SHIFTS OF GENERATORS AND DELTA SETS OF NUMERICAL MONOIDS

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Abstract. Let S be a numerical monoid with minimal generating set $\langle n_1, \ldots, n_t \rangle$. For $m \in S$, if $m = \sum_{i=1}^{t} x_i n_i$, then $\sum_{i=1}^{t} x_i$ is called a *factorization length* of m. We denote by $\mathcal{L}(m) = \{m_1, \ldots, m_k\}$ (where $m_i < m_{i+1}$ for each $1 \leq i < k$) the set of all possible factorization lengths of m. The Delta set of m is defined by $\Delta(m) = \{m_{i+1} - m_i \mid 1 \leq i < k\}$ and the Delta set of S by $\Delta(S) = \cup_{m \in S} \Delta(m)$. In this paper, we expand on the study of $\Delta(S)$ begun in [3] in the following manner. Let $r_1, r_2, \ldots, r_t$ be an increasing sequence of positive integers and $M_n = \langle n, n + r_1, n + r_2, \ldots, n + r_t \rangle$ a numerical monoid where n is some positive integer. We prove that there exists a positive integer N such that if $n > N$ then $|\Delta(M_n)| = 1$. If $t = 2$ and r_1 and r_2 are relatively prime, then we determine a value for N which is sharp.

1. Introduction

Problems involving non-unique factorizations into irreducible elements in an integral domain or monoid continue to be a popular topic in the recent mathematical literature (see the monograph [6] and the references cited therein). In this paper, we continue the study of factorization properties of numerical monoids which was begun in [3] and [1]. Before proceeding we will require some definitions. Let $\mathbb{N}$ represent the natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A numerical monoid S is a submonoid of $\mathbb{N}_0$ under regular addition. Each such S has a unique minimal generating set. When given a generating set $\{n_1, \ldots, n_k\}$, we will assume that it is minimal unless otherwise stated. If $\gcd\{n_1, \ldots, n_t\} = 1$, then $S = \langle n_1, \ldots, n_k \rangle$ is called *primitive*. It is easy to see that every numerical monoid is isomorphic to a primitive numerical monoid. A good general reference on numerical monoids is [4, Chapter 10]. It is known that for any primitive numerical monoid S there exists a positive integer k such that every $n > k$ is contained in S. The smallest such k is called the *Frobenius number* of S and is denoted $F(S)$. The problem of computing the Frobenius number has interested mathematicians for at least 100 years (the computation of the Frobenius number for a two generated numerical monoid first appeared in [8]) and the recent monograph [7] is an excellent reference on the status of the Diophatine Frobenius Problem.

We will follow the basic notation for the theory of non-unique factorizations as outlined in [6]. Let M be a commutative cancellative atomic monoid with set $\mathcal{A}(M)$ of irreducible elements and set $M^\times$ of units. For $m \in M \backslash M^\times$, set

$$\mathcal{L}(m) = \{\, t \in \mathbb{N} \mid \exists\, x_1, \ldots, x_t \in \mathcal{A}(M) \text{ with } m = x_1 \cdots x_t \,\}.$$

The set $\mathcal{L}(m)$ is called the *set of lengths* of m. For any $m \in M \backslash M^\times$, we define $L(m) = \sup \mathcal{L}(m)$ and $\ell(m) = \inf \mathcal{L}(m)$. Moreover, if $m \in M \backslash M^\times$ and $\mathcal{L}(m) = \{x_1, \ldots, x_n\}$ with $x_1 < x_2 < \cdots < x_n$, then the *delta set* of m is

$$\Delta(m) = \{x_i - x_{i-1} | 2 \leq i \leq n\},$$

2000 *Mathematics Subject Classification.* 20M14, 20D60, 11B75.

Key words and phrases. numerical monoid, numerical semigroup, non-unique factorization, delta set.

The second, third, fourth and fifth authors received support from the National Science Foundation under grant DMS-0648390.

and the *delta set* of M is

$$\Delta(M) = \bigcup_{m \in M \setminus M^\times} \Delta(m).$$

By a fundamental result of Geroldinger [5, Lemma 3], if $d = \gcd \Delta(M)$ and $|\Delta(M)| < \infty$, then

$$\{d\} \subseteq \Delta(M) \subseteq \{d, 2d, \ldots, kd\}$$

for some $k \in \mathbb{N}$. A summary of known results involving delta sets can be found in [6, Section 6.7]. Of particular interest from [3] in our current work are the following results.

Proposition 1.1. *Let $S = \langle n_1, \ldots, n_k \rangle$ be a primitive numerical monoid.*

(1) $\min \Delta(S) = \gcd\{n_i - n_{i-1} \,|\, 2 \leq i \leq k\}$ [3, Proposition 2.9].
(2) *If $S = \langle n, n+k, n+2k, \ldots, n+bk \rangle$, then $\Delta(S) = \{k\}$* [3, Theorem 3.9].
(3) *For any k and v in $\mathbb{N}$ there exists a three generated numerical monoid S with $\Delta(S) = \{k, 2k, \ldots, vk\}$* [3, Corollary 4.8].

As an example, by [3, Corollary 4.8] it follows that $S = \langle s, s+1, 2s-1 \rangle$ for $s \geq 3$ has delta set $\left\{1, 2, \ldots, \left\lfloor \frac{s}{3} \right\rfloor \right\}$. However, if we fix the successive differences between the generators and set $M_n = \langle n, n+1, n+(s-1) \rangle$, computer observations based on programming in [2] indicate that increasing n will cause the size of the delta set to diminish. For instance, if $s = 21$ we obtain the following.

n	M_n	$\Delta(M_n)$
21	$\langle 21, 22, 41 \rangle$	$\{1,2,3,4,5,6,7\}$
22	$\langle 22, 23, 42 \rangle$	$\{1,2,3,4,5\}$
53	$\langle 53, 54, 73 \rangle$	$\{1,2,3\}$
321	$\langle 321, 322, 341 \rangle$	$\{1,2\}$
$n \geq 322$	$\langle n, n+1, n+20 \rangle$	$\{1\}$

We are able to prove in Section 4 the assertion made in the last line of the table and in Section 2 that similar behavior occurs for all numerical monoids in the following sense. Let $r_1, r_2, \ldots, r_t$ be an increasing sequence of positive integers and $M_n = \langle n, n+r_1, n+r_2, \ldots, n+r_t \rangle$ a numerical monoid where n is some positive integer. We prove in Theorem 2.2 that there exists a positive integer N such that if $n > N$ then $|\,\Delta(M_n))\,| = 1$. In fact, if $\gcd(r_1, \ldots, r_t) = z$, then $\Delta(M_n) = \{\frac{z}{\gcd(n,z)}\}$ for $n > N$. Using a significant improvement of [3, Proposition 4.3] derived in Section 3, we are able to prove in Section 4 a stronger version of Theorem 2.2 when $t = 2$ and $\gcd(r_1, r_2) = 1$. Under these hypotheses, Theorem 4.1 significantly improves the bound N from Theorem 2.2 and then Proposition 4.6 shows that this value is sharp. In keeping with the spirit of the previous emphasis in the study of numerical monoids, the use of the Frobenius number is critical to several of our arguments.

2. Proof for the General Case

Given any numerical monoid M, for any $y \in \mathbb{N}$, we define

$$W_M(y) = \{x \in M \mid x \text{ has a factorization of length } y\}.$$

A closed form for $W_M(y)$ when M is a numerical monoid generated by an arithmetic sequence can be found in [1, Lemma 2.4]. Let $S = \langle r_1, \ldots, r_t \rangle$ and $M_n = \langle n, n+r_1, n+r_2, \ldots, n+r_t \rangle$. We observe that $x \in W_{M_n}(y)$ if and only if $x = yn + d$ for some $d \in S$ with $\ell_S(d) \leq y$. To see this, if $x \in W_{M_n}(y)$, we can write

$$x = a_0 n + a_1(n+r_1) + \cdots + a_t(n+r_t) = yn + \sum_{i=1}^{t} a_i r_i = yn + d,$$

so $\ell_S(d) \leq \sum_{i=1}^t a_i \leq y$. Conversely, if $x = yn + d$ where $\ell_S(d) \leq y$, then $d = \sum_{i=1}^t a_i r_i$, where $\sum_{i=1}^t a_i \leq y$. Letting $a_0 = y - \sum_{i=1}^t a_i$, we have

$$x = a_0 n + a_1(n + r_1) + \cdots + a_t(n + r_t).$$

Since $\sum_{i=0}^t a_i = y$, we have $x \in W_{M_n}(y)$.

We begin our work with a brief lemma.

Lemma 2.1. *Let $S = \langle r_1, \ldots, r_t\rangle$ be primitive. If $n \geq r_t(r_t - 1)(t - 1)$, and $x \geq n$, then $\ell_S(x) \leq \ell_S(x + n)$.*

Proof. Suppose $n \geq r_t(r_t - 1)(t - 1)$ and $x \geq n$. Note that by [6, Proposition 2.9.4], $F(S) \leq (r_1 - 1)(r_2 + \ldots + r_t) - r_1 < r_t(r_t - 1)(t - 1)$. Since $n > F(S)$, we have $n \in S$ and $x \in S$. Then $x = \sum_{i=1}^t a_i r_i$, with $a_i \in \mathbb{N}_0$ and $\sum a_i$ minimal. Note that $a_t \leq \frac{x}{r_t}$, and for all $1 \leq i \leq (t - 1)$ we have $a_i < r_t$; otherwise, we could make a trade to obtain a shorter factorization. Thus

$$\ell_S(x) \leq \frac{x}{r_t} + (r_t - 1)(t - 1) = \frac{x + r_t(r_t - 1)(t - 1)}{r_t} \leq \frac{x + n}{r_t} \leq \ell_S(x + n). \qquad \square$$

We proceed to the main result of this section.

Theorem 2.2. *Let $M_n = \langle n, n + r_1, \ldots, n + r_t\rangle$, where $\gcd(r_1, \ldots, r_t) = z$ and $S = \langle r_1, \ldots, r_t\rangle$. Then there exists $N \in \mathbb{N}$ such that for all $n > N$, $\Delta(M_n) = \left\{\frac{z}{\gcd(n,z)}\right\}$. Specifically, the statement is true for $N = r_t(r_t - 1)(t - 1) - 1$.*

Proof. We begin by proving the result when S is primitive. If S is primitive, then M_n is primitive, and by Proposition 1.1 (1), $1 \in \Delta(M_n)$. Assume $n > r_t(r_t - 1)(t - 1) - 1$. Let $y_1, y_2 \in \mathbb{N}$ with $y_2 - y_1 = c \geq 2$. Suppose $m \in M_n$, with $m \in \mathcal{W}(y_1) \cap \mathcal{W}(y_2)$. It is sufficient to show that $m \in \mathcal{W}(y_1 + 1)$.

Since $m \in \mathcal{W}(y_1)$, we have $m = y_1 n + d_1$, for some $d_1 \in S$ with $\ell_S(d_1) \leq y_1$. Similarly, since $m \in \mathcal{W}(y_2)$, we have $m = y_2 n + d_2$, for some $d_2 \in S$ with $\ell_S(d_2) \leq y_2$. Observe that

$$m = y_1 n + d_1 = (y_1 + 1)n + d_1 - n,$$

so if $d_1 - n \in S$ and $\ell_S(d_1 - n) \leq y_1 + 1$, then $m \in \mathcal{W}(y_1 + 1)$. Since $y_1 n + d_1 = y_2 n + d_2$, as $y_2 - y_1 = c$ it easily follows that $d_2 = d_1 - cn$, so $d_1 - cn \in S$. Since $n \in S$, it trivially follows that $d_1 - n \in S$.

Now since $d_1 - cn \in S$, $d_1 \geq cn \geq 2n$, and thus $d_1 - n \geq n$. By Lemma 2.1, $\ell_S(d_1 - n) \leq \ell_S(d_1) \leq y_1$. Therefore $\ell_S(d_1 - n) \leq y_1 + 1$. Hence, if m has a non-maximal factorization of length y_1, it has a factorization of length $y_1 + 1$. It follows that $\Delta(M_n) = \{1\}$, completing the argument for $z = 1$.

So suppose $z > 1$. Let $S' = \left\langle \frac{r_1}{z}, \ldots, \frac{r_t}{z}\right\rangle$. Assume $n > r_t(r_t - 1)(t - 1) - 1$. We will examine three cases.

Case 1: Suppose $\gcd(n, z) = 1$. Then M_n is primitive and $z \in \Delta(M_n)$ by Proposition 1.1 (1). Let $y_1, y_2 \in \mathbb{N}$ with $y_2 - y_1 = cz \geq 2z$. Suppose further that $m \in M_n$, with $m \in \mathcal{W}(y_1) \cap \mathcal{W}(y_2)$. It is sufficient to show that $m \in \mathcal{W}(y_1 + z)$.

Since $m \in \mathcal{W}(y_1)$, we have $m = y_1 n + d_1$, for some $d_1 \in S$ with $\ell_S(d_1) \leq y_1$. Similarly, since $m \in \mathcal{W}(y_2)$, we have $m = y_2 n + d_2$, for some $d_2 \in S$ with $\ell_S(d_2) \leq y_2$. Observe that

$$m = y_1 n + d_1 = (y_1 + z)n + d_1 - zn,$$

so if $d_1 - zn \in S$ and $\ell_S(d_1 - zn) \leq y_1 + z$, then $m \in \mathcal{W}(y_1 + z)$. Since $y_1 n + d_1 = y_2 n + d_2$, as $y_2 - y_1 = cz$ it easily follows that $d_2 = d_1 - czn$, so $d_1 - czn \in S$. By methods similar to those in the proof of Lemma 2.1, $F(S') < n$, implying $zn \in S$. It trivially follows that $d_1 - zn \in S$.

Now since $d_1 - czn \in S$, $d_1 \geq czn \geq 2zn$, and thus $d_1 - zn \geq zn$. By Lemma 2.1, $\ell_S(d_1 - zn) \leq \ell_S(d_1) \leq y_1$. Therefore $\ell_S(d_1 - zn) \leq y_1 + z$. Hence, if m has a non-maximal factorization of length y_1, it has a factorization of length $y_1 + z$. It follows that $\Delta(M_n) = \{z\}$.

Case 2: Suppose $\gcd(n, z) = z$. In this case, M_n is not primitive, but is isomorphic to the primitive monoid $M'_n = \left\langle \frac{n}{z}, \frac{n+r_1}{z}, \ldots, \frac{n+r_t}{z} \right\rangle$. Since $n > \frac{r_t}{z}\left(\frac{r_t}{z} - 1\right)(t-1) - 1$, it follows from our previous argument that $\Delta(M'_n) = \{1\}$, which implies that $\Delta(M_n) = \{1\}$.

Case 3: Suppose $\gcd(n, z) \notin \{1, z\}$. In this case, M_n is not primitive, but is isomorphic to the primitive monoid $M'_n = \left\langle \frac{n}{\gcd(n,z)}, \frac{n+r_1}{\gcd(n,z)}, \ldots, \frac{n+r_t}{\gcd(n,z)} \right\rangle$. Since $n \geq \frac{r_t}{\gcd(n,z)}\left(\frac{r_t}{\gcd(n,z)} - 1\right)(t-1) - 1$, it follows from Case 1 that $\Delta(M'_n) = \left\{\frac{z}{\gcd(n,z)}\right\}$, which completes the argument. □

The next corollary now follows immediately.

Corollary 2.3. *Let S and M_n be as above with S primitive. If $n > r_t(r_t - 1)(t-1) - 1$, then $\Delta(M_n) = \{1\}$.*

3. An Improved Upper Bound on $\Delta(M)$ in the Three Generator Case

Our aim in this section is to show that the maximum of the delta set of a primitive three-generated numerical monoid can be calculated from the delta sets of only two of its elements; specifically, a multiple of the smallest generator and a multiple of the largest generator. Theorem 3.1 below improves [3, Proposition 4.3 (2)], which was instrumental in proving the main results of [3, Section 4]. Throughout this section, let $S = \langle n_1, n_2, n_3 \rangle$. We will assume that S is primitive and minimally generated and that $n_1 < n_2 < n_3$.

We will first require some notation and terminology. Suppose that

$$m = x_1n_1 + x_2n_2 + x_3n_3 = y_1n_1 + y_2n_2 + y_3n_3 \tag{1}$$

are factorizations of $m \in S$ of different lengths. Let $v = (x_1, x_2, x_3, y_1, y_2, y_3)$ and set $\delta(v) = x_1 + x_2 + x_3 - (y_1 + y_2 + y_3)$. We may suppose (after flipping the coordinates if necessary) that $x_i \geq y_i$ for exactly one i. After canceling like factors, the vector v reduces to a new vector v' of one of the following three forms:

$$\begin{array}{rrcl} \text{(i)} & v' & = & (x'_1, 0, 0, 0, y'_2, y'_3), \\ \text{(ii)} & v' & = & (0, x'_2, 0, y'_1, 0, y'_3), \\ \text{(iii)} & v' & = & (0, 0, x'_3, y'_1, y'_2, 0). \end{array}$$

In any of these cases, we can write $\delta(v) = \delta(v') = x'_i - (y'_j + y'_k)$, for pairwise distinct i, j and k. If $i = 1$ then $x'_1n_1 = y'_2n_2 + y'_3n_3$, which implies $x'_1 > y'_2 + y'_3$ and $\delta(v) > 0$. If $i = 3$, then $x'_3n_3 = y'_1n_1 + y'_2n_2$, which implies $x'_3 < y'_1 + y'_2$ and $\delta(v) < 0$.

Now, let k_1 be the minimal positive integer such that $k_1n_1 \in \langle n_2, n_3 \rangle$. We have $k_1n_1 = a_2n_2 + a_3n_3$ for some positive integers a_2, a_3. Assume that a_2 and a_3 are chosen so their sum is maximal. Similarly, let k_3 be the minimal positive integer such that $k_3n_3 \in \langle n_2, n_3 \rangle$. We have $k_3n_3 = c_1n_1 + c_2n_2$ for some positive integers c_2, c_3. Assume that c_2 and c_3 are chosen so their sum is minimal. Let $K_1 = k_1 - (a_2 + a_3)$ and $K_3 = c_1 + c_2 - k_3$, so $K_1, K_3 > 0$. By [3, Proposition 4.3], we have that $K_1, K_3 \in \Delta(S)$. We will show the following.

Theorem 3.1. *For a primitive and minimally generated $S = \langle n_1, n_2, n_3 \rangle$, $\max(\Delta(S)) = \max\{K_1, K_3\}$.*

The proof of Theorem 3.1 will follow from Propositions 3.2, 3.3 and 3.4 in the following manner. Given a nonunique factorization of m in S of the form (1) with associated vector v, we will argue that its difference in length, $|\delta(v)|$, is either less than or equal to $\max\{K_1, K_3\}$ or there is another factorization of m into irreducibles of length strictly between $|x_1 + x_2 + x_3|$ and $|y_1 + y_2 + y_3|$. Notice that it is sufficient to argue this for the vectors of the form v' constructed above. We begin by showing this for vectors of the form (i) and (iii).

Proposition 3.2. *Let S be as in Theorem 3.1.*

1. If $(a,0,0)$ and $(0,b,c)$ are two factorizations of an_1 in S, then either $a-(b+c)\le K_1$ or there exists another factorization of $an_1 = x_1n_1+x_2n_2+x_3n_3$ such that $a > x_1+x_2+x_3 > b+c$.

2. If $(0,0,c)$ and $(a,b,0)$ are two factorizations of cn_3 in S, then either $|c-(a+b)|\le K_3$ or there exists another factorization of $cn_3 = x_1n_1+x_2n_2+x_3n_3$ such that $c < x_1+x_2+x_3 < a+b$.

Proof. We prove 1. since the proof of 2. is similar. By the minimality of k_1 we have $a \ge k_1$. Suppose $a-(b+c) > K_1$. We have $an_1 = (a-k_1)n_1 + a_2n_2 + a_3n_3$. So we have a factorization of an_1 of length a and one of length $a-k_1+a_2+a_3 = a-K_1$. Then we have a factorization of length in between a and $b+c$, completing our proof. □

The proof for vectors of the form (ii) will require two propositions.

Proposition 3.3. *Let $m \in S$ with $m = xn_2 = b_1n_1 + b_3n_3$ and $b_1+b_3-x > 0$. Then either $b_1+b_3-x \le K_1$ or there exists another factorization of $xn_2 = y_1n_1+y_2n_2+y_3n_3$ such that $x < y_1+y_2+y_3 < b_1+b_3$.*

Proof. Suppose $b_1+b_3-x > K_1$. If $b_1 = 0$ then we have $xn_2 = b_3n_3$ and $x < b_3$. Since $n_3 > n_2$ this is a contradiction. Suppose $b_1 \ge k_1$. Then we have $xn_2 = (b_1-k_1)n_1 + a_2n_2 + a_3n_3$. Either we have $x < b_1-k_1+a_2+a_3 < b_1+b_3$, and we have a factorization of intermediate length, or $b_1-k_1+a_2+a_3 \le x < b_1+b_3$ and $b_1+b_3-x \le K_1$.

So we have $0 < b_1 < k_1$. Consider the element

$$(k_1-b_1)n_1 + xn_2 = k_1n_1 + b_3n_3 = a_2n_2 + (a_3+b_3)n_3.$$

We have three factorization lengths: $k_1-b_1+x,\ k_1+b_3,\ a_2+a_3+b_3$. We have two cases. First suppose that $a_2+a_3+b_3 \le x+k_1-b_1$. Since $k_1+b_3-(a_2+a_3+b_3) = K_1$ and $k_1+b_3-(x+k_1-b_1) = b_1+b_3-x$, we have $b_1+b_3-x \le K_1$ which is a contradiction. So $x+k_1-b_1 < a_2+a_3+b_3 < k_1+b_3$. If $a_2 \ge x$ we have $(k_1-b_1)n_1 = (a_2-x)n_2 + (a_3+b_3)n_3$ and $k_1-b_1 < a_2-x+a_3+b_3$. Since $n_1 < n_2 < n_3$ this is a contradiction. So $a_2 < x$ and we have $(k_1-b_1)n_1 + (x-a_2)n_2 = (a_3+b_3)n_3$ with $k_1-b_1+x-a_2 < a_3+b_3$. Since $n_1 < n_2 < n_3$ this is a contradiction. □

We will now prove a very similar statement which involves K_3.

Proposition 3.4. *Let $m \in S$ with $m = xn_2 = b_1n_1 + b_3n_3$ and $b_1+b_3-x < 0$. Then either $x-(b_1+b_3) \le K_3$ or there exists another factorization of $xn_2 = y_1n_1+y_2n_2+y_3n_3$ such that $x > y_1+y_2+y_3 > b_1+b_3$.*

Proof. Suppose that $x-(b_1+b_3) > K_3$. If $b_3 = 0$ we have $b_1n_1 = xn_2$ and $x > b_1$. Since $n_1 < n_2$ this is a contradiction. Now suppose that $b_3 \ge k_3$. Then we have $xn_2 = b_1n_1 + b_3n_3 = (c_1+b_1)n_1 + c_2n_2 + (b_3-k_3)n_3$. We either have $b_1+b_3 < c_1+b_1+c_2+b_3-k_3 < x$, in which case we have a factorization of xn_2 of intermediate length, or $b_1+b_3 < x \le c_1+b_1+c_2+b_3-k_3$, contradicting $x-(b_1+b_3) > K_3$.

Therefore we have $0 < b_3 < k_3$. Consider the element

$$xn_2 + (k_3-b_3)n_3 = b_1n_1 + k_3n_3 = (c_1+b_1)n_1 + c_2n_2.$$

We have three factorization lengths: $x+k_3-b_3, b_1+k_3, c_1+b_1+c_2$. We have two cases. First suppose that $x+k_3-b_3 \le b_1+c_1+c_2$. Since $b_1+c_1+c_2-(b_1+k_3) = K_3$ and $x+k_3-b_3-(b_1+k_3) = x-(b_1+b_3)$, we have $x-(b_1+b_3) \le K_3$ which is a contradiction. So $b_1+k_3 < b_1+c_1+c_2 < x+k_3-b_3$. If $c_2 \ge x$ we have $(b_1+c_1)n_1 + (c_2-x)n_2 = (k_3-b_3)n_3$ and $k_3-b_3 > b_1+c_1+c_2-x$. Since $n_1 < n_2 < n_3$ this is a contradiction. So $c_2 < x$ and we have $(b_1+c_1)n_1 = (x-c_2)n_2 + (k_3-b_3)n_3$ and $b_1+c_1 < x-c_2+k_3-b_3$. Since $n_1 < n_2 < n_3$, this is a contradiction. □

Therefore given any factorization $(0, b, 0, a, 0, c)$ of bn_2 in S, then either $|b-(a+c)| \leq \max\{K_1, K_3\}$ or there is another factorization of bn_2 with length between b and $a + c$. This completes the proof of Theorem 3.1.

4. A Sharp Bound on N in the Three Generator Case

We now focus on the case where $S = \langle n, n + r, n + s\rangle$ and $\gcd(r, s) = 1$ and find the sharp value of the constant N from Theorem 2.2.

Theorem 4.1. *Let $r, s \in \mathbb{N}$, $\gcd(r, s) = 1$, $0 < r < s$. Suppose $M_n = \langle n, n + r, n + s\rangle$ where $n \in \mathbb{N}$. Then $\Delta(M_n) = \{1\}$ for all $n > \max\{rs - r - s, s^2 - rs + r - 3s\}$.*

The proof of this theorem will follow immediately from Theorem 3.1 and Lemmas 4.2, 4.3, 4.4 and 4.5.

Lemma 4.2. *If $s \leq 2r + 1$, then $K_1 = \{1\}$ for M_n when $n > rs - r - s$.*

Proof. Let $S = \langle r, s\rangle$. Suppose that $n = rs - r - s + C$, where $C \geq 1$. We first observe that the Frobenius number $F(S)$ of S is equal to $rs-r-s$, as shown by [8]. Then for any $A \in \mathbb{N}$, $An > F(S)$ and hence there exist $x, y \in \mathbb{N}_0$ such that $An = xr + ys$. For every $A \in \mathbb{N}$, choose x, y such that their sum is minimal; denote these x_A and y_A. Note that $x_A < s$, because if it were not, we could trade s r's for r s's, yielding a smaller factorization length. Clearly,

$$An = x_A r + y_A s \tag{2}$$

is equivalent to

$$(x_A + y_A + A)n = x_A(n + r) + y_A(n + s).$$

Thus $(x_A + y_A + A)n \in \langle n + r, n + s\rangle$.

Claim: $k_1 = x_1 + y_1 + 1$.

By our construction, $k_1 = \min\{x_A + y_A + A \mid A \in \mathbb{N}\}$. For $x_1 + y_1 + 1$ to be minimal, we must show that $x_1 + y_1 + 1 \leq x_A + y_A + A$ for all A. Now when $A = 1$ we have $n = x_1 r + y_1 s \geq x_1 r + y_1 r$, so

$$\frac{n}{r} \geq x_1 + y_1. \tag{3}$$

Also, for any A, $An = x_A r + y_A s \leq x_A s + y_A s$, so

$$\frac{An}{s} \leq x_A + y_A. \tag{4}$$

Next we consider several different cases.

(I) Suppose $s \leq 2r$, and hence $s \leq Ar$ for all $A \geq 2$. It follows that $\frac{1}{r} \leq \frac{A}{s}$, and thus $\frac{n}{r} \leq \frac{An}{s}$. By (3) and (4), $x_1 + y_1 \leq x_A + y_A$. Adding 1 to each side, we have $x_1 + y_1 + 1 \leq x_A + y_A + 1 \leq x_A + y_A + A$. Thus $k_1 = x_1 + y_1 + 1$ for $s \leq 2r$.

(II) Suppose $s = 2r + 1$. If $A \geq 3$, then $s \leq Ar$, so the argument from the previous case holds. Suppose $A = 2$, and for ease of notation let $x_1 = x$ and $y_1 = y$. We know $n = xr + y(2r + 1)$. Let $r \geq 3$; we will later address cases with $r < 3$.

Assume for the sake of contradiction that $x + y + 1$ is not minimal. That is there exists $c, d \in \mathbb{N}_0$ such that $2n = cr + d(2r + 1)$ with $c + d < x + y$. Then

$$2xr + 2y(2r + 1) = cr + d(2r + 1).$$

If $2x < 2r + 1$, then we are done for we cannot make the factorization smaller; the factorization on the left is of minimal length, but is longer than $x + y$. If $2x \geq 2r + 1$, we trade $2r + 1$ r's for $2r$ $(2r + 1)$'s to get

$$(2x - 2r - 1)r + (2y + r)(2r + 1) = cr + ds.$$

We can be certain that $c+d = 2x-2r-1+2y+r$ because $x < 2r+1$, so $2x < 4r+2$ and another trade to a smaller factorization cannot be made. Knowing this, $2x-2r-1+2y+r < x+y$, which simplifies to $x+y < r+1$. Now, since $c+d$ is the minimum factorization length of $2n$ in S, by [3, Proposition 3.7] we have

$$c+d = \ell_S(2n) \geq \left\lceil \frac{2n}{2r+1} \right\rceil \geq \left\lceil \frac{2r(2r+1)-2(2r+1)-2r}{2r+1} \right\rceil = \left\lceil 2r-2-\frac{2r}{2r+1} \right\rceil = 2r-2.$$

Thus, $2r-2 \leq c+d < x+y < r+1$ implies that $r < 3$, a contradiction. Thus $k_1 = x_1+y_1+1$ for $s = 2r+1$ and $r \geq 3$.

(III) It remains to prove the claim for $s = 2r+1$ when $r = 1$ or 2. Suppose $r = 1$ and hence $s = 3$ and $n = -1+C$. Observe that if $C = 1$, $n = 0$; since we are only concerned with positive values of n, it is sufficient to prove the claim for $C \geq 2$. If $A \geq 3$, then it follows that $\frac{A(-1+C)}{3} \geq -1+C$. Combining this with our previous results (3) and (4), we have

$$x_1+y_1 \leq -1+C \leq \frac{A(-1+C)}{3} \leq x_A+y_A.$$

Now suppose $A = 2$. Since $x_2 \leq 2$ and since $y_2 = \dfrac{2C-2-x_2}{3}$, we have

$$x_2+y_2+1 = \frac{2C+1+2x_2}{3} \geq \frac{2C+1}{3}.$$

Similarly, since $x_1 \leq 2$ and $y_1 = \dfrac{C-1-x_1}{3}$, we have

$$x_1+y_1 = \frac{C-1+2x_1}{3} \leq \frac{C+3}{3}.$$

Finally, since $C \geq 2$, $C+3 \leq 2C+1$ for all C. Thus we have

$$x_1+y_1 \leq \frac{C+3}{3} \leq \frac{2C+1}{3} \leq x_2+y_2+1.$$

Now suppose $r = 2$ and hence $s = 5$ and $n = 3+C$. If $A \geq 3$, then $2A \geq 5$, so $\frac{A(3+C)}{5} \geq \frac{3+C}{2}$. Again combining this with our previous results (3) and (4), we have

$$x_1+y_1 \leq \frac{3+C}{2} \leq \frac{A(3+C)}{5} \leq x_A+y_A.$$

If $A = 2$, we know that $2x_2+5y_2 = 2(3+C) = 2n$. Since $2n$ is even, y_2 must obviously be even. If x_2 is also even, $x_1+y_1 \leq x_2+y_2$ because $x_1 = \frac{x_2}{2}$ and $y_1 = \frac{y_2}{2}$.

However, if x_2 is odd, then $x_2 = 1$ or 3 (since $x_2 < 5$). As $C \geq 2$ and $x_2 \geq 1$, $2(3+C) > 10$, implying $y_2 \geq 2$. Suppose $x_2 = 1$, giving us a factorization length of y_2+1. Since $y_2 \geq 2$, we can trade two 5's for five 2's, yielding a new factorization $2n = 2x_2'+5y_2'$. So $x_2' = x_2+5 = 6$ and $y_2' = y_2-2$ and this factorization has length $x_2'+y_2' = x_2+y_2+3 = y_2+4$. Now x_2' and y_2' are even, so we can divide by 2 to obtain a factorization of n:

$$n = 2\frac{x_2'}{2}+5\frac{y_2'}{2} = 2\cdot 3+5\frac{(y_2-2)}{2}.$$

Since x_1+y_1 is the minimum factorization length of n, we have $x_1+y_1 \leq 3+\frac{y_2-2}{2} = \frac{y_2}{2}+2$. So it is sufficient to show that $\frac{y_2}{2}+2 \leq y_2+x_2 = y_2+1$. Knowing $y_2 \geq 2$, we have $y_2+4 \leq 2y_2+2$, and hence $\frac{y_2}{2}+2 \leq y_2+1$. A similar argument with $x_2 = 3$ leads to the same conclusion assuming $y_2 \geq 4$. If $x_2 = 3$ and $y_2 = 2, 2n = 16 = 2\cdot 5+3\cdot 2$, and $n = 8 = 0\cdot 5+4\cdot 2$, finishing the argument when x_2 is odd. This completes not only the proof of **(III)**, but also the proof that $k_1 = x_1+y_1+1$.

If $v = (k_1, 0, 0, , 0, x_1, y_1)$ it now clearly follows that $\delta(v) = 1$ and hence $K_1 = 1$, completing the proof. □

Lemma 4.3. *If $s > 2r+1$, then $K_1 = \{1\}$ for M_n when $n > s^2-rs+r-3s$.*

Proof. Let $S = \langle r, s\rangle$ and suppose that $n \geq s^2 - rs + r - 3s + 1$. We proceed using the notation and terminology of the proof of Lemma 4.2. We again claim that $k_1 = x_1 + y_1 + 1$ and will show this by arguing that $x_1 + y_1 + 1 \leq x_A + y_A + A$ for all A. Solving for y_A in (2), we have $y_A = \dfrac{An - x_A r}{s}$, and so

$$x_A + y_A + A = \frac{x_A s + An - x_A r + As}{s} = \frac{A(n+s) + x_A(s-r)}{s}.$$

Clearly, we now only need to show that $n + s + x_1(s-r) \leq A(n+s) + x_A(s-r)$, or equivalently $(A-1)(n+s) \geq (x_1 - x_A)(s-r)$. If $x_A \geq x_1$, then $x_1 - x_A \leq 0$, so clearly $(A-1)(n+s) \geq (x_1 - x_A)(s-r)$. If $x_1 > x_A$, then $x_1 - x_A \leq s-1$, since $x_1 < s$. Thus, $(x_1 - x_A)(s-r) \leq (s-1)(s-r)$. Now if $A \geq 3$, then

$$(A-1)(n+s) \geq 2(n+s) \geq 2s^2 - 2rs + 2r - 4s + 2 \geq (s-1)(s-r) \geq (x_1 - x_A)(s-r),$$

and we're done.

Now suppose $A = 2$ and $s - r \geq 5$. Suppose for the sake of contradiction that $x_2 + y_2 < x_1 + y_1$. Then $2x_1 r + 2y_1 s = x_2 r + y_2 s$. If $2x_1 < s$, then we are done, since a trade to a smaller factorization cannot be made, and $2x_1 + 2y_1 > x_1 + y_1$. Suppose $2x_1 \geq s$; then we can make a trade to obtain $(2x_1 - s)r + (2y_1 + r)s = x_2 r + y_2 s$. Since $x_1 < s$, $2x_1 < 2s$, we cannot make another trade. Considering the factorization lengths, we have $2x_1 - s + 2y_1 + r < x_1 + y_1$ implies that $x_1 + y_1 < s - r$. Recall from [3, Proposition 3.7] that

$$x_2 + y_2 = \ell_S(2n) \geq \left\lceil \frac{2n}{s} \right\rceil \geq \left\lceil \frac{2s^2 - 2rs + 2r - 6s + 2}{s} \right\rceil = \left\lceil 2s - 2r - 6 + \frac{2r+2}{s} \right\rceil = 2s - 2r - 5.$$

Since by assumption $x_2 + y_2 < x_1 + y_1$, it follows now that $2s - 2r - 5 \leq x_2 + y_2 < x_1 + y_1 < s - r$ which implies that $s - r < 5$, a contradiction. Thus $x_1 + y_1 \leq x_2 + y_2$, implying $x_1 + y_1 + 1 \leq x_2 + y_2 + 2$.

Finally, suppose $A = 2$ and $s - r < 5$. There are two cases:

(A) If $r = 1$ and $s = 4$, then $n = x_1 \cdot 1 + y_1 \cdot 4$ implies that $2n = 2x_1 \cdot 1 + 2y_1 \cdot 4$. If $n \equiv 0$ or $1 \pmod 4$, then $x_1 = 0$ or 1, respectively. This means that $x_2 = 2x_1 < 4$ so we cannot make a trade that yields a smaller factorization. Thus $y_2 = 2y_1$, so $x_1 + y_1 + 1 < x_2 + y_2 + 2$. If $n \equiv 2 \pmod 4$, then $x_1 = 2$, and $2x_1 = 4$. This implies that we can make a trade:

$$2n = 2x_1 \cdot 1 + 2y_1 \cdot 4 = (2x_1 - 4) \cdot 1 + (2y_1 + 1) \cdot 4.$$

Since $2x_1 - 4 = 0$ we cannot trade further, so $y_2 = 2y_1 + 1$, and the length of the factorization is $2y_1 + 1$. Thus $x_2 + y_2 + 2 = 2y_1 + 3 \geq 2 + y_1 + 1 = x_1 + y_1 + 1$. If $n \equiv 3 \pmod 4$, a similar argument leads to $x_2 = 2$ and $y_2 = 2y_1 + 2$. Thus $x_2 + y_2 + 2 = 2 + 2y_1 + 2 + 2 > 3 + y_1 + 1 = x_1 + y_1 + 1$.

(B) If $r = 1$ and $s = 5$, then by considering cases of $n \pmod 5$ in a similar fashion, it easily follows that no trade can be made for $n \equiv 0, 1, 2 \pmod 5$ and the trades in cases of $n \equiv 3, 4 \pmod 5$ do not make the factorization small enough to contradict $x_1 + y_1 + 1 \leq x_2 + y_2 + 2$. Hence, for all $A \geq 1$, and for $s > 2r + 1$, we have $x_1 + y_1 + 1 \leq x_A + y_A + A$, and the claim is proven.

As at the end of the proof of Lemma 4.2, it now easily follows that $K_1 = 1$, completing the proof. $\square$

Lemma 4.4. *If $s \leq 2r + 1$, then $K_3 = \{1\}$ for M_n when $n > rs - r - s$.*

Proof. Let $S = \langle s, s - r\rangle$. Suppose that $n = rs - r - s + C$, where $C \geq 1$. Observe that in this case, $F(S) = s^2 - rs + r - 2s$. Then for any $A \in \mathbb{N}$, $A(n+s) \geq n + s \geq rs - r + 1 \geq F(S)$ and hence there exist $x, y \in \mathbb{N}_0$ such that $A(n+s) = xs + y(s-r)$. For every $A \in \mathbb{N}$, choose x, y such that their sum is minimal; denote these x_A and y_A. Similar to our previous arguments, we have $y_A < s$. Now $A(n+s) = x_A s + y_A(s-r)$ is equivalent to

$$(x_A + y_A - A)(n+s) = x_A n + y_A(n+r).$$

Thus $(x_A + y_A - A)(n+s) \in \langle n, n+r\rangle$.

Claim: $k_3 = x_1 + y_1 - 1$.

By our construction, $k_3 = \min\{x_A + y_A - A \mid A \in \mathbb{N}\}$. For $x_1 + y_1 - 1$ to be minimal, we must show that $x_1 + y_1 - 1 \le x_A + y_A - A$ for all A. Since we easily have

$$x_A = \frac{A(n+s) - y_A(s-r)}{s}$$

for all A, it follows that $x_A + y_A - A = \dfrac{An + y_A r}{s}$. So we want to show $n + y_1 r \le An + y_A r$, or equivalently, $(y_1 - y_A)r \le (A-1)n$, for all A.

Let $A \ge 3$. First suppose $r \ge 2$. Then

$$rs - r - 2s + 2 = (s-1)(r-2) \ge 0,$$

so adding $rs - r$ to both sides of this inequality, we obtain

$$rs - r \le 2rs - 2r - 2s + 2 \le 2n \le (A-1)n$$

since $A \ge 3$. Since $0 \le y_A \le s - 1$ for all y_A,

$$(y_1 - y_A)r \le (s-1)r = rs - r.$$

Thus if $r \ge 2$, $(y_1 - y_A)r \le rs - r \le (A-1)n$ for all $A \ge 3$. If $r = 1$, then by our initial assumptions we must have $s = 3$, so $y_A \le 2$ for all A. Since we are assuming $A \ge 3$ and $n \ge 1$, we have

$$x_1 + y_1 - 1 = \frac{n + y_1}{3} \le \frac{n+2}{3} \le \frac{An + y_A}{3} = x_A + y_A - A,$$

completing our argument for $A \ge 3$.

It remains to prove $x_1 + y_1 - 1 \le x_2 + y_2 - 2$ (equivalently, $(y_1 - y_2)r \le n$), which we prove in two cases.

(I) Let $r \ge 3$. Note that $y_2 \equiv 2y_1 \pmod{s}$. If $y_1 < \frac{s}{2}$ then $y_2 = 2y_1$, in which case $(y_1 - y_2)r = -y_1 r \le 0 < n$, so we are done. So assume $y_1 \ge \frac{s}{2}$, in which case $y_2 = 2y_1 - s$. We have

$$(y_1 - y_2)r = (s - y_1)r \le \frac{rs}{2}.$$

If $s \ge 6$, we have:

$$n \ge rs - r - s + 1 \ge \frac{rs}{2} + (2r - s) + 1 \ge \frac{rs}{2} + (2r - 2r - 1) + 1 = \frac{rs}{2},$$

so $(y_1 - y_2)r \le n$. If $s < 6$, then given our initial assumptions and still keeping $r \ge 3$, there are only three possibilities for r and s: (1) $r = 3$ and $s = 4$; (2) $r = 3$ and $s = 5$; or (3) $r = 4$ and $s = 5$. In each of these cases, it is easily verifiable that if $n \ge rs - r - s + 1$, then $n \ge \frac{rs}{2}$; so in each of these cases, $(y_1 - y_2)r \le n$, completing the proof for $r \ge 3$.

(II) Let $r < 3$. Given our initial assumptions, there are only three possible combinations for r and s: $r = 1$ and $s = 3$, $r = 2$ and $s = 3$, and $r = 2$ and $s = 5$.

- Let $r = 1$ and $s = 3$. Since $x_A + y_A - A = \dfrac{An + y_A r}{s}$ for all A, we have $x_1 + y_1 - 1 \le \frac{n+2}{3}$ (as $y_A \le 2$ for all A) and $x_2 + y_2 - 2 \ge \frac{2n}{3}$. If $n \ge 2$, then $\frac{n+2}{3} \le \frac{2n}{3}$ and we are done. If $n = 1$ then $n + s = 4$, and simple calculations yield $x_1 = 0$, $y_1 = 2$, $x_2 = 2$, and $y_2 = 1$, which satisfy $x_1 + y_1 - 1 = x_2 + y_2 - 2$. Thus for all $n \ge 1$, $x_1 + y_1 - 1 \le x_2 + y_2 - 2$.
- Let $r = 2$ and $s = 3$. Note that $n \ge rs - r - s + 1 = 2$. Since $x_1 + y_1 - 1 \le \frac{n+4}{3}$ (as $r = 2$ and $y_A \le 2$ for all A) and $x_2 + y_2 - 2 \ge \frac{2n}{3}$, if $n \ge 4$ we immediately have $x_1 + y_1 - 1 \le x_2 + y_2 - 2$. If $n = 3$, $n + s = 6$, and we have $x_1 = 2$, $y_1 = 0$, $x_2 = 4$, and $y_2 = 0$, which satisfy $x_1 + y_1 - 1 < x_2 + y_2 - 2$. If $n = 2$, $n + s = 5$, and we have $x_1 = 1$, $y_1 = 2$, $x_2 = 2$, and $y_2 = 2$, which satisfy $x_1 + y_1 - 1 = x_2 + y_2 - 2$, completing the proof for all $n \ge 2$.
- Let $r = 2$ and $s = 5$. Note that $n \ge rs - r - s + 1 = 4$. Since $x_1 + y_1 - 1 \le \frac{n+8}{5}$ (as $r = 2$ and $y_A \le 4$ for all A) and $x_2 + y_2 - 2 \ge \frac{2n}{5}$, if $n \ge 8$ we immediately have $x_1 + y_1 - 1 \le x_2 + y_2 - 2$.

If $n = 7$, $n + s = 12$, and we have $x_1 = 0$, $y_1 = 4$, $x_2 = 4$, and $y_2 = 2$. If $n = 6$, $n + s = 11$, and we have $x_1 = 1$, $y_1 = 3$, $x_2 = 4$, and $y_2 = 1$. If $n = 5$, $n + s = 10$, and we have $x_1 = 2$, $y_1 = 0$, $x_2 = 4$, and $y_2 = 0$. Finally, if $n = 4$, $n + s = 9$, and we have $x_1 = 2$, $y_1 = 2$, $x_2 = 2$, and $y_2 = 4$. Simple calculations verify that all of these satisfy $x_1 + y_1 - 1 \leq x_2 + y_2 - 2$. Thus for all $n \geq 4$, $x_1 + y_1 - 1 \leq x_2 + y_2 - 2$.

Thus the claim is proved. Finally, if $v = (0, 0, k_3, x_1, y_1, 0)$, then $|\delta(v)| = 1$ and thus $K_3 = 1$, completing the proof. □

Lemma 4.5. *If $s > 2r + 1$, then $K_3 = \{1\}$ for M_n when $n > s^2 - rs + r - 3s$.*

Proof. Let $S = \langle s, s - r\rangle$ and suppose that $n = s^2 - rs + r - 3s + C$, where $C \geq 1$. We proceed using the notation and terminology of Lemma 4.4 and again claim that $k_3 = x_1 + y_1 - 1$. We will again argue that $x_1 + y_1 - 1 \leq x_A + y_A - A$ for all A.

First we consider the case where $s > 2r + 2$. Since $n = s^2 - rs + r - 3s + C$ and $C \geq 1$, and since $r < \frac{s-2}{2}$, we have

$$n \geq s^2 - rs + r - 3s + 1 > 2r(s - 1) - rs + r - 1 = rs - r - 1,$$

and so $n \geq rs - r$. Adding n to both sides of this inequality we obtain

$$\frac{n - r}{s} + r \leq \frac{2n}{s}.$$

Since $n + s = x_1 s + y_1(s - r)$, we have $x_1 = \dfrac{n + s - y_1(s - r)}{s}$. Combining this with our previous observation that $y_1 \leq s - 1$, we have

$$x_1 + y_1 - 1 = \frac{n + s - y_1(s - r) + y_1 s - s}{s} = \frac{n + y_1 r}{s} \leq \frac{n - r}{s} + r.$$

Similarly, for $A \geq 2$ we have

$$x_A + y_A - A = \frac{An + y_A r}{s} \geq \frac{An}{s} \geq \frac{2n}{s},$$

which yields

$$x_1 + y_1 - 1 \leq \frac{n - r}{s} + r \leq \frac{2n}{s} \leq x_A + y_A - A$$

for all A. Thus if $s > 2r + 2$, $k_3 = x_1 + y_1 - 1$.

Now we address the case where $s = 2r+2$. First suppose $y_2 \neq 0$. Then $x_A+y_A-A = \dfrac{An + y_A r}{s} \geq \dfrac{2n + 1}{s}$ for all $A \geq 2$. We now have:

$$n \geq s^2 - rs + r - 3s + 1 = (s - 2)(s - 1) - rs + r - 1 = 2r(s - 1) - rs + r - 1 = rs - r - 1,$$

so $n - r + rs \leq 2n + 1$, or equivalently, $\frac{n-r}{s} + r \leq \frac{2n+1}{s}$. A similar argument to that of our first case above yields $x_1 + y_1 - 1 \leq x_A + y_A - A$ for all A. Now suppose $y_2 = 0$. Then $2(n + s) = x_2 s$ is a multiple of s, so we must have $y_1 = 0$ or $\frac{s}{2}$. If $y_1 = 0$, $x_1 + y_1 - 1 = \frac{n}{s} \leq \frac{2n}{s} \leq x_A + y_A - A$ for all $A \geq 2$. If $y_1 = \frac{s}{2}$, then after solving for x_1 we have $x_1 + y_1 - 1 = \frac{n}{s} + \frac{r}{2}$. Since $s = 2r + 2$ and $C \geq 2 - r^2$ (as $C \geq 1$ and $r \geq 1$), we have:

$$n = s^2 - rs + r - 3s + C = 2r^2 + r - 2 + C \geq 2r^2 + r - 2 + 2 - r^2 = r^2 + r = r(r + 1) = \frac{rs}{2}.$$

Therefore, $n \geq \frac{rs}{2}$. That is, $\frac{r}{2} \leq \frac{n}{s}$ and so $\frac{n}{s} + \frac{r}{2} \leq \frac{2n}{s}$. The result now immediately follows. This allows us to conclude that if $s = 2r + 2$, $k_3 = x_1 + y_1 - 1$.

Thus the claim is proved. We have shown that $k_1 = x_1 + y_1 - 1$, and that $v = (0, 0, k_3, x_1, y_1, 0)$ is the minimum trade. Now $|\delta(v)| = 1$, and since $K_3 > 0$, we have $K_3 = 1$, finishing the proof. □

This completes the proof of Theorem 4.1. To show the optimality of these bounds, we have the following proposition:

Proposition 4.6. *Suppose* $r, s \in \mathbb{N}$, $\gcd(r,s) = 1$, *and* $0 < r < s$. *Let* $n = \max\{rs - r - s, s^2 - rs + r - 3s\}$, *and* $M_n = \langle n, n+r, n+s\rangle$. *Then* $\Delta(M_n) \neq \{1\}$.

Proof. First suppose $n = rs - r - s$. Let $m = n(n+r)(n+s)$. Clearly the maximum factorization length of m in M_n is $(n+r)(n+s)$, and since we trivially also have factorizations of lengths $n(n+r)$ and $n(n+s)$, we know the delta set of m is nonempty. So consider any non-maximal factorization of m, say $m = ((n+r)(n+s) - a)n + x(n+r) + y(n+s)$. Then we have $an = x(n+r) + y(n+s)$ which implies $(a - (x+y))n = xr + ys$. Since $n = rs - r - s$, $n \notin \langle r, s\rangle$, so $a - (x+y) \neq 1$; as $a - (x+y) > 0$, we have $a - (x+y) \geq 2$. Thus, the difference between the length of our non-maximal factorization and our maximal factorization is simply $a - (x+y)$, so there is no factorization of m of length $(n+r)(n+s) - 1$. Therefore there is an integer $t > 1$ with $t \in \Delta(n(n+r)(n+s)) \subset \Delta(M_n)$.

Now suppose $n = s^2 - rs + r - 3s$. Again let $m = n(n+r)(n+s)$. Clearly the minimum factorization length of m in M_n is $n(n+r)$, and since we trivially also have factorizations of lengths $n(n+s)$ and $(n+r)(n+s)$, we know the delta set of m is nonempty. So consider any non-minimal factorization of m, say $m = xn + y(n+r) + (n(n+r) - a)(n+s)$. Then we have $a(n+s) = xn + y(n+r)$. That is, $(x+y-a)(n+s) = xs + y(s-r)$. Since $n+s = s^2 - rs + r - 2s$, $n+s \notin \langle s-r, s\rangle$ and $(x+y) - a > 0$, we have $(x+y) - a \geq 2$. The difference between the length of our non-minimal factorization and our minimal factorization is simply $(x+y) - a$, so there is no factorization of m of length $n(n+r) + 1$. Therefore there is an integer $t > 1$ with $t \in \Delta(n(n+r)(n+s)) \subset \Delta(M_n)$. □

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