2)[-1], \mathsf{Pol}^{int}(A,n)[n])$$

where the right hand side is the set of morphisms of Lie algebra objects in M^{gr}.

Proof. Recall that M^{gr} is $C(k)^{gr}$-enriched, and let us consider the corresponding symmetric lax monoidal functor $R := \underline{Hom}_k^{gr}(\mathbf{1}, -) : M^{gr} \longrightarrow C(k)^{gr}$, where $\mathbf{1}$ sits in pure weight 0. From a morphism $f : \mathbf{1}(2)[-1] \longrightarrow \mathsf{Pol}^{int}(A,n)[n]$ of graded Lie algebras in M, we get a morphism of graded Lie algebras in $C(k)$

$$R(f) : k(2)[-1] \longrightarrow R(\mathsf{Pol}^{int}(A,n)[n]).$$

Now, the image under $R(f)$ of the degree 1-cycle $1 \in k$ is then a morphism

$$\varphi := R(f)(1) : \mathbf{1} \longrightarrow T^{(2)}(A,-n)[n+1]^{\Sigma_2}$$

in M. By definition of $T^{(2)}(A,-n)[n+1]^{\Sigma_2}$, the shift $\varphi[2(n-1)]$ defines a morphism in M

$$[-,-] : A[n-1] \otimes A[n-1] \longrightarrow A[n-1],$$

which is a derivation in each variable and is Σ_2-invariant. The fact that the Lie bracket is zero on $k[-1]$ implies that this bracket yields a Lie structure on A. This defines a $\mathbb{P}_n$-structure on A and we leave to the reader to verify that this is a bijection (see also [Me, Proof of Theorem 3.1]).

$\square$

Later on we will need the ∞-categorical version of the previous proposition, which is a much harder statement. For future reference we formulate this ∞-categorical version below but we refer the reader to [Me] for the details of the proof. Let $A \in \mathbf{cdga}_{\mathcal{M}}$ be a commutative dg-algebra in $\mathcal{M}$. We consider the forgetful ∞-functor

$$\mathrm{U}_{\mathbb{P}_n} : \mathbb{P}_n - \mathbf{cdga}_{\mathcal{M}} \longrightarrow \mathbf{cdga}_{\mathcal{M}}$$

sending a $\mathbb{P}_n$-algebra in $\mathcal{M}$ to its underlying commutative monoid in $\mathcal{M}$. The fiber at $A \in \mathbf{cdga}_{\mathcal{M}}$ of this ∞-functor is an ∞-groupoid and thus corresponds to a space

$$\mathbb{P}_n(A) := \mathrm{U}_{\mathbb{P}_n}^{-1}(\{A\}) \in \mathcal{T}.$$

Theorem 1.4.9 [Me, Thm. 3.2] *Suppose that A is fibrant and cofibrant in $\mathbf{cdga}_M$. There is a natural equivalence of spaces*

$$\mathbb{P}_n(A) \simeq Map_{Lie_{\mathcal{M}}^{gr}}(\mathbf{1}_M(2)[-1], \mathsf{Pol}^{int}(A,n)[n])$$

where the right hand side is the mapping space of morphisms of inside the ∞-category of Lie algebra objects in $\mathcal{M}^{gr}$.

Remark 1.4.10 Theorem 3.2 in [Me] is stated for M the model category of non-positively graded dg-modules over k, but the same proof extends immediately to our general M. The original statement seems moreover to require a restriction to those cdga's having a dualizable cotangent complex. This is due to the fact that the author uses the tangent complex (i.e. the dual of the cotangent complex) in order to identify derivations. However, the actual proof produces an equivalence between (weak, shifted) Lie brackets and (weak) biderivations. Therefore if one identifies derivations using the linear dual of the symmetric algebra of the cotangent complex, the need to pass to the tangent complex disappears, and the result holds with the same proof and without the assumption of the cotangent complex being dualizable. This is the main reason we adopted Def. 1.4.7 as our definition of internal polyvectors.

Now we give a slight enhancement of Theorem 1.4.9 and, as a corollary, we will get a strictification result (Cor. 1.4.12) that will be used in §3.3.

Let $\mathsf{Poiss}^{\mathrm{eq}}_{M,n}$ be the category whose objects are pairs (A,π) where A is a fibrant-cofibrant object in $\mathbf{cdga}_M$, and π is a map $\mathbf{1}_M[-1](2) \to \mathsf{Pol}^{int}(A,n)[n]$ in the homotopy category of Lie^{gr}_M, and whose morphisms $(A,\pi) \to (A',\pi')$ are weak equivalences $u : A \to A'$ in $\mathbf{cdga}_M$ such that the diagram

$$\begin{tikzcd} & \mathsf{Pol}^{int}(A,n)[n] \arrow[dd, "\mathsf{Pol}^{int}(u,n)[n]"] \\ \mathbf{1}_{\mathsf{M}}[-1](2) \arrow[ur, "\pi"] \arrow[dr, "\pi'"'] & \\ & \mathsf{Pol}^{int}(A',n)[n] \end{tikzcd}$$

is commutative in the homotopy category of Lie^{gr}_M. We denote the nerve of $\mathsf{Poiss}^{\mathrm{eq}}_{M,n}$ by $\mathbf{Poiss}^{\mathrm{eq}}_{M,n}$.

There is an obvious (strict) functor w from the category cofibrant-fibrant objects in $\mathbb{P}_{n+1}-\mathbf{cdga}_M$ and weak equivalences, to $\mathsf{Poiss}^{\mathrm{eq}}_{M,n}$, sending a strict $\mathbb{P}_{n+1}$-algebra B in M to the pair (B,π), where π is induced, in the standard way, by the (strict) Lie bracket on B (since the bracket is strict, it is a strict biderivation on B, and the classical construction carries over). Restriction to weak equivalences (between cofibrant-fibrant objects) in $\mathbb{P}_n - \mathbf{cdga}_M$ ensures this is a functor, and note that objects in the image of w are, by definition, *strict* pairs, i.e. maps $\pi : \mathbf{1}_M[-1](2) \to \mathsf{Pol}^{int}(A,n)[n]$ are actual morphism in Lie^{gr}_M (rather than just maps in the homotopy category). The functor w is compatible with the forgetful functors $p : \mathbb{P}_n - \mathbf{cdga}_M \to \mathbf{cdga}_M$, and $q : \mathsf{Poiss}^{\mathrm{eq}}_{M,n} \to \mathbf{cdga}_M$, and by passing to the nerves, we thus obtain a commutative diagram in $\mathcal{T}$ (where we have kept the same name for the maps)

$$\begin{tikzcd}
\mathcal{I}(\mathbb{P}_{n+1} - \mathbf{cdga}_{\mathcal{M}}) \arrow[rr, "w"] \arrow[dr, "p"'] & & \mathbf{Poiss}^{\mathrm{eq}}_{M,n} \arrow[dl, "q"] \\
& \mathcal{I}(\mathbf{cdga}_{\mathcal{M}}) &
\end{tikzcd}$$

where $\mathcal{I}(\mathcal{C})$ denotes the maximal ∞-subgroupoid of an ∞-category $\mathcal{C}$, i.e. the classifying space of $\mathcal{C}$. Note that p, and q are both surjective, since they both have a section given by choosing the trivial bracket or the trivial strict map π.

Theorem 1.4.11 *The map of spaces* $w : \mathcal{I}(\mathbb{P}_{n+1} - \mathbf{cdga}_{\mathcal{M}}) \to \mathbf{Poiss}^{\mathrm{eq}}_{n}$ *is an equivalence.*

Proof. It is enough to prove that for any cofibrant $A \in \mathbf{cdga}_M$, the map induced by w between q and p fibers over A is an equivalence. But this is exactly Theorem 1.4.9. $\square$

As an immediate consequence, we get the following useful strictification result. An arbitrary object (A, π) in $\mathsf{Poiss}^{\mathrm{eq}}_{M,n}$ will be called a *weak pair*, and we will call it a *strict pair* if π is strict, i.e. is an actual morphism $\pi : \mathbf{1}_M[-1](2) \to \mathsf{Pol}^{int}(A, n)[n]$ in Lie^{gr}_M.

Corollary 1.4.12 *Any weak pair is equivalent, inside* $\mathsf{Poiss}^{eq}_{M,n}$*, to a strict pair.*

Proof. By Theorem 1.4.11, an object $(A, \pi) \in \mathsf{Poiss}^{\mathrm{eq}}_{M,n}$ (i.e. an a priori weak pair), is equivalent to a pair of the form $w(B)$, where $B \in \mathbb{P}_{n+1} - \mathbf{cdga}_M$ (i.e. is a strict $\mathbb{P}_{n+1}$-algebra in M), whose underlying commutative algebra is weakly equivalent to A in $\mathbf{cdga}_M$. We conclude by observing that objects in the image of w are always strict pairs. $\square$

Functoriality. The assignment $A \mapsto \mathsf{Pol}^{int}(A, n)$ is not quite functorial in A, and it is therefore not totally obvious how to define its derived version. We will show however that it can be derived to an ∞-functor from a certain sub-∞-category of formally étale morphisms

$$\mathsf{Pol}^{int}(-, n) : \mathbf{cdga}^{fet}_{\mathcal{M}} \longrightarrow \mathbb{P}_{n+1} - \mathbf{cdga}^{gr}_{\mathcal{M}}.$$

We start with a (small) category I and consider the model category M^I of diagrams of shape I in M. It is endowed with the model category structure for which the cofibrations and equivalences are defined levelwise. As such, it is a symmetric monoidal model category which satisfies again our conditions $(1) - (5)$ of 1.1. For

$$(\underline{A} : i \ni I \longmapsto A_i \in Comm(M)) \in Comm(M^I) \simeq Comm(M)^I$$

an I-diagram of commutative monoids in M, we have its graded $\mathbb{P}_{n+1}$-algebra of polyvectors $\mathsf{Pol}^{int}(\underline{A}, n) \in$

$\mathbb{P}_{n+1} - \mathbf{cdga}^{gr}_{M^I} \simeq (\mathbb{P}_{n+1} - \mathbf{cdga}^{gr}_{M})^I$.

Lemma 1.4.13 *With the above notation, assume that $\underline{A}$ satisfies the following conditions*

- *$\underline{A}$ is a fibrant and cofibrant object in $Comm(M)^I$.*
- *For every morphism $i \to j$ in I, the morphism $A_i \to A_j$ induces an equivalence in $Ho(M)$*

$$\mathbb{L}_{A_i} \otimes^{\mathbb{L}}_{A_i} A_j \simeq \mathbb{L}_{A_j}.$$

Then, we have:

1. *for every object $i \in I$ there is a natural equivalence of graded $\mathbb{P}_{n+1}$-algebras*

$$\mathsf{Pol}^{int}(\underline{A}, n)_i \xrightarrow{\sim} \mathsf{Pol}^{int}(A_i, n),$$

2. *for every morphism $i \to j$ the induced morphism*

$$\mathsf{Pol}^{int}(\underline{A}, n)_i \longrightarrow \mathsf{Pol}^{int}(\underline{A}, n)_j$$

is an equivalence of graded $\mathbb{P}_{n+1}$-algebras.

Proof. Since $\underline{A}$ is fibrant and cofibrant as an object of $Comm(M)^I$, we have that for all $i \in I$ the object A_i is again fibrant and cofibrant in $Comm(M)$. As a consequence, for all $i \in I$, the A_i-module $\mathbb{L}_{A_i}$ can be represented by the strict model $\Omega^1_{A_i}$. Moreover, the second assumption implies that for all $i \to j$ in I the induced morphism

$$\Omega^1_{A_i} \otimes_{A_i} A_j \longrightarrow \Omega^1_{A_j}$$

is an equivalence in M.

As $\underline{A}$ is cofibrant, so is the $\underline{A}$-module $\Omega^1_{\underline{A}} \in \underline{A} - Mod_{M^I}$. This implies that $(\Omega^1_{\underline{A}})^{\otimes_{\underline{A}} p}$ is again a cofibrant object in $\underline{A} - Mod_{M^I}$. The graded object $\mathsf{Pol}^{int}(\underline{A}, n)$ in M^I of n-shifted polyvectors on A is thus given by

$$\bigoplus_{p \geq 0} \underline{Hom}_{\underline{A}-Mod_{M^I}}((\Omega^1_{\underline{A}})^{\otimes_{\underline{A}} p}, \underline{A}[-np])^{\Sigma_p}.$$

For all $i \in I$, and all $p \geq 0$, we have a natural evaluation-at-i morphism

$$\underline{Hom}_{\underline{A}-Mod_{M^I}}((\Omega^1_{\underline{A}})^{\otimes_{\underline{A}} p}, \underline{A}[-np])^{\Sigma_p} \longrightarrow \underline{Hom}_{A_i-Mod_M}((\Omega^1_{A_i})^{\otimes_{A_i} p}, A_i[-np])^{\Sigma_p}.$$

We now use the following sublemma

Sub-Lemma 1.4.14 *Let $\underline{A}$ be a commutative monoid in M^I. Let E and F be two $\underline{A}$-module objects, with E cofibrant and F fibrant. We assume that for all $i \to j$ in I the induced morphisms*

$$E_i \longrightarrow E_j \qquad F_i \longrightarrow F_j$$

are equivalences in M. Then, for all $i \in I$, the evaluation morphism

$$\underline{Hom}_{\underline{A}-Mod_{M^I}}(E,F)_i \longrightarrow \underline{Hom}_{A_i-Mod_M}(E_i,F_i)$$

is an equivalence in M.

Proof of sub-lemma 1.4.14. For $i \in I$, we have a natural isomorphism

$$\underline{Hom}_{\underline{A}-Mod_{M^I}}(E,F)_i \simeq \underline{Hom}_M(E_{|i},F_{|i}),$$

where $(-)_{|i} : M^I \longrightarrow M^{i/I}$ denotes the restriction functor, and $\underline{Hom}_M$ now denotes the natural enriched Hom of $M^{i/I}$ with values in M. This restriction functor preserves fibrant and cofibrant objects, so $E_{|i}$ and $F_{|i}$ are cofibrant and fibrant $A_{|i}$-modules. By assumption, if we denote by $E_i \otimes A_{|i}$ the $A_{|i}$-module sending $i \to j$ to $E_i \otimes_{A_i} A_j \in A_j - Mod_M$, the natural adjunction morphism

$$E_i \otimes A_{|i} \longrightarrow E_{|i}$$

is an equivalence of cofibrant $A_{|i}$-modules. This implies that the induced morphism

$$\underline{Hom}_M(E_{|i},F_{|i}) \longrightarrow \underline{Hom}_M(E_i \otimes A_{|i},F_{|i}) \simeq \underline{Hom}_M(E_i,F_i)$$

is an equivalence in M. $\square$

Sublemma 1.4.14 implies that the evaluation morphism $\mathsf{Pol}^{int}(\underline{A},n)_i \longrightarrow \mathsf{Pol}^{int}(A_i,n)$ is an equivalence. As this morphism is a morphism of graded $\mathbb{P}_{n+1}$-algebras, this proves assertion (1) of the lemma. Assertion (2) is proven in the same manner. $\square$

While it is not true that an arbitrary morphism $A \longrightarrow B$ in $Comm(M)$ induces a morphism $\mathsf{Pol}^{int}(A,n) \longrightarrow \mathsf{Pol}(B,n)$ (i.e. polyvectors are not functorial for arbitrary morphisms), Lemma 1.4.13 provides a way to understand a restricted functoriality of the construction $A \mapsto \mathsf{Pol}^{int}(A,n)$. In fact, let I be the sub-category of morphisms in $\mathbf{cdga}_M$ consisting of all morphisms $A \to B$ which are *formally étale* i.e. morphisms for which the induced map

$$\mathbb{L}_A^{int} \otimes_A^{\mathbb{L}} B \longrightarrow \mathbb{L}_B^{int}$$

is an isomorphism in $Ho(M)$, or equivalently in $B-Mod_M$. The category I is not small but things can be arranged by fixing universes, or bounding the cardinality of objects. We have a natural inclusion functor $I \longrightarrow \mathbf{cdga}_M$, and we choose a fibrant and cofibrant model for this functor, denoted as

$$\mathcal{A} : I \longrightarrow \mathbf{cdga}_M.$$

This functor satisfies the conditions of Lemma 1.4.13 above, and thus induces an ∞-functor after inverting equivalences

$$\mathsf{Pol}^{int}(\mathcal{A}, n) : L(I) \longrightarrow L(\mathbb{P}_{n+1} - \mathbf{cdga}_M^{gr}) = \mathbb{P}_{n+1} - \mathbf{cdga}_{\mathcal{M}}^{gr}.$$

The ∞-category $L(I)$ is naturally equivalent to the (non-full) sub-∞-category of $L(\mathbf{cdga}_M) = \mathbf{cdga}_{\mathcal{M}}$ consisting of formally étale morphisms. We denote this ∞-category by $\mathbf{cdga}_{\mathcal{M}}^{fet} \subset \mathbf{cdga}_{\mathcal{M}}$. We thus have constructed an ∞-functor

$$\mathbf{Pol}^{int}(-, n) := \mathsf{Pol}^{int}(\mathcal{A}, n) : \mathbf{cdga}_{\mathcal{M}}^{fet} \longrightarrow \mathbb{P}_{n+1} - \mathbf{cdga}_{\mathcal{M}}^{gr}.$$

Definition 1.4.15 *The ∞-functor*

$$\mathbf{Pol}^{int}(-, n) : \mathbf{cdga}_{\mathcal{M}}^{fet} \longrightarrow \mathbb{P}_{n+1} - \mathbf{cdga}_{\mathcal{M}}^{gr}$$

is called the functor of graded $\mathbb{P}_{n+1}$-algebras of internal n-shifted polyvectors *in $\mathcal{M}$.*

1. *If $A \in \mathbf{cdga}_{\mathcal{M}}$ is a commutative dg-algebra in $\mathcal{M}$, the* graded $\mathbb{P}_{n+1}$-algebra of internal n-shifted polyvectors on A *is its value* $\mathbf{Pol}^{int}(A, n) \in \mathbb{P}_n - \mathbf{cdga}_{\mathcal{M}}^{gr}$ *at A.*

2. *If $A \in \mathbf{cdga}_{\mathcal{M}}$ is a commutative dg-algebra in $\mathcal{M}$, the* graded $\mathbb{P}_{n+1}$-algebra of n-shifted polyvectors on A *is* $\mathbf{Pol}(A, n) := |\mathbf{Pol}^{int}(A, n)| \in \mathbb{P}_n - \mathbf{cdga}_k^{gr}$.

Remark 1.4.16 Note that, by lemma 1.4.13, we know that the values of the ∞-functor $\mathbf{Pol}^{int}$ at $A \in \mathbf{cdga}_M$ is naturally equivalent, inside $\mathbb{P}_n - \mathbf{cdga}_{\mathcal{M}}^{gr}$, to the graded $\mathbb{P}_{n+1}$-algebra $\mathsf{Pol}^{int}(QR(A), n)$, where $QR(A)$ is a fibrant and cofibrant model for A in $\mathbf{cdga}_M$.

1.4.3 $\mathbb{P}_n$-structures and symplectic forms.

In this section we explain how the standard relation between Poisson structures and differential forms manifests itself in our setting.

Construction ϕ_π. Let $A' \in \mathbf{cdga}_M$ be a commutative dg-algebra over M. We fix an integer $n \in \mathbb{Z}$, and we consider on one side $\mathsf{Pol}^{int}(A',n)$, the n-shifted polyvectors on A', and on the other side, $DR^{str}(A')$, the strict de Rham complex of A'. By Proposition 1.4.8, a (strict) $\mathbb{P}_n$-structure on A' is nothing else than a morphism of graded dg-Lie algebras in M

$$\pi : \mathbf{1}(2)[-1] \longrightarrow \mathsf{Pol}^{int}(A',n)[n].$$

Assume that one such $\mathbb{P}_n$-structure π is fixed on A'. We can use π in order to define a structure of a graded mixed object on $\mathsf{Pol}^{int}(A',n)$, as follows. Recall that the weight q part of $\mathsf{Pol}(A',n)$ is the object $T^{(q)}(A',-n)^{\Sigma_q}$ of Σ_q-invariant multiderivations $A'^{\otimes q} \longrightarrow A'[-nq]$. Consider the symmetric lax monoidal functor $R := \underline{Hom}_k^{gr}(\mathbf{1},-) : M^{gr} \longrightarrow C(k)^{gr}$ (where $\mathbf{1}$ sits in weight 0). Then $R(\pi) : k(2)[-1] \longrightarrow R(\mathsf{Pol}^{int}(A',n))[n]$ is a morphism of graded Lie algebras in $C(k)$. The image under $R(\pi)$ of the degree 1 cycle $1 \in k$ is then a morphism

$$\underline{\pi} := R(\pi)(1) : \mathbf{1} \longrightarrow T^{(2)}(A',-n)[n+1]^{\Sigma_2}$$

in M. The composite map

$$\epsilon_\pi : \mathbf{1} \otimes T^{(q)}(A',-n)^{\Sigma_q} \xrightarrow{\underline{\pi}\otimes \mathrm{id}} T^{(2)}(A',-n)[n+1]^{\Sigma_2} \otimes T^{(q)}(A',-n)^{\Sigma_q} \xrightarrow{[-,-]} T^{(q+1)}(A',-n)[1]^{\Sigma_{q+1}}$$

(where $[-,-]$ denotes the Lie bracket part of the graded $\mathbb{P}_{n+1}$-structure on $\mathsf{Pol}^{int}(A',n)$) defines then a mixed structure on the graded object $\mathsf{Pol}^{int}(A',n)$, making it into a graded mixed object in M. This graded mixed structure is also compatible with the multiplication and endows $\mathsf{Pol}^{int}(A',n)$ with a *graded mixed commutative dg-algebra* structure in M.

Since in weight 0 we have $\mathsf{Pol}^{int}(A',n)(0) = A'$, the identity map $A' \to A'$ induces, by Section 1.3.3, a morphism

$$\phi_{\pi,A'} : DR^{str}(A') \longrightarrow \mathsf{Pol}^{int}(A',n)$$

of graded mixed commutative algebras in M.

Remark 1.4.17 Here is an equivalent way of constructing $\phi_{\pi,A'} : DR^{str}(A') \to \mathsf{Pol}^{int}(A',n)$. The morphism $\underline{\pi}$ defines a morphism of A'-modules $\wedge^2_{A'}\Omega^1_{A'} \longrightarrow A'[1-n]$, and, by duality, a morphism of A-modules

$$\Omega^1_{A'}[-1] \longrightarrow \underline{Hom}_{A'-Mod}(\Omega^1_{A'}, A'[-n]) \simeq T^{(1)}(A',-n)$$

Since $\mathsf{Pol}^{int}(A',n) \in \mathbf{cdga}_M^{gr}$, by composing it with the map $T^{(1)}(A',-n) \to \mathsf{Pol}^{int}(A',n)$, and using adjunction, we get and induced map

$$Sym_{A'}(\Omega^1_{A'}[-1]) \longrightarrow \mathsf{Pol}^{int}(A',n)$$

of graded commutative algebras in M. Now it is enough to invoke the isomorphism $DR^{str}(A') \simeq Sym_A(\Omega^1_{A'}[-1])$ (see Section 1.3.3), to obtain a map of graded commutative algebras

$$\phi_{\pi,A'} : DR^{str}(A') \longrightarrow \mathsf{Pol}^{int}(A', n)$$

that can be verified to strictly preserve with the mixed differentials on both sides. Thus $\phi_{\pi,A'}$ is a map of graded mixed commutative algebras in M.

Let now $A \in \mathbb{P}_n - \mathbf{cdga}_{\mathcal{M}}$. Since $\mathbb{P}_n - \mathbf{cdga}_{\mathcal{M}} = L(\mathbb{P}_n - \mathbf{cdga}_M)$, we may choose A' fibrant-cofibrant in $\mathbb{P}_n - \mathbf{cdga}_M$ (a strict $\mathbb{P}_n$-algebra in M) which is equivalent to A inside $\mathbb{P}_n - \mathbf{cdga}_{\mathcal{M}}$. Since we have equivalences

$$\mathbf{DR}^{int}(A) \simeq \mathbf{DR}^{int}(A') \simeq DR^{str}(A')$$

in the ∞-category of graded mixed commutative algebras in $\mathcal{M}$, and equivalences (see Remark 1.4.16)

$$\mathbf{Pol}^{int}(A, n) \simeq \mathbf{Pol}^{int}(A', n) \simeq \mathsf{Pol}^{int}(A', n)$$

in $\mathbb{P}_{n+1} - \mathbf{cdga}^{gr}_{\mathcal{M}}$, we may run the above Construction ϕ_π on A', and use Definition 1.4.15 in order to :

- turn $\mathbf{Pol}^{int}(-, n)$ into an ∞-functor

$$\mathbf{Pol}^{int}(-, n) : (\mathbb{P}_n - \mathbf{cdga}_{\mathcal{M}})^{eq} \longrightarrow (\epsilon - \mathbf{cdga}^{gr}_{M})^{eq} \;;$$

- consider the functor

$$\mathbf{DR}^{int} : (\mathbb{P}_n - \mathbf{cdga}_{\mathcal{M}})^{eq} \longrightarrow (\epsilon - \mathbf{cdga}^{gr}_{M})^{eq}$$

  as the composition of the restriction $\mathbf{DR}^{int} : (\mathbf{cdga}_{\mathcal{M}})^{eq} \to (\epsilon - \mathbf{cdga}^{gr}_{M})^{eq}$ of the usual $\mathbf{DR}^{int}$ functor, with the forgetful functor $(\mathbb{P}_n - \mathbf{cdga}_{\mathcal{M}})^{eq} \to (\mathbf{cdga}_{\mathcal{M}})^{eq}$;

- promote the collection of all $\phi_{\pi,A}$'s to a morphism

$$\phi_\pi : \mathbf{DR}^{int} \longrightarrow \mathbf{Pol}^{int}(-, n)$$

  that is well defined in the ∞-category of ∞-functors from $(\mathbb{P}_n - \mathbf{cdga}_{\mathcal{M}})^{eq}$ to $(\epsilon - \mathbf{cdga}^{gr}_{M})^{eq}$.

Definition 1.4.18 *We say that $A \in \mathbb{P}_n - \mathbf{cdga}_{\mathcal{M}}$ is* non-degenerate *if the previous morphism*

$$\phi_{\pi,A} : \mathbf{DR}^{int}(A) \longrightarrow \mathbf{Pol}^{int}(A, n)$$

is an equivalence of graded objects in $\mathcal{M}$.

Remark 1.4.19 This definition is obviously independent of the choice of a strict model A', and hence of the corresponding map π (which is then uniquely identified by A', by Thm. 1.4.8).

For an n-Poisson commutative cdga $A \in \mathbb{P}_n - \mathbf{cdga}_{\mathcal{M}}$, we consider, as above, a strict cofibrant-fibrant model $A' \in \mathbb{P}_n - \mathbf{cdga}_M$, together with the corresponding strict map $\pi : \mathbf{1}(2)[-1] \longrightarrow \mathsf{Pol}^{int}(A', n)[n]$ of graded dg-Lie algebras in M. Such a π defines a morphism of graded mixed objects in M:

$$\omega_{\pi,A'} : \mathbf{1}(2) \longrightarrow \mathsf{Pol}^{int}(A', n)[n+1].$$

Since $\mathbf{Pol}^{int}(A, n) \simeq \mathsf{Pol}^{int}(A', n)$, we thus obtain a diagram of graded mixed objects in $\mathcal{M}$:

$$\mathbf{DR}^{int}(A)[n+1] \xrightarrow{\phi_{\pi,A}[n+1]} \mathbf{Pol}^{int}(A,n)[n+1] \xleftarrow{\omega_{\pi,A}} \mathbf{1}(2),$$

for each $A \in \mathbb{P}_n - \mathbf{cdga}_{\mathcal{M}}$, which, upon realization, produces a diagram in graded mixed k-dg modules

$$\mathbf{DR}(A)[n+1] \xrightarrow{\phi_{\pi,A}[n+1]} \mathbf{Pol}(A,n)[n+1] \xleftarrow{\omega_{\pi,A}} k(2).$$

We use $\phi_{\pi,A}[n+1]$, and $\omega_{\pi,A}$ to identify $\mathbf{DR}(A)[n+1]$ and $k(2)$ as objects in the ∞-over-category $\epsilon - \mathbf{dg}_k^{gr}/\mathbf{Pol}(A, n+1)[n+1]$, and give the following

Definition 1.4.20 *Let $A \in \mathbb{P}_n - \mathbf{cdga}_{\mathcal{M}}$. The space of* closed 2-forms compatible with the $\mathbb{P}_n$-structure on A *is the space*

$$\mathsf{Map}_{\epsilon-\mathbf{dg}_k^{gr}/\mathbf{Pol}(A,n+1)[n+1]}(k(2), \mathbf{DR}(A)[n+1]) \in \mathcal{T}.$$

In other words, the space of closed 2-forms compatible with the $\mathbb{P}_n$-structure on A consists of lifts $k(2) \longrightarrow \mathbf{DR}(A)[n+1]$ of the morphism ω_π. There is a natural forgetful morphism

$$\mathsf{Map}_{\epsilon-\mathbf{dg}_k^{gr}/\mathbf{Pol}(A,n)[n+1]}(k(2), \mathbf{DR}(A)[n+1]) \longrightarrow \mathsf{Map}_{\epsilon-\mathbf{dg}_k^{gr}}(k(2), \mathbf{DR}(A)[n+1]) \simeq \mathcal{A}^{2,cl}(A, n-1),$$

to the space of closed 2-forms on A of degree $(n-1)$.

Note that, by definition, if a $\mathbb{P}_n$-algebra A in $\mathcal{M}$ is non-degenerate, then the space of closed 2-forms compatible with the $\mathbb{P}_n$-structure on A is contractible. In particular, we obtain in this case a well defined (in $\pi_0(\mathcal{A}^{2,cl}(A, n-1))$) and canonical closed 2-form ω of degree $(n-1)$ on A. Moreover, since π is assumed to be non-degenerate, then so is the corresponding underlying 2-form. For reference, we record this observation in the following

Corollary 1.4.21 • *Let $A \in \mathbb{P}_n - \mathbf{cdga}_{\mathcal{M}}$ be non-degenerate. Then there is a unique, up to a contractible space of choices, closed and non-degenerate 2-form of degree $(n-1)$ compatible with the $\mathbb{P}_n$-structure on A.*

- *As a consequence, for any $A \in \mathbf{cdga}_{\mathcal{M}}$, there is a well-defined morphism of spaces*

$$\mathbf{W}_A : \mathbb{P}_n(A)^{nd} \longrightarrow \mathsf{Symp}(A, n-1),$$

from the space of non-degenerate $\mathbb{P}_n$-structures on A to the space of $(n-1)$-shifted symplectic structures on A.

Remark 1.4.22 Moreover, exactly as we did for Definition 1.4.15, we get that $\mathbf{W}_A$ is functorial in A, with respect to formally étale maps in $\mathbf{cdga}_{\mathcal{M}}$.

We finish this paragraph with an important corollary that will be used in §3.2.
In order to prepare for the next definition, first of all, observe that if $A \in \mathbf{cdga}_{\mathcal{M}}$ has the property that $\mathbb{L}_A^{int}$ is dualizable in $A - Mod_{\mathcal{M}}$, then we have an equivalence

$$\mathbf{Pol}^{int}(A, n) \simeq \bigoplus_{p \geq 0} Sym_A^p(\mathbb{T}_A^{int}[-n])[n]$$

in $\mathcal{M}^{gr}$. Then, recall that for any A-module P in $\mathcal{M}$, maps $\mathbf{1}_{\mathcal{M}} \to P$ in $\mathcal{M}$ are in bijection with maps $A \simeq \mathbf{1}_{\mathcal{M}} \otimes_{\mathcal{M}} A \to P$ in $A - Mod_{\mathcal{M}}$.

Definition 1.4.23 *Let $A \in \mathbf{cdga}_{\mathcal{M}}$ such that $\mathbb{L}_A^{int}$ is a dualizable A-module in $\mathcal{M}$.*

- *A morphism in $Lie_{\mathcal{M}}^{gr}$*

$$k(2)[-1] \longrightarrow \mathbf{Pol}(A, n)[n] = |\mathbf{Pol}^{int}(A, n)[n]|$$

is non-degenerate *if the map in $\mathcal{M}$ (induced by the adjunction data for $|-|$)*

$$\mathbf{1}_{\mathcal{M}} \longrightarrow Sym_A^2(\mathbb{T}_A^{int}[-n])[n+1]$$

yields, by adjunction, an equivalence of A-modules

$$\mathbb{L}_A^{int} \simeq \mathbb{T}_A^{int}[1-n].$$

- *We denote by $\mathsf{Map}^{nd}_{\mathbf{dgLie}_k^{gr}}(k(2)[-1], \mathbf{Pol}(A, n)[n])$ is the subspace of $\mathsf{Map}_{\mathbf{dgLie}_k^{gr}}(k(2)[-1], \mathbf{Pol}(A, n)[n])$ of connected components of non-degenerate morphisms.*

By Theorem 1.4.9 and Corollary 1.4.21, we get

Corollary 1.4.24 *Let $A \in \mathbf{cdga}_{\mathcal{M}}$ such that $\mathbb{L}_A^{int}$ is a dualizable A-module in $\mathcal{M}$*

1. *The map in Thm. 1.4.9 restrict to an equivalence*

$$\mathbb{P}_n(A)^{nd} \simeq \mathsf{Map}^{nd}_{\mathbf{dgLie}_k^{gr}}(k(2)[-1], \mathbf{Pol}(A,n)[n]).$$

2. *There is a natural morphism (induced by $\mathbf{W}_A$ of Cor. 1.4.21) in $\mathcal{T}$*

$$\mathsf{Map}^{nd}_{\mathbf{dgLie}_k^{gr}}(k(2)[-1], \mathbf{Pol}(A,n)[n]) \longrightarrow \mathsf{Symp}(A, n-1),$$

functorial in A with respect to formally étale morphisms.

1.5 Mixed graded modules: Tate realization

One of the most important situations in which we will use the above formalism of de Rham objects and shifted polyvectors is when $\mathcal{M}$ is itself the ∞-category of graded mixed k-dg-modules, or more generally diagrams of such. The situation gets complicated because several different graded mixed structures interact in this setting. The language of relative differential calculus developed in the previous section comes handy here and allows us to avoid confusion.

Throughout this subsection, $M = \epsilon - dg_k^{gr}$. M is a symmetric monoidal category. Recall that, unless otherwise stated, it will be endowed with the injective model structure, for which cofibrations and weak equivalences are defined on the underlying graded complexes of k-modules; as such is a symmetric monoidal model category satisfying our standing assumptions (see Section 1.1). We let $\mathcal{M} = \epsilon - \mathbf{dg}_k^{gr}$ be the corresponding ∞-category. Recall that for $M = \epsilon - dg_k^{gr}$, and $E, F \in M$, the $\mathbf{dg}_k$-enriched hom object is explicitly given by

$$\underline{Hom}_k(E,F) \equiv \underline{Hom}(E,F) := \mathrm{Z}_\epsilon(\underline{\mathrm{Hom}}_\epsilon^{gr}(E,F)(0)) \in \mathbf{dg}_k$$

where $\underline{\mathrm{Hom}}_\epsilon^{gr}$ denotes the internal hom object in M (see Section 1.1), and, for $X \in M$, we denoted by $\mathrm{Z}_\epsilon(X(0)) \in \mathbf{dg}_k$ the kernel of the map of dg-modules $\epsilon : X(0) \to X(1)[1]$. The corresponding $\mathbf{dg}_k$-tensor structure is given by

$$V \otimes E := V(0) \otimes_M E$$

where $V(0)$ is the mixed graded dg-module concentrated in weight 0 and with trivial mixed differentials, and $\otimes_M$ is the monoidal structure in M (Section 1.1). Note that the functor $\mathbf{dg}_k \to M$ sending V to $V(0)$ (in the notation just introduced) is exactly the symmetric monoidal left Quillen functor defining the $\mathbf{dg}_k$-algebra model structure on M.

The category of commutative monoids in M is simply the category $\epsilon - cdga_k^{gr}$ of graded mixed cdgas, whose corresponding ∞-category is then $\mathbf{cdga}_{\mathcal{M}} = \epsilon - \mathbf{cdga}_k^{gr}$. As already observed earlier in this section, we have a forgetful ∞-functor

$$\mathrm{U}_\epsilon : \epsilon - \mathbf{dg}_k^{gr} \longrightarrow \mathbf{dg}_k^{gr}$$

forgetting the mixed structure. This ∞-functor is induced by a left Quillen symmetric monoidal functor and thus induces a functor

$$\mathrm{U}_\epsilon : \epsilon - \mathbf{cdga}_k^{gr} \longrightarrow \mathbf{cdga}_k^{gr}$$

It is easy to see that this ∞-functor preserves de Rham objects, in the sense that, for any $A \in \mathbf{cdga}_{\mathcal{M}} = \epsilon - \mathbf{cdga}_k^{gr}$, the natural morphism[4]

$$\mathrm{U}_\epsilon(\mathbb{L}_A^{int}) \longrightarrow \mathbb{L}_{\mathrm{U}_\epsilon(A)}^{int}$$

induces an equivalence

$$\mathrm{U}_\epsilon(\mathbf{DR}^{int}(A)) \simeq \mathbf{DR}^{int}(\mathrm{U}_\epsilon(A)),$$

of graded mixed cdga inside the ∞-category $\mathbf{dg}_k^{gr}$ of graded dg-modules (note that on the left hand side the functor U_ϵ sends $\epsilon - \mathbf{cdga}_{\mathcal{M}}^{gr}$ to $\epsilon - \mathbf{cdga}_{\mathbf{dg}_k^{gr}}^{gr}$). At the level of strict models this is even simpler, as for A a graded mixed cdga, the graded mixed A-module Ω_A^1 is canonically isomorphic, as a graded A-module, to $\Omega^1_{\mathrm{U}_\epsilon(A)}$. In other words, in order to compute Ω_A^1 as a graded mixed A-module we simply compute it as a graded A-module, and then endow it with the natural mixed structure coming from the one on A.

Recall (Definition 1.3.1 with $M = \epsilon - dg_k^{gr}$) that we have defined a realization functor

$$| - | : \mathcal{M} = \epsilon - \mathbf{dg}_k^{gr} \longrightarrow \mathbf{dg}_k$$

as the ∞-functor $\mathbb{R}\underline{Hom}(1_M, -)$ associated to the right derived functor of the Quillen right adjoint to the functor $- \otimes 1_M : dg_k \to M$ (here $1_M = k(0)$ is k sitting in weight 0, degree 0, with trivial differential and trivial mixed differential). As above, M is endowed here with the injective model structure, for which the monoidal unit 1_M is cofibrant. However, M can also be given the *projective* model structure M^{proj} where fibrations and weak equivalences are defined on the underlying graded complexes of k-modules. In M^{proj} the monoidal unit 1_M is no longer cofibrant, and we have already constructed in 1.4.1 an explicit cofibrant replacement $\widetilde{k} \to 1_M$ in M^{proj}. Moreover, $\widetilde{k}$ is a counital comonoid object in M, therefore we have a Quillen pair

$$- \otimes \widetilde{k} : dg_k \longleftrightarrow M : \underline{Hom}(\widetilde{k}, -)$$

where the right adjoint is lax symmetric Quillen monoidal. The identity functor on M induces an iden-

[4]About the target, recall that $\mathbb{L}_B^{int} \simeq \mathbb{L}_B$ in $\mathbf{dg}_k$ (respectively in $\mathbf{dg}_k^{gr}$) for any $B \in \mathbf{cdga}_k$ (respectively $B \in \mathbf{cdga}_k^{gr}$).

tification (equivalence) on the associated ∞-categories, and the realization functor $|-|$ is equivalent, under this identification, to the ∞-functor induced by the right derived Quillen functor $\mathbb{R}\underline{Hom}(\widetilde{k},-)$, i.e. the derived functor with respect to the projective model structure on M. Since in M^{proj}, unlike in the injective model structure on M, every object is fibrant, we have $\mathbb{R}\underline{Hom}(\widetilde{k},-) \simeq \underline{Hom}(\widetilde{k},-)$. Thus we conclude that as ∞-functors we have an equivalence

$$\mathbb{R}\underline{Hom}(k(0),-) := |-| \simeq \underline{Hom}(\widetilde{k},-) \ : \ \epsilon - \mathbf{dg}_k^{gr} \longrightarrow \mathbf{dg}_k$$

Proposition 1.5.1 *For any $E \in M$, there is a canonical isomorphism of k-dg modules*

$$\prod_{p\geq 0} E(p) \simeq \underline{Hom}_k(\widetilde{k}, E)$$

where the source is endowed with the total differential, i.e. the sum of the cohomological and the mixed differentials.

Proof. The complex $\underline{\mathrm{Hom}}_{\epsilon}^{gr}(\widetilde{k}, E)(0) \in C(k)$ is given in degree n by

$$E(0)^n \times \prod_{p>0} (E(p)^n \times E(p)^{n+1}).$$

The map $f : \prod_{p\geq 0} E(p) \to \underline{\mathrm{Hom}}_{\epsilon}^{gr}(\widetilde{k}, E)(0)$ defined (with obvious notations) in degree n by

$$f^n : \{x_0, (x_p)_{p>0}\} \longmapsto \{x_0, (x_p, -\epsilon_E(x_{p-1}))_{p>0}\}$$

is a map of complexes, and the composite

$$\textstyle\prod_{p\geq 0} E(p) \xrightarrow{f} \underline{\mathrm{Hom}}_{\epsilon}^{gr}(\widetilde{k}, E)(0) \xrightarrow{\epsilon_{\underline{Hom}_M}} \underline{\mathrm{Hom}}_{\epsilon}^{gr}(\widetilde{k}, E)(1)[1]$$

is zero. A computation now shows that the induced map $\overline{f} : \prod_{p\geq 0} E(p) \simeq \underline{Hom}_k(\widetilde{k}, E)$ is an isomorphism of k-dg-modules. $\square$

By Proposition 1.5.1, we get that the ∞-functor

$$|-| : \epsilon - \mathbf{dg}_k^{gr} \longrightarrow \mathbf{dg}_k$$

has a canonical strict model given by

$$E \longmapsto \prod_{p\geq 0} E(p),$$

where the right hand side is endowed with the total differential = sum of the cohomological differential and the mixed structure.

Since for any $i \in \mathbb{Z}$ the $(-i)$-weight shift $\widetilde{k}((-i))$ is a cofibrant resolution of $k(i)$ (i.e. of $k[0]$ concentrated in weight i) in M^{proj}, the above computation yields the following equivalences in $\mathbf{dg}_k$

$$\mathbb{R}\underline{Hom}_k(k(i), k(i+1)) \simeq k.$$

We thus have a canonical morphism $u_i : k(i) \longrightarrow k(i+1)$ in $\epsilon - \mathbf{dg}_k$ for all $i \in \mathbb{Z}$, corresponding to $1 \in k$ in the above formula. In particular, we get a pro-object in $\epsilon - \mathbf{dg}_k$

$$k(-\infty) := \{\cdots \to k(-i) \to k(-i+1) \to \cdots \to k(-1) \to k(0)\}.$$

Definition 1.5.2 *The* Tate *or* stabilized realization ∞-functor *is defined to be*

$$|-|^t := \mathbb{R}\underline{Hom}_k(k(-\infty), -) : \epsilon - \mathbf{dg}_k^{gr} \longrightarrow \mathbf{dg}_k,$$

sending $E \in \epsilon - \mathbf{dg}_k$ *to*

$$|E|^t = colim_{i\geq 0}\mathbb{R}\underline{Hom}_k(k(-i), E) \simeq colim_{i\geq 0} \prod_{p\geq -i} E(p).$$

The natural map $k(-\infty) \longrightarrow k(0)$ of pro-objects in $\epsilon - \mathbf{dg}_k$ (where $k(0)$ is considered as a constant pro-object) provides a natural transformation

$$|-| \longrightarrow |-|^t$$

from the standard realization to the Tate realization. By definition, we see that this natural transformation induces an equivalence $|E| \simeq |E|^t$ in $\mathbf{dg}_k$, as soon as $E(p) = 0$ for all $p < 0$.

The ∞-functor $|-|$ is lax symmetric monoidal, and this endows $|-|^t$ with a canonical structure of a lax symmetric monoidal ∞-functor. This follows, for instance, from the fact that the pro-object $k(-\infty)$ defined above is a cocommutative and counital coalgebra object, which is the dual of the commutative and unital algebra $\mathrm{colim}_{i\geq 0} k(i)$. Therefore the Tate realization induces an ∞-functor on commutative algebra objects in $\mathcal{M} = \epsilon - \mathbf{dg}_k^{gr}$, and more generally on all kind of algebra-like structures in $\mathcal{M}$. In particular, we have Tate realization functors, denoted with the same symbol, for graded mixed cdgas over $\epsilon - \mathbf{dg}_k^{gr}$, as well as for graded $\mathbb{P}_{n+1}$-cdgas

$$|-|^t : \epsilon - \mathbf{cdga}^{gr}_{\epsilon - \mathbf{dg}_k^{gr}} \longrightarrow \epsilon - \mathbf{cdga}_k^{gr}$$

$$|-|^t : \mathbb{P}_{n+1} - \mathbf{cdga}^{gr}_{\epsilon - \mathbf{dg}_k^{gr}} \longrightarrow \mathbb{P}_{n+1} - \mathbf{cdga}_k^{gr}.$$

This way we get Tate versions of the de Rham and shifted polyvectors objects introduced in Def. 1.3.9 and 1.4.15.

Definition 1.5.3 *Let* $A \in \mathbf{cdga}_{\epsilon-\mathbf{dg}_k^{gr}}$ *be commutative cdga in the* ∞*-category of graded mixed complexes (i.e. a graded mixed cdga over* k*).*

1. *The* Tate de Rham complex of A *is defined by*

$$\mathbf{DR}^t(A) := |\mathbf{DR}^{int}(A)|^t \in \epsilon - \mathbf{cdga}_k^{gr}.$$

2. *The* Tate n-shifted polyvectors cof A *is defined by*

$$\mathbf{Pol}^t(A,n) := |\mathbf{Pol}^{int}(A,n)|^t \in \mathbb{P}_{n+1} - \mathbf{cdga}_k^{gr}.$$

Note that we have natural induced morphisms

$$\mathbf{DR}(A) \longrightarrow \mathbf{DR}^t(A) \qquad \mathbf{Pol}(A,n) \longrightarrow \mathbf{Pol}^t(A,n)$$

which are not always equivalences. More precisely, if $A(p) = 0$ for all $p < 0$, then $\mathbb{L}_A^{int}$ is itself only positively weighted, and we get $\mathbf{DR}(A) \simeq \mathbf{DR}^t(A)$ by the natural morphism. On the other hand, $\mathsf{Pol}(A,n)$ has in general both positive and non-positive weights, as the weights of $\mathbb{T}_A^{int}$ are dual to that of A. So, except in some very degenerate cases, $\mathsf{Pol}(A,n) \longrightarrow \mathsf{Pol}^t(A,n)$ will typically *not* be an equivalence.

To finish this section we mention the Tate analogue of the morphism constructed in Corollary 1.4.24 from the space of non-degenerate n-shifted Poisson structures to the space of n-shifted symplectic structures.

The notion of Tate realization functor, can be interpreted as a standard realization functor for a slight modification of the base ∞-category $\mathcal{M} = \epsilon-\mathbf{dg}^{gr}$. The same is true for the objects $\mathbf{DR}^t(A)$ and $\mathbf{Pol}^t(A)$ at least under some mild finiteness conditions on A. In order to see this, we let $\mathcal{M}' := \mathsf{Ind}(\mathcal{M})$ be the ∞-category of Ind-objects in $\mathcal{M}$. The ∞-category $\mathcal{M}'$ is again symmetric monoidal and possesses as a model the model category $\mathsf{Ind}(M)$ of Ind-objects in M (see [Bar-Sch, Thm. 1.5]):

$$\mathsf{Ind}(\mathcal{M}) \simeq L(\mathsf{Ind}(M)).$$

We consider the following Ind-object in $\mathcal{M}$

$$k(\infty) := \{\ k(0) \longrightarrow k(1) \longrightarrow \ldots k(i) \longrightarrow k(i+1) \longrightarrow \cdots \}$$

which is objectwise dual to the pro-object $k(-\infty)$ we have considered above. Now, the standard realization ∞-functor $|-| : \mathcal{M}' \to \mathbf{dg}_k$ for $\mathcal{M}'$ recovers the Tate realization on $\mathcal{M}$, since we have a naturally commutative diagram of ∞-functors

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "-\otimes k(\infty)"] \arrow[dr, "|-|^t"'] & & \mathcal{M}' \arrow[dl, "|-|"] \\ & \mathbf{dg}_k & \end{tikzcd}$$

Moreover, the natural equivalences $k(i) \otimes k(j) \simeq k(i+j)$ makes $k(\infty)$ into a commutative cdga in $\mathcal{M}' = \mathsf{Ind}(\mathcal{M})$. For any $A \in \mathbf{cdga}_{\mathcal{M}}$, viewed as a constant commutative cdga in $\mathcal{M}'$ via the natural functor $\mathcal{M} \to \mathsf{Ind}(\mathcal{M}) = \mathcal{M}'$, we thus have a natural object obtained by base change

$$A(\infty) := A \otimes k(\infty) \in \mathbf{cdga}_{\mathcal{M}'}.$$

Note that, as an Ind-object in $\mathcal{M}$, we have

$$A(\infty) = \{\ A \otimes k(0) \longrightarrow A \otimes k(1) \longrightarrow \ldots A \otimes k(i) \longrightarrow A \otimes k(i+1) \longrightarrow \cdots \}$$

The cdga $A(\infty)$ will be considered as a $k(\infty)$-algebra object in $\mathcal{M}'$

$$A(\infty) \in k(\infty) - \mathbf{cdga}_{\mathcal{M}'} = k(\infty)/\mathbf{cdga}_{\mathcal{M}'}.$$

It therefore has the corresponding relative de Rham and polyvector objects

$$\mathbf{DR}^{int}(A(\infty)/k(\infty)) \in \epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}'} \qquad \mathbf{Pol}^{int}(A(\infty)/k(\infty), n) \in \mathbb{P}_{n+1} - \mathbf{cdga}^{gr}_{\mathcal{M}'},$$

and, as usual, we will denote by

$$\mathbf{DR}(A(\infty)/k(\infty)) \in \epsilon - \mathbf{cdga}^{gr}_{k} \qquad \mathbf{Pol}(A(\infty)/k(\infty), n) \in \mathbb{P}_{n+1} - \mathbf{cdga}^{gr}_{k}$$

the corresponding images under the standard realization $|-| : \mathcal{M}' \to \mathbf{dg}_k$, that, recall, is lax symmetric monoidal so it sends $\epsilon - \mathbf{cdga}^{gr}_{\mathcal{M}'}$ to $\epsilon - \mathbf{cdga}^{gr}_{k}$, and $\mathbb{P}_{n+1} - \mathbf{cdga}^{gr}_{\mathcal{M}'}$ to $\mathbb{P}_{n+1} - \mathbf{cdga}^{gr}_{k}$.

The following lemma compares de Rham and polyvectors objects of $A \in \mathbf{cdga}_{\mathcal{M}}$, and of $A(\infty)$ relative to $k(\infty)$, under suitable finiteness hypotheses on A.

Lemma 1.5.4 *If $A \in \mathbf{cdga}_{\mathcal{M}}$ is such that $\mathbb{L}^{int}_A$ is a perfect (i.e. dualizable) A-module, then there are natural equivalences of graded mixed cdgas over k and, respectively, of graded $\mathbb{P}_{n+1}$-algebras over k*

$$\mathbf{DR}^t(A) \simeq \mathbf{DR}(A(\infty)/k(\infty))$$

$$\mathbf{Pol}^t(A, n) \simeq \mathbf{Pol}(A(\infty)/k(\infty), n).$$

Proof. Without any assumptions on A, we have

$$\mathbf{DR}^{int}(A) \otimes k(\infty) \simeq \mathbf{DR}^{int}(A(\infty)/k(\infty)).$$

Since, as already observed, $|-\otimes k(\infty)| \simeq |-|^t$, this shows that $\mathbf{DR}^t(A) \simeq \mathbf{DR}(A(\infty)/k(\infty))$.

For polyvectors, the dualizability condition on $\mathbb{L}_A^{int}$ implies that the natural morphism

$$\mathbf{Pol}^{int}(A,n) \otimes k(\infty) \longrightarrow \mathbf{Pol}^{int}(A(\infty)/k(\infty), n)$$

is an equivalence. So, again, we have

$$\mathbf{Pol}^t(A,n) \simeq \mathbf{Pol}(A(\infty)/k(\infty), n).$$

$\square$

We can therefore state a Tate version of Corollary 1.4.24, by working in $\mathcal{M}'$, for $A \in \mathbf{cdga}_{\mathcal{M}}$ with dualizable $\mathbb{L}_A^{int}$. In the corollary below the non-degeneracy conditions is required in $\mathcal{M}'$, that is after tensoring with $k(\infty)$. This modifies the notion of shifted symplectic structures as follows. If $\mathcal{A}^{2,cl}(A,n)$ is the space of closed 2-forms of degree n on A, we say that an element $\omega \in \pi_0\,\mathcal{A}^{2,cl}(A,n)$ is *Tate non-degenerate* if the underlying adjoint morphism in $\mathcal{M}$

$$\Theta_{\omega_0} : \mathbb{T}_A^{int} \longrightarrow \mathbb{L}_A^{int}[n]$$

induces an equivalence in $\mathcal{M}'$

$$\Theta_{\omega_0}(\infty) : \mathbb{T}_A^{int}(\infty) \longrightarrow \mathbb{L}_A^{int}(\infty)[n]$$

i.e. after tensoring with $k(\infty)$. The space $\mathsf{Symp}^t(A,n)$ of *n-shifted Tate symplectic structures* on A is then the subspace of $\mathcal{A}^{2,cl}(A,n)$ consisting of connected components of Tate non-degenerate elements. Note that by Lemma 1.5.4 we have

$$\mathsf{Symp}^t(A,n) \simeq \mathsf{Symp}(A(\infty)/k(\infty), n),$$

where the right hand side is the space of n-shifted symplectic structures on $A(\infty)$ relative to $k(\infty)$, computed in $\mathcal{M}' = \mathsf{Ind}(\mathcal{M})$.

Corollary 1.5.5 *Let $A \in \mathbf{cdga}_{\mathcal{M}}$ such that $\mathbb{L}_A^{int}$ is a dualizable A-module in $\mathcal{M}$. Then, there is a natural morphism of spaces, functorial in A with respect to formally étale morphisms*

$$\mathsf{Map}^{nd}_{\mathbf{dgLie}_k^{gr}}(k(2)[-1], \mathsf{Pol}^t(A,n)[n]) \longrightarrow \mathsf{Symp}^t(A, n-1),$$

where $\mathsf{Map}^{nd}_{\mathbf{dgLie}_k^{gr}}(k(2)[-1], \mathsf{Pol}^t(A,n)[n])$ is the subspace of $\mathsf{Map}_{\mathbf{dgLie}_k^{gr}}(k(2)[-1], \mathsf{Pol}^t(A,n)[n])$ con-

sisting of connected components of non-degenerate elements.

2 Formal localization

A commutative dg-algebra (in non-positive degrees) A over k is *almost finitely presented* if $H^0(A)$ is a k-algebra of finite type, and each $H^i(A)$ is a finitely presented $H^0(A)$-module. Notice that, in particular, such an A is *Noetherian* i.e. $H^0(A)$ is a Noetherian k-algebra (since our base $\mathbb{Q}$-algebra k is assumed to be Noetherian), and each $H^i(A)$ is a finitely presented $H^0(A)$-module.

We let $\mathbf{dAff}_k$ be the opposite ∞-category of almost finitely presented commutative dg-algebras over k concentrated in non-positive degrees. We will simply refer to its objects as *derived affine schemes* without mentioning the base k or the finite presentation condition. When writing $\mathbf{Spec}\, A$, we implicitly assume that $\mathbf{Spec}\, A$ is an object of $\mathbf{dAff}_k$, i.e. that A is almost finitely presented commutative k-algebra concentrated in non-positive degrees. The ∞-category $\mathbf{dAff}_k$ is equipped with its usual étale topology of [HAG-II, Def. 2.2.2.3], and the corresponding ∞-topos of stacks will be denoted by $\mathbf{dSt}_k$. Its objects will simply be called *derived stacks* (even though they should be, strictly speaking, called *locally almost finitely presented derived stacks over* k).

With these conventions, an algebraic derived n-stack will have a smooth atlas by objects in $\mathbf{dAff}_k$, i.e. by objects of the form $\mathbf{Spec}\, A$ where A is almost finitely presented over k. Equivalently, all our algebraic derived n-stacks will be derived n-stacks according to [HAG-II, §2], that is such stacks are defined on the category of all commutative dg-algebra concentrated in non-positive degrees. Being locally almost of finite presentation these stacks X have cotangent complexes which are in $\mathsf{Coh}(X)$ and bounded on the right.

2.1 Derived formal stacks

We start by a zoology of derived stacks with certain infinitesimal properties.

Definition 2.1.1 *A* formal derived stack *is an object $F \in \mathbf{dSt}_k$ satisfying the following conditions.*

1. *The derived stack F is* nilcomplete *i.e. for all $\mathbf{Spec}\, B \in \mathbf{dAff}_k$, the canonical map*

$$F(B) \longrightarrow \lim_k F(B_{\leq k}),$$

where $B_{\leq k}$ denotes the k-th Postnikov truncation of B, is an equivalence in $\mathcal{T}$.

2. *The derived stack F is* infinitesimally cohesive *i.e. for all cartesian squares of almost finitely presented k-cdgas in non-positive degrees*

$$\begin{tikzcd} B \arrow[r] \arrow[d] & B_1 \arrow[d] \\ B_2 \arrow[r] & B_0, \end{tikzcd}$$

such that each $\pi_0(B_i) \longrightarrow \pi_0(B_0)$ is surjective with nilpotent kernel, then the induced square

$$\begin{tikzcd} F(B) \arrow[r] \arrow[d] & F(B_1) \arrow[d] \\ F(B_2) \arrow[r] & F(B_0), \end{tikzcd}$$

is cartesian in $\mathcal{T}$.

Remark 2.1.2 Note that if one assumes that a derived stack F has a cotangent complex ([HAG-II, §1.4]), then F is a formal derived stack if and only if it is nilcomplete and satisfies the infinitesimally cohesive axiom where *at least one* of the two $B_i \to B_0$ is required to have $\pi_0(B_i) \longrightarrow \pi_0(B_0)$ surjective with nilpotent kernel ([Lu5, Proposition 2.1.13]). We also observe that, even if we omit the nilpotency condition on the kernels but keep the surjectivity, we have that the diagram obtained by applying **Spec** to the square of cdgas in 2.1.1 (2) is a homotopy push-out in the ∞-category of derived schemes, hence in the ∞-category of derived *algebraic* stacks (say for the étale topology). This is a derived analog of the fact that pullbacks along surjective maps of rings induce pushout of schemes. In particular, any derived algebraic stack F sends any diagram as in 2.1.1 (2), with the nilpotency condition possibly omitted, to pullbacks in $\mathcal{T}$, i.e. is actually *cohesive* ([Lu3, DAG IX, Corollary 6.5] and [Lu5, Lemma 2.1.7]).

There are various sources of examples of formal derived stacks.

- Any algebraic derived n-stack F, in the sense of [HAG-II, §2.2], is a formal derived stack. Nilcompleteness of F is (the easy implication of) [HAG-II, Theorem c.9 (c)], while the infinitesimally cohesive property follows from nilcompleteness, the existence of a cotangent complex for F, and the general fact that any $B_i \to B_0$ with $\pi_0(B_i) \to \pi_0(B_0)$ surjective with nilpotent kernel can be written as the limit in $\mathbf{cdga}_k/B_0$ of a tower $\cdots \to C_n \to \cdots C_1 \to C_0 := B_0$ where each C_n is a square-zero extension of C_{n-1} by some C_{n-1}-module $P_n[k_n]$, where $k_n \to +\infty$ for $n \to +\infty$ (see [Lu5, Lemma 2.1.14], or [Lu5, Proposition 2.1.13] for a full proof of the infinitesimal cohesive property for a stack that is nilcomplete and has a cotangent complex). Alternatively, one can observe ([Lu5, Lemma 2.1.7]) that any derived algebraic stack is actually cohesive (hence infinitesimally cohesive).

- For all $\mathbf{Spec}\, A \in \mathbf{dAff}_k$ we let $QCoh^{-}(A)$ be the full sub-∞-groupoid of $\mathrm{L}(A)$ consisting of A-dg-modules M with $H^i(M) = 0$ for $i >> 0$. The ∞-functor $A \mapsto QCoh^{-}(A)$ defines a derived stack which can be checked to be a formal derived stack.

- Any (small) limit, in $\mathbf{dSt}_k$, of formal derived stacks is again a formal derived stack. This follows from the fact that (by Yoneda), for any $A \in \mathbf{cdga}_k$, the functor $\mathbf{dSt}_k \to \mathcal{T}$ given by evaluation at A commutes with (small) limits, and that both convergence and infinitesimal cohesiveness are expressed by conditions on objectwise limits.

Let us consider the inclusion functor $i : \mathbf{alg}_k^{\mathrm{red}} \longrightarrow \mathbf{cdga}_k$ of the full reflective sub ∞-category of *reduced discrete* objects (i.e. $R \in \mathbf{cdga}_k$ such that R is discrete and $R \simeq H^0(R)$ is a usual reduced k-algebra). The functor i has a left adjoint

$$(-)^{red} : \mathbf{cdga}_k \longrightarrow \mathbf{alg}_k^{\mathrm{red}}\,,\ A \longmapsto A^{red} := H^0(A)/\mathrm{Nilp}(H^0(A)).$$

Moreover, it is easy to verify that i is both continuous and cocontinuous for the étale topologies on $\mathbf{cdga}_k^{op}$, and $(\mathbf{alg}_k^{\mathrm{red}})^{op}$. If we denote by $\mathbf{St}_{\mathrm{red},k}$ the ∞-category of stacks on $(\mathbf{alg}_k^{\mathrm{red}})^{op}$ for the étale topology, we thus get an induced ∞-functor

$$i^* : \mathbf{dSt}_k \longrightarrow \mathbf{St}_{\mathrm{red},k}$$

that has both a right adjoint i_*, and a left adjoint $i_!$, obeying the following properties:

- $i_* \simeq ((-)^{red})^*$ (thus $i^*\mathbf{Spec}\,A \simeq \mathrm{Spec}(A^{red})$).
- $i_!$ and i_* are fully faithful (equivalently, the adjunction maps $\mathrm{Id} \to i^*i_!$ and $i^*i_* \to \mathrm{Id}$ are objectwise equivalences).
- $i^* \simeq ((-)^{red})_!$.
- $i_!i^*$ is left adjoint to i_*i^*.

Definition 2.1.3 *1. The functor*

$$(-)_{DR} := i_*i^* : \mathbf{dSt}_k \longrightarrow \mathbf{dSt}_k$$

is called the de Rham stack functor. *By adjunction, for any $F \in \mathbf{dSt}_k$, we have a canonical natural map $\lambda_F : F \mapsto F_{DR}$.*

2. The functor

$$(-)_{red} := i_!i^* : \mathbf{dSt}_k \longrightarrow \mathbf{dSt}_k$$

is called the reduced stack functor. *By adjunction, for any $F \in \mathbf{dSt}_k$, we have a canonical natural map $\iota_F : F_{red} \mapsto F$.*

3. Let $f : F \longrightarrow G$ be a morphism in $\mathbf{dSt}_k$. We define the formal completion $\widehat{G}_f$ of G along the morphism f *as the fibered product in $\mathbf{dSt}_k$:*

$$\begin{tikzcd} \widehat{G}_f \arrow[r, "\beta_f"] \arrow[d] & F_{DR} \arrow[d, "f_{DR}"] \\ G \arrow[r, "\lambda_G"'] & G_{DR}. \end{tikzcd}$$

Since the left adjoint to i is $(-)^{red}$, then it is easy to see that

$$F_{DR}(A) \simeq F(A^{red}) \quad \text{and} \quad (\mathbf{Spec}\, A)_{\mathrm{red}} \simeq \mathbf{Spec}\,(A^{red}),$$

for any $A \in \mathbf{cdga}_k$. Therefore $\widehat{G}_f(A) = G(A) \times_{G(A^{red})} F(A^{red})$, for $f : F \to G$ in $\mathbf{dSt}_k$. We already observed that $(-)_{DR}$ is right adjoint to $(-)_{\mathrm{red}}$, as functors $\mathbf{dSt}_k \to \mathbf{dSt}_k$.

Since taking the reduced algebra is a projector, we have that the canonical map $F_{DR} \to (F_{DR})_{DR}$ is an equivalence; the same holds for $(F_{\mathrm{red}})_{\mathrm{red}} \to F_{\mathrm{red}}$. Moreover, for any $F \in \mathbf{dSt}_k$, we have $F_{DR} \simeq (\widehat{\mathbf{Spec}\, k})_f$, where $f : F \to \mathbf{Spec}\, k$ is the structure morphism (observe that for any $R \in \mathbf{alg}_k^{\mathrm{red}}$, the canonical map $\mathbf{Spec}\, R \to (\mathbf{Spec}\, R)_{DR}$ is an equivalence). We list below a few elementary properties of de Rham stacks and reduced stacks.

Proposition 2.1.4 *1. F_{DR} is a formal derived stack for any $F \in \mathbf{dSt}_k$.*

2. *If G is a formal derived stack, the formal completion $\widehat{G}_f$, along any map $f : F \to G$ in $\mathbf{dSt}_k$, is a formal derived stack.*

3. *For any $F \in \mathbf{dSt}_k$, the canonical map $\lambda_F : F \to F_{DR}$ induces an equivalence $F_{red} \to (F_{DR})_{red}$.*

4. *For any map $f : F \to G$ in $\mathbf{dSt}_k$, the canonical map $\alpha_f : F \to \widehat{G}_f$ induces an equivalence $F_{red} \to (\widehat{G}_f)_{red}$.*

5. *For any $F \in \mathbf{dSt}_k$, the canonical map $F_{red} \to F$ induces an equivalence $(F_{red})_{DR} \to F_{DR}$.*

6. *For any $F \in \mathbf{dSt}_k$, if $j : \mathsf{t}_0F \to F$ denotes the canonical map in $\mathbf{dSt}_k$ from the truncation of F to F, then the canonical map $\widehat{F}_j \to F$ is an equivalence.*

7. *If $f : F \to G$ is a map in $\mathbf{dSt}_k$ such that f_{red} is an equivalence, then the canonical map $\widehat{G}_f \to G$ is an equivalence.*

Proof. Since $F_{DR}(A) = F(A^{red})$, and $(-)^{red}$ sends cartesian squares as in Definition 2.1.1 (2) to cartesian squares of isomorphisms, (1) follows. (2) follows from (1) and the fact that formal derived stacks are closed under small limits. (3) follows from the fact that F and F_{DR} agree when restricted to $\mathbf{alg}_k^{\mathrm{red}}$ i.e. $i^*(\lambda_F) : i^*F \to i^*(F_{DR})$ is an equivalence and hence $i_!i^*(\lambda_F)$ is an equivalence as well. Let us prove (4). Both G and G_{DR} agree on $\mathbf{alg}_k^{\mathrm{red}}$, $i^*(\lambda_G) : i^*G \to i^*(G_{DR})$ is an equivalence, and i^* is right (and left) adjoint, and also $i^*(\beta_f) : i^*(\widehat{G}_f) \to i^*(F_{DR})$ is an equivalence. Furthermore $i_!$ is fully faithful, so $\beta_{f,\mathrm{red}} : (\widehat{G}_f)_{\mathrm{red}} \to (F_{DR})_{\mathrm{red}}$ is an equivalence. Now, the composite $\alpha_f \circ \beta_f$ is equal to λ_F, so we get (4) from (3). In order to prove (5) it is enough to observe that the adjunction map $i^*i_! \to \mathrm{Id}$ is an objectwise equivalence. The assertion (6) follows immediately by observing that $\mathsf{t}_0F(S) = F(S)$

for any discrete commutative k-algebra S, therefore j_{DR} is an equivalence. Finally, by (5) we get that if f_{red} is an equivalence, then so is f_{DR}, and so (7) follows. $\square$

Definition 2.1.5 *1. A formal derived stack F according to Definition 2.1.1 is called* almost affine *if $F_{red} \in \mathbf{dSt}_k$ is an affine derived scheme.*

2. An almost affine formal derived stack F in the sense above is affine *if F has a cotangent complex in the sense of [HAG-II, §1.4], and if, for all $\mathbf{Spec}\, B \in \mathbf{dAff}_k$ and all morphism $u : \mathbf{Spec}\, B \longrightarrow F$, the B-dg-module $\mathbb{L}_{F,u} \in \mathsf{L}(B)$ ([HAG-II, Definition 1.4.1.5]) is coherent and cohomologically bounded above.*

Recall our convention throughout this section, that all derived affine schemes are automatically assumed to be almost of finite presentation. Therefore, any derived affine scheme is an affine formal derived stack according to Definition 2.1.5.

Note that when F is any affine formal derived stack, there is a globally defined quasi-coherent complex $\mathbb{L}_F \in \mathsf{L}_{\mathsf{Qcoh}}(F)$ such that for all $u : \mathbf{Spec}\, B \longrightarrow F$, we have a natural equivalence of B-dg-modules

$$u^*(\mathbb{L}_F) \simeq \mathbb{L}_{F,u}.$$

The quasi-coherent complex $\mathbb{L}_F$ is then itself coherent, with cohomology bounded above.

Since $(\mathbf{Spec}\, A)_{\mathrm{red}} \simeq \mathbf{Spec}(A^{red})$, we get by Proposition 2.1.4 (4), that for any algebraic derived stack F, and any morphism in $\mathbf{dSt}_k$

$$f : \mathbf{Spec}\, A \longrightarrow F\,,$$

the formal completion $\widehat{F}_f$ of F along f is an affine formal derived stack in the sense of Definition 2.1.5 above. Moreover, the natural morphism $v : \widehat{F}_f \longrightarrow F$ is formally étale, i.e. the natural morphism

$$v^*(\mathbb{L}_F) \longrightarrow \mathbb{L}_{\widehat{F}_f}$$

is an equivalence in $\mathsf{L}_{\mathsf{Qcoh}}(\widehat{F}_f)$.

This formal completion construction along a map from an affine will be our main source of examples of affine formal derived stacks.

We will ultimately be concerned with affine formal derived stacks *over affine bases*, which we proceed to discuss.

Definition 2.1.6 *Let $X := \mathbf{Spec}\, A \in \mathbf{dAff}_k$. A* good formal derived stack over X *is an object $F \in \mathbf{dSt}_k/X$ satisfying the following two conditions.*

1. The derived stack F is an affine formal derived stack.

2. The induced morphism $F_{red} \longrightarrow (\mathbf{Spec}\,A)_{red} = \mathbf{Spec}\,A^{red}$ is an equivalence.

The full sub-∞-category of $\mathbf{dSt}_k/X$ consisting of good formal derived stacks over $X = \mathbf{Spec}\,A$ will be denoted as $\mathbf{dFSt}^g_X$, or equivalently as $\mathbf{dFSt}^g_A$.

Finally, a perfect formal derived stack F over $\mathbf{Spec}\,A$ *is a good formal derived stack over* $\mathbf{Spec}\,A$ *such that moreover its cotangent complex $\mathbb{L}_{F/\mathbf{Spec}\,A} \in \mathsf{L}_{\mathsf{Qcoh}}(F)$ is a perfect complex.*

Remark 2.1.7 Since $i_!$ is fully faithful, it is easy to see that if $F \to \mathbf{Spec}\,A$ is a good (respectively, perfect) formal derived stack, then for any $\mathbf{Spec}\,B \to \mathbf{Spec}\,A$, the base change $F_B \to \mathbf{Spec}\,B$ is again a good (respectively, perfect) formal derived stack. In this sense, good (respectively, perfect) formal derived stacks are stable under arbitrary derived affine base change.

The fundamental example of a good formal derived stack is given by an incarnation of the so-called *Grothendieck connection* (also called *Gel'fand connection* in the literature). It consists, for an algebraic derived stack $F \in \mathbf{dSt}_k$ which is locally almost of finite presentation, of the family of all formal completions of F at various points. This family is equipped with a natural flat connection, or in other words, is a crystal of formal derived stacks.

Concretely, for $F \in \mathbf{dSt}_k$ we consider the canonical map $F \to F_{DR}$ whose fibers can be described as follows.

Proposition 2.1.8 *Let $F \in \mathbf{dSt}_k$, $\mathbf{Spec}\,A \in \mathbf{dAff}_k$, and $\overline{u} : \mathbf{Spec}\,A \longrightarrow F_{DR}$, corresponding (by Yoneda and the definition of F_{DR}) to a morphism $u : \mathbf{Spec}\,A^{red} \longrightarrow F$. Then the derived stack $F \times_{F_{DR}} \mathbf{Spec}\,A$ is equivalent to the formal completion $(\widehat{\mathbf{Spec}\,A \times F})_{(i,u)}$ of the graph morphism*

$$(i,u) : \mathbf{Spec}\,A^{red} \longrightarrow \mathbf{Spec}\,A \times F,$$

where $i : \mathbf{Spec}\,A^{red} \longrightarrow \mathbf{Spec}\,A$ is the natural closed embedding.

Proof. Let $X := \mathbf{Spec}\,A$. By Proposition 2.1.4, we have $(X_{red})_{DR} \simeq X_{DR}$. Therefore the formal completion $(\widehat{X \times F})_{(i,u)}$ is in fact the pullback of the following diagram

$$\begin{tikzcd}
(\widehat{X \times F})_{(i,u)} \arrow[r] \arrow[d] & X \times F \arrow[d, "\lambda_X \times \lambda_F"] \\
X_{DR} \arrow[r, "(\mathrm{id}, u_{DR})"'] & X_{DR} \times F_{DR}
\end{tikzcd} .$$

But (i,u) is a graph, so the following diagram is cartesian

$$\begin{tikzcd}
(\widehat{X \times F})_{(i,u)} \arrow[r] \arrow[d] & X_{DR} \arrow[r, "u_{DR}"] \arrow[d] & F_{DR} \arrow[d, "\Delta"] \\
F \times X \arrow[r, "\lambda_F \times \lambda_X"'] & F_{DR \times X_{DR}} \arrow[r, "id \times u_{DR}"'] & F_{DR} \times F_{DR}.
\end{tikzcd}$$

Now recall that, in any ∞-category with products, a diagram

$$\begin{tikzcd}
D \arrow[r] \arrow[d] & A \arrow[d, "g"] \\
C \arrow[r, "h"'] & B
\end{tikzcd}$$

is cartesian iff the diagram

$$\begin{tikzcd}
D \arrow[r] \arrow[d] & B \arrow[d, "\Delta"] \\
A \times C \arrow[r, "g \times h"'] & B \times B
\end{tikzcd}$$

is cartesian, thus, in our case we conclude that

$$\begin{tikzcd}
(\widehat{X \times F})_{(i,u)} \arrow[r] \arrow[d] & F \arrow[d, "\lambda_F"] \\
X \arrow[r, "u_{DR} \circ \lambda_X"'] & F_{DR}
\end{tikzcd}$$

is cartesian.

□

By Proposition 2.1.4 (4), we get the following corollary of Proposition 2.1.8

Corollary 2.1.9 *If F is algebraic, then each fiber $F \times_{F_{DR}} \mathbf{Spec}\, A$ of $F \to F_{DR}$ is a good formal derived stack over A, according to Definition 2.1.6, which is moreover perfect when F is locally of finite presentation.*

Let us remark that in most of our applications F will indeed be locally of finite presentation (so that its cotangent complex will be perfect).

By Proposition 2.1.8, the fiber $F \times_{F_{DR}} \mathbf{Spec}\, A$ of $F \to F_{DR}$ when $A = K$ is a field, is simply the formal completion $\hat{F}_x$ of F at the point $x : \mathbf{Spec}\, K \longrightarrow F$, and corresponds to a dg-Lie algebra over K by [Lu2, Theorem 5.3] or [Lu4]. This description tells us that $F \longrightarrow F_{DR}$ is a family of good formal derived stacks over F_{DR}, and is thus classified by a morphism of derived stacks

$$F_{DR} \longrightarrow \mathbf{dFSt}^{g}_{-},$$

where the right hand side is the ∞-functor $A \mapsto \mathbf{dFSt}^{g}_{A}$. We will come back to this point of view in Section 2.4.

We conclude this section with the following easy but important observation

Lemma 2.1.10 *Let X be a derived Artin stack, and $q : X \to X_{DR}$ the associated map. Then $\mathbb{L}_X$ and $\mathbb{L}_{X/X_{DR}}$ both exist in $\mathrm{L}_{\mathrm{QCoh}}(X)$, and we have*

$$\mathbb{L}_X \simeq \mathbb{L}_{X/X_{DR}}.$$

Proof. The cotangent complex $\mathbb{L}_X$ exists because X is Artin. The cotangent complex $\mathbb{L}_{Y_{DR}}$ exists (in the sense of [HAG-II, 1.4.1]), for *any* derived stack Y, and is indeed trivial. In fact, if A is a cdga over k, and M a dg-module, then

$$Y_{DR}(A \oplus M) \simeq Y((A \oplus M)^{\mathrm{red}}) = Y(A^{\mathrm{red}}) \simeq Y_{DR}(A).$$

Hence, we may conclude by the transitivity sequence

$$0 \simeq q^{*}\mathbb{L}_{X_{DR}} \to \mathbb{L}_X \to \mathbb{L}_{X/X_{DR}}.$$

$\square$

2.2 Perfect complexes on affine formal derived stacks

For any formal derived stack F, we have its ∞-category of quasi-coherent complexes $\mathsf{L}_{\mathsf{Qcoh}}(F)$. Recall that it can described as the following limit (inside the ∞-category of ∞-categories)

$$\mathsf{L}_{\mathsf{Qcoh}}(F) := \lim_{\mathbf{Spec}\, B \longrightarrow F} \mathsf{L}(B) \in \infty - \mathbf{Cat}.$$

We can define various full ∞-categories of $\mathsf{L}_{\mathsf{Qcoh}}(F)$ by imposing appropriate finiteness conditions. We will be interested in two of them, $\mathsf{L}_{\mathsf{Perf}}(F)$ and $\mathsf{L}^{-}_{\mathsf{Qcoh}}(F)$, respectively of perfect and cohomologically bounded on the right objects. They are simply defined as

$$\mathsf{L}_{\mathsf{Perf}}(F) := \lim_{\mathbf{Spec}\, B \longrightarrow F} \mathsf{L}_{\mathsf{Perf}}(B) \qquad \mathsf{L}^{-}_{\mathsf{Qcoh}}(F) := \lim_{\mathbf{Spec}\, B \longrightarrow F} \mathsf{L}^{-}_{\mathsf{Qcoh}}(B).$$

Definition 2.2.1

- *Let $\mathbf{dFSt}^{\mathsf{aff}}_{k}$ be the full sub-∞-category of $\mathbf{dSt}_k$ consisting of all affine formal derived stacks in the sense of Definition 2.1.5.*

- *An affine formal derived stack* $F \in \mathbf{dFSt}_k^{\mathsf{aff}}$ *is* algebraisable *if there exists* $n \in \mathbb{N}$*, an algebraic derived* n*-stack* F'*, and a morphism* $f : F_{red} \longrightarrow F'$ *such that* F *is equivalent to the formal completion* $\hat{F'}_f$*.*

- *A good formal derived stack over* $X := \mathbf{Spec}\, A$ *(Definition 2.1.6) is* algebraisable over X *if there exists* $n \in \mathbb{N}$*, an algebraic derived* n*-stack* $G \longrightarrow \mathbf{Spec}\, A$*, locally of finite presentation over* $\mathbf{Spec}\, A$*, together with a morphism* $f : \mathbf{Spec}\, A_{red} \longrightarrow G$ *over* $\mathbf{Spec}\, A$*, such that* F *is equivalent, as a derived stack over* $\mathbf{Spec}\, A$*, to the formal completion* $\widehat{G}_f$*.*

In the statement of the next theorem, for $F \in \mathbf{dFSt}_k^{\mathsf{aff}}$, we will denote by A_F any k-cdga such that $F_{red} \simeq \mathbf{Spec}\, A_F$: such an A_F exists for any almost affine derived formal stack F, and is unique up to equivalence.
The rest of this subsection will be devoted to prove the following main result

Theorem 2.2.2 *There exists an* ∞*-functor*

$$\mathbb{D} : \mathbf{dFSt}_k^{\mathsf{aff}} \longrightarrow (\epsilon - \mathbf{cdga}_k^{gr})^{op}$$

satisfying the following properties

1. *If* $F \in \mathbf{dFSt}_k^{\mathsf{aff}}$ *is algebraisable, then we have an equivalence of (non-mixed) graded cdga*

$$\mathbb{D}(F) \simeq Sym_{A_F}(\mathbb{L}_{F_{red}/F}[-1]),$$

2. *For all* $F \in \mathbf{dFSt}_k^{\mathsf{aff}}$*, there exists an* ∞*-functor*

$$\phi_F : \mathsf{L}_{\mathsf{Qcoh}}(F) \longrightarrow \mathbb{D}(F) - Mod_{\epsilon-\mathbf{dg}}^{\mathsf{gr}},$$

   natural in F*, which is conservative, and induces an equivalence of* ∞*-categories*

$$\mathsf{L}_{\mathsf{Perf}}(F) \longrightarrow \mathbb{D}(F) - Mod_{\epsilon-\mathbf{dg}}^{\mathsf{gr,perf}},$$

   where the right hand side is the full sub-∞*-category of* $\mathbb{D}(F) - Mod_{\epsilon-\mathbf{dg}}^{\mathsf{gr}}$ *consisting of graded mixed* $\mathbb{D}(F)$*-modules* E *which are equivalent, as graded* $\mathbb{D}(F)$*-modules, to* $\mathbb{D}(F) \otimes_{A_F} E_0$ *for some* $E_0 \in \mathsf{L}_{\mathsf{Perf}}(A_F)$*.*

We will first prove Thm 2.2.2 for F a derived affine scheme, and then proceed to the general case.

Proof of Theorem 2.2.2: the derived affine case. We start with the special case of the theorem for the sub-∞-category $\mathbf{dAff}_k \subset \mathbf{dFSt}_k^{\mathsf{aff}}$ of derived affine schemes (recall our convention that all derived affine schemes are locally finitely presented), and construct the ∞-functor

$$\mathbb{D}:\mathbf{dAff}_k^{op}\longrightarrow \epsilon-\mathbf{cdga}_k^{gr}$$

as follows. We start by sending an object $\mathbf{Spec}\,A\in\mathbf{dAff}_k$ to the morphism $A\longrightarrow A^{red}$. This defines an ∞-functor $\mathbf{dAff}_k\longrightarrow \mathsf{Mor}(\mathbf{dAff}_k)$, from derived affine schemes to morphisms between derived affine schemes. We then compose this with the ∞-functor (see end of §1.3.2, with $\mathcal{M}=\mathbf{dg}_k$)

$$\mathbf{DR}:\mathsf{Mor}(\mathbf{cdga}_k)\longrightarrow \epsilon-\mathbf{cdga}_k^{gr},$$

sending a morphism $A\to B$ to $\mathbf{DR}(B/A)$. Recall that this second ∞-functor can be explicitly constructed as the localization along equivalences of the functor

$$DR^{str}:\mathsf{Cof}(cdga_k)\longrightarrow \epsilon-cdga_k^{gr},$$

from the category of cofibrations between cofibrant cdgas to the category of graded mixed cdgas, sending a cofibration $A\to B$ to $DR^{str}(B/A)=Sym_B(\Omega^1_{B/A}[-1])$, with mixed structure given by the de Rham differential.

Proposition 2.2.3 *The ∞-functor defined above*

$$\mathbb{D}:\mathbf{dAff}_k^{op}\longrightarrow \epsilon-\mathbf{cdga}_k^{gr}\;:\quad A\longmapsto \mathbf{DR}(A^{red}/A)$$

is fully faithful. Its essential image is contained inside the full sub-∞-category of graded mixed cdgas B satisfying the following three conditions.

1. *The cdga $B(0)$ is concentrated in cohomological degree 0, and is a reduced k-algebra of finite type.*
2. *The $B(0)$-dg-module $B(1)$ is almost finitely presented and has amplitude contained in $(-\infty,0]$.*
3. *The natural morphism*
$$Sym_{B(0)}(B(1))\longrightarrow B$$
is an equivalence of graded cdgas.

Proof. For $\mathbf{Spec}\,A\in\mathbf{dAff}_k$, we have

$$\mathbb{D}(A)=\mathbf{DR}(A_{red}/A)\simeq Sym_{A_{red}}(\mathbb{L}_{A_{red}/A}[-1]),$$

showing that conditions 1, 2, and 3 above are indeed satisfied for $\mathbb{D}(A)$ (for 2, recall that $A\to A_{red}$ being an epimorphism, we have $\pi_0(\mathbb{L}_{A_{red}/A})=0$). The fact that $\mathbb{D}$ is fully faithful is essentially the content of [Bh], stating that the relative derived de Rham cohomology of any closed immersion is the corresponding formal completion. Indeed, here $X = \mathbf{Spec}\, A$ is the formal completion of $X_{red} = (\mathbf{Spec}\, A)_{red}$ inside X. For the sake of completeness, we will provide here a new proof of this fact, for the specific closed immersion $X_{red} \longrightarrow X$.

Let $\mathbf{Spec}\, A$ and $\mathbf{Spec}\, B$ be two derived affine schemes, and consider the induced morphism of mapping spaces

$$\mathsf{Map}_{\mathbf{dSt}_k}(\mathbf{Spec}\, A, \mathbf{Spec}\, B) \simeq \mathsf{Map}_{\mathbf{cdga}_k}(B, A) \longrightarrow \mathsf{Map}_{\epsilon-\mathbf{cdga}_k^{gr}}(\mathbb{D}(B), \mathbb{D}(A)).$$

By Lemma 1.3.18, we have

$$\mathsf{Map}_{\epsilon-\mathbf{cdga}_k^{gr}}(\mathbb{D}(B), \mathbb{D}(A)) \simeq \mathsf{Map}_{\mathbf{cdga}_k}(B_{red}, A_{red}) \times_{\mathsf{Map}_{\mathbf{cdga}_k}(B, A_{red})} \mathsf{Map}_{\epsilon-\mathbf{cdga}_k^{gr}}(B, \mathbb{D}(A))$$

where B is considered as a graded mixed cdga in a trivial manner (pure of weight 0 and with zero mixed structure). But the canonical map $\mathsf{Map}_{\mathbf{cdga}_k}(B_{red}, A_{red}) \to \mathsf{Map}_{\mathbf{cdga}_k}(B, A_{red})$ is an equivalence, hence

$$\mathsf{Map}_{\epsilon-\mathbf{cdga}_k^{gr}}(\mathbb{D}(B), \mathbb{D}(A)) \simeq \mathsf{Map}_{\epsilon-\mathbf{cdga}_k^{gr}}(B, \mathbb{D}(A)).$$

Finally, by adjunction we have

$$\mathsf{Map}_{\epsilon-\mathbf{cdga}_k^{gr}}(B, \mathbb{D}(A)) \simeq \mathsf{Map}_{\epsilon-\mathbf{cdga}_k^{gr}}(k(0) \otimes_k B, \mathbb{D}(A)) \simeq \mathsf{Map}_{\mathbf{cdga}_k}(B, |\mathbb{D}(A)|)$$

where $|-| : \epsilon - \mathbf{cdga}_k^{gr} \longrightarrow \mathbf{cdga}_k$ is the realization ∞-functor of Definition 1.3.1 for commutative monoids in $\mathcal{M} = \epsilon - \mathbf{dg}_k^{gr}$. Note that the commutative $k-$dg-algebra $|\mathbb{D}(A)|$ is exactly the derived de Rham cohomology of A_{red} over A. By putting these remarks together, we conclude that, in order to prove that $\mathbb{D}$ is fully faithful, it will be enough to show that, for any $A \in \mathbf{cdga}_k$, the induced natural morphism $A \longrightarrow |\mathbb{D}(A)|$ is an equivalence, i.e. the statement is reduced to the following

Lemma 2.2.4 *For any* $\mathbf{Spec}\, B \in \mathbf{dAff}_k$ *the natural morphism* $B \longrightarrow \mathbb{D}(B)$ *of graded mixed cdgas induces an equivalence in* $\mathbf{cdga}_k$

$$B \longrightarrow |\mathbb{D}(B)|.$$

Proof of Lemma. We can assume that B is a cell non-positively graded commutative dg-algebra with finitely many cells in each dimension. As a commutative graded algebra B is a free commutative graded algebra with a finite number of generators in each degree. In particular B^0 is a polynomial k-algebra and B^i is a free B^0-module of finite rank for all i. In the same way, we chose a cofibration $B \hookrightarrow C$ which is a model for $B \longrightarrow B_{red}$. We chose moreover C to be a cell B-cdga with finitely many cells in each dimension. As B_{red} is quotient of $\pi_0(B)$ we can also chose C with no cells in degree 0.

We let $L := \Omega^1_{C/B}[-1]$, which is a cell C-dg-module with finitely many cells in each degree, and no cells in positive degrees. The commutative dg-algebra $|\mathbb{D}(B)|$ is by definition the completed symmetric cdga $\widehat{Sym}_C(L)$, with its total differential, sum of the cohomological and the de Rham differential. Note that, because L has no cells in positive degrees and only finitely many cells in each degree, the cdga $|\mathbb{D}(B)|$ is again non-positively graded. Note however that it is not clear a priori that $|\mathbb{D}(B)|$ is almost of finite presentation and thus not clear that $Spec\,|\mathbb{D}(B)| \in \mathbf{dAff}_k$.

We let C^0 be the commutative k-algebra of degree zero elements in C, and L^0 of degree zero elements in L. We have a natural commutative square of commutative dg-algebras, relating completed and non-completed symmetric algebras

$$\begin{tikzcd} Sym_C(L) \arrow[r] & \widehat{Sym}_C(L) \\ Sym_{C^0}(L^0) \arrow[r] \arrow[u] & \widehat{Sym}_{C^0}(L^0). \arrow[u] \end{tikzcd}$$

In this diagram we consider $Sym_C(L)$ and $\widehat{Sym}_C(L)$ both equipped with the total differential, sum of the cohomological and the de Rham differential (recall that $L = \Omega^1_{C/B}[-1]$).

By assumption C^0 is a polynomial k-algebra over a finite number of variables, and C^i is a free C^0-module of finite type. This implies that the diagram above is a push-out of commutative dg-algebras, and, as the lower horizontal arrow is a flat morphism of commutative rings, this diagram is moreover a homotopy push-out of cdgas. We thus have a corresponding push-out diagram of the corresponding cotangent complexes, which base changed to C provides a homotopy push-out of C-dg-modules

$$\begin{tikzcd} \mathbb{L}_{Sym_C(L)} \otimes_{Sym_C(L)} C \arrow[r] & \mathbb{L}_{\widehat{Sym}_C(L)} \otimes_{\widehat{Sym}_C(L)} C \\ \mathbb{L}_{Sym_{C^0}(L^0)} \otimes_{Sym_{C^0}(L^0)} C \arrow[r] \arrow[u] & \mathbb{L}_{\widehat{Sym}_{C^0}(L^0)} \otimes_{\widehat{Sym}_{C^0}(L^0)} C. \arrow[u] \end{tikzcd}$$

As C^0 is a polynomial algebra over k, the lower horizontal morphism is equivalent to

$$\Omega^1_{Sym_{C^0}(L^0)} \otimes_{Sym_{C^0}(L^0)} C \to \Omega^1_{\widehat{Sym}_{C^0}(L^0)} \otimes_{\widehat{Sym}_{C^0}(L^0)} C,$$

which is the base change along $C^0 \longrightarrow C$ of the morphism

$$\Omega^1_{Sym_{C^0}(L^0)} \otimes_{Sym_{C^0}(L^0)} C^0 \to \Omega^1_{\widehat{Sym}_{C^0}(L^0)} \otimes_{\widehat{Sym}_{C^0}(L^0)} C^0.$$

This last morphism is an isomorphism, and thus the induced morphism

$$\mathbb{L}_{Sym_C(L)} \otimes_{Sym_C(L)} C \longrightarrow \mathbb{L}_{\widehat{Sym}_C(L)} \otimes_{\widehat{Sym}_C(L)} C$$

is an equivalence of C-dg-modules. To put things differently, the morphism of cdgas $Sym_C(L) \longrightarrow \widehat{Sym}_C(L)$ is formally étale along the augmentation.

We deduce from this the existence of a canonical identification of C-dg-modules

$$\mathbb{L}_{B_{red}/B} \simeq \mathbb{L}_{|\mathbb{D}(B)|} \otimes_{|\mathbb{D}(B)|} B_{red}.$$

This equivalence is moreover induced by the diagram of cdgas

$$\begin{tikzcd} B \arrow[rr] \arrow[dr] & & {|\mathbb{D}(B)|} \arrow[dl] \\ & B_{red}. & \end{tikzcd}$$

Equivalently, the morphism $B \longrightarrow |\mathbb{D}(B)|$ is formally étale at the augmentation over B_{red}. By the infinitesimal lifting property, the morphism of B-cdgas $|\mathbb{D}(B)| \longrightarrow B_{red}$ can be extended uniquely to a morphism $|\mathbb{D}(B)| \longrightarrow \pi_0(B)$. Similarly, using the Postnikov tower of B, this morphism extends uniquely to a morphism of B-cdgas $|\mathbb{D}(B)| \longrightarrow B$. In other words, the adjunction morphism $i : B \longrightarrow |\mathbb{D}(B)|$ possesses a retraction up to homotopy $r : |\mathbb{D}(B)| \longrightarrow B$. We have $ri \simeq id$, and $\phi := ir$ is an endomorphism of $|\mathbb{D}(B)|$ as a B-cdga, which preserves the augmentation $|\mathbb{D}(B)| \longrightarrow B_{red}$ and is formally étale at B_{red}.

By construction, $|\mathbb{D}(B)| \simeq \lim_n |\mathbb{D}_{\leq n}(B)|$, where

$$|\mathbb{D}_{\leq n}(B)| := Sym^{\leq n}_{B_{red}}(\mathbb{L}_{B_{red}/B}[-1])$$

is the truncated de Rham complex of B_{red} over B. Each of the cdga $|\mathbb{D}_{\leq n}(B)|$ is such that $\pi_0(|\mathbb{D}_{\leq n}(B)|)$ is a finite nilpotent thickening of B_{red}, and moreover $\pi_i(|\mathbb{D}_{\leq n}(B)|)$ is a $\pi_0(|\mathbb{D}_{\leq n}(B)|)$-module of finite type. Again by the infinitesimal lifting property we see that these imply that the endomorphism ϕ must be homotopic to the identity.

This finishes the proof that the adjunction morphism $B \longrightarrow |\mathbb{D}(B)|$ is an equivalence of cdgas, and thus the proof Lemma 2.2.4. □

The lemma is proved, and thus Proposition 2.2.3 is proved as well. □

One important consequence of Proposition 2.2.3 is the following corollary, showing that quasi-coherent complexes over $\mathbf{Spec}\, A \in \mathbf{dAff}_k$ can be naturally identified with certain $\mathbb{D}(A)$-modules.

Corollary 2.2.5 *Let $\mathbf{Spec}\, A \in \mathbf{dAff}_k$ be an affine derived scheme, and $\mathbb{D}(A) := \mathbf{DR}(A_{red}/A)$ be the corresponding graded mixed cdga. There exists a symmetric monoidal stable ∞-functor*

$$\phi_A : \mathrm{L}_{\mathrm{QCoh}}(A) \hookrightarrow \mathbb{D}(A) - Mod_{\epsilon-\mathbf{dg}},$$

functorial in A, inducing an equivalence of ∞-categories

$$\mathrm{L}_{\mathsf{Perf}}(A) \simeq \mathbb{D}(A) - Mod_{\epsilon-\mathbf{dg}}^{\mathsf{Perf}},$$

where $\mathbb{D}(A) - Mod_{\epsilon-\mathbf{dg}}^{\mathsf{Perf}}$ is the full sub-∞-category consisting of mixed graded $\mathbb{D}(A)$-modules M for which there exists $E \in \mathrm{L}_{\mathsf{Perf}}(A_{red})$, and an equivalence of (non-mixed) graded modules

$$M \simeq \mathbb{D}(A) \otimes_{A_{red}} E.$$

Proof. The ∞-functor ϕ_A is defined by sending an A-dg-module $E \in \mathrm{L}(A)$ to

$$\phi_A(E) := \mathbb{D}(A) \otimes_A E \in \mathbb{D}(A) - Mod_{\epsilon-\mathrm{L}(k)^{gr}},$$

using that $\mathbb{D}(A) = \mathbf{DR}(A_{red}/A)$ is, naturally, an A-linear graded mixed cdga. This ∞-functor sends A to $\mathbb{D}(A)$ itself. In particular, we have

$$\begin{aligned}\mathsf{Map}_{\mathbb{D}(A)-Mod_{\epsilon-\mathrm{L}(k)^{gr}}}(\mathbb{D}(A), \mathbb{D}(A)) &\simeq \mathsf{Map}_{\epsilon-\mathrm{L}(k)^{gr}}(k(0), \mathbb{D}(A)) \\ &\simeq \mathsf{Map}_{\mathrm{L}(k)}(k, |\mathbb{D}(A)|) \\ &\simeq \mathsf{Map}_{A-Mod}(A, A).\end{aligned}$$

This shows that ϕ_A is fully faithful on the single object A, so, by stability, it is also fully faithful when restricted to $\mathrm{L}_{\mathsf{Perf}}(A)$, the ∞-category of perfect A-dg-modules. $\square$

Proposition 2.2.3 and Corollary 2.2.5 together prove Theorem 2.2.2 in the derived affine case. We now move to the general case.

Proof of Theorem 2.2.2 : the general case. We will extend the above relation between derived affine schemes and graded mixed cdgas to the case of affine formal derived stacks. In order to do this, we start with the ∞-functor

$$\mathbf{dAff}_k^{opp} \longrightarrow \epsilon - \mathbf{cdga}_k^{gr}$$

sending A to $\mathbb{D}(A) = \mathbf{DR}(A_{red}/A)$. This ∞-functor is a derived stack for the étale topology on $\mathbf{dAff}_k^{op}$, and thus has a right Kan extension as an ∞-functor defined on all derived stacks

$$\mathbb{D} : \mathbf{dSt}_k^{op} \longrightarrow \epsilon - \mathbf{cdga}_k^{gr}, \qquad F \longmapsto \lim_{\mathbf{Spec}\, A \to F} (\mathbb{D}(A))$$

(with the limit being taken in $\epsilon - \mathbf{cdga}_k^{gr}$), and sending colimits in $\mathbf{dSt}_k$ to limits. In general, there are no reasons to expect that $\mathbb{D}(F)$ is free as a graded cdga, and it is a remarkable property that this is the case when F is an *algebraisable* affine formal derived stack (Definition 2.2.1); we do not know if the result still holds for a general affine formal derived stack. The following Proposition establishes this, and thus point 1 of Theorem 2.2.2.

Proposition 2.2.6 *Let $F \in \mathbf{dFSt}_k^{\mathrm{aff}}$ be an algebraisable affine formal derived stack, and let $F_{red} \simeq$ $\mathbf{Spec}\, A_0$. We have a natural equivalence of (non-mixed) graded coda's*

$$Sym_{A_0}(\mathbb{L}_{F_{red}/F}[-1]) \simeq \mathbb{D}(F).$$

Proof. For all $\mathbf{Spec}\, A \longrightarrow F$, we have a commutative square

$$\begin{tikzcd} \mathbf{Spec}\, A_{red} \arrow[r] \arrow[d] & \mathbf{Spec}\, A \arrow[d] \\ F_{red} = \mathbf{Spec}\, A_0 \arrow[r] & F \end{tikzcd}$$

and, therefore, an induced a natural morphism of A_0-dg-modules

$$\mathbb{L}_{F_{red}/F} \longrightarrow \mathbb{L}_{A_{red}/A}.$$

This yields a morphism of (non mixed) graded cdgas

$$Sym_{A_0}(\mathbb{L}_{F_{red}/F}[-1]) \longrightarrow \mathbb{D}(A).$$

Taking the limit over $(\mathbf{Spec}\, A \to F) \in \mathbf{dAff}/F$, we obtain a natural morphism of (non mixed) graded cdgas

$$\phi_F : Sym_{A_0}(\mathbb{L}_{F_{red}/F}[-1]) \longrightarrow \mathbb{D}(F) = \lim_{\mathbf{Spec}\, A \to F} \mathbb{D}(A).$$

Since F is algebraisable (Definition 2.2.1), there exists an algebraic derived n-stack (for some integer n) G, a morphism $f : \mathbf{Spec}\, A \longrightarrow G$ and an equivalence $\widehat{G}_f \simeq F$. We will prove that ϕ_F is an equivalence by induction on n.

We first observe that the statement is local in the étale topology of $\mathbf{Spec}\, A_0$ in the following sense. Let $A_0 \longrightarrow A_0'$ be an étale morphism and $X' = \mathbf{Spec}\, A_0' \longrightarrow X = \mathbf{Spec}\, A_0$ be the induced morphism. We let F' be the formal completion of the morphism $X' \longrightarrow F$ (or equivalently of $X' \longrightarrow G$) so that we have a commutative square of derived stacks

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & F' \arrow[d] \\ X \arrow[r] & F. \end{tikzcd}$$

By construction this square is moreover cartesian, and induces a morphism of graded cdgas

$$\mathbb{D}(F) \longrightarrow \mathbb{D}(F').$$

Thus the assignment $X' \mapsto \mathbb{D}(F')$ defines a stack of graded cdgas over the small étale site of X, and, in the same way, $X' \mapsto Sym_{A'_0}(\mathbb{L}_{X'/F'}[-1])$ is a stack of graded cdgas on the small étale site of X. The various morphism $\phi_{F'}$ organize into a morphism of étale stacks on X. In order to prove that ϕ_F is an equivalence it is enough to prove that $\phi_{F'}$ is so after some étale covering $X' \longrightarrow X$.

The above étale locality of the statement implies that we can assume that there is an affine $Y \in \mathbf{dAff}$, a smooth morphism $Y \longrightarrow G$, such that $X \longrightarrow G$ comes equipped with a factorization through Y

$$\begin{tikzcd} X \arrow[r] \arrow[d] & G \\ Y \arrow[ur] & \end{tikzcd}$$

We let Y_* be the nerve of the morphism $Y \longrightarrow G$, which is a smooth Segal groupoid in derived stacks (see [HAG-II, §S.3.4]). Moreover, $Y_0 = Y$ is affine and Y_i is an algebraic $(n-1)$-stack. We consider the chosen lifting $X \longrightarrow Y_0$ as a morphism of simplicial objects $X \longrightarrow Y_*$, where X is considered as simplicially constant. We let $\widehat{Y}_*$ be the formal completion of Y_* along X, defined by

$$\widehat{Y}_i := \widehat{(Y_i)}_{X \to Y_i}.$$

The simplicial object $\widehat{Y}_*$ can be canonically identified with the nerve of the induced morphism on formal completions $\widehat{Y_0} \longrightarrow F = \widehat{G}$. Moreover, by construction $\widehat{Y_0} \longrightarrow F$ is an epimorphism of derived stacks, and we thus have a natural equivalence of derived stacks

$$|\widehat{Y}_*| = \mathrm{colim}_i \widehat{Y}_i \simeq F.$$

As the ∞-functor $\mathbb{D}$ sends colimits to limits we have

$$\mathbb{D}(F) \simeq \lim_i \mathbb{D}(\widehat{Y}_i).$$

Also, for each i the morphism $\widehat{Y}_i \longrightarrow Y_i$ is formally étale, and thus we have

$$\mathbb{L}_{X/\widehat{Y}_i} \simeq \mathbb{L}_{X/Y_i}.$$

Smooth descent for differential forms on G (see Appendix B) then implies that we have equivalences of A_0-dg-modules

$$\wedge^p \mathbb{L}_{X/F} \simeq \wedge^p \mathbb{L}_{X/G} \simeq \lim_i \wedge^p \mathbb{L}_{X/Y_i} \simeq \lim_i \wedge^p \mathbb{L}_{X/\widehat{Y}_i}.$$

Therefore

$$Sym_{A_0}(\mathbb{L}_{X/F}[-1]) \simeq \lim_i Sym_{A_0}(\mathbb{L}_{X/\widehat{Y}_i}[-1]).$$

The upshot is that, in order to prove that ϕ_F is an equivalence, it is enough to prove that all the ϕ_{Y_i}'s are equivalences. By descending induction on n this allows us to reduce to the case where G is a derived algebraic stack, and by further localization on G to the case where G is itself a derived affine scheme. Moreover, by refining the smooth atlas $Y \to G$ in the argument above, we may also assume that $X \longrightarrow G$ is a closed immersion of derived affine schemes.

Suppose $G = Z \in \mathbf{dAff}_k$, and $X \longrightarrow Z$ be a closed immersion; recall that this means that the induced morphism on truncations $t_0(X) = X \longrightarrow t_0(Z)$ is a closed immersion of affine schemes. We may present $X \longrightarrow Z$ by a cofibrant morphism between cofibrant cdgas $B \longrightarrow A$, and moreover we may assume that A is a cell B-algebra with finitely many cells in each degree, and that B is a cell k-algebra with finitely many cells in each degree. We let B^0 be the k-algebra of degree zero elements in B and $Z^0 = \mathbf{Spec}\, B^0$. The formal completion $\widehat{Z} = F$ of $X \longrightarrow Z$ sits in a cartesian square of derived stacks

$$\begin{tikzcd} \widehat{Z} \arrow[r] \arrow[d] & \widehat{Z^0} \arrow[d] \\ Z \arrow[r] & Z^0, \end{tikzcd}$$

where $Z \longrightarrow Z^0$ is the natural morphism induced by $B^0 \subset B$, and $\widehat{Z^0}$ is the formal completion of Z^0 along the closed immersion corresponding to the quotient of algebras

$$B^0 \longrightarrow \pi_0(B) \longrightarrow \pi_0(A) \simeq A_0.$$

We let $I \subset B^0$ be the kernel of $B^0 \longrightarrow A_0$, and we choose generators $f_1, \cdots, f_p$ for I. We set $B^0(j) := K(B^0, f_1^j, \ldots, f_p^j)$ the Koszul cdga over B^0 attached to the sequence $(f_1, \cdots, f_p)$, $Z_j^0 := \mathbf{Spec}\, B^0(j)$ and $Z_j := Z \times_{Z^0} Z_j^0$. We have a natural equivalence of derived stacks

$$F = \widehat{Z} \simeq \mathrm{colim}_j Z_j.$$

By our Appendix B we moreover know that $\widehat{Z^0}$ is equivalent to $\mathrm{colim}_j Z_j^0$ as derived prestacks, or in other words, that the above colimit of prestacks is a derived stack. By pull-back, we see that the colimit $\mathrm{colim}_j Z_j$ can be also computed in derived prestacks, and thus the equivalence $\widehat{Z} \simeq \mathrm{colim}_j Z_j$ is an equivalence of derived prestacks (i.e. of ∞-functors defined on $\mathbf{dAff}_k$). As $\mathbb{D}$ sends colimits to limits, we do have an equivalence of graded mixed cdgas

$$\mathbb{D}(F) \simeq \lim_n \mathbb{D}(Z_j).$$

The proposition follows by observing that, for any $p \geq 0$, the natural morphism

$$\wedge^p \mathbb{L}_{X/Z} \longrightarrow \lim_n \wedge^p \mathbb{L}_{X/Z_j}$$

is indeed an equivalence of dg-modules over A_0 (see Appendix B). $\square$

As a consequence of Proposition 2.2.6, if F is an algebraisable affine formal derived stack, and if $\mathbb{L}_F$ is of amplitude contained in $]-\infty, n]$ for some n, then the graded mixed cdga $\mathbb{D}(F)$ satisfies the following conditions.

1. The cdga $A := \mathbb{D}(F)(0)$ is concentrated in degree 0 and is a reduced k-algebra of finite type.

2. The A-dg-module $\mathbb{D}(F)(1)$ is almost finitely presented and of amplitude contained in $]-\infty, n]$.

3. The natural morphism
$$Sym_A(\mathbb{D}(F)(1)) \longrightarrow \mathbb{D}(F)$$
is an equivalence of graded cdgas.

We now move to the proof of point 2 in Theorem 2.2.2, i.e. we define the ∞-functor

$$\phi_F : \mathrm{L}_{\mathrm{QCoh}}(F) \longrightarrow \mathbb{D}(F) - Mod_{\epsilon-\mathbf{dg}}$$

for a general $F \in \mathbf{dFSt}_k^{\mathsf{aff}}$. This was already defined when F is an affine derived stack in Corollary 2.2.5, and for general F the ∞-functor ϕ_F will be simply defined by left Kan extension. More precisely, if $F \in \mathbf{dSt}_k$, we start with

$$\lim_{\mathbf{Spec}\, A \to F} \phi_A : \lim_{\mathbf{Spec}\, A \to F} \mathrm{L}(A) \longrightarrow \lim_{\mathbf{Spec}\, A \to F} \mathbb{D}(A) - Mod_{\epsilon-\mathbf{dg}},$$

where for each fixed A the ∞-functor $\phi_A : \mathrm{L}(A) \longrightarrow \mathbb{D}(A) - Mod_{\epsilon-\mathbf{dg}}$, is the one of our corollary 2.2.5 and sends an A-dg-module E to $E \otimes_A \mathbf{DR}(A_{red}/A)$. Finally, as $\mathbb{D}(F) = \lim_{\mathbf{Spec}\, A \to F} \mathbb{D}(A)$ there is a natural limit ∞-functor

$$lim : \lim_{\mathbf{Spec}\, A \to F} \mathbb{D}(A) - Mod_{\epsilon-\mathbf{dg}} \longrightarrow \mathbb{D}(F) - Mod_{\epsilon-\mathbf{dg}}.$$

By composing these two functors, we obtain a natural ∞-functor

$$\phi_F : \mathrm{L}_{\mathrm{QCoh}}(F) \longrightarrow \mathbb{D}(F) - Mod_{\epsilon-\mathbf{dg}},$$

which is clearly functorial in $F \in \mathbf{dSt}_k$. Note that ϕ_F exists for any F, without any extra conditions. The fact that it induces an equivalence on perfect modules only requires F_{red} to be an affine scheme, as shown in Proposition 2.2.7 below. This establishes, in particular, point 2 of Theorem 2.2.2, and thus concludes its proof.

If $B \in \epsilon - \mathbf{cdga}^{gr}$ is graded mixed cdga, a graded mixed B-dg-module $M \in B - Mod_{\epsilon-\mathbf{dg}}$ is called *perfect*, if, as a graded B-dg-module, it is (equivalent to a graded B-dg-module) of the form $B \otimes_{B(0)} E$ for $E \in \mathrm{L}_{\mathsf{Perf}}(B(0))$. Note that E is then automatically equivalent to $M(0)$. In other words, M is perfect if it is free over its degree 0 part, as a graded B-dg-module. We let $B - Mod_{\epsilon-\mathbf{dg}}^{\mathsf{Perf}}$ be the full sub-∞-category of $B - Mod_{\epsilon-\mathbf{dg}}$ consisting of perfect graded mixed B-dg-modules.

Proposition 2.2.7 *Let $F \in \mathbf{dSt}_k$, and assume that $F_{red} = \mathbf{Spec}\, A_0$ is an affine reduced scheme of finite type over k. Then, the ∞-functor*

$$\phi_F : \mathrm{L}_{\mathsf{Perf}}(F) \longrightarrow \mathbb{D}(F) - Mod_{\epsilon-\mathbf{dg}}^{\mathsf{Perf}}$$

is an equivalence of ∞-categories.

Proof. By Corollary 2.2.5, we have a natural equivalence of ∞-categories

$$\mathrm{L}_{\mathsf{Perf}}(F) \simeq \lim_{\mathbf{Spec}\, A \to F} \mathrm{L}_{\mathsf{Perf}}(A) \simeq \lim_{\mathbf{Spec}\, A \to F} \mathbb{D}(A) - Mod_{\epsilon-\mathbf{dg}}^{\mathsf{Perf}}.$$

As $\mathbb{D}(F) = \lim_{\mathbf{Spec}\, A \to F} \mathbb{D}(A)$, we have a natural adjunction of ∞-categories

$$\mathbb{D}(F) - Mod_{\epsilon-\mathbf{dg}} \longleftrightarrow \lim_{\mathbf{Spec}\, A \to F} \mathbb{D}(A) - Mod_{\epsilon-\mathbf{dg}},$$

where the right adjoint is the limit ∞-functor. Thus this adjunction induces an equivalences on perfect objects. $\square$

2.3 Differential forms and polyvectors on perfect formal derived stacks

In the previous section, we have associated to any formal affine derived stack F, a mixed graded cdga $\mathbb{D}(F)$ in such a way that $\mathrm{L}_{\mathsf{Perf}}(F) \simeq \mathbb{D}(F) - Mod_{\epsilon-\mathbf{dg}}^{\mathsf{Perf}}$. We will now compare the de Rham theories of F (in the sense of [PTVV]) and of $\mathbb{D}(F)$ (in the sense of §1.3), and prove that they are equivalent when appropriately understood.

2.3.1 De Rham complex of perfect formal derived stacks

We let $F \longrightarrow \mathbf{Spec}\, A$ be a perfect formal derived stack (Definition 2.1.6) and $\mathbb{D}(F)$ the corresponding graded mixed cdga of Theorem 2.2.2. The projection $F \longrightarrow \mathbf{Spec}\, A$ induces a morphism of graded mixed cdga $\mathbb{D}(A) \longrightarrow \mathbb{D}(F)$ which allows us to view $\mathbb{D}(F)$ as a graded mixed $\mathbb{D}(A)$-algebra. By taking $\mathcal{M} = \epsilon - \mathbf{dg}_k^{gr}$ in Proposition 1.3.16, we may consider in particular its relative de Rham object $\mathbf{DR}^{int}(\mathbb{D}(F)/\mathbb{D}(A))$ which is a graded mixed cdga over the ∞-category of graded mixed $\mathbb{D}(A)$-dg-modules. There is an equivalence

$$\mathbf{DR}^{int}(\mathbb{D}(F)/\mathbb{D}(A)) \simeq Sym_{\mathbb{D}(F)}(\mathbb{L}^{int}_{\mathbb{D}(F)/\mathbb{D}(A)}[-1])$$

of (non mixed) graded cdgas over the ∞-category of graded mixed $\mathbb{D}(A)$-dg-modules. We can consider its realization, as in Definition 1.3.1,

$$\mathbf{DR}(\mathbb{D}(F)/\mathbb{D}(A)) := |\mathbf{DR}^{int}(\mathbb{D}(F)/\mathbb{D}(A))|$$

which is thus a graded mixed cdga over $|\mathbb{D}(A)| \simeq A$ (Remark 1.3.2, and Lemma 2.2.4), i.e. an A-linear graded mixed cdga. Moreover, according to §1.5, we can also consider its Tate realization

$$\mathbf{DR}^t(\mathbb{D}(F)/\mathbb{D}(A)) := |\mathbf{DR}^{int}(\mathbb{D}(F)/\mathbb{D}(A))|^t,$$

which is, again, a graded mixed A-linear cdga.

On the other hand, it is natural to consider the following

Definition 2.3.1 *The* de Rham object of an arbitrary derived stack F over an affine derived stack $\mathbf{Spec}\, B$ *is*

$$\mathbf{DR}(F/B) := \lim_{\mathbf{Spec}\, C \to F} \mathbf{DR}(C/B) \in \epsilon - \mathbf{cdga}_B^{gr}$$

where the limit is taken in the category $\epsilon - \mathbf{cdga}_B^{gr}$ of graded mixed B-linear cdgas, and over all morphisms $\mathbf{Spec}\, C \to F$ of derived stacks over $\mathbf{Spec}\, B$.

Proposition 2.3.2 *Let $f : F \to G$ be a map in $\mathbf{dSt}/\mathbf{Spec}\, B$. There is an induced map $\mathbf{DR}(G/B) \to \mathbf{DR}(F/B)$ in $\epsilon - \mathbf{cdga}_B^{gr}$.*

Proof. Let I_F (resp. I_G) the category on which $\mathbf{DR}(F/B)$ (resp. $\mathbf{DR}(G/B)$) is defined as a limit: $\mathbf{DR}(F/B) = lim_{I_X}\mathbf{DR}^F$, where

$$\mathbf{DR}^F : I_X \to \epsilon - \mathbf{cdga}_B^{gr}, \quad I_F \ni (\mathbf{Spec}\, C \to F) \longmapsto \mathbf{DR}(C/B)$$

(resp. $\mathbf{DR}(G/B) = lim_{I_X}\mathbf{DR}^G$, where

$$\mathbf{DR}^G : I_X \to \epsilon - \mathbf{cdga}_B^{gr}, \quad I_G \ni (\mathbf{Spec}\, C \to G) \longmapsto \mathbf{DR}(C/B)\ ).$$

There is an obvious functor $\alpha_f : I_F \to I_G$, induced by composition by f, hence an induced morphism $\lim_{I_G} \mathbf{DR}^G \to \lim_{I_F} \alpha_f^*(\mathbf{DR}^G)$. But $\alpha_f^*(\mathbf{DR}^G) \simeq \mathbf{DR}^F$, hence we get a morphism $\mathbf{DR}(G/B) \to \mathbf{DR}(F/B)$. □

We now claim that the two de Rham complexes $\mathbf{DR}^t(\mathbb{D}(F)/\mathbb{D}(A))$ and $\mathbf{DR}(F/A)$ are naturally equivalent, at least when F is a perfect formal derived stack over $\mathbf{Spec}\, A$ that is moreover *algebraisable over* $\mathbf{Spec}\, A$ as in Definition 2.2.1. More precisely, we have

Theorem 2.3.3 *Let $F \longrightarrow \mathbf{Spec}\, A$ be a perfect formal derived stack. We assume that F is moreover algebraisable over $\mathbf{Spec}\, A$ (Definition 2.2.1). Then, there are natural morphisms*

$$\mathbf{DR}(\mathbb{D}(F)/\mathbb{D}(A)) \longrightarrow \mathbf{DR}^t(\mathbb{D}(F)/\mathbb{D}(A)) \longrightarrow \lim_{\mathbf{Spec}\, B \to F} \mathbf{DR}^t(\mathbb{D}(B)/\mathbb{D}(A)) \longleftarrow \mathbf{DR}(F/A)$$

that are all equivalences of graded mixed A-cdgas.

Proof. We start by defining the three natural morphisms. The first morphism on the left is induced by the natural transformation $|.| \to |.|^t$, from realization to Tate realization (see §1.5). The second morphism on the left is induced by functoriality. It remains to describe the morphism on the right

$$\mathbf{DR}(F/A) \longrightarrow \lim_{\mathbf{Spec}\, B \to F} \mathbf{DR}^t(\mathbb{D}(B)/\mathbb{D}(A)).$$

By definition 2.3.1

$$\mathbf{DR}(F/A) \simeq \lim_{\mathbf{Spec}\, B \to F} \mathbf{DR}(B/A),$$

and we have a morphism of graded mixed cdgas $B \longrightarrow \mathbb{D}(B)$, where B is considered with its trivial mixed structure of pure weight 0. This morphism is the adjoint to the equivalence $B \simeq |\mathbb{D}(B)|$ of Proposition 2.2.3. By functoriality it comes with a commutative square of graded mixed cdgas

$$\begin{tikzcd} B \arrow[r] & \mathbb{D}(B) \\ A \arrow[u] \arrow[r] & \mathbb{D}(A), \arrow[u] \end{tikzcd}$$

and thus induces a morphism on de Rham objects

$$\mathbf{DR}(B/A) \longrightarrow \mathbf{DR}(\mathbb{D}(B)/\mathbb{D}(A)) \longrightarrow \mathbf{DR}^t(\mathbb{D}(B)/\mathbb{D}(A)).$$

By taking the limit, we get the desired map

$$\mathbf{DR}(F/A) \longrightarrow \lim_{\mathbf{Spec}\, B \to F} \mathbf{DR}^t(\mathbb{D}(B)/\mathbb{D}(A)).$$

To prove the statement of Theorem 2.3.3, we first observe that all the graded mixed cdgas $\mathbb{D}(F)$ and $\mathbb{D}(B)$ are positively weighted, as they are freely generated, as graded cdgas, by their weight 1 part (see Proposition 2.2.6). The natural morphisms

$$\mathbf{DR}(\mathbb{D}(B)/\mathbb{D}(A)) \longrightarrow \mathbf{DR}^t(\mathbb{D}(B)/\mathbb{D}(A)) \qquad \mathbf{DR}(\mathbb{D}(F)/\mathbb{D}(A)) \longrightarrow \mathbf{DR}^t(\mathbb{D}(F)/\mathbb{D}(A))$$

are then equivalence by trivial weight reasons. So, it will be enough to check the following two statements

1. The descent morphism

   $$\mathbf{DR}(\mathbb{D}(F)/\mathbb{D}(A)) \longrightarrow \lim_{\mathbf{Spec}\, B \to F} \mathbf{DR}(\mathbb{D}(B)/\mathbb{D}(A))$$

   is an equivalence.

2. For any $\mathbf{Spec}\, B \longrightarrow \mathbf{Spec}\, A$, the natural morphism

$$\mathbf{DR}(B/A) \longrightarrow \mathbf{DR}(\mathbb{D}(B)/\mathbb{D}(A))$$

is an equivalence.

Statement (1) is proved using the fact that F is algebraisable completely analogously to the proof of Proposition 2.2.6. We first note that the assignment $\mathbf{Spec}\, B \mapsto \mathbf{DR}(\mathbb{D}(B)/\mathbb{D}(A))$ is a stack for the etale topology, so the right hand side in (1) is simply the left Kan extension of $\mathbf{Spec}\, B \mapsto \mathbf{DR}(\mathbb{D}(B)/\mathbb{D}(A))$ to all derived stacks. In particular, it has descent over F. We write $F = \widehat{G}_f$, for a morphism $f : \mathbf{Spec}\, A_{red} \longrightarrow G$, with G an algebraic derived n-stack locally of finite presentation over A. By localizing with respect to the étale topology on $\mathbf{Spec}\, A_{red}$, we can assume that there is an affine derived scheme U with a smooth map $U \longrightarrow G$, such that f factors through U. We let $\widehat{U_*}$ denote the formal completion of the nerve of $U \to G$ along the morphism $\mathbf{Spec}\, A_{red} \longrightarrow U_*$. We now claim that the natural morphism

$$\mathbf{DR}(\mathbb{D}(F)/\mathbb{D}(A)) \longrightarrow \lim_{n\in\Delta} \mathbf{DR}(\mathbb{D}(\widehat{U_n})/\mathbb{D}(A))$$

is an equivalence. We will actually prove the stronger statement that the induced morphism

$$\wedge^p \mathbb{L}^{int}_{\mathbb{D}(F)/\mathbb{D}(A)} \longrightarrow \lim_{n\in\Delta} \wedge^p \mathbb{L}^{int}_{\mathbb{D}(\widehat{U_n})/\mathbb{D}(A)} \qquad (*)$$

is an equivalence of non-mixed graded complexes for all p. For this, we use Proposition 2.2.6, which implies that we have equivalences of graded modules

$$\wedge^p \mathbb{L}^{int}_{\mathbb{D}(F)/\mathbb{D}(A)} \simeq \mathbb{D}(F) \otimes_{A_{red}} \wedge^p f^*(\mathbb{L}_{G/A})$$
$$\wedge^p \mathbb{L}^{int}_{\mathbb{D}(\widehat{U_n})/\mathbb{D}(A)} \simeq \mathbb{D}(\widehat{U_n}) \otimes_{A_{red}} \wedge^p f^*(\mathbb{L}_{U_n/A}).$$

Since $\mathbb{D}(F) \simeq \lim_n \mathbb{D}(\widehat{U_n})$, and tensor product of perfect modules preserves limits, we obtain $(*)$ as all $f^*(\mathbb{L}_{U_n/A})$ and $f^*(\mathbb{L}_{G/A})$ are perfect complexes of A_{red}-modules, and because differential forms satisfy descent (see Appendix B), so that

$$f^*(\mathbb{L}_{G/A}) \simeq \lim_n \wedge^p f^*(\mathbb{L}_{U_n/A}).$$

By induction on the geometric level n of G, we finally see that statement (1) can be reduced to the case where $G = \mathbf{Spec}\, B$ is affine and $f : \mathbf{Spec}\, A_{red} \longrightarrow G$ is a closed immersion. In this case, we have already seen that F can be written as $\mathrm{colim}_n \mathbf{Spec}\, B_n$, for a system of closed immersions $\mathbf{Spec}\, B_n \longrightarrow \mathbf{Spec}\, B_{n+1}$ such that $(B_n)_{red} \simeq A_{red}$. This colimit can be taken in derived prestacks, so

Appendix B B.1.3 applies. This implies statement (1), as we have

$$\wedge^p \mathbb{L}^{int}_{\mathbb{D}(F)/\mathbb{D}(A)} \simeq \mathbb{D}(F) \otimes_B \wedge^p \mathbb{L}_{B/A}$$
$$\wedge^p \mathbb{L}^{int}_{\mathbb{D}(B_n)/\mathbb{D}(A)} \simeq \mathbb{D}(B_n) \otimes_B \wedge^p \mathbb{L}_{B_n/A}.$$

It remains to prove statement (2). We need to show that the natural morphism $B \to \mathbb{D}(B)$ and $A \to \mathbb{D}(A)$ induces an equivalence

$$\wedge^p \mathbb{L}_{B/A} \longrightarrow |\wedge^p \mathbb{L}^{int}_{\mathbb{D}(B)/\mathbb{D}(A)}|.$$

This is the relative version of the following lemma, and can be in fact deduced from it.

Lemma 2.3.4 *If $F = \mathbf{Spec}\, A$ is an affine derived scheme then the natural morphism*

$$\mathbf{DR}(A/k) \longrightarrow \mathbf{DR}(\mathbb{D}(A))$$

is an equivalence of graded cdgas.

Proof of lemma. It is enough to show that the induced morphism

$$\mathcal{A}^p(A) \simeq \wedge^p \mathbb{L}_A \longrightarrow |\wedge^p \mathbb{L}_{\mathbb{D}(A)}|$$

is an equivalence of complexes, for any $p \geq 0$.

The proof will now involve strict models. We choose a cell model for A with finitely many cells in each dimension, and a factorization

$$A \longrightarrow A' \longrightarrow A_{red},$$

where $A' \longrightarrow A_{red}$ is an equivalence and A' is a cell A-algebra with finitely many cells in each dimension. Moreover, as $\pi_0(A) \longrightarrow \pi_0(A_{red})$ is surjective, we can chose A' having cells only in dimension 1 and higher (i.e. no 0-dimensional cells). With such choices, the cotangent complex $\mathbb{L}_{A_{red}/A}$ has a strict model $\Omega^1_{A'/A}$, and is itself a cell A'-module with finitely many cells in each dimension, and no 0-dimensional cell. We let $L := \Omega^1_{A'/A}$.

The graded mixed cdga $\mathbb{D}(A)$ can then be represented (§1.3.3) by the strict de Rham algebra $\mathbb{D}^{str}(A) := Sym_{A'}(L[-1])$. We consider $B := (A')^0 = A^0$ the degree 0 part of A' (which is also the degree 0 part of A because A' has no 0-dimensional cell over A), and let $V := L^{-1}$ the degree (-1) part of L. The k-algebra B is just a polynomial algebra over k, and V is a free B-module whose rank equals the number of 1-dimensional cells of A' over A.

For the sake of clarity, we introduce the following notations. For $E \in \epsilon - \mathbf{dg}^{gr}$ a graded mixed

k-dg-module, we let

$$|E| := \prod_{i \geq 0} E(i),$$

the product of the non-negative weight parts of E, endowed with its natural total differential sum of the cohomological differential and the mixed structure. In the same way, we let

$$|E|^{\oplus} := \oplus_{i \geq 0} E(i),$$

to be the coproduct of the non-negative weight parts of E, with the similar differential, so that $|E|^{\oplus}$ sits naturally inside $|E|$ as a sub-dg-module. Note that $|E|$ is a model for $\mathbb{R}\underline{Hom}_{\epsilon-\mathbf{dg}}(k(0), E)$, whereas $|E|^{\oplus}$ is a rather silly functor which is not even invariant under quasi-isomorphisms of graded mixed dg-modules.

As we have already seen in the proof of Lemma 2.2.4, there exists a strict push-out square of cdgas

$$\begin{tikzcd}
{|Sym_{A'}(L[-1])|^{\oplus}} \arrow[r] & {|Sym_{A'}(L[-1])|} \\
Sym_B(V) \arrow[r] \arrow[u] & \widehat{Sym}_B(V) \arrow[u]
\end{tikzcd}$$

where $\widehat{Sym}$ denotes the completed symmetric algebra, i.e the infinite product of the various symmetric powers. This push-out is also a homotopy push-out of cdgas because the bottom horizontal morphism is a flat morphism of commutative rings.

We have the following version of the above push-out square for modules, too. Let $M \in \mathbb{D}^{str}(A) - Mod_{\epsilon-\mathbf{dg}^{gr}}$ a graded mixed $Sym_{A'}(L[-1])$-dg-module. We assume that, as a graded dg-module, M is isomorphic to

$$M \simeq \mathbb{D}^{str}(A) \otimes_{A'} E,$$

where E is a graded A'-dg-module pure of some weight i, and moreover, E is a cell module with finitely many cells in each non-negative dimension. Under these finiteness conditions, it can be checked that there is a natural isomorphism

$$|M|^{\oplus} \otimes_{Sym_B(V)} \widehat{Sym}_B(V) \simeq |M|.$$

The same is true for any graded mixed $\mathbb{D}^{str}(A)$-dg-module M which is (isomorphic to) a successive extension of graded mixed modules as above. In particular, we can apply this to $\Omega^1_{\mathbb{D}^{str}(A)}$ as well as to $\Omega^p_{\mathbb{D}^{str}(A)}$, for any $p > 0$. Indeed, there is a short exact sequence of graded $Sym_{A'}(L[-1])$-modules

$$0 \longrightarrow \Omega^1_{A'} \otimes_{A'} Sym_{A'}(L[-1]) \longrightarrow \Omega^1_{\mathbb{D}^{str}(A)} \longrightarrow L \otimes_{A'} Sym_{A'}(L[-1])[-1] \longrightarrow 0.$$

This shows that for all $p > 0$, we have a canonical isomorphism

$$|\Omega^p_{\mathbb{D}^{str}(A)}|^{\oplus} \otimes_{Sym_B(V)} \widehat{Sym}_B(V) \simeq |\Omega^p_{\mathbb{D}^{str}(A)}|.$$

Now we notice that the natural morphism

$$|\Omega^p_{\mathbb{D}^{str}(A)}|^{\oplus} \longrightarrow |\Omega^p_{\mathbb{D}^{str}(A)}|^{\oplus} \otimes_{Sym_B(V)} \widehat{Sym}_B(V)$$

is isomorphic to

$$|\Omega^p_{\mathbb{D}^{str}(A)}|^{\oplus} \longrightarrow |\Omega^p_{\mathbb{D}^{str}(A)}|^{\oplus} \otimes_{|\mathbb{D}^{str}(A)|^{\oplus}} |\mathbb{D}^{str}(A)|.$$

Let us show that

Sub-Lemma 2.3.5 *For all $p \geq 0$ the above morphism*

$$|\Omega^p_{\mathbb{D}^{str}(A)}|^{\oplus} \longrightarrow |\Omega^p_{\mathbb{D}^{str}(A)}|^{\oplus} \otimes_{|\mathbb{D}^{str}(A)|^{\oplus}} |\mathbb{D}^{str}(A)|$$

is a quasi-isomorphism.

Proof of sub-lemma. First of all, in the push-out square of cdgas

$$\begin{array}{ccc} |\mathbb{D}^{str}(A)|^{\oplus} & \longrightarrow & |\mathbb{D}^{str}(A)| \\ \uparrow & & \uparrow \\ Sym_B(V) & \longrightarrow & \widehat{Sym}_B(V) \end{array}$$

the bottom horizontal arrow is flat. This implies that the tensor product

$$|\Omega^p_{\mathbb{D}^{str}(A)}|^{\oplus} \otimes_{|\mathbb{D}^{str}(A)|^{\oplus}} |\mathbb{D}^{str}(A)|$$

is also a derived tensor product. The sub-lemma would then follow from the fact that the inclusion

$$|\mathbb{D}^{str}(A)|^{\oplus} \hookrightarrow |\mathbb{D}^{str}(A)|$$

is a quasi-isomorphism. To see this, we consider the diagram of structure morphism over A

$$\begin{array}{ccccc} & & A & & \\ & \swarrow^{u} & & \searrow^{v} & \\ |\mathbb{D}^{str}(A)|^{\oplus} & & \longrightarrow & & |\mathbb{D}^{str}(A)|. \end{array}$$

The morphism v is an equivalence by Proposition 2.2.3 and lemma 2.2.4. The morphism u is the inclusion of A into the non-completed derived de Rham complex of A_{red} over A, and thus is also a quasi-isomorphism. $\square$

Now we can prove that the above sub-lemma implies Lemma 2.3.4. Indeed, the morphism

$$\wedge^p \mathbb{L}_A \longrightarrow | \wedge^p \mathbb{L}_{\mathbb{D}(A)}|$$

can be represented by the composition of morphisms between strict models

$$\Omega^p_A \longrightarrow |\Omega^p_{\mathbb{D}^{str}(A)}|^{\oplus} \longrightarrow |\Omega^p_{\mathbb{D}^{str}(A)}|^{\oplus} \otimes_{|\mathbb{D}^{str}(A)|^{\oplus}} |\mathbb{D}^{str}(A)| \longrightarrow |\Omega^p_{\mathbb{D}^{str}(A)}|.$$

The two rightmost morphisms are quasi-isomorphisms by what we have seen, while the leftmost one can simply be identified, up to a canonical isomorphism, with the natural morphism

$$\Omega^p_A \longrightarrow \Omega^p_{|\mathbb{D}^{str}(A)|^{\oplus}}.$$

This last morphism is again a quasi-isomorphism because it is induced by the morphism

$$A \longrightarrow |\mathbb{D}^{str}(A)|^{\oplus}$$

which is a quasi-isomorphism of quasi-free, and thus cofibrant, cdgas. $\square$

Lemma 2.3.4 is proven, and we have thus finished the proof of Theorem 2.3.3. $\square$

The following corollary is a consequence of the proof Theorem 2.3.3.

Corollary 2.3.6 *Let $F \longrightarrow$ **Spec** A be a perfect formal derived stack over* **Spec** *A, and assume that F is algebraisable. Let*

$$\phi_F : \mathrm{L}_{\mathsf{Perf}}(F) \longrightarrow \mathbb{D}(F) - Mod^{\mathsf{Perf}}_{\epsilon-\mathbf{dg}}$$

be the equivalence of Proposition 2.2.7. Then, there is a canonical equivalence of graded mixed $\mathbb{D}(F)$-modules

$$\phi_F(\mathbb{L}_{F/A}) \simeq \mathbb{L}^{int}_{\mathbb{D}(F)/\mathbb{D}(A)} \otimes_k k((1)).$$

Proof. First of all, as graded $\mathbb{D}(F)$-modules we have (Proposition 2.2.7)

$$\mathbb{L}^{int}_{\mathbb{D}(F)/\mathbb{D}(A)} \simeq \mathbb{D}(F) \otimes_{A_{red}} f^*(\mathbb{L}_{F/A}),$$

where f : **Spec** $A \longrightarrow F$ is the natural morphism, and $f^*(\mathbb{L}_{F/A})$ sits in pure weight 1, so that, according to our conventions, we should rather write

$$\mathbb{L}^{int}_{\mathbb{D}(F)/\mathbb{D}(A)} \simeq \mathbb{D}(F) \otimes_{A_{red}} f^*(\mathbb{L}_{F/A}) \otimes_k k((-1)).$$

In particular, $\mathbb{L}^{int}_{\mathbb{D}(F)/\mathbb{D}(A)} \otimes_k k((1))$ belongs to $\mathbb{D}(F) - Mod^{\mathsf{Perf}}_{\epsilon-\mathbf{dg}}$, as it is now free over its weight 0 part.

Moreover, the same proof as in Theorem 2.3.3 shows that for any perfect complex $E \in \mathrm{L}_{\mathsf{Perf}}(F)$, we have a natural equivalence, functorial in E

$$\Gamma(F, E \otimes_{\mathcal{O}_F} \mathbb{L}_{F/A}) \simeq |\phi_F(E) \otimes_{\mathbb{D}(F)} \mathbb{L}^{int}_{\mathbb{D}(F)/\mathbb{D}(A)}|.$$

We have a natural map $k = k((0)) \to k((-1))$ in the ∞-category of graded mixed complexes, represented by the map $\tilde{k} \to k((-1))$ sending x_1 to 1, in the notation of §1.4.1. Its weight-shift by 1 gives us a canonical map $k((1)) \to k$ in the ∞-category of graded mixed complexes, inducing a morphism

$$\mathbb{L}^{int}_{\mathbb{D}(F)/\mathbb{D}(A)} \otimes_k k((1)) \longrightarrow \mathbb{L}^{int}_{\mathbb{D}(F)/\mathbb{D}(A)}.$$

Finally, this morphism induces an equivalence

$$|\phi_F(E) \otimes_{\mathbb{D}(F)} \mathbb{L}^{int}_{\mathbb{D}(F)/\mathbb{D}(A)} \otimes_k k((1))| \simeq |\phi_F(E) \otimes_{\mathbb{D}(F)} \mathbb{L}^{int}_{\mathbb{D}(F)/\mathbb{D}(A)}|.$$

We thus get an equivalence

$$\Gamma(F, E \otimes_{\mathcal{O}_F} \mathbb{L}_{F/A}) \simeq |\phi_F(E) \otimes_{\mathbb{D}(F)} \mathbb{L}^{int}_{\mathbb{D}(F)/\mathbb{D}(A)} \otimes_k k((1))|,$$

functorial in E. Observe now that $\phi_F(E) \otimes_{\mathbb{D}(F)} \mathbb{L}^{int}_{\mathbb{D}(F)/\mathbb{D}(A)} \otimes_k k((1))$ is a perfect graded mixed $\mathbb{D}(F)$-module. Since E is perfect, these equivalence can also be re-written as

$$\mathbb{R}\underline{Hom}(E^\vee, \mathbb{L}_{F/A}) \simeq \mathbb{R}\underline{Hom}(\phi_F(E)^\vee, \mathbb{L}^{int}_{\mathbb{D}(F)/\mathbb{D}(A)} \otimes_k k((1))).$$

Now, ϕ_F is an equivalence, and therefore Yoneda lemma implies that $\phi_F(\mathbb{L}_{F/A})$ and $\mathbb{L}^{int}_{\mathbb{D}(F)/\mathbb{D}(A)} \otimes_k k((1))$ are naturally equivalent. $\square$

2.3.2 Shifted polyvectors over perfect formal derived stacks

We present here a version of Theorem 2.3.3 for shifted polyvectors.

Let F be a perfect formal derived stack over $\mathbf{Spec}\, A$. We have the corresponding graded mixed cdga $\mathbb{D}(F)$, which we consider as a graded mixed $\mathbb{D}(A)$-algebra. By taking $\mathcal{M} = \epsilon - \mathbf{dg}^{gr}_{\mathbb{D}(A)}$, we have the corresponding the graded $\mathbb{P}_{n+1}$-dg-algebra of n-shifted polyvectors $\mathbf{Pol}(\mathbb{D}(F), n)$ (Definition 1.4.15 (2)), as well as its Tate version $\mathbf{Pol}^t(\mathbb{D}(F), n)$ (Definition 1.5.3 (2)). To emphasize the fact that such objects are defined relative to $\mathbb{D}(A)$, we will more precisely denote them by $\mathbf{Pol}(\mathbb{D}(F)/\mathbb{D}(A), n)$, and $\mathbf{Pol}^t(\mathbb{D}(F)/\mathbb{D}(A), n)$, respectively.

On the other hand, we can give the following general

Definition 2.3.7 *Let $n \in \mathbb{Z}$, and $f : X \longrightarrow Y$ be a morphism of derived stacks, such that the relative cotangent complex $\mathbb{L}_{X/Y}$ is defined and is an object in $\mathrm{L}_{\mathsf{Perf}}(X)$. Then, we define*

$$\mathbf{Pol}(X/Y, n) := \bigoplus_p (\underline{Hom}_{\mathrm{L}_{\mathrm{QCoh}}(X)}(\otimes^p \mathbb{L}_{X/Y}, \mathcal{O}_X[pn]))^{\Sigma_p} \in \mathbf{dg}_k^{gr},$$

where $\mathrm{L}_{\mathrm{QCoh}}(X) \simeq \lim_{\mathbf{Spec}\, A \to X} \mathrm{L}(A)$ is considered as a dg-category over k, and $\underline{Hom}_{\mathrm{L}_{\mathrm{QCoh}}(X)}$ denotes its k-dg-module of morphisms.

Note that, in particular, $\mathbf{Pol}(X/Y, n)$ is defined if X and Y are derived Artin stacks locally of finite presentation over k, or if $Y = \mathbf{Spec}\, A$ and $f : X \to Y$ is a perfect formal derived stack.

Theorem 2.3.8 *If F is a perfect formal derived stack over $\mathbf{Spec}\, A$, and F is algebraisable, then there is a natural equivalence of graded k-dg-modules*

$$\mathbf{Pol}^t(\mathbb{D}(F)/\mathbb{D}(A), n) \simeq \mathbf{Pol}(F/A, n).$$

Proof. We have $\mathbb{L}_{F/A} \in \mathrm{L}_{\mathsf{Perf}}(F)$, and we consider the equivalence of Corollary 2.2.5

$$\phi_F : \mathrm{L}_{\mathsf{Perf}}(F) \longrightarrow \mathbb{D}(F) - Mod_{\epsilon-\mathbf{dg}}^{\mathsf{Perf}}.$$

By Corollary 2.3.6, there is a natural equivalence of graded mixed $\mathbb{D}(F)$-modules

$$\phi_F(\mathbb{L}_{F/A}) \simeq \mathbb{L}^{int}_{\mathbb{D}(F)/\mathbb{D}(A)} \otimes_k k((1)).$$

As ϕ_F is a symmetric monoidal equivalence, we get

$$\phi_F(Sym^p_{\mathcal{O}_F}(\mathbb{T}_{F/A}[n]) \simeq Sym^p(\mathbb{T}^{int}_{\mathbb{D}(F)/\mathbb{D}(A)}[n]) \otimes_k k((-p)),$$

for any n and p. The result then follows from the fact that ϕ_F is an equivalence together with the fact that the Tate realization is a stable realization, i.e. that, for any graded mixed $\mathbb{D}(F)$-module E, there is a natural equivalence $|E|^t \simeq |E \otimes_k k(1)|^t$. $\Box$

Remark 2.3.9 Note that corollary 2.3.6 implies that

$$\mathbb{T}^{int}_{\mathbb{D}(F)/\mathbb{D}(A)} \simeq \mathbb{D}(F) \otimes_{A_{red}} f^*(\mathbb{T}_{F/A}) \otimes_k k((-1)),$$

as a graded modules, where $f : \mathbf{Spec}\, A_{red} \longrightarrow F$ is the natural morphism. The weight-shift on the right hand side gives no chance for Theorem 2.3.8 to be true if the Tate realization $|-|^t$ is replaced by the standard one $|-|$, while this is true in the case of de Rham complexes.

2.4 Global aspects and shifted principal parts

In this last part of Section 2 we present the global aspects of what we have seen so far, namely *families* of perfect formal derived stacks and their associated graded mixed cdgas.

2.4.1 Families of perfect formal derived stacks

We start by the notion of *families* of perfect formal derived stacks.

Definition 2.4.1 *A morphism $X \longrightarrow Y$ of derived stacks is* a family of perfect formal derived stacks over Y *if, for all $\mathbf{Spec}\, A \in \mathbf{dAff}_k$ and all morphism $\mathbf{Spec}\, A \longrightarrow Y$, the fiber*

$$X_A := X \times_Y \mathbf{Spec}\, A \longrightarrow \mathbf{Spec}\, A$$

is a perfect formal derived stack over $\mathbf{Spec}\, A$ in the sense of Definition 2.1.6.

Note that, in the above definition, all derived stacks X_A have perfect cotangent complexes, for all $\mathbf{Spec}\, A$ mapping to Y. This implies that the morphism $X \longrightarrow Y$ itself has a relative cotangent complex $\mathbb{L}_{X/Y} \in \mathrm{L}_{\mathrm{QCoh}}(X)$ which is moreover perfect (see [HAG-II, §1.4.1]). In particular, for any $n \in \mathbb{Z}$, the graded k-dg-module $\mathbf{Pol}(X/Y, n)$ is well defined (Definition 2.3.7).

Definition 2.4.2 *Let $F \longrightarrow G$ be an arbitrary map of derived stacks. The* relative de Rham object of the derived stack F over G *is*

$$\mathbf{DR}(F/G) := \lim_{\mathbf{Spec}\, A \to G} \mathbf{DR}(F_A/A) \in \epsilon - \mathbf{cdga}_k^{gr}$$

where $\mathbf{DR}(F_A/A)$ is as in Def. 2.3.1, and the limit is taken in the ∞-category $\epsilon - \mathbf{cdga}_k^{gr}$ over all morphisms $\mathbf{Spec}\, A \to G$.

Proposition 2.4.3 *Let $f : F \to G$ and $g : G \to H$ be maps of derived stacks. There are canonical induced maps $\mathbf{DR}(G/H) \to \mathbf{DR}(F/H) \to \mathbf{DR}(F/G)$ in $\epsilon - \mathbf{cdga}_k^{gr}$.*

Proof. The first map $\mathbf{DR}(G/H) \to \mathbf{DR}(F/H)$ follows easily from Prop. 2.3.2. In order to produce the second map, let $J_{F/H}$ be the category on which $\mathbf{DR}(F/H)$ is defined as a limit $\mathbf{DR}(F/H) = \lim_{J_{F/H}} \mathbf{DR}^{F/H}$ where

$$\mathbf{DR}^{F/H} : J_{F/H} \to \epsilon - \mathbf{cdga}_k^{gr}\,, \quad J_{F/H} \ni (\mathbf{Spec}\, A \to H) \longmapsto \mathbf{DR}(F_{gf,A}/A) \in \epsilon - \mathbf{cdga}_k^{gr}\,,$$

where $F_{gf,A}$ denotes the base change of $g \circ f$ along $\mathbf{Spec}\, A \to H$.
Analogously, let $J_{F/G}$ be the category on which $\mathbf{DR}(F/G)$ is defined as a limit $\mathbf{DR}(F/G) = \lim_{J_{F/G}} \mathbf{DR}^{F/G}$ where

$$\mathbf{DR}^{F/G} : J_{F/G} \to \epsilon - \mathbf{cdga}_k^{gr}\,, \quad J_{F/H} \ni (\mathbf{Spec}\, B \to G) \longmapsto \mathbf{DR}(F_{f,B}/B) \in \epsilon - \mathbf{cdga}_k^{gr}\,,$$

where $F_{f,B}$ denotes the base change of f along $\mathbf{Spec}\, B \to G$.
There is an obvious functor $\alpha : J_{F/G} \to J_{F/H}$ induced by composition with g, hence a morphism $c : \lim_{J_{F/H}} \mathbf{DR}^{F/H} \to \lim_{J_{F/G}} \alpha^*(\mathbf{DR}^{F/H})\ \epsilon - \mathbf{cdga}_k^{gr}$.
Moreover, for any $(\mathbf{Spec}\, B \to G) \in J_{F/G}$, we have an induced canonical map $F_{f,B} \to F_{gf,B}$ over $\mathbf{Spec}\, B$, hence, by Prop. 2.3.2, a further induced map $\mathbf{DR}(F_{gf,B}/B) \to \mathbf{DR}(F_{f,B}/B)$. Thus, we get a morphism of functors $\varphi : \alpha^*(\mathbf{DR}^{F/H}) \to \mathbf{DR}^{F/G}$. The composition

$$\mathbf{DR}(F/H) \simeq \lim_{J_{F/H}} \mathbf{DR}^{F/H} \xrightarrow{c} \lim_{J_{F/G}} \alpha^*(\mathbf{DR}^{F/H}) \xrightarrow{\lim \varphi} \lim_{J_{F/G}} \mathbf{DR}^{F/G} \simeq \mathbf{DR}(F/G)$$

gives us the second map.

□

Remark 2.4.4 Note that, when $\mathbb{L}_{F/H}$ and $\mathbb{L}_{G/H}$ (hence $\mathbb{L}_{F/G}$) exist, the sequence in Proposition 2.4.3 becomes a fiber-cofiber sequence when considered inside $\mathbf{dg}_k^{gr}$. In fact, for a map $X \to Y$ between derived stacks having cotangent complexes, Proposition 1.3.12 implies an equivalence

$$\mathbf{DR}(X/Y) \simeq \bigoplus_{p \geq 0} \Gamma(X, Sym^p_{\mathcal{O}_X}(\mathbb{L}_{X/Y}[-1]))$$

in $\mathbf{cdga}_k^{gr}$.

Remark 2.4.5 Our main example and object of interest will be the following family of perfect formal derived stacks

$$q : X \longrightarrow X_{DR}$$

for X an Artin derived stack locally of finite presentation over k. Corollary 2.1.9 shows that this is indeed a family of perfect formal derived stacks.

Let $X \longrightarrow Y$ be a perfect family of formal derived stacks as above. The ∞-category $\mathbf{dAff}_k/Y$ of derived affine schemes over Y comes equipped with a tautological prestack of cdgas

$$\mathcal{O}_Y : (\mathbf{dAff}_k/Y)^{op} \longrightarrow \mathbf{cdga}_k, \qquad (\mathbf{Spec}\, A \to Y) \longmapsto A.$$

For each $\mathbf{Spec}\, A \to Y$, we may associate to the good formal derived stack X_A its graded mixed cdga $\mathbb{D}(X_A) \in A/\epsilon - \mathbf{cdga}_k^{gr}$ (Theorem 2.2.2). Moreover, the morphism $X_A \to \mathbf{Spec}\, A$ induces a natural $\mathbb{D}(A)$-linear structure on $\mathbb{D}(X_A)$, and we will thus consider $\mathbb{D}(X_A)$ as on object in $\mathbb{D}(A)/\epsilon - \mathbf{cdga}_k^{gr}$.

If $\mathbf{Spec}\, B \longrightarrow \mathbf{Spec}\, A$ is a morphism in $\mathbf{dAff}_k/Y$ we have an induced natural morphism of $\mathbb{D}(A)$-linear graded mixed cdgas

$$\mathbb{D}(X_A) \longrightarrow \mathbb{D}(X_B).$$

With a bit of care in the ∞-categorical constructions (e.g. by using strict models in model categories of diagrams), we obtain the following *prestacks* of graded mixed cdgas on $\mathbf{dAff}_k/Y$:

$$\begin{aligned} \mathbb{D}_Y := \mathbb{D}(\mathcal{O}_Y) :&(\mathbf{dAff}_k/Y)^{op} \longrightarrow \epsilon - \mathbf{cdga}_k^{gr}, \qquad (\mathbf{Spec}\, A \to Y) \longmapsto \mathbb{D}(A), \\ \mathbb{D}_{X/Y} :&(\mathbf{dAff}_k/Y)^{op} \longrightarrow \epsilon - \mathbf{cdga}_k^{gr}, \qquad (\mathbf{Spec}\, A \to Y) \longmapsto \mathbb{D}(X_A). \end{aligned}$$

The natural $\mathbb{D}(A)$-structure on $\mathbb{D}(X_A)$ gives a natural morphism of prestacks of graded mixed cdgas

$$\mathbb{D}_Y \longrightarrow \mathbb{D}_{X/Y},$$

which we consider as the datum of a $\mathbb{D}_Y$-linear structure on $\mathbb{D}_{X/Y}$.

Remark 2.4.6 The two prestacks $\mathbb{D}_Y$ and $\mathbb{D}_{X/Y}$ defined above, are not stacks for the induced étale topology on $\mathbf{dAff}_k/Y$. See however Remark 2.4.10 below.

By taking $\mathcal{M}$ as the ∞-category of functors $(\mathbf{dAff}_k/Y)^{op} \to \epsilon - \mathbf{dg}_k^{gr}$, we may apply to the prestacks $\mathbb{D}_Y$ and $\mathbb{D}_{X/Y}$ the constructions $\mathbf{DR}$, $\mathbf{DR}^t$ and $\mathbf{Pol}^t$ of §2.3.1 and §2.3.2, and obtain the following prestacks on $\mathbf{dAff}_k/Y$

$$\mathbf{DR}(\mathbb{D}_{X/Y}/\mathbb{D}_Y) \qquad \mathbf{DR}^t(\mathbb{D}_{X/Y}/\mathbb{D}_Y) \qquad \mathbf{Pol}^t(\mathbb{D}_{X/Y}/\mathbb{D}_Y, n).$$

The first two are prestacks of graded mixed cdgas while the last one is a prestack of graded $\mathbb{P}_{n+1}$-algebras.

The main results of Subsection 2.3, i.e. Theorem 2.3.3, Corollary 2.3.6, and Theorem 2.3.8, imply the following result for *families* of perfect formal derived stacks

Corollary 2.4.7 *Let $f : X \longrightarrow Y$ be a family of perfect formal derived stacks. We assume that for each $\mathbf{Spec}\, A \longrightarrow Y$ the perfect formal derived stack X_A is moreover algebraisable. Then*

1. *There is a natural equivalence of graded mixed cdga's over k*

$$\mathbf{DR}(X/Y) \simeq \Gamma(Y, \mathbf{DR}(\mathbb{D}_{X/Y}/\mathbb{D}_Y)) \simeq \Gamma(Y, \mathbf{DR}^t(\mathbb{D}_{X/Y}/\mathbb{D}_Y)).$$

2. *For each $n \in \mathbb{Z}$, there is a natural equivalence of graded k-dg-modules*

$$\mathbf{Pol}(X/Y, n) \simeq \Gamma(Y, \mathbf{Pol}^t(\mathbb{D}_{X/Y}/\mathbb{D}_Y, n)).$$

3. *There is a natural equivalence of ∞-categories*

$$\mathrm{L}_{\mathsf{Perf}}(X) \simeq \mathbb{D}_{X/Y} - Mod^{\mathsf{Perf}}_{\epsilon-\mathbf{dg}^{gr}},$$

where $\mathbb{D}_{X/Y} - Mod^{\mathsf{Perf}}_{\epsilon-\mathbf{dg}^{gr}}$ consists of prestacks E of graded mixed $\mathbb{D}_{X/Y}$-modules on Y satisfying the following two conditions:

   (a) For all $\mathbf{Spec}\, A \longrightarrow Y$, the graded mixed $\mathbb{D}_{X/Y}(A)$-module $E(A)$ is perfect in the sense of Theorem 2.2.2 (2).

   (b) E is quasi-coherent in the following sense: for all $\mathbf{Spec}\, B \longrightarrow \mathbf{Spec}\, A$ in $\mathbf{dAff}_k/Y$ the induced morphism

$$E(A) \otimes_{\mathbb{D}_{X/Y}(A)} \mathbb{D}_{X/Y}(B) \longrightarrow E(B)$$

   is an equivalence.

Note that in the above corollary the ∞-category $\mathbb{D}_{X/Y} - Mod^{\mathsf{Perf}}_{\epsilon-\mathbf{dg}^{gr}}$ can also be defined as the limit of ∞-categories

$$\mathbb{D}_{X/Y} - Mod^{\mathsf{Perf}}_{\epsilon-\mathbf{dg}^{gr}} := \lim_{\mathbf{Spec}\, A \to Y} \mathbb{D}_{X/Y}(A) - Mod^{\mathsf{Perf}}_{\epsilon-\mathbf{dg}^{gr}}.$$

Remark 2.4.8 Parts (1) and (2) of Corollary 2.4.7 can be made a bit more precise. We have direct image prestacks on $\mathbf{dAff}_k/Y$

$$f_*(\mathbf{DR}(-/Y)) \quad \text{and} \quad f_*(\mathbf{Pol}(-/Y, n)),$$

defined by sending $\mathbf{Spec}\, A \longrightarrow Y$ to

$$\mathbf{DR}(X_A/A) \quad \text{and} \quad \mathbf{Pol}(X_A/A, n).$$

These are prestacks of graded mixed cdgas and of graded $\mathbb{P}_{n+1}$-algebras, respectively, and are indeed stacks for the étale topology (being direct images of stacks). Corollary 2.4.7 can be refined to the existence of equivalences of prestacks over $\mathbf{dAff}_k/Y$

$$f_*(\mathbf{DR}(-/Y)) \simeq \mathbf{DR}(\mathbb{D}_{X/Y}/\mathbb{D}_Y) \qquad f_*(\mathbf{Pol}(-/Y, n)) \simeq \mathbf{Pol}^t(\mathbb{D}_{X/Y}/\mathbb{D}_Y, n)$$

before taking global sections (i.e. one recovers Corollary 2.4.7 (1) and (2) from these equivalences of prestacks by taking global sections, i.e. by applying $\lim_{\mathbf{Spec}\, A \to Y}$).

As a consequence of Remark 2.4.8, we get the following corollary

Corollary 2.4.9 *The prestacks* $\mathbf{DR}(\mathbb{D}_{X/Y}/\mathbb{D}_Y)$ *and* $\mathbf{Pol}^t(\mathbb{D}_{X/Y}/\mathbb{D}_Y, n)$ *are stacks over* $\mathbf{dAff}_k/Y$*.*

We have a similar refinement also for statement (3) of Corollary 2.4.7. The ∞-category $\mathbb{D}_{X/Y} - Mod^{\mathsf{Perf}}_{\epsilon-\mathbf{dg}^{gr}}$ can be localized to a prestack of ∞-categories on $\mathbf{dAff}_k/Y$

$$\mathbb{D}_{X/Y} - \underline{Mod}^{\mathsf{Perf}}_{\epsilon-\mathbf{dg}^{gr}} : (\mathbf{Spec}\, A \to Y) \mapsto \mathbb{D}_{X/Y}(A) - Mod^{\mathsf{Perf}}_{\epsilon-\mathbf{dg}^{gr}}.$$

And we have an equivalence of prestacks of ∞-categories on $\mathbf{dAff}_k/Y$

$$f_*(\mathrm{L}_{\mathsf{Perf}}(-)) \simeq \mathbb{D}_{X/Y} - \underline{Mod}^{\mathsf{Perf}}_{\epsilon-\mathbf{dg}^{gr}}.$$

Remark 2.4.10 Even though the prestacks $\mathbb{D}_Y$ and $\mathbb{D}_{X/Y}$ are not stacks for the induced étale topology, the associated constructions we are interested in, namely their de Rham complex, shifted polyvectors and perfect modules, are in fact stacks. In a sense, this shows that the defect of stackiness of $\mathbb{D}_Y$ and $\mathbb{D}_{X/Y}$ is somehow artificial, and irrelevant for our purposes.

2.4.2 Shifted principal parts on a derived Artin stack.

We will be mainly interested in applying the results of §2.4.1 to the special family

$$q : X \longrightarrow X_{DR},$$

for X an Artin derived stack locally of finite presentation over k. As already observed, this is a family of perfect formal derived stacks by Corollary 2.1.9.

Definition 2.4.11 *Let* X *be a derived Artin stack locally of finite presentation over* k*, and* $q : X \longrightarrow X_{DR}$ *the natural projection.*

1. *The prestack* $\mathbb{D}_{X_{DR}}$ *of graded mixed cdgas on* $\mathbf{dAff}_k/X_{DR}$ *will be called the* shifted crystalline structure sheaf of X*.*

2. *The prestack* $\mathbb{D}_{X/X_{DR}}$ *of graded mixed cdgas under* $\mathbb{D}_{X_{DR}}$ *will be called the* shifted principal parts of X*. It will be denoted by*
$$\mathcal{B}_X := \mathbb{D}_{X/X_{DR}}.$$

The prestack shifted crystalline structure sheaf $\mathbb{D}_{X_{DR}}$ (which is not a stack) is a graded mixed model for the standard crystalline structure sheaf $\mathcal{O}_{X_{DR}}$ on $\mathbf{dAff}_k/X_{DR}$. Indeed, by Corollary 2.4.7, we have

$$|\mathbb{D}_{X_{DR}}| \simeq \mathbf{DR}(\mathbb{D}_{X_{DR}}/\mathbb{D}_{X_{DR}}) \simeq \mathcal{O}_{X_{DR}}.$$

Analogously, $\mathbb{D}_{X/X_{DR}}$ is a graded mixed model for the standard sheaf of principal parts. Indeed, we have

$$|\mathbb{D}_{X/X_{DR}}| \simeq q_*(\mathcal{O}_X).$$

The value of the sheaf $q_*(\mathcal{O}_X)$ on $\mathbf{dAff}_k/X_{DR}$ on $\mathbf{Spec}\, A \to X_{DR}$ is the ring of functions on X_A, and recall (Proposition 2.1.8) that X_A can be identified with the formal completion of $X \times \mathbf{Spec}\, A$ along the graph of the morphism $\mathbf{Spec}\, A_{red} \to X$. When X is a smooth scheme over $\mathbf{Spec}\, k$, the sheaf $\pi_*(\mathcal{O}_X)$ is the usual sheaf of principal parts on X ([Gr, 16.7]), endowed with its natural crystalline structure (i.e. descent data with respect to the map $q : X \to X_{DR}$). We may view $\mathcal{B}_X$ as controlling the formal completion of X along the diagonal, together with its natural Grothendieck-Gel'fand connection.

Also recall (Lemma 2.1.10) that for $q : X \to X_{DR}$, we have

$$\mathbb{L}_X \simeq \mathbb{L}_{X/X_{DR}}.$$

In the special case of the perfect family of formal derived stacks $q : X \to X_{DR}$, Corollary 2.4.7 thus yields the following

Corollary 2.4.12 *Let X be an Artin derived stack locally of finite presentation over k.*

1. *There is a natural equivalence of graded mixed cdgas over k*

$$\mathbf{DR}(X/X_{DR}) \simeq \mathbf{DR}(X/k) \simeq \Gamma(X_{DR}, \mathbf{DR}(\mathcal{B}_X/\mathbb{D}_{X_{DR}})) \simeq \Gamma(X_{DR}, \mathbf{DR}^t(\mathcal{B}_X/\mathbb{D}_{X_{DR}})).$$

2. *For each $n \in \mathbb{Z}$, there is a natural equivalence of graded complexes over k*

$$\mathbf{Pol}(X/X_{DR}, n) \simeq \mathbf{Pol}(X, n) \simeq \Gamma(X_{DR}, \mathbf{Pol}^t(\mathcal{B}_X/\mathbb{D}_{X_{DR}}, n)).$$

3. *There is a natural equivalence of ∞-categories*

$$\mathrm{L}_{\mathsf{Perf}}(X) \simeq \mathcal{B}_X - Mod^{\mathsf{Perf}}_{\epsilon-\mathbf{dg}^{gr}}.$$

4. *The natural ∞-functor*

$$\mathcal{B}_X - Mod^{\mathsf{Perf}}_{\epsilon-\mathbf{dg}^{gr}} \longrightarrow \mathcal{B}_X(\infty) - Mod^{\mathsf{Perf}}_{k(\infty)-Mod},$$

induced by the base change $(-) \otimes k(\infty)$, is an equivalence.

Proof. The first equivalence in (1) is just the statement that the natural map $u : \mathbf{DR}(X/k) \to \mathbf{DR}(X/X_{DR})$ (Prop. 2.4.3) is an equivalence of mixed graded cdga's over k. In fact, by Prop. 1.3.12 we have equivalences of graded cdga's over k

$$\mathbf{DR}(X/X_{DR}) \simeq \bigoplus_{p\geq 0} \Gamma(X, Sym^p_{\mathcal{O}_X}(\mathbb{L}_{X/X_{DR}}[-1]))\,,$$

$$\mathbf{DR}(X/k) \simeq \bigoplus_{p\geq 0} \Gamma(X, Sym^p_{\mathcal{O}_X}(\mathbb{L}_{X}[-1]))$$

Hence the map u becomes an equivalence in $\mathbf{cdga}_k^{gr}$, by Prop. 2.1.10, and is therefore itself an equivalence. The other equivalences in (1) follows immediately from Corollary 2.4.7. The proof of (2) is analogous to the proof of (1). Point (3) follows immediately from the corresponding result in Corollary 2.4.7. Only point (4) requires some further explanations, and an explicit proof. First of all $k(\infty)$ is a cdga in the ∞-category $\mathsf{Ind}(\epsilon - \mathbf{dg}_k^{gr})$ of Ind-objects in graded mixed complexes over k. The notation $\mathcal{B}_X(\infty)$ stands for $\mathcal{B}_X \otimes_k k(\infty)$, which is a prestack on X_{DR} with values in cdgas inside $Ind(\epsilon - \mathbf{dg}_k^{gr})$. As usual $\mathcal{B}_X(\infty) - Mod_{k(\infty)-Mod}$ denotes the ∞-category of prestacks of $\mathcal{B}_X(\infty)$-modules. Finally, $\mathcal{B}_X(\infty) - Mod^{\mathsf{Perf}}_{k(\infty)-Mod}$ is defined as for $\mathcal{B}_X - Mod^{\mathsf{Perf}}_{\epsilon - \mathbf{dg}^{gr}}$: it is the full sub-$\infty$-category of $\mathcal{B}_X(\infty)$-modules E satisfying the following two conditions

1. For all $\mathbf{Spec}\, A \longrightarrow X_{DR}$, the $\mathcal{B}_X(\infty)$-module $E(A)$ is of the form

$$E(A) \simeq E_A \otimes_{\mathcal{B}_X(A)} \mathcal{B}_X(\infty)(A),$$

   for E_A a perfect $\mathcal{B}_X(A)$-graded mixed module in the sense of Theorem 2.2.2.

2. For all $\mathbf{Spec}\, B \longrightarrow \mathbf{Spec}\, A$ in $\mathbf{dAff}_k/Y$, the induced morphism

$$E(A) \otimes_{\mathcal{B}_X(\infty)(A)} \mathcal{B}_X(\infty)(B) \longrightarrow E(B)$$

   is an equivalence of Ind-objects in $\epsilon - \mathbf{dg}_k^{gr}$

From this description, the natural ∞-functor of point (4) is obtained by a limit of ∞-functors

$$\lim_{\mathbf{Spec}\, A \to X_{DR}} (\mathcal{B}_X(A) - Mod^{\mathsf{Perf}}_{\epsilon - \mathbf{dg}_k^{gr}} \longrightarrow \mathcal{B}_X(\infty)(A) - Mod^{\mathsf{Perf}}_{k(\infty)-Mod}).$$

We will now prove that, for each A, the ∞-functor

$$\mathcal{B}_X(A) - Mod^{\mathsf{Perf}}_{\epsilon - \mathbf{dg}_k^{gr}} \longrightarrow \mathcal{B}_X(\infty)(A) - Mod^{\mathsf{Perf}}_{k(\infty)-Mod}$$

is an equivalence. It is clearly essentially surjective by definition. As both the source and the target of this functor are rigid symmetric monoidal ∞-categories, and the ∞-functor is symmetric monoidal, fully faithfulness will follow from the fact that for any object $E \in \mathcal{B}_X(A) - Mod^{\mathsf{Perf}}_{\epsilon-\mathbf{dg}_k^{gr}}$ the induced morphism of spaces

$$\mathsf{Map}_{\mathcal{B}_X(A)-Mod^{\mathsf{Perf}}_{\epsilon-\mathbf{dg}_k^{gr}}}(\mathbf{1}, E) \longrightarrow \mathsf{Map}_{\mathcal{B}_X(\infty)(A)-Mod^{\mathsf{Perf}}_{k(\infty)-Mod}}(\mathbf{1}, E(\infty))$$

is an equivalence. By definition, E is perfect, so is freely generated over $\mathcal{B}_X(A)$ by its weight 0 part. By Proposition 2.2.6 $\mathcal{B}_X(A)$ is free over its part of degree 1, as a graded cdga. Therefore, both $\mathcal{B}_X(A)$ and E has no non-trivial negative weight components. The natural morphism of Ind-objects

$$E \longrightarrow E(\infty)$$

induces an equivalence on realizations $|E| \simeq |E(\infty)| \simeq |E|^t$. This achieves the proof of Corollary, as we have natural identifications

$$\mathsf{Map}_{\mathcal{B}_X(A)-Mod^{\mathsf{Perf}}_{\epsilon-\mathbf{dg}_k^{gr}}}(\mathbf{1}, E) \simeq \mathsf{Map}_{\mathbf{dg}_k}(\mathbf{1}, |E|)$$

$$\mathsf{Map}_{\mathcal{B}_X(\infty)(A)-Mod^{\mathsf{Perf}}_{k(\infty)-Mod}}(\mathbf{1}, E(\infty)) \simeq \mathsf{Map}_{\mathbf{dg}_k}(\mathbf{1}, |E(\infty)|).$$

$\square$

Remark 2.4.13 We describe what happens over a *reduced* point $f : \mathbf{Spec}\, A_{red} = \mathbf{Spec}\, A \longrightarrow X$. The graded mixed cdga $\mathbb{D}_{X_{DR}}(A)$ reduces here to A (with trivial mixed structure and pure weight 0). Therefore, $\mathcal{B}_X(A)$ is here an A-linear graded mixed cdga together with an augmentation $\mathcal{B}_X(A) \longrightarrow A$ (as a map of graded mixed cdgas). Moreover, as a graded cdga, we have (Proposition 2.2.6)

$$\mathcal{B}_X(A) \simeq Sym_A(f^*\mathbb{L}_X).$$

This implies that $f^*(\mathbb{T}_X)[-1]$ is endowed with a natural structure of a dg-Lie algebra over A. This is the tangent Lie algebra of [Hen]. Moreover, $\mathcal{B}_X - Mod^{\mathsf{Perf}}_{\epsilon-\mathbf{dg}^{gr}}$ is here equivalent to the ∞-category of perfect Lie $f^*(\mathbb{T}_X)[-1]$-dg-modules, and we recover the equivalence

$$\mathsf{L}_{\mathsf{Perf}}(X_A) \simeq f^*(\mathbb{T}_X)[-1] - Mod^{\mathsf{Perf}},$$

between perfect complexes on the formal completion of $X \times \mathbf{Spec}\, A$ along the graph $\mathbf{Spec}\, A \longrightarrow X \times \mathbf{Spec}\, A$, and perfect A-dg-modules with an action of the dg-Lie algebra $f^*(\mathbb{T}_X)[-1]$ (see [Hen]).

The situation over *non-reduced* points is more complicated. In general, the graded mixed cdga $\mathcal{B}_X(A)$ has no augmentation to A, as the morphism $X_A \longrightarrow \mathbf{Spec}\, A$ might have no section (e.g. if the point $\mathbf{Spec}\, A \longrightarrow X_{DR}$ does not lift to X itself). In particular $\mathcal{B}_X(A)$ cannot be the Chevalley complex of an A-linear dg-Lie algebra anymore. It is, instead, more accurate to think of $\mathcal{B}_X(A)$ as the Chevalley complex of a dg-Lie *algebroid* over $\mathbf{Spec}\, A_{red}$, precisely the one given by the nerve groupoid of the morphism $\mathbf{Spec}\, A_{red} \longrightarrow X_A$. However, the lack of perfection of the cotangent complexes involved implies that this dg-Lie algebroid is not the kind of objects studied in [Vez]. Finally, the action of $\mathbb{D}(A)$ on $\mathcal{B}_X(A)$ for a non-reduced cdga A, encodes the action of the Grothendieck connection on the formal derived stack X_A.

Shifted symplectic structures on derived stacks. In this paragraph we make a link between [PTVV] and the setting of this paper.
Recall that for a derived Artin stack X, we have defined $\mathbf{DR}(X/k)$ (Def. 2.3.1), a mixed graded cdga over k, and for a map $X \to Y$ of derived Artin stacks over k, we have $\mathbf{DR}(X/Y)$ (Def. 2.4.2), again a mixed graded cdga over k. The definitions of [PTVV], can be rephrased as follows.

Definition 2.4.14 *Let X be a derived Artin stack over k, $p \in \mathbb{N}$, and $n \in \mathbb{Z}$.*

- *the* space of closed p-forms of degree n on X *is*

$$\mathcal{A}^{p,cl}(X,n) := \mathsf{Map}_{\epsilon-\mathbf{dg}_k^{gr}}(k(p)[-p-n], \mathbf{DR}(X/k)) \in \mathcal{T}.$$

- *The* space of p-forms of degree n on X *is defined by*

$$\mathcal{A}^{p}(X,n) := \mathsf{Map}_{\mathbf{dg}_k}(k[-n], \Gamma(X, \wedge^p_{\mathcal{O}_X}\mathbb{L}_X)) \in \mathcal{T}.$$

- *By Proposition 1.3.12, for any $\mathbf{Spec}\, A \to X$, there is a natural map*

$$\mathcal{A}^{p,cl}(\mathbf{Spec}\, A, n) \simeq \mathsf{Map}_{\epsilon-\mathbf{dg}_k^{gr}}(k(p)[-p-n], \mathbf{DR}(A)) \to \mathsf{Map}_{\mathbf{dg}_k^{gr}}(k(p)[-p-n], \mathbf{DR}(A)) \simeq$$

$$\simeq \mathsf{Map}_{\mathbf{dg}_k^{gr}}(k(p)[-p-n], Sym_A(\mathbb{L}_A[-1]) \simeq \mathsf{Map}_{\mathbf{dg}_k}(k[-p-n], Sym^p_A(\mathbb{L}_A[-1]) \simeq \mathcal{A}^p(\mathbf{Spec}\, A, n)$$

  which induces, by passing to the limit, a map

$$\mathcal{A}^{p,cl}(X,n) \to \mathcal{A}^p(X,n)$$

  called the underlying-form map.

- *If the cotangent complex $\mathbb{L}_X \in L_{Perf}(X)$, then the* space of n-shifted symplectic structures *on X is $\mathsf{Symp}(X,n)$ is the subspace of $\mathcal{A}^{2,cl}(X,n)$ consisting of non-degenerate forms, i.e. elements of $\pi_0(\mathcal{A}^{2,cl}(X,n))$ whose underlying form induces an equivalence $\mathbb{T}_X \to \mathbb{L}_X[n]$.*

Proposition 2.4.15 *Let X be a derived Artin stack over k, and $n \in \mathbb{Z}$. There are canonical equivalences in $\mathcal{T}$*

$$\mathsf{Symp}(X,n) \simeq \mathsf{Symp}(\mathcal{B}_X/\mathbb{D}_{X_{DR}},n) \simeq \mathsf{Symp}(\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty),n)$$

where $\mathsf{Symp}(\mathcal{B}_X/\mathbb{D}_{X_{DR}},n)$ *and* $\mathsf{Symp}(\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty),n)$ *are defined as in Def. 1.4.4.*

Proof. By Cor. 2.4.12 we have equivalences

$$\mathbf{DR}(X/k) \simeq \Gamma(X_{DR}, \mathbf{DR}(\mathcal{B}_X/\mathbb{D}_{X_{DR}})) \simeq \Gamma(X_{DR}, \mathbf{DR}^t(\mathcal{B}_X/\mathbb{D}_{X_{DR}})),$$

and by Lemma 1.5.4 we deduce the further equivalence

$$\Gamma(X_{DR}, \mathbf{DR}^t(\mathcal{B}_X/\mathbb{D}_{X_{DR}})) \simeq \Gamma(X_{DR}, \mathbf{DR}(\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty))).$$

Moreover,

$$\mathsf{Map}_{\epsilon-\mathbf{dg}^{gr}_{\mathcal{M}}}(1(p)[-p-n], \mathbf{DR}(\mathcal{B}_X/\mathbb{D}_{X_{DR}})) \simeq \mathsf{Map}_{\epsilon-\mathbf{dg}^{gr}_k}(k(p)[-p-n], |\mathbf{DR}(\mathcal{B}_X/\mathbb{D}_{X_{DR}})|),$$

and

$$|\mathbf{DR}(\mathcal{B}_X/\mathbb{D}_{X_{DR}})| \simeq \Gamma(X_{DR}, \mathbf{DR}(\mathcal{B}_X/\mathbb{D}_{X_{DR}})),$$

where $\mathcal{M}$ is the ∞-category of functors from $\mathbf{dAff}/X_{DR}$ to $\epsilon-\mathbf{dg}^{gr}_k$. Analogously, we get

$$\mathsf{Map}_{\epsilon-\mathbf{dg}^{gr}_{\mathcal{M}'}}(1(p)[-p-n], \mathbf{DR}(\mathcal{B}_X/\mathbb{D}_{X_{DR}})) \simeq \mathsf{Map}_{\epsilon-\mathbf{dg}^{gr}_k}(k(p)[-p-n], \Gamma(X_{DR}, \mathbf{DR}(\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty))))$$

where $\mathcal{M}'$ is the ∞-category of functors from $\mathbf{dAff}/X_{DR}$ to $Ind(\epsilon-\mathbf{dg}^{gr}_k)$. Since the non-degeneracy conditions match, we conclude.

$\square$

3 Shifted Poisson structures and quantization

3.1 Shifted Poisson structures: definition and examples

Let X be a derived Artin stack locally of finite presentation. In the previous section (see Definition 2.4.11) we constructed the prestack $\mathbb{D}_{X_{DR}}$, the shifted crystalline structure sheaf on X_{DR}, and the prestack $\mathcal{B}_X$ of shifted principal parts, which is a prestack of graded mixed $\mathbb{D}_{X_{DR}}$-cdgas on X_{DR}. This gives us a prestack of $\mathcal{O}_{X_{DR}}$-linear graded $\mathbb{P}_{n+1}$-algebras $\mathbf{Pol}^t(\mathcal{B}_X/\mathbb{D}_{X_{DR}}, n)$ defined in Remark 2.4.8 (see also Corollary 2.4.9):

$$\mathbf{Pol}^t(\mathcal{B}_X/\mathbb{D}_{X_{DR}}, n) : (\mathbf{Spec}\, A \to X_{DR}) \longmapsto \mathbf{Pol}^t(\mathcal{B}_X/(A)\mathbb{D}_{X_{DR}}(A), n).$$

We will define $\mathbf{Pol}(X, n)$ as the graded $\mathbb{P}_{n+1}$-algebra obtained by taking its global sections on X_{DR}:

$$\mathbf{Pol}(X, n) := \Gamma(X_{DR}, \mathbf{Pol}^t(\mathcal{B}_X/\mathbb{D}_{X_{DR}}, n)) = \lim_{\mathbf{Spec}\, A \to X_{DR}} \mathbf{Pol}^t(\mathcal{B}_X/\mathbb{D}_{X_{DR}}, n)(A),$$

and call it the *n-shifted polyvectors on X*. Note that, by Theorem 2.3.8, the underlying graded complex is

$$\bigoplus_{p \geq 0} \Gamma(X, Sym^p_{\mathcal{O}_X}(\mathbb{T}_X[-n])),$$

so our notation $\mathbf{Pol}(X, n)$ should be unambiguous. The reader should just keep in mind that from now on, unless otherwise stated, we view $\mathbf{Pol}(X, n)$ with its full structure of graded $\mathbb{P}_{n+1}$-algebra over k. In particular, $\mathbf{Pol}(X, n+1)[n+1]$ is a graded dg-Lie algebra over k.

Definition 3.1.1 *In the notations above, the space of* n-shifted Poisson structures on X *is*

$$\mathsf{Poiss}(X, n) := \mathsf{Map}_{\mathbf{dgLie}^{gr}_k}(k(2)[-1], \mathbf{Pol}(X, n+1)[n+1]),$$

where $\mathbf{dgLie}^{gr}_k$ is the ∞-category of graded k-linear dg-Lie algebras.

As a direct consequence of this definition and of the main theorem of [Me], we get the following important result (see §1.5 for the relation between Tate realization and twists by $k(\infty)$). In the theorem below, $\mathbb{D}_{X_{DR}}(\infty)$ is a prestack of commutative monoids in the ∞-category of Ind-objects in graded mixed complexes, $\mathcal{B}_X(\infty)$ is a prestack of commutative monoids in the ∞-category of Ind-objects in graded mixed complexes, and we have a canonical morphism

$$\mathbb{D}_{X_{DR}}(\infty) \longrightarrow \mathcal{B}_X(\infty).$$

Theorem 3.1.2 *There is a canonical equivalence of spaces*

$$\mathsf{Poiss}(X, n) \simeq \mathbb{P}_{n+1} - (\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty)),$$

where the right hand side is the space of $\mathbb{P}_{n+1}$-structures on $\mathcal{B}_X(\infty)$ compatible with its fixed structure of commutative monoid in the ∞-category of prestacks of graded mixed $\mathbb{D}_{X_{DR}}(\infty)$-dg-modules.

Proof. Let $\mathcal{M}'$ be the ∞-category of prestacks on $\mathbf{dAff}/X_{DR}$ with values in $\mathrm{Ind}(\epsilon - \mathbf{dg}_k^{gr})$, and $\mathcal{M}$ be the ∞-category of $\mathbb{D}_{X_{DR}}(\infty)$-modules inside $\mathcal{M}'$. Recall that $\mathcal{B}_X(\infty)$ is a commutative monoid in $\mathcal{M}$. Let us first consider the space

$$\mathsf{Map}_{\mathbf{Lie}_{\mathcal{M}}^{gr}}(1_{\mathcal{M}}[-1](2), \mathbf{Pol}^{int}(\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty), n+1)[n+1]).$$

This space is, on one hand, equivalent to $\mathbb{P}_{n+1} - (\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty))$ (i.e. the rhs of Thm 3.1.2) by Thm 1.4.9, and on the other hand, equivalent to

$$\mathsf{Map}_{\mathbf{dgLie}_k^{gr}}(k[-1](2), |\mathbf{Pol}^{int}(\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty), n+1)[n+1]|)$$

by the definition of the realization functor $|-| : \mathbf{Lie}_{\mathcal{M}}^{gr} \to \mathbf{dgLie}_k^{gr}$ as a right adjoint. We want to show that the previous space is equivalent to the space

$$K := \mathsf{Map}_{\mathbf{dgLie}_k^{gr}}(k(2)[-1], \Gamma(X_{DR}, \mathbf{Pol}^t(\mathcal{B}_X/\mathbb{D}_{X_{DR}}, n+1))[n+1]).$$

Since the forgetful functor $\mathbb{P}_{n+2} - \mathbf{cdga}_k^{gr} \to \mathbf{dgLie}_k^{gr}$ commutes with limits, we have

$$K \simeq \lim_{\mathbf{Spec}\, A \in \mathbf{dAff}/X_{DR}} \mathsf{Map}_{\mathbf{dgLie}_k^{gr}}(k[-1](2), \mathbf{Pol}^t(\mathcal{B}_X(A)/\mathbb{D}_{X_{DR}}(A), n+1))[n+1]).$$

By Lemma 1.5.4

$$\mathbf{Pol}^t(\mathcal{B}_X(A)/\mathbb{D}_{X_{DR}}(A), n+1))[n+1] \simeq |\mathbf{Pol}^{int}(\mathcal{B}_X(\infty)(A)/\mathbb{D}_{X_{DR}}(\infty)(A), n+1))[n+1]|\,,$$

so that

$$K \simeq \mathsf{Map}_{\mathbf{dgLie}_k^{gr}}(k[-1](2), \lim_{\mathbf{Spec}\, A \in \mathbf{dAff}/X_{DR}} |\mathbf{Pol}^{int}(\mathcal{B}_X(\infty)(A)/\mathbb{D}_{X_{DR}}(\infty)(A), n+1)[n+1]|).$$

We are thus reduced to proving an equivalence

$$\lim_{\mathbf{Spec}\, A \in \mathbf{dAff}/X_{DR}} |\mathbf{Pol}^{int}(\mathcal{B}_X(\infty)(A)/\mathbb{D}_{X_{DR}}(\infty)(A), n+1)[n+1]| \simeq |\mathbf{Pol}^{int}(\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty), n+1)[n+1]|,$$

and this follows immediately from the general fact that the enriched Hom in a category of diagrams $I^{op} \to C$, in an enriched symmetric monoidal category C, satisfies

$$\underline{Hom}_{C^{I^{op}}}(\mathbf{1}_{C^{I^{op}}}, F) \simeq \lim_{x \in I^{op}} \underline{Hom}_C(\mathbf{1}_C, F(x)),$$

since the monoidal unit $\mathbf{1}_{C^{I^{op}}}$ is given by the constant I^{op}-diagram at $\mathbf{1}_C$. It's enough to apply this to our case $I = \mathbf{dAff}/X_{DR}$, and $C = \mathrm{Ind}(\epsilon - \mathbf{dg}_k^{gr})$.

□

We describe below what shifted Poisson structures look like on smooth schemes and classifying stacks of reductive groups. We will see more advanced examples later on.

Smooth schemes. Let X be a smooth scheme over k. The $(n+1)$-shifted polyvectors can be sheafified over X_{Zar} in an obvious way, and yield a stack of graded dg-Lie algebras $\mathbf{Pol}(X, n+1)[n+1]$ on X_{Zar}. As a stack of graded $\mathcal{O}_X$-dg-modules, this is just $\oplus_p Sym_{\mathcal{O}_X}(\mathbb{T}_X[-1-n])[n+1]$. As the weight grading is compatible with the cohomological grading, this stack of graded dg-Lie algebras is formal, and coincides with the standard sheaf of shifted polyvectors with its (shifted) Schouten bracket. By theorem 3.1.2, we know that the space of n-shifted Poisson structures on X as defined in definition 3.1.1 is equivalent to the space of $\mathbb{P}_{n+1}$-structures on the sheaf $\mathcal{O}_X$. When $n = 0$, this recovers the standard notion of algebraic Poisson structure on the smooth scheme X.

Classifying stacks. Let G be a reductive group over k with Lie algebra $\mathfrak{g}$. Again, as a graded k-dg-module $\mathbf{Pol}(BG, n+1)$ is

$$\mathbf{Pol}(BG, n+1)[n+1] \simeq \bigoplus_p Sym_k^p(\mathfrak{g}[-n])^G[n+1].$$

Again because the weight grading is compatible with the cohomological grading, $\mathbf{Pol}(BG, n+1)$ is formal as a graded dg-Lie algebra, and the bracket is here trivial. Using the explicit formulas for the description of $\mathsf{Map}_{\mathbf{dgLie}_k^{gr}}(k(2)[-1], -)$, we get

$$\begin{aligned}
\pi_0(\mathsf{Poiss}(BG, 2)) &\simeq Sym_k^2(\mathfrak{g})^G \\
\pi_0(\mathsf{Poiss}(BG, 1)) &\simeq \wedge_k^3(\mathfrak{g})^G \\
\pi_0(\mathsf{Poiss}(BG, n)) &\simeq * \qquad \text{if } n \neq 1, 2.
\end{aligned}$$

3.2 Non-degenerate shifted Poisson structures

Let X be a derived Artin stack locally of finite presentation over k, and $p \in \pi_0\mathsf{Poiss}(X, n)$ an n-shifted Poisson structure on X in the sense of Definition 3.1.1. So, p is a morphism

$$p : k(2)[-1] \longrightarrow \mathbf{Pol}(X, n+1)[n+1],$$

in the ∞-category of graded dg-Lie algebras over k, and, in particular, it induces a morphism in the ∞-category of graded k-dg-modules

$$p_0 : k(2) \longrightarrow \mathbf{Pol}(X, n+1)[n+2].$$

Since by Thm. 2.3.8, $\mathbf{Pol}(X, n+1)[n+2] \simeq \oplus_p \Gamma(X, Sym^p_{\mathcal{O}_X}(\mathbb{T}_X[-n-1])[n+2])$ in the ∞-category of graded k-dg-modules, p_0 defines an element in

$$p_0 \in H^{-n}(X, \Phi^{(2)}_n(\mathbb{T}_X)),$$

where

$$\Phi^{(2)}_n(\mathbb{T}_X) := \begin{cases} Sym^2_{\mathcal{O}_X} \mathbb{T}_X, & \text{if n is odd} \\ \wedge^2_{\mathcal{O}_X} \mathbb{T}_X, & \text{if n is even.} \end{cases}$$

Hence p_0 induces, by adjunction, a map $\Theta_{p_0} : \mathbb{L}_X \to \mathbb{T}_X[-n]$ of perfect complexes.

Definition 3.2.1 *With the notations above, the n-shifted Poisson structure p is called* non-degenerate *if the induced map*

$$\Theta_{p_0} : \mathbb{L}_X \to \mathbb{T}_X[-n]$$

is an equivalence of perfect complexes on X.

By Theorem 3.1.2, the datum of $p \in \pi_0 \mathsf{Poiss}(X, n)$ is equivalent to the datum of a compatible $\mathbb{P}_{n+1}$-structure on the prestack of Tate principal parts $\mathcal{B}_X(\infty)$ on X_{DR}, relative to $\mathbb{D}_{X_{DR}}(\infty)$. The bracket of this induced $\mathbb{P}_{n+1}$-structure provides a bi-derivation, relative to $\mathbb{D}_{X_{DR}}(\infty)$,

$$[\cdot, \cdot] : \mathcal{B}_X(\infty) \otimes_{\mathbb{D}_{X_{DR}}(\infty)} \mathcal{B}_X(\infty) \longrightarrow \mathcal{B}_X(\infty),$$

and thus a morphism of prestacks of $\mathcal{B}_X(\infty)$-modules on X_{DR}

$$\mathbb{T}^{int}_{\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty)} \otimes \mathbb{T}^{int}_{\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty)} \longrightarrow \mathcal{B}_X(\infty).$$

By Corollary 2.3.6 and 2.4.12, we know that $\mathbb{T}^{int}_{\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty)}$ can be naturally identified with the image of $\mathbb{T}_X$ by the equivalence

$$\phi_X : \mathrm{L}_{\mathsf{Perf}}(X) \simeq \mathcal{B}_X(\infty) - Mod^{\mathsf{Perf}}_{k(\infty)-Mod}$$

of Corollary 2.4.12. As a consequence, we obtain the following

Corollary 3.2.2 *Let X be a derived Artin stack locally of finite presentation over k, and $n \in \mathbb{Z}$. An n-shifted Poisson structure $p \in \pi_0 \mathsf{Poiss}(X, n)$ is non-degenerate in the sense of Definition 3.2.1 if and only if the corresponding $\mathbb{P}_{n+1}$-structure on the $\mathbb{D}_{X_{DR}}(\infty)$-cdga $\mathcal{B}_X(\infty)$ is non-degenerate in the sense* *of Definition 1.4.18.*

Remark 3.2.3 We note that a similar corollary applies to the symplectic case. More precisely, if $\omega \in \mathcal{A}^{2,cl}(X,n)$ is an n-shifted closed 2-form on X, it defines a canonical n-shifted closed 2-form ω' on $\mathcal{B}_X(\infty)$ relative to $\mathbb{D}_{X_{DR}}(\infty)$. Then, ω is non-degenerate if and only if ω' is non-degenerate.

We are now ready to state the main theorem of this section. Let $\mathsf{Poiss}^{nd}(X,n)$ the subspace of $\mathsf{Poiss}(X,n)$ of connected components of non-degenerate n-shifted Poisson structures on X. By Corollary 3.2.2, we get that the equivalence of Theorem 3.1.2 induces an equivalence

$$\mathsf{Poiss}^{nd}(X,n) \simeq \mathbb{P}^{nd}_{n+1}(\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty))$$

in $\mathcal{T}$, where $\mathbb{P}^{nd}_{n+1}(\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty))$ is the space of non-degenerate $\mathbb{P}_{n+1}$-structures on $\mathcal{B}_X(\infty)$ relative to $\mathbb{D}_{X_{DR}}(\infty)$. On the other hand, if we take $\mathcal{M}$ to be the category of prestacks over $\mathbf{dAff}_k/X_{DR}$ of Ind-objects in mixed graded $\mathbb{D}_{X_{DR}}(\infty)$-modules, then Corollary 1.4.24 (2) applied to A equal to the prestack $\mathcal{B}_X(\infty)$ of $\mathbb{D}_{X_{DR}}(\infty)$-linear cdgas, provides a morphism of spaces

$$\psi : \mathbb{P}^{nd}_{n+1}(\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty)) \longrightarrow \mathsf{Symp}(\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty)).$$

By Proposition 2.4.15, ψ then induces a well defined morphism in $\mathcal{T}$

$$\psi : \mathsf{Poiss}^{nd}(X,n) \longrightarrow \mathsf{Symp}(X,n).$$

Theorem 3.2.4 *The morphism constructed above*

$$\psi : \mathsf{Poiss}^{nd}(X,n) \longrightarrow \mathsf{Symp}(X,n)$$

is an equivalence in $\mathcal{T}$.

Note: A version of this theorem for Deligne-Mumford derived stacks was recently proven by J. Pridham by a different method [Pri].

This theorem will be a consequence of the following finer statement, which implies Theorem 3.2.4 by taking global sections.

Theorem 3.2.5 *Let $q : X \longrightarrow X_{DR}$ be the natural projection. Then, the induced morphism*

$$\psi : q_*(\mathsf{Poiss}^{nd}(-,n)) \longrightarrow q_*(\mathsf{Symp}(-,n))$$

is an equivalence of stacks on $\mathbf{dAff}_k/X_{DR}$.

Note that the stacks $q_*(\mathsf{Poiss}^{nd}(-,n))$ and $q_*(\mathsf{Symp}(-,n))$ have values, respectively,

$$\mathbb{P}^{nd}_{n+1}(\mathcal{B}_X(\infty)(A)/\mathbb{D}_{X_{DR}}(\infty)(A)) \simeq \mathsf{Poiss}^{nd}(X_A, n),$$

and

$$\mathsf{Symp}(\mathcal{B}_X(\infty)(A)\mathbb{D}_{X_{DR}}(\infty)(A), n) \simeq \mathsf{Symp}(X_A, n)$$

on $\mathbf{Spec}\, A \to X_{DR}$.

The proof of Theorem 3.2.5 is rather long and will be given in the next subsection. Before that, we give some important consequences of Theorem 3.2.5. The following corollary is obtained from the construction of a canonical symplectic structure on certain mapping derived stacks ([PTVV, Theorem 2.5]).

Corollary 3.2.6 *Let Y be a derived Artin stack locally of finite presentation and endowed with an n-shifted symplectic structure. Let X be an $\mathcal{O}$-compact and oriented derived stack of dimension d in the sense of* [PTVV]. *We assume that the derived stack $\mathbb{R}\mathbf{Map}(X, Y)$ is a Artin derived stack. Then, $\mathbb{R}\mathbf{Map}(X, Y)$ carries a canonical $(n-d)$-shifted Poisson structure.*

The main context of application of the above corollary is when $Y = BG$ for G a reductive group endowed with a non-degenerate G-invariant scalar product on its Lie algebra $\mathfrak{g}$. The corollary implies existence of natural shifted Poisson structures on derived moduli stacks of G-bundles on oriented spaces of various sorts: projective CY manifolds, compact oriented topological manifolds, de Rham shapes of smooth and projective varieties, etc. (see [PTVV] for a discussion of these examples).

Theorem 3.2.5 together with [PTVV, Theorem 2.12] yield the following

Corollary 3.2.7 *The derived stack* **Perf** *of perfect complexes carries a natural 2-shifted Poisson structure.*

More generally, via Theorem 3.2.5, all the examples of shifted symplectic derived stacks constructed in [PTVV], admit corresponding shifted Poisson structures.

Remark 3.2.8 More generally we expect suitable generalizations of the main results in [PTVV] to hold in the (not necessarily non-degenerate) shifted Poisson case. For example, Theorem 3.2.6 should hold when the target is a general n-shifted Poisson derived stack, yielding a canonical $(n-d)$-shifted Poisson structure on $\mathbb{R}\mathbf{Map}(X, Y)$. The same result should be true for derived intersections of coisotropic maps (see §3.4 for a definition of coisotropic structure on a map) into a general shifted Poisson Artin derived stack locally of finite presentation over k. Both of these problems are currently being investigated by V. Melani.

3.3 Proof of Theorem 3.2.5

The proof of this theorem will take us some time and will occupy the rest of this section. Before going into the details of the proof, we present its basic steps.

1. The map ψ induces an isomorphisms on all homotopy sheaves π_i for $i > 0$.
2. The derived stacks $\mathsf{Poiss}(\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty), n)$ and $\mathsf{Symp}(\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty), n)$ are formal derived stacks in the sense of Definition 2.1.1.
3. When A is reduced, the π_0-sheaves of $q_*(\mathsf{Poiss}^{nd}(-, n))$ and $q_*(\mathsf{Symp}(-, n))$, restricted to $(\mathbf{Spec}\, A)_{Zar}$, can be described in terms of pairing and co-pairing on L_∞-algebras.
4. (2) and (3) imply that the morphism ψ also induces an isomorphism on the sheaves π_0, by reducing to the case of a reduced base.

In the remaining subsections, we will give the proof of Theorem 3.2.5, following the above outline.

3.3.1 Derived stacks associated with graded dg-Lie and graded mixed complexes

We will discuss here the general form of the derived stacks $q_*(\mathsf{Poiss}(-, n))$, and $q_*(\mathsf{Symp}(-, n))$ on $\mathbf{dAff}_k/X_{DR}$ (where $q : X \to X_{DR}$ is the canonical map). We will see that this will easily lead us to proving that the morphism ψ of Theorem 3.2.5 induces isomorphisms on all higher homotopy sheaves. The case of the sheaves π_0 will require more work: it will be a consequence of the results of this subsection together with a Darboux type statement proved in Lemma 3.3.11.

Derived stacks associated with graded dg-Lie algebras. We work over the ∞-site $\mathbf{dAff}_k/Y$, of derived affine schemes over some base derived stack Y (it will be $Y = X_{DR}$ later on). We assume given a stack of $\mathcal{O}_Y$-linear graded dg-Lie algebras $\mathcal{L}$ on $\mathbf{dAff}_k/Y$. Here we do not assume $\mathcal{L}$ to be quasi-coherent, so $\mathcal{L}$ is a graded dg-Lie algebra inside the ∞-category $\mathrm{L}(\mathcal{O}_Y)$ of all (not necessarily quasi-coherent) $\mathcal{O}_Y$-modules on $\mathbf{dAff}_k/Y$.

We define the *stack associated with* $\mathcal{L}$ to be the ∞-functor

$$\mathbb{V}(\mathcal{L}) : (\mathbf{dAff}_k/Y)^{op} \longrightarrow \mathcal{T}$$

sending $(\mathbf{Spec}\, A \to Y)$ to the space

$$\mathbb{V}(\mathcal{L})(A) := \mathsf{Map}_{\mathbf{dgLie}_k^{gr}}(k(2)[-1], \mathcal{L}(A)).$$

Note that as $\mathcal{L}$ is a stack of graded dg-Lie algebras, the definition above makes $\mathbb{V}(\mathcal{L})$ into a stack of spaces on $\mathbf{dAff}_k/Y$, because $\mathsf{Map}_{\mathbf{dgLie}_k^{gr}}(k(2)[-1], -)$ preserves limits.

We are now going to describe the tangent spaces to the derived stack $\mathbb{V}(\mathcal{L})$. For this, let

$$p : k(2)[-1] \longrightarrow \mathcal{L}(A)$$

be an A-point of $\mathbb{V}(\mathcal{L})$ that is given by a strict morphism in the usual (non ∞-) category of graded dg-Lie algebras over k. Such a morphism p is thus completely characterized by an element $p \in \mathcal{L}(A)(2)^1$, of cohomological degree 1 and weight 2, satisfying $[p,p] = dp = 0$. We associate to such a p a graded mixed A-dg-module $(\mathcal{L}(A), p)$ as follows. The underlying graded complex will be $\mathcal{L}(A)$ together with its cohomological differential, while the mixed structure is defined to be $[p, -]$. We will write, as usual,

$$T_p^i(\mathbb{V}(\mathcal{L})(A)) := \mathrm{hofib}(\mathbb{V}(\mathcal{L})(A \oplus A[i]) \longrightarrow \mathbb{V}(\mathcal{L})(A); p).$$

The graded mixed complex $(\mathcal{L}(A), p)$ is then directly related to the tangent space of the derived stack $\mathbb{V}(\mathcal{L})$ at p, as shown by the following lemma.

Lemma 3.3.1 *Assume that for all i, the natural morphism*

$$\mathcal{L}(A) \otimes_A (A \oplus A[i]) \longrightarrow \mathcal{L}(A \oplus A[i])$$

is an equivalence of graded dg-Lie algebras. Then, there is a canonical equivalence of spaces

$$T_p^i(\mathbb{V}(\mathcal{L}))(A) \simeq \mathsf{Map}_{\epsilon-\mathbf{dg}_k^{gr}}(k(2)[-1], (\mathcal{L}(A), p)[i]).$$

Proof. This is a direct check, using the explicit way of [Me] to describe elements in $\mathsf{Map}_{\mathbf{dgLie}_k^{gr}}(k(2)[-1], \mathcal{L}(A))$. With such a description, we see that the space of lifts

$$k(2)[-1] \longrightarrow \mathcal{L}(A \oplus A[i]) \simeq \mathcal{L}(A) \oplus \mathcal{L}(A)[i]$$

of the morphism p, consists precisely of the data giving a morphism of graded mixed complexes $k(2)[-1] \longrightarrow (\mathcal{L}(A), p)[i]$. Namely, any such a lift is given by a family of elements $(q_0, \ldots, q_j, \ldots)$, where q_j is an element of cohomological degree $(1 + i)$ and weight $(2 + i)$ in $\mathcal{L}(A)$, such that the equation

$$[p, q_j] + d(q_{j+1}) = 0$$

holds for all $j \geq 0$. $\square$

Derived stacks associated with graded mixed complexes. We work in the same context as before, over the ∞-site $\mathbf{dAff}_k/Y$, but now we start with a stack of $\mathcal{O}_Y$-linear graded mixed dg-modules

$\mathcal{E}$ on $\mathbf{dAff}_k/Y$. We define the derived stack associated to $\mathcal{E}$ as

$$\mathbb{V}(\mathcal{E}) : (\mathbf{dAff}_k/Y)^{op} \longrightarrow \mathcal{T}$$

sending $\mathbf{Spec}\, A \mapsto Y$ to the space

$$\mathbb{V}(\mathcal{E})(A) := \mathbb{R}\mathbf{Map}_{\epsilon-\mathbf{dg}_k^{gr}}(k(2)[-1], \mathcal{E}(A)).$$

Let

$$\omega : k(2)[-1] \longrightarrow \mathcal{E}(A)$$

be an A-point of $\mathbb{V}(\mathcal{E})$, and

$$T^i_\omega(\mathbb{V}(\mathcal{E})(A)) := \mathrm{hofib}(\mathbb{V}(\mathcal{L})(A \oplus A[i]) \longrightarrow \mathbb{V}(\mathcal{E})(A); \omega).$$

Lemma 3.3.1 has the following version in this case, with a straightforward proof.

Lemma 3.3.2 *With the notations above, and assuming that for all $i \geq 0$ the natural morphism*

$$\mathcal{E}(A) \otimes_A (A \oplus A[i]) \longrightarrow \mathcal{E}(A \oplus A[i])$$

is an equivalence of graded mixed A-dg-modules. Then, there is a canonical equivalence of spaces

$$T^i_\omega(\mathbb{V}(\mathcal{E}))(A) \simeq \mathsf{Map}_{\epsilon-\mathbf{dg}_k^{gr}}(k(2)[-1], \mathcal{E}(A)[i]).$$

Trivial square zero extensions. Here is an easy variation on the two previous lemmas 3.3.1 and 3.3.2.

Lemma 3.3.3 *Let $\mathcal{L}$ be a graded dg-Lie algebra over $\mathbf{Spec}\, A$ and $p : k(2)[-1] \longrightarrow \mathcal{L}$ a strict morphism of graded dg-Lie algebras over k. For all $i \in \mathbb{Z}$, we have a natural equivalence of derived stacks over $\mathbf{Spec}\, A$*

$$\mathbb{V}(\mathcal{L} \oplus \mathcal{L}[i]) \times_{\mathbb{V}(\mathcal{L})} \mathbf{Spec}\, A \simeq \mathbb{V}((\mathcal{L}, p)[i]),$$

where $(\mathcal{L}, p)$ is the graded mixed dg-module associated to $\mathcal{L}$ and p.

3.3.2 Higher automorphisms groups

In this subsection we use the descriptions of the tangent spaces given in §3.3.1 in order to conclude that the morphism ψ of Theorem 3.2.5 induces an isomorphisms on all π_i-sheaves, for $i > 0$.

Let $\mathbf{Spec}\, A \longrightarrow X_{DR}$ and let us fix a non-degenerate n-shifted Poisson structure p on the corresponding base change X_A of $q : X \to X_{DR}$. We already know that p corresponds to a non-degenerate

$\mathbb{P}_{n+1}$-structure on $\mathcal{B}_X(\infty)(A)$ relative to $\mathbb{D}_{X_{DR}}(\infty)(A) = \mathbb{D}(A)(\infty)$. We first compute the derived stack of loops of $\mathsf{Poiss}(X, n)$ based at p.

We represent $\mathcal{B}_X(\infty)(A)$ by a strict $\mathbb{P}_{n+1}$-algebra C, inside the category of $\mathbb{D}(A)(\infty)$-modules (note that everything here is happening inside the category of Ind-objects in $\epsilon - \mathbf{dg}_k^{gr}$). The Poisson structure p is then given by a strict morphism of graded dg-Lie algebras

$$k(2)[-1] \longrightarrow \mathbf{Pol}(C/\mathbb{D}(A)(\infty), n+1)[n+1].$$

Moreover, the derived stack $q_*(\mathsf{Poiss}(-, n))$is, by definition of n-shifted Poisson structures, given by

$$q_*(\mathsf{Poiss}(-, n))_{|\mathbf{Spec}\, A} \simeq \mathbb{V}(\underline{\mathbf{Pol}}(\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty), n+1)[n+1]),$$

where $\underline{\mathbf{Pol}}(\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty), n+1)$ is the sheafified version of $\mathbf{Pol}(\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty), n+1)$ on X_{DR}. i.e.

$$\underline{\mathbf{Pol}}(\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty), n+1) : (\mathbf{Spec}\, A \to X_{DR}) \mapsto \mathbf{Pol}(\mathcal{B}_X(\infty)(A)/\mathbb{D}_{X_{DR}}(\infty)(A), n+1).$$

We consider the based loop stack

$$\Omega_p q_*(\mathsf{Poiss}(-, n)),$$

which is a derived stack over $\mathbf{Spec}\, A$. The strict morphism p induces a graded mixed structure on the complex

$$\mathbf{Pol}(C/\mathbb{D}(A)(\infty), n+1)[n+1] \simeq \underline{\mathbf{Pol}}(\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty), n+1)(A),$$

and we denote the corresponding graded mixed complex by $(\mathcal{L}, p)$.

Lemma 3.3.4 *There is a natural equivalence of derived stacks over* $\mathbf{Spec}\, A$

$$\Omega_p \pi_*(\mathsf{Poiss}(-, n)) \simeq \mathbb{V}((\mathcal{L}, p)[-1]).$$

Proof. This is a general fact. If $\mathcal{L}$ is a graded dg-Lie over $\mathbf{Spec}\, A$, then there is a natural equivalence

$$Map(S^1, \mathbb{V}(\mathcal{L})) \simeq \mathbb{V}(\mathcal{L}^{S^1}),$$

where $\mathcal{L}^{S^1}$ is the S^1-exponentiation in the ∞-category of graded dg-Lie algebras. As a graded dg-Lie algebra this exponentiation is equivalent to $\mathcal{L} \otimes_k C^*(S^1)$, where $C^*(S^1)$ is the cdga of cochains on S^1. As $C^*(S^1)$ is naturally equivalent to $k \oplus k[-1]$, we find that

$$Map(S^1, \mathbb{V}(\mathcal{L})) \simeq \mathbb{V}(\mathcal{L} \oplus \mathcal{L}[-1]).$$

The statement now follows from Lemma 3.3.3. □

Corollary 3.3.5 *The morphism ψ of Theorem 3.2.5 induces an equivalence on based loop stacks, i.e. for each*

$$p : \mathbf{Spec}\, A \longrightarrow q_*(\mathsf{Poiss}^{nd}(X, n)),$$

the induced morphism

$$\Omega_p q_*(\mathsf{Poiss}^{nd}(X, n)) \longrightarrow \Omega_{\psi(p)} q_*(\mathsf{Symp}(X, n))$$

is an equivalence of derived stacks over $\mathbf{Spec}\, A$*.*

Proof. Lemma 3.3.4 describes $\Omega_p q_*(\mathsf{Poiss}^{nd}(X, n))$ as $\mathbb{V}(\mathcal{L}, p)[-1]$, where $(\mathcal{L}, p)$ is the graded mixed complex given by $\mathbf{Pol}(C/\mathbb{D}(A)(\infty), n + 1)[n + 1]$ with the mixed structure being $[p, -]$ (and where as above C is a strict $\mathbb{P}_{n+1}$-algebra over $\mathbb{D}(A)(\infty)$ representing p). The strict morphism p induces a morphism of graded mixed complexes

$$\phi_p : \mathbf{DR}(C/\mathbb{D}(A)(\infty)) \longrightarrow \mathbf{Pol}(C/\mathbb{D}(A)(\infty), n + 1)[n + 1].$$

But, p being non-degenerate, this morphism is an equivalence. By Lemma 3.3.4, we get

$$\Omega_p q_*(\mathsf{Poiss}^{nd}(X, n)) \simeq \mathbb{V}(\mathbf{DR}(C/\mathbb{D}(A)(\infty))[-1]).$$

Now, we have a canonical identification (see Lemma 3.3.4)

$$\mathbb{V}(\mathbf{DR}(C/\mathbb{D}(A)(\infty))[-1]) \simeq \Omega_{\psi(p)} q_*(\mathsf{Symp}(X, n)).$$

Thus we find an equivalence of derived stacks over $\mathbf{Spec}\, A$

$$\Omega_p q_*(\mathsf{Poiss}^{nd}(X, n)) \simeq \Omega_{\psi(p)} q_*(\mathsf{Symp}(X, n)),$$

which can be easily checked to be exactly the morphism induced by the map ψ in Theorem 3.2.5. $\square$

Corollary 3.3.6 *The morphism*

$$\psi : \mathsf{Poiss}^{nd}(X, n) \longrightarrow \mathsf{Symp}(X, n)$$

of Theorem 3.2.4 has discrete homotopy fibers.

So, we are left to proving that ψ of theorem 3.2.5 induces an isomorphism also on π_0-sheaves. In order to do this, we will need some preliminary reductions.

3.3.3 Infinitesimal theory of shifted Poisson and symplectic structures

In this section we prove a result that enables us to reduce Theorem 3.2.5 to a question over reduced base rings. Let $\mathbf{dAff}_k^{red}/X_{DR}$ be the sub ∞-site of $\mathbf{dAff}_k/X_{DR}$ consisting of $\mathbf{Spec}\, A \longrightarrow X_{DR}$ with $A = A_{red}$. The ∞-site $\mathbf{dAff}_k^{red}/X_{DR}$ is equivalent to the big ∞-site $\mathbf{dAff}_k^{red}/X_{red}$ of reduced affine schemes over X_{red}, and it comes equipped with an inclusion ∞-functor

$$j : \mathbf{dAff}_k^{red}/X_{red} \hookrightarrow \mathbf{dAff}_k/X_{DR}.$$

The result we need is then the following

Proposition 3.3.7 *The morphism*

$$\psi : q_*(\mathsf{Poiss}^{nd}(-,n)) \longrightarrow q_*(\mathsf{Symp}(-,n))$$

of Theorem 3.2.5 is an equivalence of stacks if and only if the induced morphism

$$j^*\psi : j^*q_*(\mathsf{Poiss}^{nd}(-,n)) \longrightarrow j^*q_*(\mathsf{Symp}(-,n))$$

is an equivalence of stacks over $\mathbf{dAff}_k^{red}/X_{red}$.

Proof. We will use a deformation theory argument. We have to prove that if $\mathbf{Spec}\, A \longrightarrow X_{DR}$ is an object in $\mathbf{dAff}_k/X_{DR}$, then

$$\psi_A : \pi_*(\mathsf{Poiss}(-,n))(A) \longrightarrow \pi_*(\mathsf{Symp}(-,n))(A)$$

is an equivalence as soon as

$$\psi_{A_{red}} : \pi_*(\mathsf{Poiss}(-,n))(A_{red}) \longrightarrow \pi_*(\mathsf{Symp}(-,n))(A_{red})$$

is an equivalence.

Lemma 3.3.8 *The two derived stacks* $q_*(\mathsf{Symp}(-,n))$ *and* $q_*(\mathsf{Poiss}^{nd}(-,n))$ *are nilcomplete and infinitesimally cohesive in the sense of Definition 2.1.1.*

Proof of the lemma. Remind that nilcomplete and infinitesimally cohesive for F a derived stack over X_{DR}, means the following two conditions.

1. For all $\mathbf{Spec}\, B \longrightarrow X_{DR} \in \mathbf{dAff}_k/X_{DR}$, the canonical map

   $$F(B) \longrightarrow \lim_k F(B_{\leq k}),$$

   where $B_{\leq k}$ denotes the k-th Postnikov truncation of B, is an equivalence in $\mathcal{T}$.

2. For all fibered product of almost finite presented k-cdgas in non-positive degrees

$$\begin{tikzcd} B \arrow[r] \arrow[d] & B_1 \arrow[d] \\ B_2 \arrow[r] & B_0, \end{tikzcd}$$

such that each $\pi_0(B_i) \longrightarrow \pi_0(B_0)$ is surjective with nilpotent kernels, and all morphism $\mathbf{Spec}\, B \longrightarrow X_{DR}$, the induced square

$$\begin{tikzcd} F(B) \arrow[r] \arrow[d] & F(B_1) \arrow[d] \\ F(B_2) \arrow[r] & F(B_0), \end{tikzcd}$$

is cartesian in $\mathcal{T}$.

To prove the lemma we write the two derived stacks $q_*(\mathsf{Poiss}(-,n))$ and $q_*(\mathsf{Symp}(-,n))$ in the form (see §3.3.1)

$$q_*(\mathsf{Poiss}(-,n)) \simeq \mathbb{V}(\mathcal{L}) \qquad q_*(\mathsf{Symp}(-,n)) \simeq \mathbb{V}(\mathcal{E}).$$

Here,

$$\mathcal{L} = \underline{\mathbf{Pol}}(\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty), n+1)[n+1]$$

is the stack of ($\mathcal{O}_{X_{DR}}$-linear) graded dg-algebras of $(n+1)$-shifted polyvectors on $\mathcal{B}_X(\infty)$ relative to $\mathbb{D}_{X_{DR}}(\infty)$, and

$$\mathcal{E} = \underline{\mathbf{DR}}(\mathcal{B}_X(\infty)/\mathbb{D}_{X_{DR}}(\infty))[n+1].$$

The fact that $\mathbb{V}(\mathcal{L})$ and $\mathbb{V}(\mathcal{E})$ are both nilcomplete and infinitesimally cohesive will result from the fact that both $\mathcal{L}$ and $\mathcal{E}$, considered as stacks of complexes, are themselves nilcomplete and infinitesimally cohesive. By looking at weight graded components, this will follow from the fact that the two stacks of complexes on X_{DR}

$$q_*(Sym^p(\mathbb{T}_X[-n-1]) \quad \text{and} \quad q_*(Sym^p(\mathbb{L}_X[-1])$$

are themselves nilcomplete and infinitesimally cohesive. Let us prove that this is the case for

$$q_*(Sym^p(\mathbb{T}_X[-n-1]),$$

the other case being established by the same argument (since $\mathbb{T}_X$ is perfect).

The stack $q_*(Sym^p(\mathbb{T}_X[-n-1])$ can be described explicitly as follows. Given a map $\mathbf{Spec}\, A \longrightarrow X_{DR}$, we let, as usual,

$$X_A := X \times_{X_{DR}} \mathbf{Spec}\, A.$$

The derived stack X_A is the formal completion of $\mathbf{Spec}\, A_{red} \longrightarrow X \times \mathbf{Spec}\, A$, and it comes equipped with a natural morphism $u : X_A \longrightarrow X$.

The value of the derived stack $q_*(Sym^p(\mathbb{T}_X[-n-1])$ at A is then

$$q_*(Sym^p(\mathbb{T}_X[-n-1])(A) = \Gamma(X_A, u^*(Sym^p(\mathbb{T}_X[-n-1])).$$

The lemma then follows from the following elementary fact whose proof we leave to the reader.

Sub-Lemma 3.3.9 *Let $f : Y \longrightarrow \mathbf{Spec}\, A$ be any derived stack over $\mathbf{Spec}\, A$ and $E \in \mathrm{L}_{\mathsf{Perf}}(F)$ be a perfect complex over Y. Then, the stack of complexes $f_*(E)$ over $\mathbf{Spec}\, A$ is nilcomplete and infinitesimally cohesive.*

The Sub-Lemma achieves the proof of Lemma 3.3.8. □

We are now able to finish the proof of Proposition 3.3.7. By Lemma 3.3.8 and the standard Postnikov decomposition argument, we will be done once we prove the following statement. Suppose that $\mathbf{Spec}\, A \longrightarrow X_{DR}$ is such that the induced morphism

$$\psi_A : q_*(\mathsf{Poiss}(-,n))(A) \longrightarrow q_*(\mathsf{Symp}(-,n))(A)$$

is an equivalence. Let M be a module of finite type over A_{red}, $i \geq 0$ and $A \oplus M[i]$ the trivial square zero extension of A by $M[i]$. We have to prove that the induced morphism

$$\psi_{A\oplus M[i]} : q_*(\mathsf{Poiss}(-,n))(A \oplus M[i]) \longrightarrow q_*(\mathsf{Symp}(-,n))(A \oplus M[i])$$

is again an equivalence. This morphism fibers over the morphism ψ_A, which is an equivalence by assumption and it is then enough to check that the morphism induced on the fibers is an equivalence. But this is identical to the computation carried out in subsection 3.3.2. □

3.3.4 Completion of the proof of Theorem 3.2.5

We are now in a position to conclude the proof of Theorem 3.2.5. We consider the morphism

$$\psi : q_*(\mathsf{Poiss}^{nd}(-,n)) \longrightarrow q_*(\mathsf{Symp}(-,n))$$

of the theorem. This is a morphism of derived stacks over the big ∞-site $\mathbf{dAff}_k/X_{DR}$, of derived affine schemes over X_{DR}, and, by Corollary 3.3.5, we know that it induces equivalences on all based loop stacks, hence on all higher homotopy sheaves. It remains to prove that the induced morphism

$$\pi_0(q_*(\mathsf{Poiss}^{nd}(-,n))) \longrightarrow \pi_0(q_*(\mathsf{Symp}(-,n)))$$

is an isomorphism of sheaves of sets on $\mathbf{dAff}_k/X_{DR}$. By Proposition 3.3.7 it is enough to show that the restriction of this morphism to *reduced* affine schemes over X_{DR} is an isomorphism of sheaves of sets.

We thus fix a reduced affine scheme $S = \mathbf{Spec}\, A$ with a morphism $S \longrightarrow X_{DR}$; by definition of X_{DR}, this corresponds to a morphism $u : S \longrightarrow X$. We consider

$$X_A := X \times_{X_{DR}} \mathbf{Spec}\, A,$$

which is naturally identified with the formal completion of the graph morphism $S \longrightarrow X \times S$ (Proposition 2.1.8). We have natural projection

$$q^A : X_A \longrightarrow S,$$

and we consider the induced sheaves of sets on the small Zariski site S_{Zar}

$$\pi_0(q_*^A(\mathsf{Poiss}^{nd}(-,n))) \quad \text{and} \quad \pi_0(q_*^A(\mathsf{Symp}(-,n)))$$

as well as the morphism induced by ψ

$$\psi_A : \pi_0(q_*^A(\mathsf{Poiss}^{nd}(-,n))) \longrightarrow \pi_0(q_*^A(\mathsf{Symp}(-,n))).$$

We will prove that ψ_A is an isomorphism of sheaves on S_{Zar}. This will be achieved by using certain *minimal models* for graded mixed cdgas over A in order to reconstruct $\mathbb{P}_{n+1}$-structures out of symplectic structures. We start by discussing such models.

The perfect formal derived stack X_A has a corresponding graded mixed cdga $\mathbb{D}(X_A)$. Since A is reduced, we note that $\mathbb{D}(X_A)$ here is an A-linear graded mixed cdga which, as a non-mixed graded cdga, is of the form (see Proposition 2.2.6)

$$\mathbb{D}(X_A) \simeq Sym_A(u^*(\mathbb{L}_X)),$$

where $u^*(\mathbb{L}_X)$ is the pull back of the cotangent complex of X along the morphism $u : S \longrightarrow X$ (note that $\mathbb{L}_{(X_A)_{red}/A}$ is trivial here, so $u^*(\mathbb{L}_X) \simeq \mathbb{L}_{(X_A)_{red}/X_A}[-1]$).

We introduce a strict model for $\mathbb{D}(X_A)$ as follows. We choose a model L for $u^*(\mathbb{L}_X)$ as a bounded complex of projective A-modules of finite rank, and we consider the graded cdga $B := Sym_A(L)$. We also fix a strict model C for $\mathbb{D}(X_A)$, as a cofibrant graded mixed cdga. As B is a cofibrant graded cdga (and C is automatically fibrant), we can chose an equivalence of graded cdgas

$$v : B \longrightarrow C.$$

The mixed structure on C can be transported to a *weak* mixed structure on B as follows. The equivalence v induces a canonical isomorphism inside the homotopy category $\mathrm{Ho}(\mathbf{dgLie}_k^{gr})$ of graded dg-Lie

$$v : Der^{gr}(B,B) \simeq Der^{gr}(C,C),$$

where Der^{gr} denotes the graded dg-Lie algebra of graded derivations. The mixed structure on C defines a strict morphism of graded dg-Lie algebras

$$k(1)[-1] \longrightarrow Der^{gr}(C,C),$$

which can be transported by the equivalence v into a morphism in $\mathrm{Ho}(\mathbf{dgLie}_k^{gr})$

$$\ell : k(1)[-1] \longrightarrow Der^{gr}(B,B).$$

The morphism ℓ determines the data of an $\mathcal{L}_\infty$-structure on $L^\vee[-1]$, that is a family of morphisms of complexes of A-modules

$$[\cdot,\cdot]_i : L \longrightarrow Sym_A^i(L),$$

for $i \geq 2$ satisfying the standard equations (see e.g. [Ko1, 4.3]).

We thus consider L equipped with this $\mathcal{L}_\infty$-structure. It induces a Chevalley differential on the commutative cdga B making it into a mixed cdga. Note that the mixed structure is not strictly compatible with the weight grading, so B is not a graded mixed cdga for the Chevalley differential, it is however a filtered mixed cdga for the natural filtration on B associated to the weight grading. By taking the total differential, sum of the cohomological and and the Chevalley differential, we end up with a well defined commutative A-cdga

$$|B| := \prod_{i\geq 0} Sym_A^i(L).$$

Note that $|B|$ is also the completed Chevalley complex $\widehat{C}^*(\mathcal{L}^\vee[-1])$ of the $\mathcal{L}_\infty$-algebra $\mathcal{L}^\vee[-1]$.

We define explicit de Rham and polyvector objects, which are respectively a graded mixed complex and a graded dg-Lie algebra over k, as follows. We let

$$\mathbf{DR}^{ex}(B) := \bigoplus_p |B| \otimes_A Sym_A^p(L[-1]).$$

The object $\mathbf{DR}^{ex}(B)$ is first of all a graded dg-module over k, by using the total differential sum of the cohomological and Chevalley differential. Put differently, each $|B| \otimes_A Sym_A^p(L[-1])$ can be identified with the Chevalley complex with coefficient in the $\mathcal{L}_\infty$-$L^\vee[-1]$-module $Sym_A^p(L[-1])$. Moreover, $\mathbf{DR}^{ex}(B)$ comes equipped with a de Rham differential

$$dR : |B| \otimes_A Sym_A^p(L[-1]) \longrightarrow |B| \otimes_A Sym_A^{p+1}(L[-1]),$$

making it into a graded mixed complex over A.

The case of polyvectors is treated similarly. We set

$$\mathbf{Pol}^{ex}(B,n) := \bigoplus_p |B| \otimes_A Sym_A^p(L^\vee[-n]).$$

We consider $\mathbf{Pol}^{ex}(B,n)$ endowed with the total differential, sum of the cohomological and the Chevalley differential for the $\mathcal{L}_\infty$-L-module $L^\vee[-n]$. Moreover, $\mathbf{Pol}^{ex}(B,n)$ is also equipped with a natural bracket making it into a a graded $\mathbb{P}_{n+1}$-algebra. In particular, $\mathbf{Pol}^{ex}(B,n)[n]$ has a natural structure of graded dg-Lie algebra over A.

The next Lemma shows that $\mathbf{DR}^{ex}(B)$ and $\mathbf{Pol}^{ex}(B)$ provide strict models.

Lemma 3.3.10 *We have natural equivalences of*

1. $\mathbf{DR}^{ex}(B) \simeq \mathbf{DR}(\mathbb{D}_X(A)/A)$
2. $\mathbf{Pol}^{ex}(B) \simeq \mathbf{Pol}^t(\mathbb{D}_X(A)/A)$.

Proof. We consider $k(1)[-1]$ (i.e. k sitting in pure weight 1 and in pure cohomological degree 1), as a graded dg-Lie algebra with zero differential, and with bracket of weight 0. Beware that this is different from the standard convention used in the rest of the paper. Note that the graded Lie dg-modules over $k(1)[-1]$ are exactly graded mixed complexes.

We now consider the canonical quasi-free resolution of $k(1)[-1]$ as graded dg-Lie algebras $k[f_*] \simeq k(1)[-1]$ described in [Me]. Here for $i \geq 0$, f_0 is a generator of cohomological degree -1 (set $f_i = 0$ for $i < 0$), pure of weight $(i+1)$. We moreover impose equations for all $i \geq -1$

$$df_{i+1} + \frac{1}{2}\sum_{a+b=i}[f_a, f_b] = 0.$$

The graded dg-Lie $k[f_*]$ is a cofibrant model for $k(1)[-1]$. The ∞-category of graded $k(1)[-1]$-dg-modules is thus equivalent to the ∞-category of graded Lie-$k[f_*]$-dg-modules. We denote this second ∞-category by

$$w - \epsilon - \mathbf{dg}^{gr} := k[f_*] - \mathbf{dg}_k^{gr}.$$

Objects in this second ∞-category will be simply called *weak graded mixed dg-modules*, where *weak* refers here to the mixed structure. In concrete terms, an object in $w - \epsilon - \mathbf{dg}^{gr}$ consists of a graded complex $E = \oplus_p E(p)$, together with family of morphism of complexes (for $i \geq 0$)

$$\epsilon_i : E(p) \longrightarrow E(p+i+1)[1],$$

such that

$$d\epsilon_{i+1} + \frac{1}{2}\sum_{a+b=i}[\epsilon_a, \epsilon_b] = 0$$

holds inside $\underline{End}^{gr}(E)$, the graded dg-Lie algebra of graded endomorphisms of E.

We can now do differential calculus inside the ∞-category $\mathcal{M} := w - \epsilon - \mathbf{dg}^{gr}$ as we have done in §1, and more precisely inside the model category of weak graded mixed dg-modules. By construction, our cdga $B = Sym_A(L)$ in the lemma is endowed with a structure of weak graded mixed cdga over A. As such, its de Rham object is precisely given by our explicit complex $\mathbf{DR}^{ex}(B)$. In the same way, $\mathbf{Pol}^{ex}(B,n)$ identifies with the polyvector objects of B considered as a weak graded mixed cdga over A. Moreover, B is, as a weak graded A-cgda, equivalent to $\mathbb{D}_X(A)$, so the lemma holds simply because the natural inclusion from graded mixed complexes to weak graded mixed complexes induces an equivalence of symmetric monoidal model categories. $\square$

Because of Lemma 3.3.10 we can now work with the explicit de Rham and polyvector objects $\mathbf{DR}^{ex}(B)$ and $\mathbf{Pol}^{ex}(B,n)$ constructed above. Now, Corollary 1.4.24 provides a morphism of spaces

$$\psi : \mathsf{Map}_{\mathbf{dgLie}_k^{gr}}(k(2)[-1], \mathbf{Pol}^{ex}(B, n+1)[n+1]) \longrightarrow \mathsf{Map}_{\epsilon-\mathbf{dg}^{gr}}(k(2)[-n-2], \mathbf{DR}^{ex}(B)).$$

This morphism can be stackified over S_{Zar}, where $S = \mathbf{Spec}\, A$, by sending an open $\mathbf{Spec}\, A' \subset \mathbf{Spec}\, A$ to the map

$$\begin{array}{c} \mathsf{Map}_{\mathbf{dgLie}_k^{gr}}(k(2)[-1], \mathbf{Pol}^{ex}(B, n+1)[n+1] \otimes_A A') \\ \big\downarrow \psi_{A'} \\ \mathsf{Map}_{\epsilon-\mathbf{dg}^{gr}}(k(2)[-n-2], \mathbf{DR}^{ex}(B) \otimes_A A'). \end{array}$$

We already know that this morphism of stacks induces equivalences on all higher homotopy sheaves, so it only remains to show that it also induces an isomorphism on the sheaf π_0.

In order to prove this, we start by the following strictification result. Recall that a morphism of graded dg-Lie algebras

$$p : k(2)[-1] \longrightarrow \mathbf{Pol}^{ex}(B, n+1)[n+1]$$

is non-degenerate if the morphism induced by using the augmentation $|B| \to A$

$$k \to |B| \otimes_k Sym^2(L^\vee[-n-1])[n] \longrightarrow Sym^2(L^\vee[-n-1])[n]$$

induces an equivalence of complexes of A-modules $L \simeq L^\vee[-n-2]$.

The following lemma is an incarnation of the Darboux lemma for shifted symplectic and shifted Poisson structures. It is inspired by the Darboux lemma for $\mathcal{L}_\infty$-algebras of Costello-Gwilliam [Co-Gwi, Lemma 11.2.0.1].

Lemma 3.3.11 *Assume that the complex L is minimal at a point $p \in \mathbf{Spec}\, A$, in the sense that its differential vanishes on $L \otimes_A k(p)$.*

1. *Any morphism in the ∞-category of graded mixed complexes*

$$\omega : k(2)[-2-n] \longrightarrow \mathbf{DR}^{ex}(B),$$

is homotopic to a strict morphism of graded mixed complexes.

2. *For any morphism in the ∞-category of graded dg-Lie algebras*

$$\pi : k(2)[-1] \longrightarrow \mathbf{Pol}^{ex}(B, n+1)[n+1],$$

which is non-degenerate at p, there is a Zariski open neighborood $\mathbf{Spec}\, A' \subseteq \mathbf{Spec}\, A$ with $p \in \mathbf{Spec}\, A'$, such that

$$\pi'_A : k(2)[-1] \longrightarrow \mathbf{Pol}^{ex}(B, n+1)[n+1] \otimes_A A'$$

is homotopic to a strict morphism of graded dg-Lie algebras.

Proof. (1) The de Rham cohomology of the weak graded mixed cdga B is acyclic, because B is a free cdga. In other words, the natural augmentation

$$|\mathbf{DR}^{ex}(B)| \longrightarrow A$$

is an equivalence (where $|\mathbf{DR}^{ex}|$ denotes the standard realization of the graded mixed complex $\mathbf{DR}^{ex}$). By using the Hodge filtration, we find an equivalence of spaces

$$\mathsf{Map}_{\epsilon-\mathbf{dg}_k^{gr}}(k(2)[-2-n], \mathbf{DR}^{ex}(B)) \simeq \mathsf{Map}_{\mathbf{dg}_k}(k, |\mathbf{DR}^{ex}(B)/A|^{\leq 1}[1+n].$$

To put things differently, any closed 2-form of degree n on B can be represented by an element ω' of the form $dR(\eta)$ for $\eta \in (|B| \otimes_k L)^n$, such that there exists $f \in (|B|/A)^{n-1}$ with $d(f) + dR(\eta) = 0$. In particular, ω' is an element of cohomological degree $(n+2)$ in $\mathbf{DR}^{ex}(B)$ which is both d and dR-closed. It is thus determined by a strict morphism of graded dg-modules

$$k(2)[-2-n] \longrightarrow \mathbf{DR}^{ex}(B).$$

(2) Let $\pi : k(2)[-1] \longrightarrow \mathbf{Pol}^{ex}(B, n+1)[n+1]$ be non-degenerate at p. We represent π by a strict morphism of graded dg-Lie algebras

$$p : k[f_*] \longrightarrow \mathbf{Pol}^{ex}(B, n+1)[n+1].$$

As L is minimal at p, there is a Zariski open $p \in \mathbf{Spec}\, A' \subset \mathbf{Spec}\, A$ such that π'_A is strictly non-degenerate, i.e. the induced morphism

$$L \otimes_A A' \simeq L^\vee \otimes_A A'[-n-2]$$

is an isomorphism. By replacing A by A', we can assume that π is in fact strictly non-degenerate over A.

The morphism π consists of a family of elements

$$\{p_i \in \mathbf{Pol}^{ex}(B, n+1)^{n+2}\}_{i \geq 0},$$

of cohomological degree $(n+2)$, with p_i pure of weight $(i+2)$, satisfying the equation

$$dp_{i+1} + \frac{1}{2} \sum_{a+b=i} [p_a, p_b] = 0.$$

We consider

$$p_0 \in |B| \otimes_k Sym^2(L^\vee[-n])^{n+2},$$

and we write it as $p_0 = q + p_0'$, with respect to the direct sum decomposition coming from $|B| \simeq A \oplus |B| \geq 1$. The element q of $|B| \otimes_k Sym^2(L^\vee[-n])^{n+2}$ has now constant coefficients, and satisfies $d(q) = [q, q] = 0$. Therefore, it defines a strict morphism of graded dg-Lie algebras

$$q : k(2)[-1] \longrightarrow \mathbf{Pol}^{ex}(B, n+1)[n+1],$$

which is the leading term of π.

The strict morphism q defines a strict $\mathbb{P}_{n+1}$-structure on the weak graded mixed cdga B, which is strictly non-degenerate. It induces, in particular, an isomorphism of graded objects

$$\phi_q : \mathbf{DR}^{ex}(B) \simeq \mathbf{Pol}^{ex}(B, n+1).$$

The isomorphism ϕ_q is moreover an isomorphism of graded mixed objects where the mixed structure on the right hand side is given by $[q, -]$. After Tate realization, we obtain a filtered isomorphism of filtered complexes

$$|\phi_q|^t : |\mathbf{DR}^{ex}(B)|^t[n+1] \longrightarrow |(\mathbf{Pol}^{ex}(B, n+1), [q, -])|^t[n+1].$$

We will only be interested in the part of weight higher than 2, that is the induced isomorphism

$$|\phi_q|^t : |\mathbf{DR}^{ex}(B)^{\geq 2}|^t[n+1] \longrightarrow |(\mathbf{Pol}^{ex}(B, n+1)^{\geq 2}, [q, -])|^t[n+1].$$

We are now going to modify the filtrations on $|\mathbf{DR}^{ex}(B)|^t$ and $\mathbf{Pol}^{ex}(B, n+1)$ by also taking into account the natural filtration on $|B|$ induced by the augmentation ideal $I \subset |B|$. We have

$$|\mathbf{DR}^{ex}(B)|^t = \bigoplus_p |B| \otimes_A Sym^p(L[-1]),$$

and we set

$$F^i|\mathbf{DR}^{ex}(B)|^t := \bigoplus_{p\geq 0} I^{i-p} \otimes_A Sym^p(L[-1]) \subset |\mathbf{DR}^{ex}(B)|^t.$$

This defines a descending filtration on $|\mathbf{DR}^{ex}(B)|^t$ which is complete. In the same way, we have

$$|\mathbf{Pol}^{ex}(B, n+1)|^t = \bigoplus_p |B| \otimes_A Sym^p(L^\vee[-n])$$

and we set

$$F^i|(\mathbf{Pol}^{ex}(B, n+1), [q, -])|^t := \bigoplus_{p\geq 0} I^{i-p} \otimes_A Sym^p(L^\vee[-n]) \subset |(\mathbf{Pol}^{ex}(B, n+1), [q, -])|^t,$$

which is a complete filtration of $\mathbb{P}_{n+2}$-algebras. The isomorphism $|\phi_q|^t$ constructed above is compatible with these filtrations $F*$, and thus induces a filtered isomorphisms

$$f_1 : F^3|\mathbf{DR}^{ex}(B)^{\geq 2}|^t[n+1] \longrightarrow F^3(\mathbf{Pol}^{ex}(B, n+1)^{\geq 2}, [q, -])|^t[n+1].$$

Note that we have

$$F^3|\mathbf{DR}^{ex}(B)^{\geq 2}|^t = I \otimes_A Sym^2(L[-1]) \oplus \bigoplus_{p\geq 3} |B| \otimes_A Sym^p(L[-1]),$$

and as well for the polyvector sides.

By the results of Fiorenza-Manetti [Fi-Ma, Corollary 4.6], the morphism f_1 is the leading term of a filtered $\mathcal{L}_\infty$-isomorphism

$$f_* : F^3|\mathbf{DR}^{ex}(B)^{\geq 2}|^t[n+1] \longrightarrow F^3(\mathbf{Pol}^{ex}(B, n+1)^{\geq 2}, [q, -])|^t[n+1]$$

of dg-lie algebras, where the lie bracket on the left hand side is taken to be zero. This $\mathcal{L}_\infty$-isomorphism is moreover obtained by exponentiating an explicit bilinear operator obtained as the commutator of the cup product of differential forms and of the contraction by the Poisson bivector q. In particular, the $\mathcal{L}_\infty$-isomorphism f_* induces an isomorphism on the spaces of Mauer-Cartan elements (here we use that the filtrations are complete, see [Ya])

$$MC(F^3|\mathbf{DR}^{ex}(B)^{\geq 2}|^t[n+1]) \simeq MC(F^3(\mathbf{Pol}^{ex}(B, n+1)^{\geq 2}, [q, -])|^t[n+1]).$$

The MC elements on the left hand side are simply 1-cocycles in $F^3|\mathbf{DR}^{ex}(B)^{\geq 2}|^t[n+1]$, and thus are closed 2-forms of degree n with no constant terms in $Sym^2(L[-1]) \subset |B| \otimes_A Sym^2(L[-1])$. Moreover, by the explicit form of the $\mathcal{L}_\infty$-isomorphism f_* we see that closed 2-forms of degree n which are strict (i.e. pure of weight 2), corresponds in $MC(F^3(\mathbf{Pol}^{ex}(B, n+1)^{\geq 2}, [q, -])|^t[n+1])$ to MC elements which are also pure of weight 2.

We are now back to our Poisson structure π, given by the family of elements p_i. Recall that q is the constant term of p_0, let us write $p_0 = q + p_0'$. The family of elements $p_0', p_1, \dots, p_n, \dots$ defines an element in $MC(F^3(\mathbf{Pol}^{ex}(B, n+1)^{\geq 2}, [q, -])|^t[n+1])$, denoted by π'. In other terms we have

$$d\pi' + [q, \pi'] + \frac{1}{2}[\pi', \pi'] = 0,$$

which is another way to write the original MC equation satisfied by π. By the $\mathcal{L}_\infty$-isomorphism above this element π' provides a closed 2-form ω'. By the point (1) of the lemma 3.3.11, ω' is equivalent to a strict closed 2-form ω'', which by the $\mathcal{L}_\infty$-isomorphism gives a new MC element π'' in $F^3(\mathbf{Pol}^{ex}(B, n+1)^{\geq 2}, [q, -])|^t[n+1])$. This MC element is pure of weight 2, so the equation

$$d\pi'' + [q, \pi''] + \frac{1}{2}[\pi'', \pi''] = 0$$

implies that

$$d\pi'' = 0 \qquad [q, \pi''] + \frac{1}{2}[\pi'', \pi''].$$

In other words, $q + \pi''$ is a strict $\mathbb{P}_{n+1}$-structure on B, which by construction is equivalent to the original structure π. $\square$

We come back to our morphism

$$\psi_A : \pi_0(\pi_*(\mathsf{Poiss}^{nd}(-, n))) \longrightarrow \pi_0(\pi_*(\mathsf{Symp}(-, n)))$$

of sheaves on the small Zariski site of $S = Spec\, A$. Lemma 3.3.11 (1) easily implies that this morphism has local sections. Indeed, locally on S_{Zar} any n-shifted symplectic structure can be represented by a strictly non-degenerate strict symplectic structure, which can be dualized to a strict $\mathbb{P}_{n+1}$-structure. Moreover, the point (2) of the lemma 3.3.11 implies that these local sections are locally surjective. This implies that ψ_A is an isomorphism of sheaves of sets.

This, finally proves Theorem 3.2.5.

3.4 Coisotropic structures

In this Subsection, we propose a notion of *coisotropic structure* in the shifted Poisson setting. Our approach here is based on the so-called *additivity theorem*, a somewhat folkloric operadic result which should be considered as a Poisson analogue of Deligne's conjecture as proved in [Lu6]. N. Rozenblyum has comunicated to us a very nice argument for a proof of this additivity theorem, based on the duality between chiral and factorization algebras. For future reference we state the additivity theorem as Theorem 3.4.1 below. Since the details of Rozenblyum's argument are not yet publicly available we also give some conceptual explanations of why such a statement should be true (see Remark 3.4.2).

The dg-operad $\mathbb{P}_n$ is a Hopf operad, i.e. it comes equipped with a comultiplication morphism

$$\nabla : \mathbb{P}_n \longrightarrow \mathbb{P}_n \otimes_k \mathbb{P}_n,$$

making it into a cocommutative coalgebra object inside the category of dg-operads over k. We recall that $\mathbb{P}_n$ is the homology of the E_n-operad (for $n > 1$), and the morphism ∇ is simply defined by the diagonal morphism of E_n. For our base model category M (as in §1.1), this implies that the category of $\mathbb{P}_n$-algebra objects in M has a natural induced symmetric monoidal structure. The tensor product of two $\mathbb{P}_n$-algebras A and B is defined as being the tensor product in M together with the $\mathbb{P}_n$-structure induced by the following compositions

$$\mathbb{P}_n(p) \otimes (A \otimes B)^{\otimes p} \xrightarrow{\nabla} (\mathbb{P}_n(p) \otimes_k \mathbb{P}_n(p)) \otimes (A^{\otimes p} \otimes B^{\otimes p}) \xrightarrow{a \otimes b} A \otimes B,$$

where a and b are the $\mathbb{P}_n$-structures of A and B respectively.

This construction defines a natural symmetric monoidal structure on the ∞-category $\mathbb{P}_n - \mathbf{cdga}_{\mathcal{M}}$ for $\mathcal{M} = L(M)$, the ∞-category associated to M, such that the forgetful ∞-functor

$$\mathbb{P}_n - \mathbf{cdga}_{\mathcal{M}} \longrightarrow \mathcal{M}$$

has a natural structure of symmetric monoidal ∞-functor. In particular, it makes sense to consider the ∞-category $\mathsf{Alg}(\mathbb{P}_n - \mathbf{cdga}_{\mathcal{M}})$ of unital and associative monoids in $\mathbb{P}_n - \mathbf{cdga}_{\mathcal{M}}$ (in the sense of [Lu6, 4.1]).

The additivity property of Poisson operads, proven by N. Rozenblyum, can then be stated as follows.

Theorem 3.4.1 *For any $n \geq 1$ and any ∞-category $\mathcal{M} = L(M)$ as in Section 1.1, there exists an equivalence of ∞-categories*

$$\mathrm{Dec}_{n+1} : \mathbb{P}_{n+1} - \mathbf{cdga}_{\mathcal{M}} \longrightarrow \mathsf{Alg}(\mathbb{P}_n - \mathbf{cdga}_{\mathcal{M}})$$

satisfying the following two properties

1. *The ∞-functor Dec_{n+1} is natural, with respect to symmetric monoidal ∞-functors, in the variable $\mathcal{M}$.*

2. *The ∞-functor Dec_{n+1} commutes with the forgetful ∞-functors to $\mathcal{M}$.*

Remark 3.4.2 Before going further, we make a few comments about this theorem. As the additivity for the operad Lie is rather straightforward, Theorem 3.4.1 can be made even more precise by requiring the compatibility of Dec_{n+1} with respect to the forgetful maps induced from the inclusion of the (shifted) Lie operad inside $\mathbb{P}_n$. We can, moreover, require compatibility with respect to the inclusion of the commutative algebras operad **Comm** into $\mathbb{P}_n$, as, again, the additivity property for **Comm** is straightforward. Indeed, the main difficulty in proving Theorem 3.4.1 is in constructing the ∞-functor *Dec*. Once it is constructed and it is shown to satisfy these various compatibilities, it is rather easy to check that it has to be an equivalence.

As a second comment, we should mention that there is an indirect proof to this theorem based on formality. Indeed, as we are in characteristic zero, we are entitled to chose equivalences of dg-operads

$$\alpha_n : \mathbb{E}_n \simeq \mathbb{P}_n$$

for each $n > 1$. These equivalences can be actually chosen as equivalences of Hopf dg-operads. Now, the solution to the Deligne's conjecture given in [Lu6] implies the existence of a natural equivalence of ∞-categories

$$\mathrm{Dec}^{\mathbb{E}}_{n+1} : \mathbb{E}_{n+1} - \mathbf{cdga}_{\mathcal{M}} \longrightarrow \mathsf{Alg}(\mathbb{E}_n - \mathbf{cdga}_{\mathcal{M}}),$$

satisfying all the required properties. Then, we can simply define Dec by transporting $\mathrm{Dec}^{\mathbb{E}}_{n+1}$ through the equivalences α_n and α_{n+1}. This proof is however not explicit, depends on the choices of the α_n's, and thus is not very helpful for us.

For our purposes, the importance of Theorem 3.4.1 is that it allows for a notion of $\mathbb{P}_{n+1}$-structure on a *morphism* between cdgas. Indeed, we can consider the ∞-category $\mathbb{P}_{(n+1,n)} - \mathbf{cdga}_{\mathcal{M}}$, whose objects consist of pairs (A, B) where A is an object in $\mathsf{Alg}(\mathbb{P}_n - \mathbf{cdga}_{\mathcal{M}})$ and B is an A-module in $\mathbb{P}_n - \mathbf{cdga}_{\mathcal{M}}$. Theorem 3.4.1 implies that this ∞-category $\mathbb{P}_{(n+1,n)} - \mathbf{cdga}_{\mathcal{M}}$ comes equipped with two forgetful ∞-functors

$$\mathbb{P}_{n+1} - \mathbf{cdga}_{\mathcal{M}} \longleftarrow \mathbb{P}_{(n+1,n)} - \mathbf{cdga}_{\mathcal{M}} \longrightarrow \mathbb{P}_n - \mathbf{cdga}_{\mathcal{M}}.$$

Moreover, $\mathbb{P}_{(n+1,n)} - \mathbf{cdga}_{\mathcal{M}}$ has a forgetful ∞-functor to the ∞-category $\mathbb{E}_{(1,0)}(\mathbf{cdga}_{\mathcal{M}})$ of pairs (A, B), where $A \in \mathsf{Alg}(\mathbf{cdga}_{\mathcal{M}})$ and B is an A-module in $\mathbf{cdga}_{\mathcal{M}}$. It is easy to see that the ∞-category $\mathbb{E}_{(1,0)}(\mathbf{cdga}_{\mathcal{M}})$ is equivalent to the ∞-category $\mathsf{Mor}(\mathbf{cdga}_{\mathcal{M}})$ of morphisms between cdgas in $\mathcal{M}$. We are then able to give the following definition of $\mathbb{P}_{(n+1,n)}$-structure on a given morphism between cdgas.

Definition 3.4.3 *Let $f : A \longrightarrow B$ be a morphism between cdgas in $\mathcal{M}$. The* space of $\mathbb{P}_{(n+1,n)}$-

structures on f *is the fiber at* f *of the forgetful* ∞*-functor constructed above*

$$\mathbb{P}_{(n+1,n)} - \mathbf{cdga}_{\mathcal{M}} \longrightarrow \mathsf{Mor}(\mathbf{cdga}_{\mathcal{M}}).$$

It will be denoted by

$$\mathbb{P}_{(n+1,n)}{-}\mathsf{Str}(f) := \mathbb{P}_{(n+1,n)} - \mathbf{cdga}_{\mathcal{M}} \times_{\mathsf{Mor}(\mathbf{cdga}_{\mathcal{M}})} \{f\}.$$

Note that, for a morphism $f : A \to B$, the space $\mathbb{P}_{(n+1,n)}{-}\mathsf{Str}(f)$ has two natural projections

$$\mathbb{P}_{n+1}(A) \longleftarrow \mathbb{P}_{(n+1,n)}{-}\mathsf{Str}(f) \longrightarrow \mathbb{P}_n(B),$$

where $\mathbb{P}_{n+1}(A)$ (respectively, $\mathbb{P}_n(B)$) denotes the space of $\mathbb{P}_{n+1}$-structures (resp. $\mathbb{P}_n$-structures) on the given cdga A (resp. B). Loosely speaking, a $\mathbb{P}_{(n+1,n)}$-structure on a given f consists of a $\mathbb{P}_{n+1}$-structure on A, a $\mathbb{P}_n$-structure on B, together with some compatibility data between these structures. These data not only express the fact that B is an A-module in $\mathbb{P}_n$-algebras, through the ∞-equivalence Dec_{n+1} of Theorem 3.4.1, but also that this module structure induces the given morphism f between the corresponding cdgas.

We are now able to use Definition 3.4.3 in order to introduce the important notion of *shifted coisotropic structures*. Let $f : X \longrightarrow Y$ be a morphism of derived Artin stacks locally of finite presentation over k. Recall (Definition 2.4.11) that we have constructed stacks of graded mixed cdgas $\mathbb{D}_{X_{DR}}$ and $\mathbb{D}_{Y_{DR}}$, the shifted crystalline structure sheaves of, respectively, X and Y. These are stacks of graded mixed cdgas on X_{DR} and Y_{DR}, respectively. The morphism f obviously induces a pull-back morphism (where we simply write f^* for f^*_{DR})

$$f^*(\mathbb{D}_{Y_{DR}}) \longrightarrow \mathbb{D}_{X_{DR}}$$

which is an equivalence of stacks of graded mixed cdgas over X_{DR}.

By Definition 2.4.11, we also have the shifted principal parts $\mathcal{B}_X$ and $\mathcal{B}_Y$, of, respectively, X and Y, which are stacks of graded mixed $\mathbb{D}_{X_{DR}}$ and $\mathbb{D}_{Y_{DR}}$ algebras. respectively. The morphism f induces a pull-back map

$$f^*(\mathcal{B}_Y) \longrightarrow \mathcal{B}_X,$$

which is a morphism of graded mixed $\mathbb{D}_{X_{DR}}$-cdgas. Over an affine $\mathbf{Spec}\, A \longrightarrow X_{DR}$, corresponding to a morphism $\mathbf{Spec}\, A_{red} \longrightarrow X$, the morphism

$$f^*(\mathcal{B}_Y)(A) \longrightarrow \mathcal{B}_X(A)$$

is the image by the ∞-functor $\mathbb{D}$ of the morphism of perfect formal derived stacks over $\mathbf{Spec}\, A$

$$X_A \longrightarrow Y_A,$$

where X_A is the formal completion of the morphism $\mathbf{Spec}\, A_{red} \longrightarrow \mathbf{Spec}\, A \times X$, and, similarly, Y_A is the formal completion of the morphism $\mathbf{Spec}\, A_{red} \longrightarrow \mathbf{Spec}\, A \times Y$. By tensoring with $k(\infty)$, we obtain a morphism of stacks of Ind-objects in graded mixed $\mathbb{D}_{X_{DR}}$-cdgas on X_{DR}

$$f^*(\mathcal{B}_Y(\infty)) \longrightarrow B_X(\infty).$$

If we suppose that Y is endowed with an n-shifted Poisson structure, then $\mathcal{B}_Y(\infty)$ comes equipped with a $\mathbb{P}_{n+1}$-structure, and is thus a stack of graded mixed $\mathbb{D}_{Y_{DR}}(\infty) - \mathbb{P}_{n+1}$-cdgas on Y_{DR}. The pull-back $f^*(\mathcal{B}_Y(\infty))$ is therefore a stack of graded mixed $\mathbb{D}_{X_{DR}}(\infty) - \mathbb{P}_{n+1}$-cdgas on X_{DR}

Definition 3.4.4 *Let $f : X \longrightarrow Y$ be a morphism of derived Artin stacks locally of finite presentation over k, and assume that Y is equipped with an n-shifted Poisson structure p. We denote by*

$$f^*_{\mathcal{B}} : f^*(\mathcal{B}_Y(\infty)) \longrightarrow \mathcal{B}_X(\infty)$$

the induced morphism of $\mathbb{D}_{X_{DR}}(\infty)$-algebras. The space of coisotropic structures on f relative to p *is defined as*

$$\mathsf{Coiso}(f, p) := \mathbb{P}_{(n+1,n)} - \mathsf{Str}(f^*_{\mathcal{B}}) \times_{\mathbb{P}_{n+1} - \mathsf{Str}(f^*_{\mathcal{B}}(\mathcal{B}_Y(\infty))/\mathbb{D}_{X_{DR}}(\infty))} \{p\}.$$

In the above definition, $f^*(\mathcal{B}_Y(\infty))$ acquires an induced $\mathbb{D}_{X_{DR}}$-linear $\mathbb{P}_{n+1}$-structure coming from the n-shifted Poisson structure p. The datum of a coisotropic structure on f consists of the datum of a $\mathbb{D}_{X_{DR}}(\infty)$-linear $\mathbb{P}_n$-structure on $\mathcal{B}_X(\infty)$ together with a suitably compatible structure of module over $f^*(\mathcal{B}_Y(\infty))$, inside the ∞-category of $\mathbb{D}_{X_{DR}}(\infty)$-linear graded mixed $\mathbb{P}_n$-algebras on X_{DR}. We note, in particular, that a coisotropic structure on $f : X \to Y$, with Y n-shifted Poisson, trivially induces an $(n-1)$-shifted Poisson structure on the target X itself.

We end this subsection by the following statement, which is a relative version of our comparison Theorem 3.2.5. We state it now as a conjecture as we have not yet carried out all the details.

Conjecture 3.4.5 *Let Y be a derived Artin stack with an n-shifted symplectic structure ω, and $f : X \longrightarrow Y$ be a morphism of derived Artin stacks. Let p denote the n-shifted Poisson structure corresponding to ω via Theorem 3.2.5. Then, there exists a natural equivalence of spaces*

$$\mathsf{Lag}(f, \omega) \simeq \mathsf{Coiso}(f, p)^{nd},$$

between the space of Lagrangian structures on f with respect to ω (in the sense of [PTVV, 2.2]) and an appropriate space of non-degenerate *coisotropic structures on f relative to p.*

Note that the above conjecture recovers Theorem 3.2.5, by taking $Y = \mathbf{Spec}\,k$ (and $\omega = 0$).

Remark 3.4.6 We expect the Lagrangian intersection theorem [PTVV, Theorem 2.9] to extend to shifted Poisson structures as follows. Let (X, p) be a n-shifted Poisson Artin stack locally of finite presentation over k, and $f_i : Y_i \to X$, $i = 1, 2$ be maps of derived Artin stacks, each endowed with a coisotropic structure relative to p. Then, we expect the existence of a $(n-1)$-shifted Poisson structure on the derived pullback $Y_1 \times_X Y_2$, suitably compatible with the given coisotropic structures on f_1 and f_2. A first evidence of this result comes from [Gi-Ba], which basically treats (on the cohomological level) the case $n = 0$, for X, Y_1 and Y_2 smooth schemes. The general case is currently being investigated by V. Melani.

3.5 Existence of quantization

We propose here a notion of *deformation quantization* of n-shifted Poisson structures on derived Artin stacks, and prove that they always exist as soon as $n \neq 0$. The special case of $n = 0$ would require further investigations and will not be treated in this paper. Also, the more general, and more delicate, problem of quantization of coisotropic structures will not be addressed here.

3.5.1 Deformation quantization problems

Let FM_n be Fulton-MacPherson's topological operad: given a finite set I, the space of operations with entries labelled by I is the compactified configuration space $FM_n(I)$ of I-indexed configurations of points in $\mathbb{R}^n$. The corresponding chain k-dg-operad is a model for $\mathbb{E}_n$:

$$\mathbb{E}_n = C_{-*}(FM_n, k)\,.$$

If $n \geq 1$ one can construct a filtration on $\mathbb{E}_n$ with associated graded being the n-shifted Poisson graded k-dg-operad $\mathbb{P}_n$.

Let us start with the case $n = 1$, which is a bit special. First of all observe that $\mathbb{E}_1$ is equivalent to its cohomology:

$$\mathbb{E}_1 \cong H_0(FM_1, k) = \mathbb{A}s\,,$$

where $\mathbb{A}s$ is the associative k-operad. Note that the k-module $\mathbb{A}s(I)$ of operations with entries labelled by I is the free associative algebra in I generators:

$$\mathbb{A}s(I) =< x_i | i \in I >= U\big(Lie(x_i | i \in I)\big)\,,$$

where $Lie(-)$ denoted the "free Lie algebra generated by". Being a universal envelopping algebra, $\mathbb{A}s(I)$ is filtered with associated graded

$$Sym\big(Lie(x_i | i \in I)\big) = \mathbb{P}_1(I)\,.$$

Applying the Rees construction we get a $k[\hbar]$-dg-operad $\mathbb{BD}_1$.

If $n \geq 2$ then we consider the filtration of $\mathbb{E}_n$ given by a (functorial) Postnikov Tower of FM_n. Applying the Rees construction one gets a $k[\hbar]$-linear-dg-operad $\mathbb{BD}_n$. Note that in this case the associated graded is the homology operad of FM_n, which is $\mathbb{P}_n$.

So, for $n \geq 1$ we get a $k[\hbar]$-dg-operad operad $\mathbb{BD}_n$ such that $\mathbb{BD}_n \otimes_{k[\hbar]} k \cong \mathbb{P}_n$ and $\mathbb{BD}_n \otimes_{k[\hbar]} k[\hbar, \hbar^{-1}] \cong \mathbb{E}_n[\hbar, \hbar^{-1}]$.

Remark 3.5.1 The story in the case $n = 0$ is even more special than for $n = 1$, and is discussed extensively in [Co-Gwi]. Namely, one introduces a $k[\hbar]$-dg-operad $\mathbb{BD}_0$ defined as follows: its underlying graded operad is $\mathbb{P}_0$ and the differential sends the degree 1 generating operation $\{-,-\}$ to 0 and the degree 0 generating operation $- \cdot -$ to $\hbar\{-,-\}$. Again, one has that $\mathbb{BD}_0 \otimes_{k[\hbar]} k \cong \mathbb{P}_0$ and $\mathbb{BD}_0 \otimes_{k[\hbar]} k[\hbar, \hbar^{-1}] \cong \mathbb{E}_0[\hbar, \hbar^{-1}]$.

Whenever one has a $\mathbb{P}_n$-algebra object A_0 in a symmetric monoidal k-linear (i.e. $\mathbb{E}_\infty$-monoidal) ∞-category $\mathcal{C}$, the *deformation quantization problem* reads as follows:

Question 3.5.2 (Deformation quantization problem) *Does there exists a $\mathbb{BD}_n$-algebra object A in $\mathcal{C} \otimes_k k[\hbar]$ such that $A \otimes_{k[\hbar]} k \cong A_0$ as $\mathbb{P}_n$-algebra objects in $\mathcal{C}$? If this happens we say that A is a* deformation quantization *of A_0.*

Observe that one can also consider the *formal deformation quantization problem*, where one replaces $k[\hbar]$ by $k[\![\hbar]\!]$ everywhere.

Let (X, p) be a derived Artin stack locally of finite presentation over k, endowed with an n-shifted Poisson structure p, with $n \geq -1$. By Thm. 3.1.2, p corresponds to a $\mathbb{D}_{X_{DR}}(\infty)$-linear $\mathbb{P}_{n+1}$-structure on the stack (over X_{DR}) $\mathcal{B}_X(\infty)$ of Ind-objects in graded mixed k-cdgas.

In this context one can

- either pose the question of the existence of a (formal) deformation quantization of the $\mathbb{P}_{n+1}$-algebra object $\mathcal{B}_X(\infty)$, whenever $n \geq -1$ (here $\mathcal{C}$ is the ∞-category of prestacks of Ind-objects in mixed graded $\mathbb{D}_{X_{DR}}(\infty)$-dg-modules).
- or pose the question of the existence of a (formal) deformation quantization of the $\mathbb{P}_n$-monoidal ∞-category
$$\mathsf{Perf}(X) \cong \Gamma(X_{DR}, \mathcal{B}_X(\infty) - \mathsf{Mod}^{\mathsf{Perf}}),$$
whenever $n \geq 0$ (here $\mathcal{C}$ is the ∞-category $\mathbf{dgCat}_{/k}$ of dg-categories over k or, equivalently, the ∞-category of stable k-linear ∞-categories).

Remark 3.5.3 The second variant makes use of the equivalence Dec_{n+1} from Theorem 3.4.1 as well as the version of $\mathbb{P}_n$ for k-dg-categories from [To1].

3.5.2 Solution to the deformation quantization problem

Let (X, p) be a derived Artin stack locally of finite presentation over k, endowed with an n-shifted Poisson structure p.

Theorem 3.5.4 *If $n > 0$ then there exists a deformation quantization of the stack $\mathcal{B}_X(\infty)$ of $\mathbb{D}_{X_{DR}}(\infty)$-linear graded mixed $\mathbb{P}_{n+1}$-algebras over X_{DR} from Thm. 3.1.2.*

Proof. Since $n > 0$ we can choose a *formality* equivalence of k-dg-operads

$$\alpha_{n+1} : \mathbb{E}_{n+1} \simeq \mathbb{P}_{n+1} .$$

It induces an equivalence $\mathbb{BD}_{n+1} \simeq \mathbb{P}_{n+1} \otimes_k k[\hbar]$ which is the identity mod $\hbar$.

Therefore one can consider $\mathcal{B}_X(\infty) \otimes_k k[\hbar]$ as a stack of $\mathbb{D}_{X_{DR}}(\infty)$-linear graded mixed $\mathbb{BD}_{n+1}$-algebras on X_{DR}. It is a deformation quantization of $\mathcal{B}_X(\infty)$. $\square$

Putting $\hbar = 1$ in the above Theorem (or directly using the formality equivalence α_{n+1}) we can consider $\mathcal{B}_X(\infty)$ as a stack of $\mathbb{D}_{X_{DR}}(\infty)$-linear graded mixed $\mathbb{E}_{n+1}$-algebras on X_{DR}.

We denote by $\mathcal{B}_X(\infty) - \mathsf{Mod}_p^{\mathsf{Perf}}$ the stack of perfect $\mathcal{B}_X(\infty)$-modules on X_{DR}, where $\mathcal{B}_X(\infty)$ is viewed as a stack of $\mathbb{D}_{X_{DR}}(\infty)$-linear graded mixed $\mathbb{E}_{n+1}$-algebras on X_{DR}. By [Lu6, 5.1.2.2 and 5.1.2.8], $\mathcal{B}_X(\infty) - \mathsf{Mod}_p^{\mathsf{Perf}}$ is endowed with the structure of a stack of $\mathbb{E}_n$-monoidal ∞-categories on X_{DR}. We denote this stack by $\mathcal{B}_X(\infty) - Mod_{\mathbb{E}_n,p}^{\mathsf{Perf}}$.

Definition 3.5.5 *With the notation above, and $n > 0$,* the quantization of X with respect to p *is the $\mathbb{E}_n$-monoidal ∞-category*

$$\mathsf{Perf}(X, p) := \Gamma(X_{DR}, \mathcal{B}_X(\infty) - \mathsf{Mod}_{\mathbb{E}_n,p}^{\mathsf{Perf}}).$$

Remark 3.5.6 Technically speaking $\mathsf{Perf}(X, p)$ also depends on the choice of the formality equivalence α_{n+1}. However, as α_{n+1} can be chosen independently of all X and p, we simply assume that such a choice has been made and will omit to mention it in our notation.

Now observe that the underlying ∞-category of $\mathsf{Perf}(X, p)$ is exactly the category of sections $\Gamma(X_{DR}, \mathcal{B}_X(\infty) - \mathsf{Mod}^{\mathsf{Perf}})$ which coincides with the ∞-category $\mathsf{Perf}(X)$ of perfect $\mathcal{O}_X$-modules on X. In other words, $\mathsf{Perf}(X, p)$ consists of the *datum of a $\mathbb{E}_n$-monoidal structure on* $\mathsf{Perf}(X)$.

This $\mathbb{E}_n$-monoidal structure can also be understood as a *deformation* of the standard symmetric monoidal (i.e. $\mathbb{E}_\infty$-) structure on $\mathsf{Perf}(X)$ by considering the family, parametrized by the affine line $\mathbb{A}^1_k$, of n-shifted Poisson structure $\hbar \cdot p$, with $\hbar \in k$.

Conjecture 3.5.7 *The quantization of X with respect to p is indeed a deformation quantization of the $\mathbb{P}_n$-monoidal structure on* $\mathsf{Perf}(X)$.

This conjecture is actually a consequence of a result that has been announced by N. Rozenblyum, stating that Dec_{n+1} can be lifted to an equivalence of ∞-categories

$$\mathbb{BD}_{n+1} - \mathbf{cdga}_{\mathcal{M}} \longrightarrow \mathsf{Alg}(\mathbb{BD}_n - \mathbf{cdga}_{\mathcal{M}})$$

such that one recovers the Dunn-Lurie additivity from [Lu6, 5.1.2.2] by evaluating at $\hbar = 1$ and the equivalence Dec_{n+1} by evaluating at $\hbar = 0$.

Quantization for negative n.

Let us now treat the case of a n-shifted Poisson structure p on X, with $n < 0$. Let $\hbar_{2n}$ a formal variable of cohomological degree $2n$, and consider

$$\mathcal{B}_X(\infty)[\hbar_{2n}],$$

which is now a stack, on X_{DR}, of Ind-objects in graded $k(\infty)[\hbar_{2n}]$-linear mixed cdgas. It comes equipped with a natural $k(\infty)[\hbar_{2n}]$-linear $\mathbb{P}_{1-n}$-structure, induced by $\hbar_{2n} \cdot p$. Since $n < 0$, we are back to the situation of positively shifted Poisson structures.

Namely, we may choose a formality equivalence of k-dg-operads

$$\alpha_{1-n} : \mathbb{E}_{1-n} \simeq \mathbb{P}_{1-n},$$

and thus view $\mathcal{B}_X(\infty)[\hbar_{2n}]$ as a an $k(\infty)[\hbar_{2n}]$-linear $\mathbb{E}_{1-n}$-algebra. Again by using [Lu6, 5.1.2.2 and 5.1.2.8], the associated stack $\mathcal{B}_X(\infty)[\hbar_{2n}] - \mathsf{Mod}_p^{\mathsf{Perf}}$ of perfect $\mathcal{B}_X(\infty)[\hbar_{2n}]$-modules comes equipped with a natural $\mathbb{E}_{-n}$-monoidal structure which will be denoted by

$$\mathcal{B}_X(\infty)[\hbar_{2n}] - \mathsf{Mod}_{\mathbb{E}_{-n},p}^{\mathsf{Perf}}.$$

Definition 3.5.8 *With the notation above, and $n < 0$, the* quantization of X with respect to p *is the $\mathbb{E}_{-n}$-monoidal ∞-category*

$$\mathsf{Perf}(X,p) := \Gamma(X_{DR}, \mathcal{B}_X(\infty)[\hbar_{2n}] - \mathsf{Mod}_{\mathbb{E}_{-n},p}^{\mathsf{Perf}}).$$

Now observe that, by construction, the underlying ∞-category of $\mathsf{Perf}(X,p)$ is

$$\mathsf{Perf}(X) \otimes_k k[\hbar_{2n}] =: \mathsf{Perf}(X)[\hbar_{2n}].$$

The quantization of Definition 3.5.8 consists then of the *datum of a $\mathbb{E}_{-n}$-monoidal structure on* $\mathsf{Perf}(X)[\hbar_{2n}]$. As above, such a quantization can be considered as a deformation of the standard symmetric monoidal (i.e. $\mathbb{E}_\infty$-) structure on $\mathsf{Perf}(X)[\hbar_{2n}]$. Note that this standard symmetric monoidal structure on $\mathsf{Perf}(X)[\hbar_{2n}]$ recovers the standard symmetric monoidal structure on $\mathsf{Perf}(X)$ after base change along the canonical map $k[\hbar_{2n}] \to k$.

3.6 Examples of quantizations

3.6.1 Quantization formally at a point

Let X be an Artin derived stack and $x : * := \mathbf{Spec}\, k \to X$ a closed point. We start with an obvious observation.

Lemma 3.6.1 $\mathbf{Pol}(\widehat{X}_x, n+1) = \mathbf{Pol}^t(\mathcal{B}_X/\mathbb{D}_{X_{DR}}, n+1)(x)$

Proof. Observe that $(\widehat{X}_x)_{DR} = \mathbf{Spec}\, k$. Therefore,

$$\mathbf{Pol}(\widehat{X}_x, n+1) = \mathbf{Pol}^t(\mathcal{B}_{\widehat{X}_x}, n+1) = \mathbf{Pol}^t(\mathbb{D}_{\widehat{X}_x}, n+1) = \mathbf{Pol}^t(\mathcal{B}_X/\mathbb{D}_{X_{DR}}, n+1)(x)\,.$$

$\square$

In particular, we get a dg-lie algebra morphism

$$\mathbf{Pol}(X, n+1) = \Gamma\big(X_{DR}, \mathbf{Pol}^t(\mathcal{B}_X/\mathbb{D}_{X_{DR}}, n+1)\big) \to \mathbf{Pol}^t(\mathcal{B}_X/\mathbb{D}_{X_{DR}}, n+1)(x) = \mathbf{Pol}(\widehat{X}_x, n+1)\,.$$

Therefore, any n-shifted Poisson structure on X induces an n-shifted Poisson structure on the formal completion $\widehat{X}_x$ at x.

Recall from Theorem 2.2.2 that, as a (non-mixed) graded cdga over k, $\mathcal{B}_{\widehat{X}_x}$ is equivalent to

$$Sym(\mathbb{L}_{*/\widehat{X}_x}[-1]) \cong Sym(x^*\mathbb{L}_{\widehat{X}_x}) \cong Sym(x^*\mathbb{L}_X)\,.$$

We therefore get a graded mixed $\mathbb{P}_{n+1}$-algebra structure on $Sym(x^*\mathbb{L}_X)$, whose underlying graded mixed cdgas is the one from $\mathcal{B}_{\widehat{X}_x}$. After a choice of formality α_{n+1}, we get a graded mixed $\mathbb{E}_{n+1}$-structure on $Sym(x^*\mathbb{L}_X)$ whenever $n > 0$.

We would like to make the above $\mathbb{E}_{n+1}$-structure on $Sym(x^*\mathbb{L}_X)$ rather explicit for a large class of examples.

Before doing so, let us recall very briefly Kontsevich's construction of an equivalence α_{n+1} [Ko2]. Let FM_{n+1} be the Fulton-MacPherson operad of compactified configuration spaces of points in $\mathbb{R}^{n+1}$ (which is a topological model for the operad $\mathbb{E}_{n+1}$: $\mathbb{E}_{n+1} = C_{-*}(FM_{n+1}, k)$ and $\mathbb{P}_{n+1} = H_{-*}(FM_{n+1}, k)$). The equivalence α_{n+1} comes from a zig-zag of explicit equivalences, which can be easily understood on the dual cooperads:

$$C^*(FM_{n+1}, k) \longleftarrow Graphs_{n+1} \longrightarrow H^*(FM_{n+1}, k)\,.$$

Here $Graphs_{n+1}$ is a certain cooperad in quasi-free cdgas: generators of $Graphs_{n+1}(I)$ are certain connected graphs, with external and internal vertices, having their external vertices labeled by I. The morphism $Graphs_{n+1}(I) \to H^*(FM_{n+1}(I), k)$ sends

- the connected graph without internal vertex and linking i to j, to the pull-back a_{ij} of the fundamental class of $FM_{n+1}(2) \cong S^n$ along the map $FM_{n+1}(I) \to FM_{n+1}(2)$ that forgets all points but i and j.

- all other generators, to zero.

The morphism $Graphs_{n+1}(I) \to C^*(FM_{n+1}(I), k)$ is transcendental in nature: it sends a graph Γ to the form

$$\int_{\text{internal vertices}} \bigwedge_{\text{edges }(i,j)} \omega_{ij} \,,$$

where ω_{ij} is the pull-back of the $SO(n+1)$-invariant volume form on $FM_{n+1}(2) \cong S^n$ along the map $FM_{n+1}(I) \to FM_{n+1}(2)$ that forgets all points but i and j.

Let us now chose a minimal model L for $x^*\mathbb{L}_X$. As we already observed, we get a weak mixed structure on the graded cdga $B := Sym(L)$, that is equivalent to $\mathcal{B}_{\widehat{X}_x}$. This weak mixed structure induces (and is actually equivalent to) the data of an $\mathcal{L}_\infty$-structure on $L^\vee[-1]$.

If we further assume that the n-shifted Poisson structure on X we started with is non-degenerate at x, then Lemma 3.3.11 tells us that the induced Poisson structure on $\widehat{X}_x$ is homotopic to a strict morphism of graded dg-lie algebras

$$k(2)[-1] \longrightarrow \mathbf{Pol}^{ex}(B, n+1) \,.$$

Let us assume for simplicity that the strict degree $-n$ Poisson bracket q we get that way on B is constant (meaning, as in the proof of Lemma 3.3.11, that q is a degree $n+2$ element in $Sym^2(L^\vee[-n-1]) \subset |B| \otimes Sym^2(L^\vee[-n-1])$). In this case the corresponding strict $\mathbb{P}_{n+1}$-structure on B has the following remarkable description: structure maps

$$B^{\otimes I} \longrightarrow B \otimes H^*(FM_{n+1}(I), k)$$

are given by

$$B^{\otimes I} \xrightarrow{exp(a)} B^{\otimes I} \otimes H^*(FM_{n+1}(I), k) \xrightarrow{m \otimes id} B \otimes H^*(FM_{n+1}(I), k) \,,$$

where m is the multiplication on B and

$$a := \sum_{i \neq j} \partial_p^{i,j} \otimes a_{ij} \,.$$

It can be checked that this formula lifts to graphs without modification whenever p is constant, and thus the induced $\mathbb{E}_{n+1}$-structure on B can be described by structure maps

$$B^{\otimes I} \overset{exp(A)}{\longrightarrow} B^{\otimes I}\widehat{\otimes}C^*(FM_{n+1}(I),k) \overset{m\otimes id}{\longrightarrow} B\widehat{\otimes}C^*(FM_{n+1}(I),k)\,,$$

where

$$A := \sum_{i\neq j} \partial_p^{i,j} \otimes \omega_{ij}\,. \tag{1}$$

Of course A is a formal sum, but when evaluated on chains it becomes finite and makes perfect sense.

We recover that way the Weyl n-algebras that were recently defined by Markarian (see [Mar]).

3.6.2 Quantization of BG

Let now $X = BG$, where G is an affine group scheme, and observe that $X_{DR} = B(G_{DR})$. Let $x : * \to BG$ be the classifying map of the unit $e : * \to G$. We have a fiber sequence of groups

$$\widehat{G}_e \longrightarrow G \longrightarrow G_{DR}\,,$$

so that $\widehat{BG}_x \simeq B(\widehat{G}_e)$.

We have already seen in the previous § that the pull-back of $\mathcal{B}_X$ along $x_{DR} : * \to BG_{DR}$ is $\mathcal{B}_{\widehat{X}_x}$. Therefore we get that the symmetric monoidal ∞-category

$$\mathsf{Perf}(BG) \simeq \mathcal{B}_X - Mod^{Perf}_{\epsilon-\mathbf{dg}^{gr}}$$

is equivalent to the symmetric monoidal ∞-category of G_{DR}-equivariant objects in

$$\mathcal{B}_{\widehat{X}_x} - Mod^{Perf}_{\epsilon-\mathbf{dg}^{gr}} \simeq \mathsf{Perf}(B\widehat{G}_e)\,.$$

Therefore, given an n-shifted Poisson structure p on BG, the quantization we get is completely determined by the G_{DR}-equivariant graded mixed $\mathbb{E}_{n+1}$-algebra structure on $\mathcal{B}_{\widehat{X}_x}$ obtained from the equivalence $\alpha_{n+1} : \mathbb{P}_{n+1} \simeq \mathbb{E}_{n+1}$. This shall have a fairly explicit description as $\mathcal{B}_{\widehat{X}_x} \simeq \mathbb{D}(B\widehat{G}_e)$ is equivalent to $Sym(x^*\mathbb{L}_{BG}) \simeq Sym(\mathfrak{g}^\vee[-1])$ as a graded (non-mixed) cdga, where $\mathfrak{g} := e^*T_G$.

Before going further, let us prove that $\mathbb{D}(B\widehat{G}_e)$ is actually equivalent to the Chevalley-Eilenberg graded mixed cdga of the Lie algebra $\mathfrak{g}$. The proof mainly goes in two steps:

- we first prove that equivalences classes graded mixed cdga structures on $Sym(V^\vee[-1])$, for V a discrete projective k-module of finite type, are in bijection with isomorphisms classes of strict Lie algebra structures on V.

- we then show that the Lie algebra structure on $\mathfrak{g}$ coming from the above mixed structure on $Sym(\mathfrak{g}^\vee[-1])$ is isomorphic to the standard Lie algebra structure on $\mathfrak{g} = e^*T_G$.

For $C \in \mathbf{cdga}_k^{gr}$, we will denote by $\epsilon - \mathbf{cdga}_k^{gr}(C)$ the fiber product

$$\begin{tikzcd} \epsilon - \mathbf{cdga}_k^{gr}(C) \arrow[r] \arrow[d] & \epsilon - \mathbf{cdga}_k^{gr} \arrow[d, "U"] \\ \{*\} \arrow[r, "C"'] & \mathbf{cdga}_k^{gr} \end{tikzcd}$$

where U denotes the forgetful functor, and C the given graded cdga structure. We then define $\epsilon - \mathrm{cdga}_k^{gr}(C) := \pi_0(\epsilon - \mathbf{cdga}_k^{gr} C)$. For V a k-module, we write $\mathrm{LieAlg}^{\mathrm{str}}(V)$ for the set of isomorphism classes of Lie algebra structures on V.

Proposition 3.6.2 *Let V be a discrete projective k-module of finite type.*

1. *for $B \in \epsilon - \mathrm{cdga}_k^{gr}(\mathrm{Sym}(V^\vee[-1]))$, let $H(B)$ be the graded mixed cdga defined by*

$$H(B)(p) := H^p(B(p))[-p] \, , \, p \in \mathbb{Z}$$

with mixed differential induced by $H^*(\epsilon_B)$. Then there is a canonical equivalence $B \simeq H(B)$ in $\epsilon - \mathbf{cdga}_k^{gr}$ (i.e. B is formal *as a graded* mixed *cdga).*

2. *there is a bijection*

$$\mathrm{Lie} : \epsilon - \mathrm{cdga}_k^{gr}(\mathrm{Sym}(V^\vee[-1])) \longrightarrow \mathrm{LieAlg}^{\mathrm{str}}(V)$$

whose inverse

$$\mathrm{Mix} : \mathrm{LieAlg}^{\mathrm{str}}(V) \longrightarrow \epsilon - \mathrm{cdga}_k^{gr}(\mathrm{Sym}(V^\vee[-1]))$$

is given by the (strict) Chevalley-Eilenberg construction.

Proof. (1) Let $B \in \epsilon - \mathrm{cdga}_k^{gr}(\mathrm{Sym}(V^\vee[-1]))$, and $u : B \simeq \mathrm{Sym}(V^\vee[-1])$ an equivalence in $\mathbf{cdga}_k^{gr}$. Since the differential in $\mathrm{Sym}(V^\vee[-1])$ is zero, $\mathrm{Sym}(V^\vee[-1])$ is a formal graded cdga, and we have

$$H^*(B(p)) = 0 \, , \text{ for any } p < 0,$$

$$H^i(B(p)) = 0 \, , \text{ for any } p \geq 0 \, , i \neq p,$$

and u induces k-module isomorphisms

$$H^p(B(p)) \simeq \wedge^p V^\vee \, , \text{ for any } p \geq 0 \, .$$

We may also consider $\tau_{\leq}(B)$ as

$$\tau_{\leq}(B)(p) := \tau_{\leq p}(B(p)) \, , \, p \in \mathbb{Z} \, ,$$

where $\tau_{\leq p}(E)$ denotes the good truncation of a dg-module E. One can check that the graded mixed cdga structure on B induces a graded mixed cdga structure on $\tau_{\leq}(B)$, and that the obvious dg-modules maps define a strict diagram of graded mixed cdgas

$$B \xleftarrow{h} \tau_{\leq}(B) \xrightarrow{g} H(B).$$

By our computation of $H(B)$ above, we deduce that both g and h are graded quasi-isomorphisms, hence that B is equivalent to $H(B)$ in $\epsilon-\mathbf{cdga}_k^{gr}$, i.e. any $B \in \epsilon-\mathrm{cdga}_k^{gr}(\mathrm{Sym}(V^\vee[-1]))$ is formal as a graded *mixed* cdga.

(2) For B as above, we now consider the mixed differential $\epsilon_1 : B(1) \to B(2)[1]$, for $p \geq 0$. It induces on H^1 a map

$$V^\vee \simeq H^1(B(1)) \to H^2(B(2)) \simeq \wedge^2 V^\vee$$

whose dual

$$\langle \, , \, \rangle_u : \wedge^2 V \to V$$

can easily be checked to define a Lie bracket on V. If $B' \in \epsilon-\mathrm{cdga}_k^{gr}(\mathrm{Sym}(V^\vee[-1]))$, $u' : B' \simeq \mathrm{Sym}(V^\vee[-1])$ an equivalence in $\mathbf{cdga}_k^{gr}$, and $B \simeq B'$ in $\epsilon - \mathbf{cdga}_k^{gr}$, then $\langle \, , \, \rangle_u$ and $\langle \, , \, \rangle_{u'}$ defines the same element in $\mathrm{LieAlg}^{\mathrm{str}}(V)$. Thus, we have a well defined map

$$\mathrm{Lie} : \epsilon - \mathrm{cdga}_k^{gr}(\mathrm{Sym}(V^\vee[-1])) \longrightarrow \mathrm{LieAlg}^{\mathrm{str}}(V)\,.$$

Let us show that Lie is injective. Let us recall (e.g. [Xu, Lemma 2.2]) that the map Lie^{str} sending a strict graded mixed cdga structure $\{\epsilon_p : \wedge^p V^\vee[-p] \to \wedge^{p+1} V^\vee[-p]\}$ to $(\epsilon_1)^\vee : \wedge^2 V \to V$ defines a bijection between strict isomorphism classes of (strict) graded mixed cdga structures on $\mathrm{Sym}(V^\vee[-1])$ and $\mathrm{LieAlg}^{\mathrm{str}}(V)$. We denote its inverse by strMix. Let B and B' be such that $\mathrm{Lie}(B) = \mathrm{Lie}(B')$. By definition of Lie, and the bijection just mentioned, we have strict isomorphisms of graded mixed cdgas

$$H(B) \simeq (\mathrm{Sym}(V^\vee[-1]), \mathrm{strMix}(\mathrm{Lie}(B)))$$

$$H(B') \simeq (\mathrm{Sym}(V^\vee[-1]), \mathrm{strMix}(\mathrm{Lie}(B')))\,.$$

But $\mathrm{Lie}(B) = \mathrm{Lie}(B')$, so we get a strict isomorphism of graded mixed cdgas $H(B) \simeq H(B')$. Since we have proved that B and $H(B)$ (respectively, B' and $H(B')$) are equivalent as graded mixed cdgas, we conclude that Lie is injective.

Now, the (strict) Chevalley-Eilenberg construction yields a map

$$\mathrm{Mix} : \mathrm{LieAlg}^{\mathrm{str}}(V) \longrightarrow \epsilon - \mathrm{cdga}_k^{gr}(\mathrm{Sym}(V^\vee[-1]))$$

which is easily checked to be a left inverse to Lie; therefore Lie is surjective, hence bijective with inverse Mix. □

Recall that $\mathfrak{g}$ is the Lie algebra of G, and denote by $[\,,\,]$ its Lie bracket. As we have already seen in, we have a canonical equivalence

$$u : \mathbb{D}(B\widehat{G}_e) \simeq \mathrm{Sym}(\mathfrak{g}^{\vee}[-1])$$

in $\mathbf{cdga}_k^{gr}$. Since $\mathbb{D}(B\widehat{G})$ has a canonical structure of graded mixed cdga, let $\langle\,,\,\rangle_u$ the Lie bracket induced on $\mathfrak{g}$ according to Proposition 3.6.2.

Proposition 3.6.3 *With the above notation, and assume that k is a field, we have*

1. $(\mathfrak{g}, [\,,\,])$ *and* $(\mathfrak{g}, \langle\,,\,\rangle_u)$ *are isomorphic Lie algebras.*

2. *There is an equivalence*

$$\mathbb{D}(B\widehat{G}) \simeq \big(\mathrm{Sym}(\mathfrak{g}^{\vee}[-1]), \epsilon := d_{\mathrm{CE},[\,,\,]}\big) =: \mathsf{CE}\big(\mathfrak{g}, [\,,\,]\big)$$

in $\epsilon - \mathbf{cdga}_k^{gr}$.

Proof. (1) Recall the equivalence of symmetric monoidal ∞-categories

$$\mathbb{D}(B\widehat{G}_e) - Mod^{Perf}_{\epsilon-\mathbf{dg}^{gr}} \simeq \mathsf{Perf}(B\widehat{G}_e$$

Let $\mathbb{D}(B\widehat{G}_e) - Mod^{qffd}_{\epsilon-\mathbf{dg}^{gr}}$ be the full sub-∞-category of $\mathbb{D}(B\widehat{G}_e) - Mod^{Perf}_{\epsilon-\mathbf{dg}^{gr}}$ consisting of *quasi-free finite dimensional modules*; i.e. those $\mathbb{D}(B\widehat{G}_e)$-modules which are equivalent as graded modules to $\mathbb{D}(B\widehat{G}_e)\otimes V$, where V is a discrete finite dimensional k-vector space that is concentrated in pure weight 0. The above equivalence then restricts to an equivalence of tensor k-linear (discrete) categories

$$\mathbb{D}(B\widehat{G}_e) - Mod^{qffd}_{\epsilon-\mathbf{dg}^{gr}} \simeq \mathsf{Rep}^{fd}(\widehat{G}_e)\,,$$

where $\mathsf{Rep}^{fd}(\widehat{G}_e)$ is the tensor k-linear category of finite dimensional representations of $\widehat{G}_e$. Observe that this equivalence commutes with the obvious fiber functors to $\mathsf{Vect}(k)$ (whose geometric origin is simply the pull-back x^* along the point $x : * \to B\widehat{G}_e$), where $\mathsf{Vect}(k)$ is the category of vector spaces. In particular, the above equivalence is an equivalence of neutral Tannakian categories, and we therefore have the following chain of equivalences between neutral Tannakian categories:

$$\mathsf{Rep}^{fd}\big(\mathfrak{g}, \langle\,,\,\rangle_u\big) \simeq \mathsf{CE}\big(\mathfrak{g}, \langle\,,\,\rangle_u\big) - Mod^{qffd}_{\epsilon-\mathbf{dg}^{gr}} \simeq \mathbb{D}(B\widehat{G}_e) - Mod^{qffd}_{\epsilon-\mathbf{dg}^{gr}} \simeq \mathsf{Rep}^{fd}(\widehat{G}_e) \simeq \mathsf{Rep}^{fd}\big(\mathfrak{g}, [\,,\,]\big)\,.$$

We refer to [De-Mi] for general facts about the Tannakian formalism, which tells us that we therefore have the following sequence of Lie algebra morphisms:

$$\big(\mathfrak{g}, \langle\,,\,\rangle_u\big) \longrightarrow \mathbf{End}(f_{\langle\,,\,\rangle_u}) \cong \mathbf{End}(f_{[\,,\,]}) \longleftarrow \big(\mathfrak{g}, [\,,\,]\big)\,, \tag{2}$$

where $\mathbf{End}(f)$ is the Lie k-algebra of natural transformations of a given fiber functor f (endowed with the commutator as Lie bracket), and $f_{\langle\,,\,\rangle_u}$ and $f_{[\,,\,]}$ are the fiber functors of $\mathsf{Rep}^{fd}\big(\mathfrak{g}, \langle\,,\,\rangle_u\big)$ and $\mathsf{Rep}^{fd}\big(\mathfrak{g}, [\,,\,]\big)$, respectively. It is a general fact that the leftmost and rightmost morphisms in (2) are injective. Moreover, $(\mathfrak{g}, [\,,\,])$ being algebraic, the leftmost morphism is actually an isomorphism. Therefore we get an injective Lie algebra morphism $\big(\mathfrak{g}, \langle\,,\,\rangle_u\big) \to \big(\mathfrak{g}, [\,,\,]\big)$, which must be an isomorphism for obvious dimensional reasons.
(2) To ease notations, we will write $B := \mathbb{D}(B\widehat{G})$ as a graded mixed cdga, and ϵ_B its mixed differential. Since $B \in \epsilon - \mathrm{cdga}_k^{gr}(\mathrm{Sym}(V^\vee[-1]))$, by Proposition 3.6.2 we have

$$\langle\,,\,\rangle_u = \mathrm{Lie}(H(B)) = \mathrm{Lie}(B)\,.$$

By (1), and, again, Proposition 3.6.2, we get

$$\mathsf{CE}\big(\mathfrak{g}, [\,,\,]\big) = \mathrm{Mix}([\,,\,]) = \mathrm{Mix}(\langle\,,\,\rangle_u) = H(B) = B\,,$$

where the equalities are in $\epsilon - \mathrm{cdga}_k^{gr}(C) := \pi_0(\epsilon - \mathbf{cdga}_k^{gr} C)$. In particular, B and $\mathsf{CE}\big(\mathfrak{g}, [\,,\,]\big)$ are equivalent in $\epsilon - \mathbf{cdga}_k^{gr}$. □

Remark 3.6.4 Let us give an alternative, less elementary but direct proof of (2). As observed in §3.2.5, an equivalence of graded cdgas $v : B \simeq \mathrm{Sym}(\mathfrak{g}^\vee[-1])$ induces a *weak* mixed structure (see proof of Lemma 3.3.10) on $C := \mathrm{Sym}(\mathfrak{g}^\vee[-1])$, i.e. a family of strict maps

$$\epsilon_i : C(p) \longrightarrow C(p+i+1)[1]\,,\; i \geq 0$$

satisfying a Maurer-Cartan-like equation. In our case

$$\epsilon_i : (\wedge^p \mathfrak{g}^\vee)[-p] \longrightarrow (\wedge^{p+i+1} \mathfrak{g}^\vee)[-p-i]$$

hence $\epsilon_i = 0$ for $i > 0$, because $\mathfrak{g}$ sits in cohomological degree 0. The only non-trivial remaining map is ϵ_0, and the Maurer-Cartan equation tells us exactly that it defines a strict graded mixed cdga structure on $\mathrm{Sym}(\mathfrak{g}^\vee[-1])$, and that, with such structure, the equivalence $v : B \simeq \mathrm{Sym}(\mathfrak{g}^\vee[-1])$ is indeed an equivalence of graded *mixed* cdgas.

The case $n = 1$ for a reductive G. We have seen in §3.1 that equivalences classes of 1-shifted Poisson structures on BG, for a reductive group G, are in bijection with elements $Z \in \wedge^3(\mathfrak{g})^G$. The induced 1-shifted Poisson structure on the graded mixed cdga $\mathsf{CE}(\mathfrak{g})$ is then very explicit in terms of a so-called semi-strict $\mathbb{P}_{n+1}$-structure (see [Me]): all structure 2-shifted polyvectors are trivial except for the 3-ary one which is constant and given by Z.

Our deformation quantization in particular leads to a deformation of $\mathsf{Rep}^{fd}(\mathfrak{g})$ as a monoidal category.

Example 3.6.5 *Given a non-degenerate invariant pairing* $<,>$ *on* $\mathfrak{g}$*, such an element can be obtained from the* G*-invariant linear form*

$$\wedge^3\mathfrak{g} \longrightarrow k \quad , \quad (x,y,z) \longmapsto < x,[y,z] > \ .$$

Alternatively, any invariant symmetric 2*-tensor* $t \in Sym^2(\mathfrak{g})^G$ *leads to such an element* $Z = [t^{1,2}, t^{2,3}] \in \wedge^3(\mathfrak{g})^G$*. In this case the deformation of* $\mathsf{Rep}^{fd}(\mathfrak{g})$ *as a monoidal category can be obtained by means of a deformation of the associativity constraint only (see [Dr1]), which then looks like*

$$\Phi = 1^{\otimes 3} + \hbar^2 Z + o(\hbar^2) \in U(\mathfrak{g})^{\otimes 3}[[\hbar]] \,.$$

Remark 3.6.6 Note that even in the case when G is not reductive, every element $Z \in \wedge^3(\mathfrak{g})^G$ lead to a 1-shifted Poisson structure on BG as well (but we have a map $\wedge^3(\mathfrak{g})^G \to \pi_0\mathsf{Pois}(BG,1)$ rather than a bijection). The above reasoning works as well for these 1-shifted Poisson structures.

The case $n = 2$ for a reductive G. We have seen in §3.1 that equivalences classes of 2-shifted Poisson structures on BG, for a reductive group G, are in bijection with elements $t \in Sym^2(\mathfrak{g})^G$. The induced 2-shifted Poisson structure on the graded mixed cdga $\mathsf{CE}(\mathfrak{g})$ is strict and constant. The graded mixed $\mathbb{E}_3$-structure on $\mathsf{CE}(\mathfrak{g})$ given by our deformation quantization then takes the form of a Weyl 3-algebra, as described in §3.6.1 (one simply has to replace p by t in (1)).

Note that, as we already mentioned, this graded mixed $\mathbb{E}_3$-structure is G_{DR}-equivariant by construction, so that it leads to an $\mathbb{E}_2$-monoidal deformation of $\mathsf{Perf}(BG)$. This in particular leads to a braided monoidal deformation of $\mathsf{Rep}^{fd}(\mathfrak{g})$.

Remark 3.6.7 Note that even in the case when G is not reductive, elements $t \in Sym^2(\mathfrak{g})^G$ are exactly 2-shifted Poisson structure on BG (i.e. we have a map $Sym^2(\mathfrak{g})^G \cong \pi_0\mathsf{Pois}(BG,2)$). The above reasoning works as well for these 2-shifted Poisson structures.

Such deformation quantizations of BG have already been constructed:

- when $\mathfrak{g}$ is reductive and t is non-degenerate, by means of purely algebraic methods: the quantum group $U_\hbar(\mathfrak{g})$ is an explicit deformation of the enveloping algebra $U(\mathfrak{g})$ as a quasi-triangular Hopf algebra.

- without any assumption, by Drinfeld [Dr2], using transcendental methods similar to the ones that are crucial in the proof of the formality of $\mathbb{E}_2$.

It is known that Drinfeld's quantization is equivalent to the quantum group one in the semi-simple case (see e.g. [Ka] and references therein).

Remark 3.6.8 It is remarkable that our quantization relies on the formality of $\mathbb{E}_3$ rather than on the formality of $\mathbb{E}_2$. It deserves to be compared with Drinfeld's one, but this task is beyond the scope of the present paper.

Appendix A

This Appendix contains a few technical results needed in Sect. 1.

Proposition A.1.1 *Any $C(k)$-model category is a stable model category.*

Proof. Let N be a $C(k)$-model category, and let $\underline{\mathrm{Hom}}_k(-,-)$ be its enriched hom-complex. There is a unique map $0 \to \underline{\mathrm{Hom}}_k(*, \emptyset)$ in $C(k)$, where $*$ (respectively, $\emptyset$) is the final (respectively, initial) object in N. By Composing with the map $k \to 0$ in $C(k)$, we get a map in N from its final to its initial object: hence N is pointed. Let us denote by $\Sigma : \mathrm{Ho}(N) \to \mathrm{Ho}(N)$ the corresponding suspension functor. For $X \in N$ cofibrant we have that $X \otimes_k k[1] \simeq \Sigma(X)$ (since $X \otimes_k (-)$ preserves homotopy pushouts and $k[1]$ is the suspension of k in $C(k)$). Therefore, the suspension functor Σ is an equivalence, its quasi inverse being given by $(-) \otimes_k^{\mathbb{L}} k[-1]$. $\square$

Proposition A.1.2 *Let M be a symmetric monoidal combinatorial model category satisfying the standing assumptions $(1)-(5)$ of Section 1.1, and let $A \in Comm(M)$. Then the symmetric monoidal combinatorial model category $A - Mod_M$ also satisfies the standing assumptions $(1)-(5)$.*

Proof. Left to the reader. $\square$

Proposition A.1.3 *Let M be a symmetric monoidal combinatorial model category satisfying the standing assumptions $(1)-(5)$ of Section 1.1. If $w : A \to B$ is a weak equivalence in $Comm(M)$, then the Quillen adjunction*

$$w^* = - \otimes_A B \ : A - Mod_M \longleftrightarrow B - Mod_M : w_*$$

is a Quillen equivalence.

Proof. Since w_* reflects weak equivalences, w^* is a Quillen equivalence iff for any cofibrant A-module N, the natural map $i : \mathrm{id}_N \otimes w : N \simeq N \otimes_A A \to N \otimes_A B$ is a weak equivalence. Since N is cofibrant, we may write it as $\mathrm{colim}_{\beta\leq\alpha} N_\beta$ (colimit in $A - Mod_M$) where α is an ordinal, $N_0 = 0$ and each map $N_\beta \to N_{\beta+1}$ is obtained as a pushout in $A - Mod_M$

$$\begin{tikzcd} A \otimes X \arrow[r, "\mathrm{id}\otimes u"] \arrow[d] & A \otimes Y \arrow[d] \\ N_\beta \arrow[r] & N_{\beta+1} \end{tikzcd}$$

where $u : X \to Y$ belongs to the set I of generating cofibrations of M (all assumed with M-cofibrant domain, by standing assumption (3)). In order to prove that $i : N \simeq N \otimes_A A \to N \otimes_A B$ is a weak equivalence, we will prove, by transfinite induction, that each $i_\beta : N_\beta \simeq N_\beta \otimes_A A \to N_\beta \otimes_A B$ is a weak equivalence.

Since $N_0 = 0$, the induction can start. Let us suppose that i_β is a weak equivalence, and consider the pushout diagram P defining $N_\beta \to N_{\beta+1}$

$$\begin{tikzcd} A \otimes X \arrow[r, "\mathrm{id}\otimes u"] \arrow[d] & A \otimes Y \arrow[d] \\ N_\beta \arrow[r] & N_{\beta+1}. \end{tikzcd}$$

Now, let us apply the functor w^* to this pushout. We obtain the diagram P'

$$\begin{tikzcd} B \otimes X \arrow[r, "\mathrm{id}\otimes u"] \arrow[d] & B \otimes Y \arrow[d] \\ N_\beta \otimes_A B \arrow[r] & N_{\beta+1} \otimes_A B \end{tikzcd}$$

which is again a pushout in $B - Mod_M$ (since w^* is left adjoint). There is an obvious map of diagrams from P to P' induced by the maps $w \otimes \mathrm{id}_X : A \otimes X \to B \otimes X$, $i_\beta : N_\beta \to N_\beta \otimes_A B$, and $w \otimes \mathrm{id}_Y : A \otimes Y \to B \otimes Y$. All these three maps are weak equivalences (i_β by induction hypothesis, and the other two by standing assumption (3), since X is cofibrant, and so is Y, u being a cofibration). Since the forgetful functor $A - Mod_M \to M$ has right adjoint the internal hom-functor $\underline{Hom}_M(A, -)$, both P and P' are pushouts in M, too. Thus ([Hir, Proposition 13.5.10]) also the induced map $i_{\beta+1} : N_{\beta+1} \to N_{\beta+1} \otimes_A B$ is a weak equivalence (in M) as the two diagrams P and P' are also homotopy pushouts, by standing assumption (2) on M. We are done with the successor ordinal case and left to prove the limit ordinal case. The family of maps $\{i_\beta\}$ are all weak equivalences and define a map of sequences $\{N_\beta\} \to \{N_\beta \otimes_A B\}$, where each map $N_\beta \to N_{\beta+1}$ is a cofibration (as pushout of a cofibration), and the same is true for each map $N_\beta \otimes_A B \to N_{\beta+1} \otimes_A B$ (since w^* is left Quillen). Moreover, each N_β is cofibrant (since $N_0 = 0$ is and each $N_\beta \to N_{\beta+1}$ is a cofibration), and the same is true for each $N_\beta \otimes_A B$ (since w^* is left Quillen). Therefore the map induced on the (homotopy) colimit is a weak equivalence too.

$\square$

Proposition A.1.4 *Let M be a symmetric monoidal combinatorial model category satisfying the standing assumptions $(1)-(5)$ of Section 1.1. Then the forgetful functor $Comm(M) \to M$ preserves fibrant-cofibrant objects.*

Proof. The forgetful functor is right Quillen, so it obviously preserves fibrant objects. The $C(k)$-enrichment, together with $\mathrm{char}(k) = 0$, implies that M is freely powered in the sense of [Lu6, Definition 4.5.4.2]. By [Lu6, Lemma 4.5.4.11], M satisfies the strong commutative monoidal axiom of [Wh, Definition 3.4]. Then, the statement follows from our standing assumption (1) and from [Wh, Corollary 3.6].

$\square$

Appendix B

We prove here several technical statement about differential forms and formal completions in the derived setting, needed in Sect. 2.

Lemma B.1.1 *Let $X \longrightarrow U \longrightarrow Y$ be morphisms of derived algebraic n-stacks. Let U_* be the nerve of the morphism $U \longrightarrow Y$. Then, for all p there is a natural equivalence*

$$\Gamma(X, \wedge^p \mathbb{L}_{X/Y}) \simeq \lim_{n \in \Delta} \Gamma(X, \wedge^p \mathbb{L}_{X/U_n}).$$

Proof. For $F \in \mathbf{dSt}_k$ we consider the shifted tangent derived stack

$$T^1(F) := \mathbb{R}\mathbf{Map}(\mathbf{Spec}\, k[\epsilon_{-1}], F),$$

the internal Hom object, where $k[\epsilon_{-1}] = k \oplus k[1]$ is the free cdga over one generator in degree -1. The natural augmentation $k[\epsilon_{-1}] \to k$ induces a projection $T^1(F) \longrightarrow F$. Moreover, if F is an algebraic derived n-stack then $T^1(F)$ is an algebraic derived $(n+1)$-stack.

For a morphism $F \longrightarrow G$, we let

$$T^1(F/G) := T^1(F) \times_{T^1(G)} G,$$

as a derived stack over F. The multiplicative group $\mathbb{G}_m$ acts on $T^1(F/G)$, and thus we can consider $\Gamma(T^1(F/G), \mathcal{O})$ as a graded complex. As such, its part of weight p is

$$\Gamma(F, \wedge^p \mathbb{L}_{F/G})[-p].$$

In order to conclude, we observe that the induced morphism, which is naturally $\mathbb{G}_m$-equivariant

$$T^1(X/U) \longrightarrow T^1(X/F)$$

is an epimorphism of derived stacks. The nerve of this epimorphism is the simplicial object $n \mapsto T^1(X/U_n)$. By descent for functions of weight p we see that the natural morphism

$$\Gamma(X, \wedge^p \mathbb{L}_{X/F}) \longrightarrow \lim_n \Gamma(X, \wedge^p \mathbb{L}_{X/U_n})$$

is an equivalence. $\square$

For the next lemma, we will use Koszul commutative dg-algebras. For a commutative k-algebra B, and $f_1, \ldots, f_p$ a family of elements in B, we let $K(B, f_1, \ldots, f_p)$ be the commutative dg-algebra freely generated over B by variables $X_1, \ldots, X_p$ with $deg(X_i) = -1$, and with $dX_i = f_i$. When $f_1, \ldots, f_p$ form a regular sequence in B, then $K(B, f_1, \ldots, f_p)$ is a cofibrant model for $B/(f_1, \ldots, f_p)$ considered as a B-algebra. In general, $\pi_i(K(B, f_1, \ldots, f_p)) \simeq Tor_i^B(B/(f_1), \ldots, B/(f_p))$ are possibly non zero only when $i \in [0, p]$.

Lemma B.1.2 *Let B be a commutative (non-dg) k-algebra of finite type and $I \subset B$ an ideal generated by $(f_1, \ldots, f_p)$. Let $f : X = \mathbf{Spec}\, B/I \longrightarrow Y = \mathbf{Spec}\, B$ be the induced morphism of affine schemes, and $X_n := \mathbf{Spec}\, K(B, f_1^n, \ldots, f_p^n)$. Then, the natural morphism*

$$\mathrm{colim}_n X_n \longrightarrow \widehat{Y}_f$$

is an equivalence of derived prestacks: for all $\mathbf{Spec}\, A \in \mathbf{dAff}_k$ we have an equivalence

$$\mathrm{colim}_n(X_n(A)) \simeq \widehat{Y}_f(A).$$

Proof. We let F be the colimit prestack $\mathrm{colim}_n X_n$. There is a natural morphism of derived prestacks

$$\phi : F \longrightarrow \widehat{Y}_f.$$

For any k-algebra A of finite type, the induced morphism of sets

$$F(A) \longrightarrow \widehat{Y}_f(A)$$

is bijective. Indeed, the left hand side is equivalent to the colimit of sets $\mathrm{colim}_n Hom_{k-Alg}(B/I(n), A)$, where $I(n)$ is the ideal generated by the n-th powers of the f_i's, whereas the right hand side consists of the subset of $Hom_{k-Alg}(B, A)$ of maps $f : B \longrightarrow A$ sending I to the nilpotent radical of A. In order to prove that the morphism ϕ induces an equivalences for all $\mathbf{Spec}\, A \in \mathbf{dAff}_k$ we use a Postnikov decomposition of A

$$A \longrightarrow \dots \longrightarrow A_{\leq k} \longrightarrow A_{\leq k-1} \longrightarrow \dots \longrightarrow A_{\leq 0} = \pi_0(A).$$

As prestacks, i.e. as ∞-functors on $\mathbf{dAff}_k^{op}$, both F and $\widehat{Y}_f$ satisfy the following two properties.

- For all $\mathbf{Spec}\, A \in \mathbf{dAff}_k$, we have equivalences

$$F(A) \simeq \lim_k F(A_{\leq k}) \qquad \widehat{Y}_f(A) \simeq \lim_k \widehat{Y}_f(A_{\leq k})$$

- For all fibered product of almost finite presented k-cdgas in non-positive degrees

$$\begin{tikzcd} B \arrow[r] \arrow[d] & B_1 \arrow[d] \\ B_2 \arrow[r] & B_0, \end{tikzcd}$$

  such that $\pi_0(B_i) \longrightarrow \pi_0(B_0)$ are surjective with nilpotent kernels, the induced square

$$\begin{tikzcd} F(B) \arrow[r] \arrow[d] & F(B_1) \arrow[d] \\ F(B_2) \arrow[r] & F(B_0), \end{tikzcd}$$

  is cartesian in $\mathcal{T}$.

The above two properties are clear for $\widehat{Y}_f$, because $\widehat{Y}_f$ is a formal stack. The second property is also clear for F because filtered colimits preserve fiber products. Finally, the first property is satisfied for F because for each fixed n, and each fixed $i \geq 0$ the projective system of homotopy groups

$$\pi_i(X_n(A)) \longrightarrow \dots \longrightarrow \pi_i(X_n(A_{\leq k})) \longrightarrow \pi_i(X(A_{\leq k-1})) \longrightarrow \dots \longrightarrow \pi_i(X(A_{\leq 0}))$$

stabilizes (this is because $K(B, f_1^n, \dots, f_p^n)$ are cell B-cdga with finitely many cells and thus with a perfect cotangent complex).

By these above two properties, and by Postnikov decomposition, we are reduced to prove that for any non-dg k-algebra A of finite type, any A-module M of finite type, and any $k \geq 1$ the induced morphism

$$F(A \oplus M[k]) \longrightarrow \widehat{Y}_f(A \oplus M[k])$$

is an equivalence. We can fiber this morphism over $F(A) \simeq \widehat{Y}_f(A)$ and thus are reduced to compare cotangent complexes of F and $\widehat{Y}_f$.

By replacing X by one of the X_n, we can assume that $\mathbf{Spec}\, A = X$ and thus that $A = B/I$. We thus consider the morphism induced on cotangent complexes for the morphism $X \longrightarrow F \longrightarrow \widehat{Y}_f$

$$\mathbb{L}_{X/F} \longrightarrow \mathbb{L}_{X/\widehat{Y}_f}.$$

Here, $\mathbb{L}_{X/F}$ is not quite an A-dg-module but is a pro-object in $L^{\leq 0}_{coh}(A)$ which represents the adequate ∞-functor. This pro-object is explicitly given by

$$\mathbb{L}_{X/F} \simeq \text{"}\lim_n\text{"} \mathbb{L}_{X/X_n}.$$

We have to prove that the morphism of pro-objects

$$\text{"}\lim_n\text{"} \mathbb{L}_{X/X_n} \longrightarrow \mathbb{L}_{X/\widehat{Y}_f},$$

where the right hand side is a constant pro-object, is an equivalence. Equivalently, using various exact triangles expressing cotangent complexes we must prove that the natural morphism

$$\text{"}\lim_n\text{"} u_n^*(\mathbb{L}_{X_n/Y}) \longrightarrow u^*(\mathbb{L}_{\widehat{Y}_f/Y})$$

is an equivalence of pro-objects, where $u_n : X \longrightarrow X_n$ and $u : X \longrightarrow Y$ are the natural maps. The right hand side vanishes because $\widehat{Y}_f \longrightarrow Y$ is formally étale. Finally, the left hand side is explicitly given by the projective systems of $A = B/I$-dg-modules "$\lim_n(A^p[1])$ (because $K(B, f_1^n, \dots, f_p^n) \otimes_B A$ is freely generated over A by p cells of dimension 1). Here the transition morphisms are obtained by multiplying the i-th coordinate of A^p by f_i and thus are the zero morphisms. This pro-object is therefore equivalent to the zero pro-object, and this finishes the proof of the lemma. $\square$

Lemma B.1.3 *Let X be an affine formal derived stack. We assume that, as a derived prestack X is of the form $X \simeq \mathrm{colim}_{n\geq 0} X_n$, with $X_n \in \mathbf{dAff}_k$ for all n. Then, for all p, the natural morphism*

$$\wedge^p \mathbb{L}_{X_{red}/X} \simeq \lim_n \wedge^p \mathbb{L}_{X_{red}/X_n}$$

is an equivalence in $\mathsf{L}_{\mathsf{Qcoh}}(X_{red})$.

Proof: We consider the ∞-functor co-represented by $\mathbb{L}_{X_{red}/X}$

$$\mathsf{Map}(\mathbb{L}_{X_{red}/X}, -) : L^{\leq 0}_{coh}(X_{red}) \longrightarrow \mathcal{T}.$$

Note that because X is a colimit of derived schemes its cotangent complex $\mathbb{L}_{X_{red}/X}$ sits itself in $L^{\leq 0}_{coh}(X_{red})$. Moreover, as X is the colimit of the X_n as derived prestacks, the ∞-functor $\mathsf{Map}(\mathbb{L}_{X_{red}/X}, -)$

is also pro-representable by the pro-object in $L^{\leq 0}_{coh}(X_{red})$

$$"\lim_n"\{\mathbb{L}_{X_{red}/X_n}\}.$$

Therefore, this pro-object is equivalent, in the ∞-category of pro-objects in $L^{\leq 0}_{coh}(X_{red})$, to the constant pro-object $\mathbb{L}_{X_{red}/X}$. Passing to wedge powers, we see that for all p the pro-object $"\lim_n"\{\wedge^p\mathbb{L}_{X_{red}/X_n}\}$ is also equivalent to the constant pro-object $\wedge^p\mathbb{L}_{X_{red}/X}$, and the lemma follows. $\square$

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