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SHINTANI FUNCTIONS AND AUTOMORPHIC L-FUNCTIONS FOR $GL(n)$

Dedicated to Professor Hideo Shimizu on his sixtieth birthday

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Abstract. We introduce certain special functions ("Shintani functions") on $GL(n)$ over a non-Archimedean local field. We prove the uniqueness, existence and partial explicit formula of Shintani functions. We give several applications of these local results to the theory of automorphic L-functions for $GL(n)$.

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Introduction. In the study of automorphic L-functions, various special functions on reductive groups have been playing fundamental roles. Among others, the spherical function and the Whittaker function have been studied by many mathematicians. The aim of this paper is to introduce and study a new kind of special functions for $GL(n)$ that we call Shintani functions. We investigate their local properties, which is similar to those of the spherical and Whittaker functions, and give several applications to the theory of automorphic L-functions for $GL(n)$.

Shintani functions were first introduced by Shintani for the symplectic groups [Shin 2] in order to study the automorphic L-functions of Siegel (or Jacobi) modular forms. Several properties conjectured by him were studied in [M-S 1] and [Mu]. The notion of Shintani functions was later generalized to the case of orthogonal and unitary

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groups and used to obtain a new integral expression of automorphic L-functions for classical groups (cf. [M-S 2]).

To explain our results more precisely, let $G_o = GL(n-1)$ and $G = GL(n)$ be the general linear groups over an algebraic number field E and embed G_o into G via

$$g_o \mapsto \begin{bmatrix} g_o & 0 \\ 0 & 1 \end{bmatrix}.$$

Let f and F be automorphic forms on G_o and G, respectively. If f is square integrable over $G_{o,E}\backslash G^1_{o,A}$ (cf. §8.6) and F is cuspidal, then the integral

$$W_{f,F}(g) = \int_{G_{o,E}\backslash G_{o,A}} f(x)F(xg)dx \qquad (g \in G_A) \tag{0.1}$$

is absolutely convergent. We call $W_{f,F}$ the *global Shintani function* attached to (f, F). Let $\mathscr{H}_{G_o,v} = \mathscr{H}(G_o(E_v), G_o(\mathfrak{o}_v))$ and $\mathscr{H}_{G,v} = \mathscr{H}(G(E_v), G(\mathfrak{o}_v))$ be the Hecke algebras of G_o and G at a finite prime v of E, where E_v denotes the completion of E at v and $\mathfrak{o}_v$ its integer ring. Assume that f and F are common eigenfunctions under the action of the Hecke algebras $\mathscr{H}_{G_o,v}$ and $\mathscr{H}_{G,v}$ for every v. Let ξ_v (resp. Ξ_v) be the Satake parameter at v of f (resp. of F) and denote by $\xi_v^\wedge$ (resp. by $\Xi_v^\wedge$) the corresponding $\boldsymbol{C}$-algebra homomorphism of $\mathscr{H}_{G_o,v}$ (resp. of $\mathscr{H}_{G,v}$) to $\boldsymbol{C}$ (cf. §1). By definition, for $\varphi_v \in \mathscr{H}_{G_o,v}$ and $\Phi_v \in \mathscr{H}_{G,v}$ we have $f * \varphi_v = \xi_v^\wedge(\varphi_v)f$ and $F * \Phi_v = \Xi_v^\wedge(\Phi_v)F$. Then the restriction W of $W_{f,F}$ to $G_v = G_{E_v}$ is a common eigenfunction under the action of $\mathscr{H}_{G_o,v}$ on the left and that of $\mathscr{H}_{G,v}$ on the right:

$$\varphi_v * W * \Phi_v = \xi_v^\wedge(\varphi_v)\Xi_v^\wedge(\Phi_v)W, \tag{0.2}$$

where we put

$$(\varphi_v * W * \Phi_v)(g) = \int_{G_{o,v}} dx \int_{G_v} dy\, \varphi_v(x)W(x^{-1}gy)\Phi_v(y) \qquad (g \in G_v).$$

The space $\mathrm{Sh}(\xi_v, \Xi_v)$ of $\boldsymbol{C}$-valued functions W on $G_o(\mathfrak{o}_v)\backslash G(E_v)/G(\mathfrak{o}_v)$ satisfying (0.2) is called the space of *local Shintani functions* attached to (ξ_v, Ξ_v). One of our main results asserts that the dimension of $\mathrm{Sh}(\xi_v, \Xi_v)$ is equal to one. This implies that the global Shintani function defined by the integral (0.1) splits into the product of local Shintani functions. Moreover we present several integral formulas for local Shintani functions, which yield new integral expressions of automorphic L-functions for $GL(n)$.

We now explain a relation between a recent work of Prasad [Pr] and ours. Let (π_o, V_o) and (π, V) be admissible representations of $G_o = GL(n-1, E)$ and $G = GL(n, E)$, respectively, where E is a non-Archimedean local field and V_o (resp. V) is the representation space of π_o (resp. of π). Assume that there exists a non-zero G_o-equivariant linear mapping T of V to $V_o^\sim$, where $V_o^\sim$ is the representation space of the contragredient $\pi_o^\sim$ of π_o. For example, the assumption holds if both of π_o and π are irreducible and generic (cf. [Pr, Theorem 3]). Suppose that π_o and π are of class 1. Let v_o (resp. v) be a $G_o(\mathfrak{o}_E)$-fixed (resp. $G(\mathfrak{o}_E)$-fixed) vector in V_o (resp. in V) and $\langle\ ,\ \rangle$ the canonical pairing of $V_o^{\sim} \times V_o$. Let ξ (resp. Ξ) be the Satake parameter corresponding to π_o (resp. π). Then the function $W(g)=\langle T(\pi(g)v), v_o\rangle$ on G is a non-zero local Shintani function for (ξ, Ξ) in our sense. This implies that

$$\dim_{\mathbf{C}} \mathrm{Sh}(\xi, \Xi) \geq 1 \tag{0.3}$$

holds at least for a pair (ξ, Ξ) for which both of π_o and π are irreducible and generic. We note that our proof of the fact (0.3) in §4 is different from the above argument and applies for all the pairs (ξ, Ξ), though our consideration is restricted to the case of class 1 representations.

The paper is organized as follows. In Part I, we study the local Shintani functions for $GL(n)$. From §1 to §4, we consider the non-Archimedean case. In §1, after fixing notation, we introduce the notion of local Shintani functions for $GL(n)$ and state the main result of Part I: the uniqueness and existence of local Shintani functions. The object of §2 is to study the structure of the coset space $G_o(\mathfrak{o}_v)\backslash G(E_v)/G(\mathfrak{o}_v)$, which is crucial to the proof of the uniqueness. In §3, following the method of Shintani [Shin 1] and Kato [Ka], we study the system of difference equations satisfied by the values of Shintani functions. This enables us to reduce the proof of the uniqueness theorem (Theorem 3.1) to a certain integral formula proved in §6. In §4, we prove the existence theorem (Theorem 4.10) by giving an integral expression of Shintani functions. The local Shintani functions in the Archimedean case are defined and studied in §5. In this case, the uniqueness and existence theorems (in an appropriate form) have not yet been established. The aim of the next two sections is to show two integral formulas for local Shintani functions. The first one proved in §6 together with the results of §3 establishes the uniqueness theorem. Both formulas are later used to study certain global integrals of Rankin-Selberg type (cf. §9 and §11).

The theme of Part II is a global application of the local results of Part I. In §8, after recalling the notion of automorphic forms on $GL(n)$, we define the global Shintani function $W_{f,F}$ attached to (f, F), where f is an automorphic form on $G_o=GL(n-1)$ with $\int_{G_{o,E}\backslash G^1_{o,A}} |f(x)|^2 dx<\infty$ and F is a cusp form on $G=GL(n)$. We also define a twisted global Shintani function, which is needed in the next section. The first global application of Shintani functions is given in §9. To be more precise, we let P and Q be the standard maximal parabolic subgroups of G of types $(n-1, 1)$ and $(1, n-1)$, respectively. Since Levi subgroups of P and Q are isomorphic to $GL(1)\times G_o$, we can define the (normalized) Eisenstein series $E^*(g; s; f; P)$ (resp. $E^*(g; s; \mathbf{1}; Q)$) attached to f (resp. $\mathbf{1}$) with respect to P (resp. Q) on G_A. The main result (Theorem 9.4) of §9 asserts that the integral

$$\int_{Z_A G_E\backslash G_A} F(g)E^*(g; s_1; f; P)E^*(g; s_2; \mathbf{1}; Q)dg \tag{0.4}$$

is expressed essentially in terms of the standard L-function $L(\tilde{F}; s)$ of $\tilde{F}(g)=F({}^t g^{-1})$ and the tensor L-function $L(f \otimes F; s)$ up to certain local factors at the infinite primes. This fact may be considered as an analog of Shimura's result on the Hecke L-functions for $GL(2)$ (cf. [Shim, p. 799]). The proof is based on the first integral formula given in §6 and the fact that the integral (0.4) is equal to a certain integral of the (twisted) global Shintani function $W_{f,F}(*; s)$ over $N_{\boldsymbol{A}}$, where N is the unipotent radical of P. In the remaining part of the paper (§§10–11), we give another global application, which may be viewed as an analog of the results of our previous paper [M-S 2]. Let $G_1=GL(n+1)$ and embed G into G_1 via

$$g \mapsto \begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix}.$$

Let P_1 be the standard parabolic subgroup of G_1 corresponding to the partition $n+1=1+(n-1)+1$. In §10, we study the orbit structure of $P_1 \backslash G_1/G$, which is needed in the proof of the basic identity in the next section. In §11, after recalling the definition of the normalized Eisenstein series $\mathscr{E}^*(g_1; s, s'; f)$ $(g_1 \in G_{1,\boldsymbol{A}}, s, s' \in \boldsymbol{C})$ attached to f with respect to P_1, we prove the following results (Theorem 11.4):

(i) Let $\{\upsilon_j\}$ be a sequence in $C_c^\infty(\boldsymbol{R}_+^\times)$ with $0<\upsilon_1(x)\leq\upsilon_2(x)\leq\cdots\leq 1$ converging to the constant function $\mathbf{1}$. Assume that $\mathrm{Re}(s)$, $\mathrm{Re}(s')$ are sufficiently large. Then, as $j\to\infty$, the integral

$$\mathscr{Z}^*_{f,F}(s, s'; \upsilon_j)=\int_{G_E\backslash G_{\boldsymbol{A}}} F(g)\mathscr{E}^*(g; s, s'; f)\upsilon_j(|\det g|_{\boldsymbol{A}})dg$$

absolutely converges to a value independent of the choice of $\{\upsilon_j\}$.

(ii) The limit is expressed in terms of the standard L-functions $L(F; s)$, $L(\tilde{F}; s)$ and the initial value $W_{f,F}(1)$.

The key of the proof is the second local integral formula proved in §7 and the basic identity (Proposition 11.6) asserting that $\mathscr{Z}^*_{f,F}(s, s'; \upsilon_j)$ is expressed as an integral of the (modified) global Shintani function over $G_{o,\boldsymbol{A}}\backslash G_{\boldsymbol{A}}$.

Recently S. Kato and the first named author have proved an explicit formula for Shintani functions on $GL(n)$ in the non-Archimedean case. Details will appear in a forthcoming paper.

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NOTATION. We denote by $\mathrm{diag}(t_1, \ldots, t_n)$ the diagonal matrix with entries t_i $(1\leq i\leq n)$:

$$\mathrm{diag}(t_1, \ldots, t_n)=\begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{bmatrix}.$$

For a matrix A, we denote by ${}^tA$ the transpose of A.

Part I. Local theory.

1. Local Shintani functions.

1.1. Let E be a non-Archimedean local field with the integer ring $\mathfrak{o}=\mathfrak{o}_E$. We fix a prime element π of E and put $q_E=\sharp(\mathfrak{o}/\pi\mathfrak{o})$. We normalize the Haar measure dx on E by $\int_{\mathfrak{o}}dx=1$. For $a\in E^\times$, put $|a|_E=d(ax)/dx$. Then $|\pi|_E=q_E^{-1}$. Define $\mathrm{ord}_E\colon E^\times\to \boldsymbol{Z}$ by $|a|_E=q_E^{-\mathrm{ord}_E(a)}$ $(a\in E^\times)$. Throughout this paper, we normalize the Haar measure dg on $GL(r, E)$ by $\int_{GL(r,\mathfrak{o})}dg=1$. Fix an integer $n\geq 2$ and put $G=GL(n, E)$ and $K=GL(n, \mathfrak{o})$. Let B denote the subgroup of G consisting of upper triangular matrices. Let δ_B be the module of B defined by

$$\delta_B(b)=\prod_{i=1}^{n}|t_i|_E^{n+1-2i} \qquad \text{for} \quad b=\begin{bmatrix} t_1 & & * \\ & \ddots & \\ 0 & & t_n \end{bmatrix}\in B\,.$$

Let $T=\{\mathrm{diag}(t_1,\ldots,t_n)\,|\,t_1,\ldots,t_n\in E^\times\}$ be a maximal split torus of G. The group of unramified characters of T is denoted by $X_{\mathrm{unr}}(T)$. For $\Xi\in X_{\mathrm{unr}}(T)$, let Ξ_i be the i-th component of Ξ:

$$\Xi(\mathrm{diag}(t_1,\ldots,t_n))=\prod_{i=1}^{n}\Xi_i(t_i)\,.$$

1.2. We recall several basic facts about the Hecke algebra $\mathscr{H}_G=\mathscr{H}(G, K)$ (cf. [Ta]; see also [Sa]). By definition, $\mathscr{H}_G$ is the $\boldsymbol{C}$-algebra of compactly supported bi-K-invariant functions on G. Let $\Xi\in X_{\mathrm{unr}}(T)$ and extend it to a character of B in a natural way. Let ϕ_Ξ be the function on G given by

$$(1.1)\qquad \phi_\Xi(bk)=(\Xi\delta_B^{1/2})(b)\quad (b\in B, k\in K)\,.$$

Define a $\boldsymbol{C}$-algebra homomorphism $\Xi^\wedge$ of $\mathscr{H}_G$ to $\boldsymbol{C}$ by

$$(1.2)\qquad \Xi^\wedge(\Phi)=\int_G \Phi(g)\phi_\Xi(g)dg \qquad (\Phi\in\mathscr{H}_G)\,.$$

Then $\mathrm{Hom}_{\boldsymbol{C}}(\mathscr{H}_G, \boldsymbol{C})=\{\Xi^\wedge\,|\,\Xi\in X_{\mathrm{unr}}(T)/W_G\}$, where the Weyl group $W_G=N_G(T)/T\cong\mathfrak{S}_n$ (the symmetric group of degree n) acts on $X_{\mathrm{unr}}(T)$ in a natural manner. Furthermore, if F is a bi-K-invariant function on G and satisfies

$$(1.3)\qquad F*\Phi(g):=\int_G F(gy)\Phi(y)dy=\Xi^\wedge(\Phi)\cdot F(g) \qquad (g\in G)$$

for every $\Phi\in\mathscr{H}_G$, then we have $F(g)=F(1)\cdot\Psi_\Xi(g)$. Here Ψ_Ξ is the zonal spherical function on G attached to Ξ given by

(1.4) $$\Psi_{\Xi}(g)=\int_K \phi_{\Xi}(kg)dk\,.$$

It is well-known that

(1.5) $$\Psi_{\Xi}(g^{-1})=\Psi_{\Xi^{-1}}(g)$$

(1.6) $$\int_K \Psi_{\Xi}(gkg')dk=\Psi_{\Xi}(g)\Psi_{\Xi}(g') \qquad g, g'\in G\,.$$

1.3. Let $G_o=GL(n-1, E)$ and $K_o=GL(n-1, \mathfrak{o})$. We often regard G_o as a subgroup of G via the embedding

$$g_o \mapsto \begin{bmatrix} g_o & 0 \\ 0 & 1 \end{bmatrix}.$$

Let T_o be the group of diagonal matrices in G_o. For $\xi\in X_{\mathrm{unr}}(T_o)$, we define $\xi^\wedge\in \mathrm{Hom}_{\boldsymbol{C}}(\mathscr{H}_{G_o}, \boldsymbol{C})$ in a manner similar to that in §1.2. For $(\xi, \Xi)\in X_{\mathrm{unr}}(T_o)\times X_{\mathrm{unr}}(T)$, let

(1.7) $$\mathrm{Sh}(\xi, \Xi)=\{W\colon K_o\backslash G/K\to \boldsymbol{C} \mid \varphi * W * \Phi=\xi^\wedge(\varphi)\Xi^\wedge(\Phi)W \quad (\varphi\in\mathscr{H}_{G_o}, \Phi\in\mathscr{H}_G)\}$$

where

$$(\varphi * W * \Phi)(g)=\int_{G_o} dx\int_G dy\, \varphi(x)W(x^{-1}gy)\Phi(y)\,.$$

We call $\mathrm{Sh}(\xi, \Xi)$ the space of *Shintani functions attached to* (ξ, Ξ). Note that

(1.8) $$W\left(\begin{bmatrix} t'1_{n-1} & 0 \\ 0 & 1 \end{bmatrix} g\cdot t1_n\right)=\omega^{-1}(t')\Omega(t)W(g) \qquad (t', t\in E^\times, g\in G)\,,$$

where we put

(1.9) $$\omega=\xi_1\cdots\xi_{n-1}\,, \qquad \Omega=\Xi_1\cdots\Xi_n\,.$$

In particular, we have

(1.10) $$W\left(\begin{bmatrix} 1_{n-1} & 0 \\ 0 & t \end{bmatrix} g\right)=\Omega\omega(t)W(g) \qquad (t\in E^\times, g\in G)\,.$$

We are now ready to state one of the main results of the paper.

1.4. THEOREM. *For every* $(\xi, \Xi)\in X_{\mathrm{unr}}(T_o)\times X_{\mathrm{unr}}(T)$, *we have* $\dim_{\boldsymbol{C}}\mathrm{Sh}(\xi, \Xi)=1$.

2. Coset decomposition.

2.1. Let $Z_o=\{t1_{n-1} \mid t\in E^\times\}$ and $Z=\{t1_n \mid t\in E^\times\}$ be the centers of G_o and G, respectively. In this section, we study the orbit structure of $Z_oK_o\backslash G/ZK$, which is crucial to the proof of the uniqueness of Shintani functions.

2.2. For $(\lambda, \mu)=(\lambda_1, \dots, \lambda_{n-1}, \mu_1, \dots, \mu_{n-1}) \in \boldsymbol{Z}^{n-1} \times \boldsymbol{Z}^{n-1}$, we put

$$g(\lambda, \mu)=\begin{bmatrix} \pi^{\lambda_1} & & & \\ & \ddots & & \\ & & \pi^{\lambda_{n-1}} & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \pi^{-\mu_1} \\ & \ddots & & \vdots \\ & & 1 & \pi^{-\mu_{n-1}} \\ & & & 1 \end{bmatrix} \in G. \tag{2.1}$$

Let

$$\Lambda=\{(\lambda, \mu) \in \boldsymbol{Z}^{n-1} \times \boldsymbol{Z}^{n-1} \mid \lambda_{n-1}=0, \mu_1 \geq \cdots \geq \mu_{n-1} \geq 0, \\ \lambda_1-\mu_1 \geq \cdots \geq \lambda_{n-1}-\mu_{n-1}\}. \tag{2.2}$$

Note that $\lambda_1 \geq \cdots \geq \lambda_{n-1}$ if $(\lambda, \mu) \in \Lambda$. We endow Λ with the lexicographic order. The aim of this section is to show the following result.

2.3. **Proposition.** *We have*

$$G=\coprod_{(\lambda,\mu)\in\Lambda} Z_o K_o \cdot g(\lambda, \mu) \cdot ZK \quad (\textit{disjoint union}).$$

Proof. The assertion is easily verified in the case $n=2$, hence we assume $n \geq 3$. To simplify the notation, we write $g_1 \sim g_2$ if $g_1 \in Z_o K_o \cdot g_2 \cdot ZK$. Let g be an arbitrary element of G. By the Iwasawa decomposition for G and the Cartan decomposition for G_o, there exists $(\lambda, \mu') \in \boldsymbol{Z}^{n-1} \times \boldsymbol{Z}^{n-1}$ with $\lambda_1 \geq \cdots \geq \lambda_{n-1}=0$, $\mu'_i \geq 0$ $(i=1, \dots, n-1)$ such that $g \sim g(\lambda, \mu')$. We claim that $g(\lambda, \mu') \sim g(\lambda, \mu)$ for some $(\lambda, \mu) \in \Lambda$. First assume that $n=3$. Let $\mu'=(\mu'_1, \mu'_2)$. If $\mu'_1<\mu'_2$, we have

$$g(\lambda, \mu') \sim \begin{bmatrix} 1 & \pi^{\lambda_1-\lambda_2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} g(\lambda, \mu') \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$=\begin{bmatrix} \pi^{\lambda_1} & 0 & 0 \\ 0 & \pi^{\lambda_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & \pi^{-\mu'_1}+\pi^{-\mu'_2} \\ 0 & 1 & \pi^{-\mu'_2} \\ 0 & 0 & 1 \end{bmatrix} \sim g(\lambda_1, \lambda_2, \mu'_2, \mu'_2).$$

Hence we may assume $\mu'_1 \geq \mu'_2 \geq 0$. If $\lambda_1-\mu'_1<\lambda_2-\mu'_2$, we put $\mu_1=\mu'_1$ and $\mu_2=\mu'_1-\lambda_1+\lambda_2>\mu'_2$. Then $(\lambda, \mu) \in \Lambda$ and

$$g(\lambda, \mu') \sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} g(\lambda, \mu') \begin{bmatrix} 1 & 0 & 0 \\ -\pi^{\lambda_1-\lambda_2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$=\begin{bmatrix} \pi^{\lambda_1} & 0 & 0 \\ 0 & \pi^{\lambda_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & \pi^{-\mu_1} \\ 0 & 1 & \pi^{-\mu'_2}+\pi^{-\mu_2} \\ 0 & 0 & 1 \end{bmatrix} \sim g(\lambda, \mu),$$

which proves the claim in this case. We can prove the claim for $n\geq 4$ by repeating the above argument.

To prove the disjointness, let (λ, μ), $(\lambda', \mu')\in\Lambda$ and suppose that $g(\lambda, \mu)\sim g(\lambda', \mu')$ and $(\lambda, \mu)<(\lambda', \mu')$. It is obvious that $\lambda=\lambda'$. Let $\lambda=(\alpha_1, \dots, \alpha_1, \dots, \alpha_r, \dots, \alpha_r)$, where $\alpha_1>\cdots>\alpha_r=0$ and each α_i appears n_i times $(n_1+\cdots+n_r=n-1)$. Then we have $\mu=(\beta_1, \dots, \beta_1, \dots, \beta_r, \dots, \beta_r)$ and $\mu'=(\beta'_1, \dots, \beta'_1, \dots, \beta'_r, \dots, \beta'_r)$, where each of β_i and β'_i appears n_i times $(1\leq i\leq r)$ and

$$\begin{aligned}(2.3)\qquad &\beta_1\geq\cdots\geq\beta_r\geq 0\,, \quad \alpha_1-\beta_1\geq\cdots\geq\alpha_r-\beta_r\\ &\beta'_1\geq\cdots\geq\beta'_r\geq 0\,, \quad \alpha_1-\beta'_1\geq\cdots\geq\alpha_r-\beta'_r\,.\end{aligned}$$

We put

$$\Pi_\lambda=\begin{bmatrix}\pi^{\lambda_1} & & \\ & \ddots & \\ & & \pi^{\lambda_{n-1}}\end{bmatrix}=\begin{bmatrix}\pi^{\alpha_1}\cdot 1_{n_1} & & \\ & \ddots & \\ & & \pi^{\alpha_r}\cdot 1_{n_r}\end{bmatrix}.$$

By the assumption, there exist $k_o, k'_o\in K_o$ and $X\in\mathfrak{o}^{n-1}$ such that

$$g(\lambda, \mu)\begin{bmatrix}k_o & X\\ 0 & 1\end{bmatrix}=\begin{bmatrix}k'_o & 0\\ 0 & 1\end{bmatrix}g(\lambda, \mu')\,.$$

Then we have

$$(2.4)\qquad \Pi_\lambda k_o=k'_o\Pi_\lambda$$

$$(2.5)\qquad \Pi_\lambda\left(\begin{bmatrix}\pi^{-\mu_1}\\ \vdots\\ \pi^{-\mu_{n-1}}\end{bmatrix}+X\right)=k'_o\Pi_\lambda\begin{bmatrix}\pi^{-\mu'_1}\\ \vdots\\ \pi^{-\mu'_{n-1}}\end{bmatrix}$$

and hence

$$(2.6)\qquad \begin{bmatrix}\pi^{-\mu_1}\\ \vdots\\ \pi^{-\mu_{n-1}}\end{bmatrix}\equiv k_o\begin{bmatrix}\pi^{-\mu'_1}\\ \vdots\\ \pi^{-\mu'_{n-1}}\end{bmatrix}\qquad (\mathrm{mod}\ \mathfrak{o}^{n-1})\,.$$

Let $k_o=(\kappa_{ij})_{1\leq i,j\leq r}$ $(\kappa_{ij}\in M_{n_i,n_j}(\mathfrak{o}))$ be the block decomposition of k_o according to the partition $n-1=n_1+\cdots+n_r$. By (2.4),

$$(2.7)\qquad \kappa_{ii}\in GL_{n_i}(\mathfrak{o})\quad\text{and}\quad \kappa_{ij}\in\pi^{-\alpha_i+\alpha_j}M_{n_i,n_j}(\mathfrak{o})\quad\text{if}\quad i>j\,.$$

By the assumption $(\lambda, \mu)<(\lambda, \mu')$, there exist integers c and d $(1\leq c\leq d\leq r)$ such that

$$(2.8)\qquad \beta'_i=\beta_i\ (1\leq i\leq c-1)\,,\quad \beta'_c>\beta_c\quad\text{and}\quad \beta'_c=\cdots=\beta'_d>\beta'_{d+1}\,.$$

For j $(1\leq j\leq r)$, put $\varepsilon_j={}^t(1, 1, \dots, 1)\in\mathfrak{o}^{n_j}$. The congruence (2.6) implies

$$(2.9)\qquad \sum_{j=1}^{r} \kappa_{dj}\cdot\varepsilon_j\cdot\pi^{-\beta'_j}\equiv\varepsilon_d\cdot\pi^{-\beta_d}\pmod{\mathfrak{o}^{n_d}}.$$

Put $\eta_j=\kappa_{dj}\cdot\varepsilon_j\cdot\pi^{\beta'_d-\beta'_j}$ $(1\le j\le r)$. By (2.3) and (2.7), we have $\eta_j\in\mathfrak{o}^{n_d}$. Since $\beta'_d>\beta_d\ge 0$, we have $\sum_{j=1}^{r}\eta_j\equiv\varepsilon_d\cdot\pi^{\beta'_d-\beta_d}\equiv 0\ (\mathrm{mod}(\pi\mathfrak{o})^{n_d})$ by (2.9). Observe that $\eta_j\in(\pi\mathfrak{o})^{n_d}$ if $c\le j<d$ or if $j>d$. Since $\eta_d=\kappa_{dd}\cdot\varepsilon_d\notin(\pi\mathfrak{o})^{n_d}$, there exists an integer i $(1\le i\le c-1)$ such that $\eta_i\notin(\pi\mathfrak{o})^{n_d}$. It follows that $\alpha_i-\alpha_d+\beta'_d-\beta'_i=0$. Since $\alpha_i-\alpha_d+\beta_d-\beta_i\ge 0$, we have $(\beta_d-\beta_i)-(\beta'_d-\beta'_i)\ge 0$, which contradicts (2.8). q.e.d.

3. Uniqueness of Shintani functions.

3.1. Throughout this section, we keep the notation of §§1–2. Let $\xi\in X_{\mathrm{unr}}(T_o)$ and $\Xi\in X_{\mathrm{unr}}(T)$. In this section we prove:

THEOREM. *Let* $W\in\mathrm{Sh}(\xi,\Xi)$. *If* $W(1)=0$, *then* W *is identically equal to zero. In particular, we have* $\dim_{\boldsymbol{C}}\mathrm{Sh}(\xi,\Xi)\le 1$.

3.2. For $W\in\mathrm{Sh}(\xi,\Xi)$ and $(\lambda,\mu)\in\Lambda$, we write $W(\lambda,\mu)$ for $W(g(\lambda,\mu))$ to simplify notation. Let $\Lambda(\lambda,\mu)$ be the set of $(\lambda',\mu')\in\Lambda$ with $(\lambda',\mu')<(\lambda,\mu)$. We denote by $\boldsymbol{C}[\xi,\xi^{-1},\Xi,\Xi^{-1}]$ the $\boldsymbol{C}$-algebra of polynomial functions in $\xi_i(\pi)$, $\xi_i^{-1}(\pi)$, $\Xi_j(\pi)$, $\Xi_j^{-1}(\pi)$ $(1\le i\le n-1,\ 1\le j\le n)$. In view of Proposition 2.3, the proof of Theorem 3.1 is reduced to the following:

3.3. PROPOSITION. *Let* $W\in\mathrm{Sh}(\xi,\Xi)$. *For any* $(\lambda,\mu)\in\Lambda$, *we have*

$$(3.1)\qquad W(\lambda,\mu)=\sum_{(\lambda',\mu')} c_{\lambda',\mu'}(\xi,\Xi)W(\lambda',\mu'),$$

where (λ',μ') *runs over the set* $\Lambda(\lambda,\mu)$ *and* $c_{\lambda',\mu'}(\xi,\Xi)$ *is an element of* $\boldsymbol{C}[\xi,\xi^{-1},\Xi,\Xi^{-1}]$ *depending only on* (λ',μ').

3.4. COROLLARY. *Let* $W\in\mathrm{Sh}(\xi,\Xi)$ *with* $W(1)=1$. *Then the value* $W(\lambda,\mu)$ *belongs to* $\boldsymbol{C}[\xi,\xi^{-1},\Xi,\Xi^{-1}]$ *for any* $(\lambda,\mu)\in\Lambda$.

3.5. To prove Proposition 3.3, we need preparations. Let

$$N_o=\left\{\begin{bmatrix}1 & & *\\ & \ddots & \\ 0 & & 1\end{bmatrix}\in G_o\right\},\quad N'_o=\left\{\begin{bmatrix}1 & & 0\\ & \ddots & \\ * & & 1\end{bmatrix}\in G_o\right\},\quad \Pi_\alpha=\begin{bmatrix}\pi^{\alpha_1} & & 0\\ & \ddots & \\ 0 & & \pi^{\alpha_{n-1}}\end{bmatrix}\in G_o$$

for $\alpha=(\alpha_1,\dots,\alpha_{n-1})\in\boldsymbol{Z}^{n-1}$. Let $I=\{0,1\}$ and put $I(d)=\{\varepsilon=(\varepsilon_1,\dots,\varepsilon_{n-1})\in I^{n-1}\,|\,\varepsilon_1+\cdots+\varepsilon_{n-1}=d\}$ for $d\ (0\le d\le n-1)$. For $\varepsilon\in I^{n-1}$, put $N_o(\varepsilon)=\{v=(v_{ij})\in N_o(\mathfrak{o})\,|\,$for i,j $(1\le i<j\le n-1)$, $v_{ij}\in\mathfrak{o}/\pi\mathfrak{o}$ if $\varepsilon_i>\varepsilon_j$ and $v_{ij}=0$ otherwise$\}$ and $N'_o(\varepsilon)=\{v'=(v'_{ij})\in N'_o(\mathfrak{o})\,|$ for i,j $(1\le j<i\le n-1)$, $v'_{ij}\in\mathfrak{o}/\pi\mathfrak{o}$ if $\varepsilon_i<\varepsilon_j$ and $v'_{ij}=0$ otherwise$\}$. Put $L=\mathfrak{o}^{n-1}$ (the set of column vectors of size $n-1$). The following fact is easily verified.

3.6. LEMMA. (i) *For* d *with* $0\le d\le n-1$, *we have*

$$K_o \cdot \begin{bmatrix} \pi 1_d & 0 \\ 0 & 1_{n-1-d} \end{bmatrix} \cdot K_o = \coprod_{\varepsilon \in I(d)} \coprod_{v' \in N'_o(\varepsilon)} K_o \cdot \Pi_\varepsilon v' \quad (\textit{disjoint union}).$$

(ii) *For d with $0 \leq d \leq n$, we have*

$$K \cdot \begin{bmatrix} \pi 1_d & 0 \\ 0 & 1_{n-d} \end{bmatrix} \cdot K = \coprod_{\varepsilon \in I(d)} \coprod_{v \in N_o(\varepsilon)} \coprod_{y \in L/\Pi_\varepsilon L} \begin{bmatrix} v & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1_{n-1} & y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Pi_\varepsilon & 0 \\ 0 & 1 \end{bmatrix} K$$

$$\cup \coprod_{\varepsilon \in I(d-1)} \coprod_{v \in N_o(\varepsilon)} \begin{bmatrix} v & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Pi_\varepsilon & 0 \\ 0 & \pi \end{bmatrix} K \quad (\textit{disjoint union}).$$

3.7. For $d\,(1 \leq d \leq n-1)$, we denote by φ_d (resp. Φ_d) the characteristic function of

$$K_o \begin{bmatrix} \pi 1_d & 0 \\ 0 & 1_{n-1-d} \end{bmatrix} K_o \quad \left(\text{resp. } K \begin{bmatrix} \pi 1_d & 0 \\ 0 & 1_{n-d} \end{bmatrix} K\right).$$

For d with $1 \leq d \leq n-2$, we set

$$\Lambda^+(d) = \{(\lambda, \mu) \in \Lambda \mid \lambda_1 = \cdots = \lambda_d > \lambda_{d+1}, \lambda_d - \mu_d = \lambda_{d+1} - \mu_{d+1}\},$$

$$\Lambda^-(d) = \{(\lambda, \mu) \in \Lambda \mid \lambda_1 = \cdots = \lambda_d > \lambda_{d+1}, \lambda_d - \mu_d > \lambda_{d+1} - \mu_{d+1}\},$$

and

$$\Lambda(n-1) = \{(\lambda, \mu) \in \Lambda \mid \lambda_1 = \cdots = \lambda_{n-1} = 0, \mu_1 = \cdots = \mu_{n-1}\}.$$

Then $\Lambda = \bigcup_{1 \leq d \leq n-2} (\Lambda^+(d) \cup \Lambda^-(d)) \cup \Lambda(n-1)$ (disjoint union). Put $x_\mu = {}^t(\pi^{-\mu_1}, \ldots, \pi^{-\mu_{n-1}}) \in E^{n-1}$ for $\mu = (\mu_1, \ldots, \mu_{n-1}) \in \boldsymbol{Z}^{n-1}$.

3.8. The proof of Proposition 3.3. Let $W \in \mathrm{Sh}(\xi, \Xi)$. First let $(\lambda, \mu) \in \Lambda^+(d)$. Note that $\mu_d > \mu_{d+1}$. Put $\lambda' = (\lambda_1 - 1, \ldots, \lambda_d - 1, \lambda_{d+1}, \ldots, \lambda_{n-1})$ and $\mu' = (\mu_1 - 1, \ldots, \mu_d - 1, \mu_{d+1}, \ldots, \mu_{n-1})$. Then $(\lambda', \mu') \in \Lambda(\lambda, \mu)$. By Lemma 3.6 and (1.10), we have

$$\begin{aligned}
&\Xi^\wedge(\Phi_d) \cdot W(\lambda', \mu') = (W * \Phi_d)(g(\lambda', \mu')) \\
&= \sum_{\varepsilon \in I(d)} \sum_{v \in N_o(\varepsilon)} \sum_{y \in L/\Pi_\varepsilon L} W\left(\begin{bmatrix} \Pi_{\lambda'} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1_{n-1} & x_{\mu'} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1_{n-1} & y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Pi_\varepsilon & 0 \\ 0 & 1 \end{bmatrix}\right) \\
&\quad + \sum_{\varepsilon \in I(d-1)} \sum_{v \in N_o(\varepsilon)} W\left(\begin{bmatrix} \Pi_{\lambda'} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1_{n-1} & x_{\mu'} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Pi_\varepsilon & 0 \\ 0 & \pi \end{bmatrix}\right) \\
&= \sum_{\varepsilon \in I(d)} \sum_{v \in N_o(\varepsilon)} \sum_{y \in L/\Pi_\varepsilon L} W\left(\begin{bmatrix} \Pi_{\lambda'+\varepsilon} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1_{n-1} & \Pi_\varepsilon^{-1}(v^{-1} x_{\mu'} + y) \\ 0 & 1 \end{bmatrix}\right) \\
&\quad + \sum_{\varepsilon \in I(d-1)} \sum_{v \in N_o(\varepsilon)} \Omega\omega(\pi) \cdot W\left(\begin{bmatrix} \Pi_{\lambda'+\varepsilon} & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1_{n-1} & \pi \cdot \Pi_\varepsilon^{-1} v^{-1} x_{\mu'} \\ 0 & 1 \end{bmatrix}\right)
\end{aligned}$$

(note that $\Pi_{\lambda'} v \Pi_{\lambda'}^{-1} \in K_o$ for $v \in N_o(\mathfrak{o})$). Observe that $\Pi_\varepsilon^{-1}(v^{-1} x_{\mu'} + y) \in \Pi_{\mu'+\varepsilon}^{-1} L$ for $v \in N_o(\mathfrak{o})$ and $y \in L$. We put $\varepsilon^d = (1^{(d)}, 0^{(n-1-d)}) \in I^{n-1}$, where $1^{(d)} = (1, \ldots, 1)$ (1 repeated d times) and $0^{(n-1-d)}=(0,\dots,0)$ (0 repeated $n-1-d$ times). Then we have $\lambda'+\varepsilon^d=\lambda$ and $\Pi_{\varepsilon^d}^{-1}x_{\mu'}=x_\mu$. It follows that

$$\Xi^\wedge(\Phi_d)\cdot W(\lambda',\mu')=a(\lambda,\mu)W(\lambda,\mu)+\sum_{(\lambda'',\mu'')}a(\lambda,\mu,\lambda'',\mu'')W(\lambda'',\mu''),$$

where (λ'',μ'') runs over the set $\Lambda(\lambda,\mu)$, and $a(\lambda,\mu)$ (resp. $a(\lambda,\mu,\lambda'',\mu'')$) is an element of $\boldsymbol{C}^\times$ (resp. $\boldsymbol{C}[\xi,\xi^{-1},\Xi,\Xi^{-1}]$) depending only on (λ,μ) (resp. $(\lambda,\mu,\lambda'',\mu'')$). Since $\Xi^\wedge(\Phi_d)\in\boldsymbol{C}[\Xi,\Xi^{-1}]$, we have proved (3.1) for $(\lambda,\mu)\in\Lambda^+(d)$.

Next let $(\lambda,\mu)\in\Lambda^-(d)$ and put $\lambda'=(\lambda_1-1,\dots,\lambda_d-1,\lambda_{d+1},\dots,\lambda_{n-1})$ and $\mu'=\mu$. Then $(\lambda',\mu')\in\Lambda(\lambda,\mu)$. By Lemma 3.6, we have

$$\begin{aligned}\omega^{-1}(\pi)\xi^\wedge(\varphi_{n-1-d})\cdot W(\lambda',\mu')&=\int_{G_o}\varphi_d(x)W(x\cdot g(\lambda',\mu))dx\\&=\sum_{\varepsilon\in I(d)}\sum_{v'\in N_o'(\varepsilon)}W\left(\begin{bmatrix}\Pi_\varepsilon v'&0\\0&1\end{bmatrix}\begin{bmatrix}\Pi_{\lambda'}&0\\0&1\end{bmatrix}\begin{bmatrix}1_{n-1}&x_\mu\\0&1\end{bmatrix}\right)\\&=\sum_{\varepsilon\in I(d)}\sum_{v'\in N_o'(\varepsilon)}W\left(\begin{bmatrix}\Pi_{\lambda'+\varepsilon}&0\\0&1\end{bmatrix}\begin{bmatrix}1_{n-1}&\Pi_{\lambda'}^{-1}v'\Pi_{\lambda'}\cdot x_\mu\\0&1\end{bmatrix}\right)\end{aligned}$$

(note that $\Pi_{\lambda'}^{-1}v'\Pi_{\lambda'}\in K_o$ for $v'\in N_o'(\mathfrak{o})$). Since $\lambda_j-\lambda_i-\mu_j\geq-\mu_i$ for $j\leq i$, we see that $(\Pi_{\lambda'}^{-1}v'\Pi_{\lambda'}x_\mu)_i=\sum_{j<i}\pi^{\lambda_j-\lambda_i-\mu_j}\cdot v'_{ij}+\pi^{-\mu_i}\in\pi^{-\mu_i}\mathfrak{o}$. By an argument similar to that above, we have

$$\omega^{-1}(\pi)\xi^\wedge(\varphi_{n-1-d})\cdot W(\lambda',\mu')=b(\lambda,\mu)W(\lambda,\mu)+\sum_{(\lambda'',\mu'')}b(\lambda,\mu,\lambda'',\mu'')W(\lambda'',\mu''),$$

where (λ'',μ'') runs over the set of $\Lambda(\lambda,\mu)$, and $b(\lambda,\mu)$ (resp. $b(\lambda,\mu,\lambda'',\mu'')$) is an element of $\boldsymbol{C}^\times$ (resp. $\boldsymbol{C}[\xi,\xi^{-1},\Xi,\Xi^{-1}]$) depending only on (λ,μ) (resp. $(\lambda,\mu,\lambda'',\mu'')$). Since $\xi^\wedge(\varphi_{n-1-d})\in\boldsymbol{C}[\xi,\xi^{-1}]$, we have proved (3.1) for $(\lambda,\mu)\in\Lambda^-(d)$.

We postpone the proof of (3.1) for $(\lambda,\mu)\in\Lambda(n-1)$ until §6 (see the remark after Theorem 6.4). q.e.d.

4. Existence of Shintani functions.

4.1. In this section, we prove the existence of Shintani functions by using an integral expression. We keep the notation of the preceding sections. Let w_τ be the permutation matrix corresponding to $\tau\in\mathfrak{S}_n$ (the symmetric group of degree n). We write w_l for

$$w_{\tau_l}=\begin{bmatrix}0&&1\\&\cdot^{\cdot^{\cdot}}&\\1&&0\end{bmatrix},$$

where τ_l is the longest element of $\mathfrak{S}_n$. Let B (resp. B_o) be the subgroup of G (resp. G_o) consisting of upper triangular matrices. Let N (resp. N_o) be the unipotent radical of B

(resp. B_o), and put $N'=\{{}^t n \mid n\in N\}$, $N'_o=\{{}^t n_o \mid n_o\in N_o\}$. Denote by T (resp. T_o) the group of diagonal matrices in G (resp. G_o). The following result is a direct consequence of the Bruhat decomposition for G.

4.2. LEMMA. *We have*

$$G=\bigcup_{\tau\in\mathfrak{S}_n}\bigcup_{\eta\in\{0,1\}^{n-1}} B_o\cdot\begin{bmatrix} 1_{n-1} & \eta \\ 0 & 1 \end{bmatrix} w_\tau\cdot B.$$

REMARK. The decomposition is not a disjoint union in general.

4.3. For $g=(g_{ij})\in G$ and for $i_1,\dots,i_r,j_1,\dots,j_r$ $(1\le i_1<\cdots<i_r\le n,\ 1\le j_1<\cdots<j_r\le n)$, put $\Delta_{i_1\cdots i_r,j_1\cdots j_r}(g)=\det(g_{i_k j_l})_{1\le k,l\le r}$. We define

$$\begin{aligned}(4.1)\qquad \alpha_i(g)&=\Delta_{1\cdots i,1\cdots i}(w_l g) &&(1\le i\le n)\\ \beta_j(g)&=\Delta_{2\cdots j+1,1\cdots j}(w_l g) &&(1\le j\le n-1)\end{aligned}$$

and make a convention that $\alpha_0(g)=1$ and $\beta_0(g)=-1$. Note that $\alpha_n(g)=(-1)^{[n/2]}\det g\neq 0$, where $[n/2]$ is an integer satisfying $0\le n/2-[n/2]<1$. Put

$$g_l=\begin{bmatrix} 1 & & 0 & 1 \\ & \ddots & & \vdots \\ 0 & & 1 & 1 \\ 0 & \cdots & 0 & 1 \end{bmatrix} w_l.$$

4.4. LEMMA. (i) *$g\in B_o g_l B$ if and only if $\alpha_i(g)\neq 0$, $\beta_i(g)\neq 0$ $(1\le i\le n-1)$.*

(ii) *Let $g\in B_o g_l B$ and write*

$$(4.2)\qquad g=\begin{bmatrix} t'_1 & & * & 0 \\ & \ddots & & \vdots \\ 0 & & t'_{n-1} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} g_l \begin{bmatrix} t_1 & & * \\ & \ddots & \\ 0 & & t_n \end{bmatrix}.$$

Then the t'_i, $t_j\in E^\times$ are uniquely determined by g and given as follows:

$$t'_i=(-1)^{n-i+1}\frac{\beta_{n-i}(g)}{\alpha_{n-i}(g)}\quad(1\le i\le n-1),\qquad t_j=(-1)^j\frac{\alpha_j(g)}{\beta_{j-1}(g)}\quad(1\le j\le n).$$

PROOF. First note that $B_o g_l B\subset Bw_l B$. It is well-known that g belongs to $Bw_l B$ if and only if $\alpha_1(g)\cdots\alpha_{n-1}(g)\neq 0$. Let g be an element of $Bw_l B$ and write

$$g=\begin{bmatrix} v & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1_{n-1} & x \\ 0 & 1 \end{bmatrix} w_l b,$$

where $v\in N_o$, $x={}^t(x_1,\dots,x_{n-1})\in E^{n-1}$ and $b\in B$. It is easily verified that $g\in B_o g_l B$ if and only if $x_1\cdots x_{n-1}\neq 0$. Since

$$w_l g = \begin{bmatrix} 1 & 0 \\ 0 & v' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x' & 1_{n-1} \end{bmatrix} b$$

with some $v' \in N_o'$ and $x' = {}^t(x_{n-1}, \ldots, x_1)$, we obtain

$$\alpha_i(g) = y_1 \cdots y_i\,, \qquad \beta_i(g) = (-1)^{i+1} x_{n-i} y_1 \cdots y_i\,, \tag{4.3}$$

where y_i is the i-th diagonal component of b $(1 \le i \le n)$. This implies that $x_1 \cdots x_{n-1} \neq 0$ if and only if $\beta_1(g) \cdots \beta_{n-1}(g) \neq 0$, which proves the first assertion (i). To prove the second one, let

$$g = \begin{bmatrix} t_1' & & * & 0 \\ & \ddots & & \vdots \\ 0 & & t_{n-1}' & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} g_l \begin{bmatrix} t_1 & & * \\ & \ddots & \\ 0 & & t_n \end{bmatrix} \in B_o g_l B\,.$$

Then we have

$$g = \begin{bmatrix} v & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & 0 & t_1' \\ & \ddots & & \vdots \\ 0 & & 1 & t_{n-1}' \\ 0 & \cdots & 0 & 1 \end{bmatrix} w_l \begin{bmatrix} t_1 & & & * \\ & t_{n-1}'t_2 & & \\ & & \ddots & \\ 0 & & & t_1't_n \end{bmatrix}$$

with $v \in N_o$. By (4.3), we get $\alpha_i(g) = t_1 \cdot \prod_{k=2}^{i} t_{n+1-k}' t_k$ and $\beta_i(g) = (-1)^{i+1} t_{n-i}' \alpha_i(g)$, which proves the assertion (ii). q.e.d.

4.5. Let δ_{B_o} and δ_B be the modules of B_o and B, respectively (see, §1.1). For $\xi \in X_{\mathrm{unr}}(T_o)$ and $\Xi \in X_{\mathrm{unr}}(T)$, let $Y_{\xi,\Xi}$ be a function on G satisfying the following three conditions:

(4.4) The support of $Y_{\xi,\Xi}$ is $B_o g_l B$.

(4.5) $Y_{\xi,\Xi}(b_o g b) = (\xi^{-1}\delta_{B_o}^{1/2})(b_o)(\Xi \delta_B^{-1/2})(b) Y_{\xi,\Xi}(g) \quad (b_o \in B_o,\ b \in B,\ g \in G)$.

(4.6) $Y_{\xi,\Xi}(g_l) = 1$.

4.6. Lemma. *For $g \in B_o g_l B$, we have*

$$Y_{\xi,\Xi}(g) = (\Xi_n |\ |_E^{(n-1)/2})(\det g) \cdot \prod_{i=1}^{n-1} (\xi_{n-i}\Xi_i |\ |_E^{-1/2})(\alpha_i(g))(\xi_{n-i}^{-1}\Xi_{i+1}^{-1} |\ |_E^{-1/2})(\beta_i(g))\,.$$

Proof. This follows from Lemma 4.4 (ii).

4.7. Corollary. *Assume that (ξ, Ξ) satisfies*

$$|(\xi_{n-i}\Xi_i)(\pi)|_E < q_E^{-1/2} \quad \textit{and} \quad |(\xi_{n-i}^{-1}\Xi_{i+1}^{-1})(\pi)|_E < q_E^{-1/2} \qquad (1 \le i \le n-1)\,. \tag{4.7}$$

Then $Y_{\xi,\Xi}$ is continuous on G.

4.8. For $(\xi, \Xi) \in X_{\mathrm{unr}}(T_o) \times X_{\mathrm{unr}}(T)$, we set

$$W_{\xi,\Xi}(g) = \int_{K_o} dk_o \int_K dk\, Y_{\xi,\Xi}(k_o g k)\,. \tag{4.8}$$

If $(\xi, \Xi) \in X_{\mathrm{unr}}(T_o) \times X_{\mathrm{unr}}(T)$ satisfies (4.7), the integral (4.8) is absolutely convergent by Corollary 4.7.

4.9. LEMMA. *If* $(\xi, \Xi) \in X_{\mathrm{unr}}(T_o) \times X_{\mathrm{unr}}(T)$ *satisfies* (4.7), *we have* $W_{\xi,\Xi} \in \mathrm{Sh}(\xi, \Xi)$.

PROOF. For simplicity, we write Y and W for $Y_{\xi,\Xi}$ and $W_{\xi,\Xi}$. Let $\varphi \in \mathscr{H}_{G_o}$. Then we have

$$(\varphi * W)(g) = \int_{G_o} dx \int_{K_o} dk_o \int_K dk\, \varphi(x) Y(k_o x^{-1} g k) = \int_{G_o} dx \int_K dk\, \varphi(x) Y(x^{-1} g k)\,.$$

Decompose $x \in G_o$ into $k_o^{-1} b_o$ ($k_o \in K_o$, $b_o \in B_o$). Then $dx = \delta_{B_o}(b_o) d_l b_o dk_o$, where $d_l b_o$ is a left invariant measure on B_o. Then

$$\begin{aligned}(\varphi * W)(g) &= \int_{B_o} d_l b_o \int_{K_o} dk_o \int_K dk\, \varphi(b_o) Y(b_o^{-1} k_o g k) \delta_{B_o}(b_o) \\ &= \int_{B_o} \varphi(b_o)(\xi \delta_{B_o}^{1/2})(b_o) d_l b_o \int_{K_o} dk_o \int_K dk\, Y(k_o g k) = \xi^{\wedge}(\varphi) \cdot W(g)\,.\end{aligned}$$

The equality $W * \Phi = \Xi^{\wedge}(\Phi) W$ for $\Phi \in \mathscr{H}_G$ is proved in a similar way. q.e.d.

4.10. We can now prove the existence of Shintani functions, which completes the proof of Theorem 1.4. The proof was suggested to us by Fumihiro Sato.

THEOREM. *For every* $(\xi, \Xi) \in X_{\mathrm{unr}}(T_o) \times X_{\mathrm{unr}}(T)$, *there exists a* $W \in \mathrm{Sh}(\xi, \Xi)$ *with* $W(1) = 1$.

PROOF. Let X_o be the set of $(\xi, \Xi) \in X_{\mathrm{unr}}(T_o) \times X_{\mathrm{unr}}(T)$ satisfying (4.7) and $\xi_i(\pi)$, $\Xi_j(\pi) > 0$ $(1 \le i \le n-1,\ 1 \le j \le n)$. Then X_o is a $(2n-1)$-dimensional real submanifold of the complex manifold $X_{\mathrm{unr}}(T_o) \times X_{\mathrm{unr}}(T)$. Let $(\xi, \Xi) \in X_o$. Since $Y_{\xi,\Xi}$ is positive on an open dense subset $B_o g_l B$ of G, we have $W_{\xi,\Xi}(1) > 0$. By Lemma 4.9, $W_{\xi,\Xi}^{\sim}(g) = W_{\xi,\Xi}(g)/W_{\xi,\Xi}(1) \in \mathrm{Sh}(\xi, \Xi)$. It follows from Corollary 3.4 that $W_{\xi,\Xi}^{\sim}(g)$ is a polynomial function in $\xi_1^{\pm 1}(\pi), \ldots, \xi_{n-1}^{\pm 1}(\pi), \Xi_1^{\pm 1}(\pi), \ldots, \Xi_n^{\pm 1}(\pi)$ for each $g \in G$. This implies that, for each $g \in G$, the function $(\xi, \Xi) \mapsto W_{\xi,\Xi}^{\sim}(g)$ on X_o can be continued to a holomorphic function on $X_{\mathrm{unr}}(T_o) \times X_{\mathrm{unr}}(T)$. By analytic continuation, we see that $W_{\xi,\Xi}^{\sim}(1) = 1$ and $W_{\xi,\Xi}^{\sim} \in \mathrm{Sh}(\xi, \Xi)$ for $(\xi, \Xi) \in X_{\mathrm{unr}}(T_o) \times X_{\mathrm{unr}}(T)$, which completes the proof of the theorem.

q.e.d.

5. Shintani functions at the infinite primes.

5.1. In this section, we let $E = \boldsymbol{R}$ or $\boldsymbol{C}$. We normalize the Haar measure dx on E

as follows:

$$dx=\begin{cases}\text{the usual Lebesgue measure} & \text{if } E=\boldsymbol{R}\\ 2d(\operatorname{Re}x)d(\operatorname{Im}x) & \text{if } E=\boldsymbol{C}.\end{cases}$$

For $a\in E^\times$, put $|a|_E=d(ax)/dx$. Let $G=GL(n, E)$ and

$$K=\begin{cases}O(n,\boldsymbol{R}) & \text{if } E=\boldsymbol{R}\\ U(n) & \text{if } E=\boldsymbol{C}.\end{cases}$$

The Haar measure dk on K is always normalized so that the total measure of K is equal to 1. We normalize the Haar measure dg on G by

$$\int_G f(g)dg=\int_N dn\int_{(E^\times)^n} d^\times t_1\cdots d^\times t_n\int_K dk f\left(n\begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_n\end{bmatrix}k\right)\prod_{i=1}^n |t_i|_E^{-n+2i-1}$$

for $f\in C_c^\infty(G)$, where $dn=\prod_{i<j}dn_{ij}$ is the Haar measure on $N=\{n=(n_{ij})\in G\,|\,n_{ii}=1, n_{ij}=0$ if $i>j\}$. Let Lie(G) be the Lie algebra of G and $\mathscr{U}_G$ the universal enveloping algebra of $\mathrm{Lie}(G)\otimes_{\boldsymbol{R}}\boldsymbol{C}$. We denote by $\mathscr{Z}_G$ the center of $\mathscr{U}_G$. For $X\in\mathrm{Lie}(G)$ and $f\in C^\infty(G)$, put

$$R_X f(g)=\frac{d}{dt}f(g\cdot\exp(tX))\bigg|_{t=0},\qquad L_X f(g)=\frac{d}{dt}f(\exp(-tX)\cdot g)\bigg|_{t=0}. \tag{5.1}$$

These actions of Lie(G) on $C^\infty(G)$ extend to those of $\mathscr{U}_G$ in a natural way.

5.2. Let T be the group of diagonal matrices in G and $\mathscr{U}_T$ the universal enveloping algebra of $\mathrm{Lie}(T)_{\boldsymbol{C}}=\mathrm{Lie}(T)\otimes_{\boldsymbol{R}}\boldsymbol{C}$. Then

$$\mathscr{Z}_G\cong(\mathscr{U}_T)^{W_G} \tag{5.2}$$

via the Harish-Chandra isomorphism, where $W_G=N_G(T)/T$ (for example, see [G-V, §2.6]). Denote by $X_{\mathrm{unr}}(T)$ the group of continuous homomorphisms of T to $\boldsymbol{C}^\times$ trivial on $T^1=\{\mathrm{diag}(\varepsilon_1,\dots,\varepsilon_n)\,|\,\varepsilon_i\in E^\times, |\varepsilon_i|_E=1\ (1\le i\le n)\}$. The differential $d\Xi\in(\mathrm{Lie}(T)_{\boldsymbol{C}})^*$ of $\Xi\in X_{\mathrm{unr}}(T)$ determines an element $\Xi^\wedge$ of $\mathrm{Hom}_{\boldsymbol{C}}(\mathscr{Z}_G,\boldsymbol{C})$ via the isomorphism (5.2). It is known that $\mathrm{Hom}_{\boldsymbol{C}}(\mathscr{Z}_G,\boldsymbol{C})=\{\Xi^\wedge\,|\,\Xi\in X_{\mathrm{unr}}(T)/W_G\}$.

5.3. For $\Xi\in X_{\mathrm{unr}}(T)$, we set

$$\Psi_\Xi(g)=\int_K\phi_\Xi(kg)dk, \tag{5.3}$$

where $\phi_\Xi(g)$ is defined as in (1.1). We now recall several well-known facts about the spherical function $\Psi_\Xi(g)$:

$$\Psi_\Xi(1)=1. \tag{5.4}$$

$$\Psi_\Xi(kgk')=\Psi_\Xi(g)\qquad k,k'\in K,\quad g\in G. \tag{5.5}$$

$$\int_K \Psi_\Xi(gkg')dk = \Psi_\Xi(g)\Psi_\Xi(g') . \tag{5.6}$$

$$\Psi_\Xi(g^{-1}) = \Psi_{\Xi^{-1}}(g) . \tag{5.7}$$

$$R_Z\Psi_\Xi = \Xi^\wedge(Z)\cdot\Psi_\Xi \qquad Z\in\mathscr{Z}_G . \tag{5.8}$$

5.4. LEMMA. *Let $\Xi\in X_{\mathrm{unr}}(T)$ and $F\in C^\infty(K\backslash G/K)$. If*

$$R_ZF = \Xi^\wedge(Z)\cdot F \tag{5.9}$$

holds for any $Z\in\mathscr{Z}_G$, then we have $F(g)=F(1)\Psi_\Xi(g)$.

PROOF. This follows from [G-V, Theorem 3.2.3] and [H, Proposition 5.32].

5.5. Let $G_o=GL(n-1, E)$ and define T_o, K_o similarly as T and K. Let $\xi\in X_{\mathrm{unr}}(T_o)$, $\Xi\in X_{\mathrm{unr}}(T)$ and put $\omega=\xi_1\cdots\xi_{n-1}$, $\Omega=\Xi_1\cdots\Xi_n\in X_{\mathrm{unr}}(E^\times)$. We now define the space $\mathrm{Sh}(\xi, \Xi)$ of *Shintani functions* on G attached to ξ and Ξ to be the space of $W\in C^\infty(K_o\backslash G/K)$ satisfying

$$L_zR_ZW = \xi^\wedge(z)\Xi^\wedge(Z)\cdot W \qquad z\in\mathscr{Z}_{G_o},\quad Z\in\mathscr{Z}_G . \tag{5.10}$$

$$W\left(\begin{bmatrix} t'1_{n-1} & 0 \\ 0 & 1\end{bmatrix}\cdot g\cdot t1_n\right) = \omega^{-1}(t')\Omega(t)\cdot W(g) \qquad t, t'\in E^\times . \tag{5.11}$$

5.6. REMARK. It is an open problem to compute $\dim_{\boldsymbol{C}}\mathrm{Sh}(\xi, \Xi)$ in the Archimedean case.

6. Integral formula (I).

6.1. Let E be a local field (either Archimedean or non-Archimedean). In this section, we show an integral formula for Shintani functions on $G=GL(n, E)$, which is crucial to the proof of the uniqueness theorem in the non-Archimedean case (Theorem 3.1). We use the same notation as in §1 (resp. §5) in the non-Archimedean (resp. Archimedean) case.

6.2. Let $\zeta_E(s)$ be the local zeta function of E:

$$\zeta_E(s) = \begin{cases} (1-q_E^{-s})^{-1} & \text{if } E \text{ is non-Archimedean} \\ \pi^{-s/2}\Gamma\left(\dfrac{s}{2}\right) & \text{if } E=\boldsymbol{R} \\ (2\pi)^{1-s}\Gamma(s) & \text{if } E=\boldsymbol{C} . \end{cases} \tag{6.1}$$

Let $\Xi=(\Xi_1, \dots, \Xi_n)\in X_{\mathrm{unr}}(T)$. Denote by $L_E(\Xi; s)$ the standard L-factor attached to Ξ given as follows:

$$L_E(\Xi; s)=\begin{cases}\prod_{i=1}^{n}(1-\Xi_i(\pi)q_E^{-s})^{-1} & \text{if } E \text{ is non-Archimedean}\\ \prod_{i=1}^{n}\zeta_E(s+\mu_i) & \text{if } E \text{ is Archimedean}\,,\end{cases} \tag{6.2}$$

where $\Xi_i(t)=|t|_E^{\mu_i}(\mu_i\in \boldsymbol{C})$ in the Archimedean case. For $\chi\in X_{\mathrm{unr}}(E^\times)$, we put

$$\chi\otimes\Xi=(\chi\Xi_1,\ldots,\chi\Xi_n)\in X_{\mathrm{unr}}(T)\,. \tag{6.3}$$

6.3. For $s\in \boldsymbol{C}$, we define a function $\nu_{E,s}$ on E^{n-1} as follows:
If E is non-Archimedean,

$$\nu_{E,s}(X)=\begin{cases}1 & \text{if } X\in L\\ q_E^{-ls} & \text{if } X\in\pi^{-l}L_{\mathrm{prim}}\ (l>0)\,,\end{cases} \tag{6.4a}$$

where $L=\mathfrak{o}_E^{n-1}$ and $L_{\mathrm{prim}}=L-\pi L$. If E is Archimedean, then

$$\nu_{E,s}(X)=\begin{cases}(1+{}^tXX)^{-s/2} & \text{if } E=\boldsymbol{R}\\ (1+{}^t\bar{X}X)^{-s} & \text{if } E=\boldsymbol{C}\,.\end{cases} \tag{6.4b}$$

Note that, in the non-Archimedean case, $\nu_s(X)=q_E^{-ls}$ if and only if

$$\begin{bmatrix}1_{n-1} & X\\ 0 & 1\end{bmatrix}\sim g(0^{(n-1)}, l^{(n-1)})$$

(cf. §2). The main result of this section is as follows:

6.4. THEOREM (the first integral formula). *Let* $\xi\in X_{\mathrm{unr}}(T_o)$, $\Xi\in X_{\mathrm{unr}}(T)$ *and assume that* $\mathrm{Re}(s)$ *is sufficiently large. For* $W\in \mathrm{Sh}(\xi,\Xi)$, *we have*

$$\int_{E^{n-1}}W\left(\begin{bmatrix}1_{n-1} & X\\ 0 & 1\end{bmatrix}\right)\nu_{E,s+(n-1)/2}(X)dX=\frac{L_E(\Omega\omega\otimes\Xi^{-1}; s)}{\zeta_E\left(s+\dfrac{n-1}{2}\right)L_E\left(\Omega\omega\otimes\xi; s+\dfrac{1}{2}\right)}W(1)\,.$$

Here $\Omega=\Xi_1\cdots\Xi_n$ *and* $\omega=\xi_1\cdots\xi_{n-1}$.

REMARK. Consider the non-Archimedean case and put $v_l=\mathrm{vol}(\{X\in E^{n-1}\,|\,\nu_s(X)=q_E^{-ls}\})$. Then we have

$$v_l=\begin{cases}1 & \text{if } l=0\\ q_E^{l(n-1)}(1-q_E^{1-n}) & \text{if } l>0\end{cases}$$

and the integral of the theorem is equal to $\sum_{l=0}^{\infty}v_lW(0^{(n-1)}, l^{(n-1)})q_E^{-l(s+(n-1)/2)}$. This implies that $W(0^{(n-1)}, l^{(n-1)})$ is uniquely determined by ξ, Ξ and $W(1)$, and that $W(0^{(n-1)}, l^{(n-1)})\in \boldsymbol{C}[\xi,\xi^{-1},\Xi,\Xi^{-1}]$ if $W(1)=1$, which completes the proof of Proposition 3.3 (and hence Theorem 3.1).

6.5. Throughout the remainder of this section, we assume that Re(s) is sufficiently large. For $r, r' \geq 1$, let $\sigma_{r,r'}$ be a function on $M_{r,r'}(E)$ given as follows: If E is non-Archimedean, $\sigma_{r,r'}$ is the characteristic function of $M_{r,r'}(\mathfrak{o}_E)$. If E is Archimedean,

$$\sigma_{r,r'}(X)=\begin{cases}\exp(-\pi\,\mathrm{tr}({}^tXX)) & \text{if } E=\boldsymbol{R}\\ \exp(-2\pi\,\mathrm{tr}({}^t\bar{X}X)) & \text{if } E=\boldsymbol{C}.\end{cases}$$

If $r=r'$, we write σ_r for $\sigma_{r,r}$. We often omit the subscripts (r, r') and r if there is no fear of confusion. The following result is elementary.

6.6. LEMMA. (i) *If*

$$X=\begin{array}{c c} & \begin{array}{cc} r & r' \end{array} \\ \begin{array}{c} r \\ r' \end{array} & \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{array} \in M_{r+r'}(E),$$

we have $\sigma_{r+r'}(X)=\sigma_r(a)\sigma_{r,r'}(b)\sigma_{r',r}(c)\sigma_{r'}(d)$.

(ii) $\int_{M_{r,r'}(E)}\sigma_{r,r'}(X)dX=1$.

(iii) $\int_{E^\times}|t|_E^s\sigma_1(t)d^\times t=\zeta_E(s)\quad(d^\times t=dt/|t|_E)$.

(iv) *Let* $\Xi\in X_{\mathrm{unr}}(T)$ *and define* $\phi_\Xi\colon G\to\boldsymbol{C}$ by (1.1). *Then*

$$\int_G\phi_\Xi(g)\sigma_n(g)|\det g|_E^{s+(n-1)/2}dg=L_E(\Xi; s).$$

6.7. LEMMA. *For* $X\in E^{n-1}$, *we have*

$$\int_{E^\times}\sigma_{n-1,1}(tX)\sigma_1(t)|t|^s d^\times t=\zeta_E(s)v_{E,s}(X). \tag{6.5}$$

PROOF. The proof in the Archimedean case is straightforward and we omit it. Suppose that E is non-Archimedean. The assertion is obvious if $X\in L=\mathfrak{o}_E^{n-1}$. Let $X\in\pi^{-1}L_{\mathrm{prim}}$ $(l>0)$. Since both sides of (6.5) is left $GL_{n-1}(\mathfrak{o}_E)$-invariant as a function of $X\in E^{n-1}$, we may assume that $X={}^t(\pi^{-l}, 0, \ldots, 0)$. Then (6.5) follows from an elementary formula $\int_{E^\times}\sigma(t\pi^{-l})\sigma(t)|t|^s d^\times t=\zeta_E(s)\cdot q_E^{-ls}$ $(l>0)$. q.e.d.

6.8. LEMMA. *Let* $W\in\mathrm{Sh}(\xi, \Xi)$ *and let* Φ (*resp.* φ) *be a bi-K-* (*resp. bi-*K_o*-*) *invariant function on G* (*resp.* G_o). *Then, for any* $g\in G$, *we have*

$$\int_G W(gy)\Phi(y)dy=W(g)\int_G\phi_\Xi(y)\Phi(y)dy, \tag{6.6}$$

$$\int_{G_o}W\left(\begin{bmatrix} x & 0\\ 0 & 1\end{bmatrix}g\right)\varphi(x)dx=W(g)\int_{G_o}\phi_{\xi^{-1}}(x)\varphi(x)dx, \tag{6.7}$$

if the integrals are convergent.

PROOF. The left-hand side of (6.6) is equal to $\int_G\Phi(y)F_g(y)dy$, where $F_g(y)=\int_K W(gky)dk$ $(y\in G)$. Observe that $F_g(y)$ satisfies (1.3) (resp. (5.9)) in the non-Archimedean case (resp. in the Archimedean case) and $F_g(1)=W(g)$. Then, by the uniqueness of spherical functions (see §1.2 and Lemma 5.4), we have $F_g(y)=W(g)\Psi_{\Xi}(y)$, which implies (6.6). The assertion (6.7) is proved similarly. q.e.d.

6.9. THE PROOF OF THEOREM 6.4. Let $W\in \mathrm{Sh}(\xi, \Xi)$. To prove the theorem, we calculate the integral

$$I_W(s)=\int_G W(g)\sigma(g^{-1})\Omega\omega(\det g^{-1})|\det g|_E^{-(s+(n-1)/2)}dg$$

in two ways. We first apply (6.6) to $I_W(s)$ and get

$$\begin{aligned}I_W(s)&=W(1)\int_G \phi_{\Omega^{-1}\omega^{-1}\otimes\Xi}(g)\sigma(g^{-1})|\det g|_E^{-(s+(n-1)/2)}dg\\&=W(1)\int_G \phi_{\Omega\omega\otimes\Xi^{-1}}(g)\sigma(g)|\det g|_E^{s+(n-1)/2}dg\,.\end{aligned}$$

By Lemma 6.6 (iv), we have

$$I_W(s)=W(1)L_E(\Omega\omega\otimes\Xi^{-1};s)\,. \tag{6.8}$$

Next decompose $g\in G$ into

$$g=\begin{bmatrix} g_o & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix} 1_{n-1} & X\\ 0 & t\end{bmatrix}k \quad (g_o\in G_o,\ X\in E^{n-1},\ t\in E^\times,\ k\in K)\,.$$

Then $dg=dg_o dX d^\times t dk$ and we have

$$\begin{aligned}I_W(s)=\int_{G_o}dg_o\int_{E^{n-1}}dX\int_{E^\times}d^\times t\, W\left(\begin{bmatrix} g_o & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix} 1_{n-1} & X\\ 0 & t\end{bmatrix}\right)\sigma(g_o^{-1})\sigma(t^{-1})\sigma(t^{-1}X)\\ \times(\Omega\omega)^{-1}(t\cdot\det g_o)|t|_E^{-(s+(n-1)/2)}|\det g_o|_E^{-(s+(n-1)/2)}\,.\end{aligned}$$

It follows from (6.7), Lemma 6.6 and (1.10) that

$$\begin{aligned}I_W(s)&=\int_{G_o}\phi_{\Omega\omega\otimes\xi}(g_o)\sigma(g_o)|\det g_o|_E^{s+(n-1)/2}dg_o\\&\quad\times\int_{E^{n-1}}dX\int_{E^\times}d^\times t\, W\left(\begin{bmatrix} 1_{n-1} & X\\ 0 & t\end{bmatrix}\right)\sigma(t^{-1})\sigma(t^{-1}X)(\Omega\omega)^{-1}(t)|t|_E^{-(s+(n-1)/2)}\\&=L_E\left(\Omega\omega\otimes\xi;s+\frac{1}{2}\right)\int_{E^{n-1}}\left(\int_{E^\times}\sigma(t)\sigma(tX)|t|_E^{s+(n-1)/2}d^\times t\right)W\left(\begin{bmatrix} 1_{n-1} & X\\ 0 & 1\end{bmatrix}\right)dX\,.\end{aligned}$$

By Lemma 6.7, we have

$$I_W(s)=\zeta_E\left(s+\frac{n-1}{2}\right)L_E\left(\Omega\omega\otimes\xi;s+\frac{1}{2}\right)\int_{E^{n-1}}W\left(\begin{bmatrix} 1_{n-1} & X\\ 0 & 1\end{bmatrix}\right)\nu_{E,s+(n-1)/2}(X)dX\,. \tag{6.9}$$

The theorem now follows from (6.8) and (6.9). q.e.d.

7. Integral formula.

7.1. We keep the notation of §6. Let $G_1=GL(n+1, E)$ and embed $G=GL(n, E)$ into G_1 via

$$\iota: g\mapsto\begin{bmatrix}1&0\\0&g\end{bmatrix}.$$

Note that

$$\iota(g_o)=\begin{bmatrix}1&&0\\&g_o&\\0&&1\end{bmatrix}$$

for $g_o\in G_o=GL(n-1, E)$. Let $P_1=N_1M_1$ be the standard parabolic subgroup of G_1 corresponding to the partition $n+1=1+(n-1)+1$, where

$$N_1=\left\{\begin{bmatrix}1&*&*\\0&1_{n-1}&*\\0&0&1\end{bmatrix}\in G_1\right\},\qquad M_1=\left\{\begin{bmatrix}t&&0\\&g_o&\\0&&t'\end{bmatrix}\middle|\, t, t'\in E^\times, g_o\in G_o\right\}.$$

Then we have the Iwasawa decomposition $G_1=P_1K_1$, where

$$K_1=\begin{cases}GL(n+1, \mathfrak{o}) & \text{if } E \text{ is non-Archimedean}\\ O(n+1, \boldsymbol{R}) & \text{if } E=\boldsymbol{R}\\ U(n+1) & \text{if } E=\boldsymbol{C}.\end{cases}$$

Put

$$\Upsilon_o=\begin{bmatrix}1&&0\\&1_{n-1}&\\1&&1\end{bmatrix}\in G_1 .$$

For $g\in G$, we decompose $\Upsilon_o\cdot\iota(g)$ into

$$n_1(g)\begin{bmatrix}\alpha(g)&&0\\&\beta(g)&\\0&&\alpha'(g)\end{bmatrix}k_1(g),$$

where $n_1(g)\in N_1$, $\alpha(g)$, $\alpha'(g)\in E^\times$, $\beta(g)\in G_o$ and $k_1(g)\in K_1$. For $W\in \mathrm{Sh}(\xi, \Xi)$ and $s, s'\in \boldsymbol{C}$, we define the integral

$$(7.1)\qquad Z_W(s, s')=\int_{G_o\backslash G} W(\beta(g)^{-1}g)|\alpha(g)|_E^{s+(n-1)/2}|\alpha'(g)|_E^{-(s'+(n-1)/2)}dg .$$

Note that the integrand does not depend on the choice of $\alpha(g)$, $\alpha'(g)$ and $\beta(g)$, and is left G_o-invariant.

7.2. THEOREM (the second integral formula). *Let* $W \in \mathrm{Sh}(\xi, \Xi)$ *and assume that* $\mathrm{Re}(s)$ *and* $\mathrm{Re}(s')$ *are sufficiently large. Then we have*

$$Z_W(s, s') = \frac{1}{\zeta_E(s+s')} \frac{L_E(\Xi; s) L_E(\Xi^{-1}; s')}{L_E\left(\xi^{-1}; s+\frac{1}{2}\right) L_E\left(\xi; s'+\frac{1}{2}\right)} W(1) .$$

7.3. To prove the theorem, we define a function $N_{G,s}$ on G as follows ($s \in \boldsymbol{C}$). Let

$$g = k \begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{bmatrix} k' \quad (k, k' \in K, t_1, \dots, t_n \in E^\times)$$

be a Cartan decomposition of $g \in G$. We may assume that $t_1, \dots, t_n > 0$ in the Archimedean case. If E is non-Archimedean, we put

$$N_{G,s}(g) = \prod_{\mathrm{ord}_E(t_i) < 0} |t_i|_E^{-s} . \tag{7.2a}$$

If E is Archimedean, we put

$$N_{G,s}(g) = \begin{cases} \prod_{i=1}^{n} (1+t_i^2)^{-s/2} & \text{if} \quad E = \boldsymbol{R} \\ \prod_{i=1}^{n} (1+t_i^2)^{-s} & \text{if} \quad E = \boldsymbol{C} . \end{cases} \tag{7.2b}$$

We define a function $N_{G_o,s}\colon G_o \to \boldsymbol{C}$ in a similar manner. It is easy to see that

$$N_{G,s}(kgk') = N_{G,s}(g) \tag{7.3}$$

$$N_{G,s}({}^t g^{-1}) = |\det g|_E^s \cdot N_{G,s}(g) \tag{7.4}$$

$$N_{G,s}(g) N_{G,s'}(g) = N_{G,s+s'}(g) \tag{7.5}$$

($g \in G$, $k, k' \in K$, $s, s' \in \boldsymbol{C}$). The following integral expression of $N_{G,s}$ is well-known.

7.4. LEMMA. *Assume that* $\mathrm{Re}(s)$ *is sufficiently large. For* $g \in G$, *we have*

$$\int_G \sigma(yg)\sigma(y) |\det y|_E^s dy = \zeta_E^{(n)}(s) N_{G,s}(g) ,$$

where $\zeta_E^{(n)}(s) = \prod_{i=0}^{n-1} \zeta_E(s-i)$.

7.5. LEMMA. *Let* $\Xi \in X_{\mathrm{unr}}(T)$ *and* $s_1, s_2 \in \boldsymbol{C}$. *Assume that* $\mathrm{Re}(s_1)$ *and* $\mathrm{Re}(s_2 - s_1)$ *are sufficiently large. Then we have*

(7.6) $$\int_G \phi_\Xi(g)|\det g|_E^{s_1} N_{G,s_2}(g)dg = \frac{L_E\left(\Xi; s_1 - \frac{n-1}{2}\right) L_E\left(\Xi^{-1}; s_2 - s_1 - \frac{n-1}{2}\right)}{\zeta_E^{(n)}(s_2)}.$$

PROOF. We write $A(\Xi; s_1, s_2)$ for the left-hand side of (7.6). Let $\Psi_\Xi(g) = \int_K \phi_\Xi(kg)dk$ be the zonal spherical function attached to Ξ (cf. §1.2 and §5.3). By Lemma 7.4, we have

$$\begin{aligned}\zeta_E^{(n)}(s_2)A(\Xi; s_1, s_2) &= \int_G dg \int_G dy\, \phi_\Xi(g)|\det g|_E^{s_1}\sigma(yg)\sigma(y)|\det y|_E^{s_2} \\ &= \int_G dg \int_G dy\, \Psi_\Xi(y^{-1}g)\sigma(g)\sigma(y)|\det g|_E^{s_1}|\det y|_E^{s_2-s_1}.\end{aligned}$$

By (1.6) (or (5.6) in the Archimedean case), $\zeta_E^{(n)}(s_2)A(\Xi; s_1, s_2)$ equals

$$\int_G \Psi_\Xi(g)\sigma(g)|\det g|_E^{s_1}dg \int_G \Psi_\Xi(y^{-1})\sigma(y)|\det y|_E^{s_2-s_1}dy.$$

The proposition now follows from Lemma 6.6 (iv) (note that $\Psi_\Xi(y^{-1}) = \Psi_{\Xi^{-1}}(y)$).

q.e.d.

7.6. PROPOSITION. *For $g \in G$, we have*

$$N_{G,s}(g) = N_{G_o,s}(\beta(g)) \cdot |\alpha'(g)|_E^{-s}$$
$$N_{G,s}({}^t g^{-1}) = N_{G_o,s}({}^t\beta(g)^{-1}) \cdot |\alpha(g)|_E^{s}.$$

PROOF. The latter formula is an immediate consequence of the former, since $|\det g|_E = |\alpha(g)\alpha'(g)\det\beta(g)|_E$. To prove the first formula, we may suppose that $\mathrm{Re}(s)$ is sufficiently large. For $g_1 \in G_1$, put

$$B_s(g_1) = \int_{G_o} dx \int_{E^\times} d^\times t \int_{E^{n-1}} dX\, \sigma\left(\begin{bmatrix} 0 & x & X \\ 0 & 0 & t \end{bmatrix} g_1\right)\sigma(x)|\det x|_E^s |t|_E^{s-n+1}.$$

Let

$$g_1 = \begin{bmatrix} \alpha & * & * \\ 0 & \beta & * \\ 0 & 0 & \alpha' \end{bmatrix} k_1 \quad (\alpha, \alpha' \in E^\times, \beta \in G_o, k_1 \in K_1)$$

be an Iwasawa decomposition of g_1. Applying Lemma 7.4 to G_o, we have

$$\begin{aligned}B_s(g_1) &= \int_{G_o} dx \int_{E^\times} d^\times t \int_{E^{n-1}} dX\, \sigma\left(\begin{bmatrix} 0 & x\beta & \alpha' X + * \\ 0 & 0 & \alpha' t \end{bmatrix}\right)\sigma(x)|\det x|_E^s |t|_E^{s-n+1} \\ &= \int_{G_o} \sigma(x\beta)\sigma(x)|\det x|_E^s dx \int_{E^\times} \sigma(\alpha' t)|t|_E^{s-n+1} d^\times t \int_{E^{n-1}} \sigma(\alpha' X)dX \\ &= \zeta_E^{(n-1)}(s) N_{G_o,s}(\beta)|\alpha'|_E^{-s}\zeta_E(s-n+1) = \zeta_E^{(n)}(s) N_{G_o,s}(\beta)|\alpha'|_E^{-s}.\end{aligned}$$

In view of Lemma 7.4, it now remains to show

(7.7) $$B_s(\Upsilon_o \cdot \iota(g)) = \int_G \sigma(yg)\sigma(y)|\det y|_E^s dy .$$

Since both sides of (7.7) is right K-invariant as functions in $g \in G$, we may suppose that g is of the form

$$\begin{bmatrix} g_o & b \\ 0 & a \end{bmatrix} \quad (g_o \in G_o, a \in E^\times, b \in E^{n-1}) .$$

Then the left-hand side of (7.7) equals

$$\int_{G_o} dx \int_{E^\times} d^\times t \int_{E^{n-1}} dX \, \sigma\left(\begin{bmatrix} X & xg_o & xb+aX \\ t & 0 & at \end{bmatrix}\right)\sigma(x)|\det x|_E^s |t|_E^{s-n+1} .$$

On the other hand, decomposing $y \in G$ into

$$k\begin{bmatrix} x & X \\ 0 & t \end{bmatrix} \quad (k \in K, x \in G_o, t \in E^\times) ,$$

we see that the right-hand side of (7.7) equals

$$\int_{G_o} dx \int_{E^\times} d^\times t \int_{E^{n-1}} dX \, \sigma\left(\begin{bmatrix} x & X \\ 0 & t \end{bmatrix} g\right)\sigma\left(\begin{bmatrix} x & X \\ 0 & t \end{bmatrix}\right)|\det x|_E^s |t|_E^{s-n+1}$$
$$= \int_{G_o} dx \int_{E^\times} d^\times t \int_{E^{n-1}} dX \, \sigma\left(\begin{bmatrix} xg_o & xb+aX \\ 0 & at \end{bmatrix}\right)\sigma\left(\begin{bmatrix} x & X \\ 0 & t \end{bmatrix}\right)|\det x|_E^s |t|_E^{s-n+1} .$$

This proves the proposition. q.e.d.

7.7. We now finish the proof of Theorem 7.2. Let $W \in \mathrm{Sh}(\xi, \Xi)$ and assume that $\mathrm{Re}(s)$ and $\mathrm{Re}(s')$ are sufficiently large. We calculate the integral

(7.8) $$J_W(s, s') = \int_G W(g) N_{G,s'+(n-1)/2}(g) N_{G,s+(n-1)/2}({}^t g^{-1}) dg$$

in two ways. By (7.4) and Lemma 6.8, $J_W(s, s')$ equals

$$\int_G W(g)|\det g|_E^{s+(n-1)/2} N_{G,s+s'+n-1}(g) dg$$
$$= W(1)\int_G \phi_\Xi(g)|\det g|_E^{s+(n-1)/2} N_{G,s+s'+n-1}(g) dg .$$

From Lemma 7.5, we get

(7.9) $$J_W(s, s') = \frac{L_E(\Xi; s) L_E(\Xi^{-1}; s')}{\zeta_E^{(n)}(s+s'+n-1)} W(1) .$$

On the other hand, by Proposition 7.6 and (7.4), $J_W(s, s')$ equals

$$\int_{G_o \backslash G} dg \int_{G_o} dg_o W(g_o g) N_{G_o, s+s'+n-1}(g_o \cdot \beta(g)) \\ \times |\alpha(g) \cdot \det(g_o \cdot \beta(g))|_E^{s+(n-1)/2} |\alpha'(g)|_E^{-(s'+(n-1)/2)}$$

(note that $\alpha(g_o g)=\alpha(g)$, $\alpha'(g_o g)=\alpha'(g)$ and $\beta(g_o g)=g_o \cdot \beta(g)$ for $g_o \in G_o$ and $g \in G$). Changing the variable g_o into $g_o \cdot \beta(g)^{-1}$ and applying Lemma 6.8 and Lemma 7.5, we get

$$J_W(s, s')=Z_W(s, s') \times \frac{L_E\left(\xi^{-1}; s+\frac{1}{2}\right) L_E\left(\xi; s'+\frac{1}{2}\right)}{\zeta_E^{(n-1)}(s+s'+n-1)}. \tag{7.10}$$

The theorem is a consequence of (7.9) and (7.10). q.e.d.

Part II. Global theory.

8. Global Shintani functions attached to automorphic forms.

8.1. Throughout Part II, we fix a finite extension E of $\boldsymbol{Q}$. Let $\mathscr{P}$ be the set of primes of E and $\mathscr{P}_f$ (resp. $\mathscr{P}_\infty$) the set of finite (resp. infinite) primes of E. For $v \in \mathscr{P}$, E_v stands for the completion of E at v and write $|\ |_v$ for the normalized valuation $|\ |_{E_v}$ (see §1.1 and §5.1). For $v \in \mathscr{P}_f$, let $\mathfrak{o}_v$ be the ring of integers of E_v and fix a prime element π_v of $\mathfrak{o}_v$. We put $q_v = \#(\mathfrak{o}_v/\pi_v \mathfrak{o}_v)$. The adele ring $A = A_E$ (resp. the idele group $A^\times = A_E^\times$) of E is the restricted direct product of E_v (resp. $E_v^\times$) with respect to $\prod_{v \in \mathscr{P}_f} \mathfrak{o}_v$ (resp. $\prod_{v \in \mathscr{P}_f} \mathfrak{o}_v^\times$). We write $|a|_A$ for the idele norm of $a \in A^\times$: $|a|_A = \prod_{v \in \mathscr{P}} |a|_v$. Denote by d_E the discriminant of E. We set

$$\xi_E(s) = |d_E|^{s/2} \prod_{v \in \mathscr{P}} \zeta_{E_v}(s), \qquad \xi_E^{(n)}(s) = \prod_{i=0}^{n-1} \xi_E(s-i) \tag{8.1}$$

(for the definition of $\zeta_{E_v}(s)$, see §6.2). The (completed) Dedekind zeta function $\xi_E(s)$ is holomorphic except at simple poles $s=0$ and $s=1$, and satisfies the functional equation $\xi_E(s) = \xi_E(1-s)$.

8.2. We consider $G = GL(n)$ as a linear algebraic group defined over E and denote by $G_A = G(A)$ the adelization of G over E. Throughout Part II, we define the Haar measure dg on G_A to be the product measure $\prod_{v \in \mathscr{P}} dg_v$, where each Haar measure dg_v on G_{E_v} is normalized as in §1.1 and §5.1. For $v \in \mathscr{P}$, let K_v be a maximal compact subgroup of $G_v = G(E_v)$ given by

$$K_v = \begin{cases} GL(n, \mathfrak{o}_v) & \text{if } v \in \mathscr{P}_f \\ O(n, \boldsymbol{R}) & \text{if } E_v = \boldsymbol{R} \\ U(n) & \text{if } E_v = \boldsymbol{C}. \end{cases}$$

Denote by $C^\infty(G_E \backslash G_A / K_A)$ the space of smooth functions on $G_E \backslash G_A / K_A$, where $K_A = \prod_{v \in \mathscr{P}} K_v$. Let T be the group of diagonal matrices in G and $W_G = N_G(T)/T$ the Weyl group of (G, T).

8.3. Let $v \in \mathscr{P}_\infty$. Denote by $\mathscr{Z}_{G_v}$ the center of the universal enveloping algebra of $\mathrm{Lie}(G_v) \otimes_{\boldsymbol{R}} \boldsymbol{C}$. Then $\mathscr{Z}_{G_v}$ acts on $C^\infty(G_E \backslash G_{\boldsymbol{A}}/K_{\boldsymbol{A}})$ via right translations. Recall that $\mathrm{Hom}_{\boldsymbol{C}}(\mathscr{Z}_{G_v}, \boldsymbol{C}) = \{\Xi^\wedge \mid \Xi \in X_{\mathrm{unr}}(T_v)/W_G\}$ (cf. §5.2). Next let $v \in \mathscr{P}_f$. The Hecke algebra $\mathscr{H}(G_v, K_v)$ acts on $C^\infty(G_E \backslash G_{\boldsymbol{A}}/K_{\boldsymbol{A}})$ via

$$(F * \Phi)(g) = \int_{G_v} F(gy)\Phi(y)dy \qquad (F \in C^\infty(G_E \backslash G_{\boldsymbol{A}}/K_{\boldsymbol{A}}),\ \Phi \in \mathscr{H}(G_v, K_v)),$$

Recall that $\mathrm{Hom}_{\boldsymbol{C}}(\mathscr{H}(G_v, K_v), \boldsymbol{C}) = \{\Xi^\wedge \mid \Xi \in X_{\mathrm{unr}}(T_v)/W_G\}$ (cf. §1.2).

8.4. Let Ω be a Hecke character of E unramified everywhere. By definition, Ω is a continuous homomorphism of $E^\times \backslash \boldsymbol{A}^\times$ to $\boldsymbol{C}^\times$ trivial on $\prod_{v \in \mathscr{P}_f} \mathfrak{o}_v^\times \times \prod_{v \in \mathscr{P}_\infty} E_v^1$, where $E_v^1 = \{t \in E_v^\times \mid |t|_v = 1\}$ for $v \in \mathscr{P}_\infty$. Denote by $C^\infty(G_E \backslash G_{\boldsymbol{A}}/K_{\boldsymbol{A}}; \Omega)$ the space of $F \in C^\infty(G_E \backslash G_{\boldsymbol{A}}/K_{\boldsymbol{A}})$ satisfying $F(tg) = \Omega(t)F(g)$ $(g \in G_{\boldsymbol{A}},\ t \in \boldsymbol{A}^\times)$. Let $\Xi = (\Xi_v) \in \prod_{v \in \mathscr{P}} X_{\mathrm{unr}}(T_v)$. Under the assumption $\Xi_{v,1} \cdots \Xi_{v,n} = \Omega_v$ for every $v \in \mathscr{P}$, we let $\mathscr{A}(G_E \backslash G_{\boldsymbol{A}}/K_{\boldsymbol{A}}; \Omega; \Xi)$ be the space of $F \in C^\infty(G_E \backslash G_{\boldsymbol{A}}/K_{\boldsymbol{A}}; \Omega)$ satisfying the following conditions:

(8.2) For every $v \in \mathscr{P}_f$, we have $F * \Phi = \Xi_v^\wedge(\Phi)F \quad (\Phi \in \mathscr{H}(G_v, K_v))$.

(8.3) For every $v \in \mathscr{P}_\infty$, we have $R_Z F = \Xi_v^\wedge(Z)F \quad (Z \in \mathscr{Z}_{G_v})$.

(8.4) F is slowly increasing on $G_E \backslash G_{\boldsymbol{A}}$ (cf. [G-J, §10]).

We call $\mathscr{A}(G_E \backslash G_{\boldsymbol{A}}; \Omega; \Xi)$ *the space of automorphic forms on G with eigenvalues* Ξ. By definition, $F \in \mathscr{A}(G_E \backslash G_{\boldsymbol{A}}/K_{\boldsymbol{A}}; \Omega; \Xi)$ is *cuspidal* if $\int_{N_E \backslash N_{\boldsymbol{A}}} F(ng)dn = 0$ $(g \in G_{\boldsymbol{A}})$ for the unipotent radical N of any proper parabolic subgroup of G. Let $\mathscr{A}_{\mathrm{cusp}}(G_E \backslash G_{\boldsymbol{A}}/K_{\boldsymbol{A}}; \Omega; \Xi) = \{F \in \mathscr{A}(G_E \backslash G_{\boldsymbol{A}}/K_{\boldsymbol{A}}; \Omega; \Xi) \mid F \text{ is cuspidal}\}$. If $F \in \mathscr{A}_{\mathrm{cusp}}(G_E \backslash G_{\boldsymbol{A}}/K_{\boldsymbol{A}}; \Omega; \Xi)$, F is rapidly decreasing on $G_E \backslash G_{\boldsymbol{A}}$ (cf. [G-J, §10]).

8.5. Let $F \in \mathscr{A}(G_E \backslash G_{\boldsymbol{A}}/K_{\boldsymbol{A}}; \Omega; \Xi)$. We define the (completed) standard L-function $\xi(F; s)$ by

$$\xi(F; s) = |d_E|^{ns/2} \prod_{v \in \mathscr{P}} L_{E_v}(\Xi_v; s), \tag{8.5}$$

where $L_{E_v}(\Xi_v; s)$ is defined by (6.2). Put

$$\tilde{F}(g) = F({}^t g^{-1}). \tag{8.6}$$

Then we see that $\tilde{F} \in \mathscr{A}(G_E \backslash G_{\boldsymbol{A}}/K_{\boldsymbol{A}}; \Omega^{-1}; \Xi^{-1})$ with $\Xi^{-1} = (\Xi_v^{-1})_{v \in \mathscr{P}}$ and that $\xi(\tilde{F}; s) = \prod_{v \in \mathscr{P}} L_{E_v}(\Xi_v^{-1}; s)$. It is known (cf. [G-J, §13]) that $\xi(F; s)$ is continued to a meromorphic function of s on $\boldsymbol{C}$ and satisfies the functional equation

$$\xi(F; s) = \xi(\tilde{F}; 1 - s). \tag{8.7}$$

Moreover, $\xi(F; s)$ is entire if F is a cusp form. For a Hecke character χ of E, define the twisted L-function of F by $\xi(\chi \otimes F; s) = |d_E|^{ns/2} \prod_{v \in \mathscr{P}} L_{E_v}(\chi_v \otimes \Xi_v; s)$ (cf. §6.2).

8.6. Recall that $G_o = GL(n-1)$ is embedded into G via

$$g_o \mapsto \begin{bmatrix} g_o & 0 \\ 0 & 1 \end{bmatrix}.$$

Let (Ω, ω) be a pair of Hecke characters of E both unramified everywhere. We put

$$C^\infty_{L^2}(G_{o,E}\backslash G_{o,A}/K_{o,A};\ \omega)=\left\{f\in C^\infty(G_{o,E}\backslash G_{o,A}/K_{o,A};\ \omega)\ \middle|\ \int_{G_{o,E}\backslash G^1_{o,A}} |f(x)|^2 dx<\infty\right\},$$

where $G^1_{o,A}=\{x\in G_{o,A} \mid |\det x|_A=1\}$.

LEMMA. *Let $f\in C^\infty_{L^2}(G_{o,E}\backslash G_{o,A}/K_{o,A};\ \omega)$ and $F\in C^\infty(G_E\backslash G_A/K_A;\ \Omega)$. Assume that F is rapidly decreasing on $G_E\backslash G_A$. Then the integral*

$$W_{f,F}(g)=\int_{G_{o,E}\backslash G_{o,A}} f(x)F(xg)dx \tag{8.8}$$

converges absolutely and uniformly for g in a compact subset of G_A.

PROOF. Let C be a compact subset of G_A. The estimate due to Jacquet and Shalika [J-S 2, p. 799] asserts that for every $N>0$ there exists a positive constant $c=c(C, N)$ depending only on C and N such that

$$|F(xg)|\le c\cdot \mathrm{Inf}(|\det x|_A^{-N}, |\det x|_A^N) \qquad (x\in G_{o,A}, g\in C).$$

Then the integral (8.8) is majorized by

$$c'\left\{\int_{G_{o,E}\backslash G^1_{o,A}} |f(x)|^2 dx\right\}^{1/2} \times \int_0^\infty t^{\mu_\omega}\cdot \mathrm{Inf}\{t^{-(n-1)N}, t^{(n-1)N}\}d^\times t,$$

where c' is a positive constant depending only on C and N, and $\mu_\omega\in\boldsymbol{R}$ is defined by $|\omega(a)|=|a|_A^{\mu_\omega}$ $(a\in A^\times)$. If we take N sufficiently large, the last integral is convergent and we are done. q.e.d.

8.7. Suppose that $\Xi=(\Xi_v)\in\prod_{v\in\mathscr{P}} X_{\mathrm{unr}}(T_v)$ and $\xi=(\xi_v)\in\prod_{v\in\mathscr{P}} X_{\mathrm{unr}}(T_{o,v})$ satisfy $\Xi_{v,1}\cdots\Xi_{v,n}=\Omega_v$ and $\xi_{v,1}\cdots\xi_{v,n-1}=\omega_v$ for every $v\in\mathscr{P}$ with certain Hecke characters Ω and ω, where T (resp. T_o) is the group of diagonal matrices in G (resp. G_o). We set

$$\mathscr{A}_{L^2}(G_{o,E}\backslash G_{o,A}/K_{o,A};\ \omega;\ \xi)=\mathscr{A}(G_{o,E}\backslash G_{o,A}/K_{o,A};\ \omega;\ \xi)\cap C^\infty_{L^2}(G_{o,E}\backslash G_{o,A}/K_{o,A};\ \omega).$$

Let $F\in\mathscr{A}_{\mathrm{cusp}}(G_E\backslash G_A/K_A;\ \Omega;\ \Xi)$ and $f\in\mathscr{A}_{L^2}(G_{o,E}\backslash G_{o,A}/K_{o,A};\ \omega;\ \xi)$. We call the function $W_{f,F}(g)$ on G_A defined by (8.8) *the global Shintani function attached to* (f, F). Since the restriction of $W_{f,F}$ to G_{E_v} is in $\mathrm{Sh}(\xi_v, \Xi_v)$, the uniqueness of local Shintani functions at the non-Archimedean primes (Theorem 3.1) implies

$$W_{f,F}(g)=W_\infty(g_\infty)\prod_{v\in\mathscr{P}_f} W_v(g_v), \tag{8.9}$$

where $g=g_\infty\prod_{v\in\mathscr{P}_f} g_v\in G_A$ with $g_\infty\in G_\infty=\prod_{v\in\mathscr{P}_\infty} G_v$, W_∞ is the restriction of $W_{f,F}$ to G_∞ and, for $v\in\mathscr{P}_f$, W_v is the element of $\mathrm{Sh}(\xi_v, \Xi_v)$ with $W_v(1)=1$.

8.8. Let $f \in \mathscr{A}(G_{o,E} \backslash G_{o,A}/K_{o,A}; \omega; \xi)$ and $F \in \mathscr{A}_{\mathrm{cusp}}(G_E \backslash G_A/K_A; \Omega; \Xi)$ (we do not suppose that $f \in \mathscr{A}_{L^2}$). In the next section, we need the following twisted form of the global Shintani function:

$$W_{f,F}(g; s) = \int_{G_{o,E} \backslash G_{o,A}} f(x)F(xg)|\det x|_A^s dx \qquad (g \in G_A,\ s \in \boldsymbol{C}). \tag{8.10}$$

Due to the results of Piatetski-Shapiro [PS, §2], we have the following:

(i) The integral (8.10) is absolutely convergent in the half plane $\mathrm{Re}(s) > c$ for some c. (Note that the integral is absolutely convergent for any $s \in \boldsymbol{C}$ if $f \in \mathscr{A}_{L^2}(G_{o,E} \backslash G_{o,A}/K_{o,A}; \omega; \xi)$ (cf. Lemma 8.6).)

(ii) The function $W_{f,F}(g; s)$ is continued to an entire function of s on $\boldsymbol{C}$ and satisfies the functional equation $W_{f,F}(g; s) = W_{\bar{f},\bar{F}}({}^t g^{-1}; -s)$.

(iii) We have

$$W_{f,F}(g; s) = \int_{N_{o,A} \backslash G_{o,A}} W_{f,\bar{\psi}}(x) W_{F,\psi}\left(\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} g\right)|\det x|_A^s dx ,$$

where $W_{f,\bar{\psi}}$ and $W_{F,\psi}$ are the usual Whittaker functions attached to f and F:

$$W_{f,\bar{\psi}}(x) = \int_{N_{o,E} \backslash N_{o,A}} f(nx)\bar{\psi}\left(\sum_{i=1}^{n-2} n_{i,i+1}\right) dn ,$$

$$W_{F,\psi}(g) = \int_{N_E \backslash N_A} F(ng)\psi\left(\sum_{i=1}^{n-1} n_{i,i+1}\right) dn$$

(ψ is a nontrivial additive character of $E \backslash A$). Note that this implies $W_{f,F}(g; s) \equiv 0$ unless f is generic.

Moreover, we have the following Euler product for $W_{f,F}(1; s)$ (cf. [J-S 2]; see also [Bu]):

$$W_{f,F}(1; s) = Z_\infty(f \otimes F; s) L\left(f \otimes F; s + \frac{1}{2}\right), \tag{8.11}$$

where

$$Z_\infty(f \otimes F; s) = \int_{N_{o,\infty} \backslash G_{o,\infty}} W_{f,\bar{\psi}}(x) W_{F,\psi}\left(\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}\right)|\det x|_\infty^s dx$$

and $L(f \otimes F; s)$ is the tensor L-function of the pair (f, F):

$$L(f \otimes F; s) = \prod_{v \in \mathscr{P}_f} \left\{ \prod_{1 \le i \le n-1} \prod_{1 \le j \le n} (1 - \xi_{i,v}(\pi_v) \Xi_{j,v}(\pi_v) q_v^{-s}) \right\}^{-1} .$$

9. Rankin-Selberg convolution (I).

9.1. In this section, we give an application of the first integral formula stated in

§6 to an integral expression of the automorphic L-functions for $GL(n)$. For $s\in \boldsymbol{C}$ and $X=\prod_{v\in\mathscr{P}} X_v \in \boldsymbol{A}^{n-1} (X_v\in E_v^{n-1})$, we put

$$(9.1)\qquad v_s(X)=\prod_{v\in\mathscr{P}} v_{E_v,s}(X_v)\,,$$

where $v_{E_v,s}: E_v^{n-1}\to \boldsymbol{C}$ is given by (6.4).

9.2. PROPOSITION. *Let* $F\in\mathscr{A}_{\mathrm{cusp}}(G_E\backslash G_{\boldsymbol{A}}/K_{\boldsymbol{A}};\Omega;\Xi)$ *and* $f\in\mathscr{A}(G_{o,E}\backslash G_{o,\boldsymbol{A}}/K_{o,\boldsymbol{A}};\omega;\xi)$. *If* $\mathrm{Re}(s_1)$ *and* $\mathrm{Re}(s_2)$ *are sufficiently large, then*

$$\int_{\boldsymbol{A}^{n-1}} W_{f,F}\left(\begin{bmatrix} 1_{n-1} & X \\ 0 & 1\end{bmatrix}; s_1\right) v_{s_2}(X)dX$$

$$=|d_E|^{n-1}\frac{\xi\left(\Omega\omega\otimes\tilde{F};(n-1)s_1+s_2-\dfrac{n-1}{2}\right)}{\xi_E(s_2)\xi\left(\Omega\omega\otimes f; ns_1+s_2-\dfrac{n}{2}+1\right)}\times W_{f,F}(1;s_1)\,.$$

(For the definition of $W_{f,F}(g;s)$, see §8.8.)

PROOF. Observe that the restriction of $W_{f,F}(*;s)$ to G_{E_v} belongs to $\mathrm{Sh}(\xi_v^s,\Xi_v)$, where $\xi_v^s=(\xi_{v,i}\cdot|\ |_{E_v}^s)_{1\le i\le n-1}$. Then the proposition is an immediate consequence of Theorem 6.4. q.e.d.

9.3. In the remainder of this section, we let $F\in\mathscr{A}_{\mathrm{cusp}}(G_E\backslash G_{\boldsymbol{A}}/K_{\boldsymbol{A}};\Omega;\Xi)$ and $f\in\mathscr{A}_{L^2}(G_{o,E}\backslash G_{o,\boldsymbol{A}}/K_{o,\boldsymbol{A}};\omega;\xi)$. To define a Rankin-Selberg convolution, we introduce certain Eisenstein series on $G=GL(n)$. Let P and Q be the standard maximal parabolic subgroups of G of types $(n-1,1)$ and $(1,n-1)$, respectively. Namely, $P=N_PM_P$ and $Q=N_QM_Q$ where

$$N_P=\left\{\begin{bmatrix} 1_{n-1} & X\\ 0 & 1\end{bmatrix}\right\},\quad M_P=\left\{\begin{bmatrix} g_o & 0\\ 0 & t\end{bmatrix}\middle|\, g_o\in G_o=GL(n-1),\ t\in GL(1)\right\},$$

$$N_Q=\left\{\begin{bmatrix} 1 & X'\\ 0 & 1_{n-1}\end{bmatrix}\right\},\quad M_Q=\left\{\begin{bmatrix} t & 0\\ 0 & g_o\end{bmatrix}\middle|\, g_o\in G_o,\ t\in GL(1)\right\}.$$

Note that P and Q are not conjugate in G if $n\ge 3$. Let δ_P and δ_Q be the modules of $P_{\boldsymbol{A}}$ and $Q_{\boldsymbol{A}}$. We use the same letters δ_P and δ_Q to denote their natural extensions to $G_{\boldsymbol{A}}$. Namely,

$$(9.2)\qquad \delta_P\left(\begin{bmatrix} g_o & *\\ 0 & t\end{bmatrix}k\right)=\left|\frac{\det g_o}{t^{n-1}}\right|_{\boldsymbol{A}}$$

$$(9.3)\qquad \delta_Q\left(\begin{bmatrix} t & *\\ 0 & g_o\end{bmatrix}k\right)=\left|\frac{t^{n-1}}{\det g_o}\right|_{\boldsymbol{A}}$$

($g_o \in G_{o,A}$, $t \in A^\times$, $k \in K_A$). For $g \in G_A$ and $s \in C$, we get

(9.4)
$$\phi(g; s; f, \Omega^{-1}; P) = (\omega\Omega)^{-1}(t) f(g_o) \delta_P(g)^{s/n+1/2}$$
$$\left(g = \begin{bmatrix} g_o & * \\ 0 & t \end{bmatrix} k,\ n \in N_A,\ g_o \in G_{o,A},\ t \in A^\times,\ k \in K_A\right)$$

(9.5)
$$\phi(g; s; \mathbf{1}; Q) = \delta_Q(g)^{s/n+1/2}.$$

Define the Eisenstein series as follows (cf. [J-S1]):

(9.6)
$$E(g; s; f, \Omega^{-1}; P) = \sum_{\gamma \in P_E \backslash G_E} \phi(\gamma g; s; f, \Omega^{-1}; P)$$

(9.7)
$$E(g; s; \mathbf{1}; Q) = \sum_{\gamma \in Q_E \backslash G_E} \phi(\gamma g; s; \mathbf{1}; Q).$$

The series are absolutely convergent if $\mathrm{Re}(s)$ is sufficiently large. Put

(9.8)
$$E^*(g; s; f, \Omega^{-1}; P) = \xi(\Omega\omega \otimes f; s+1) E(g; s; f, \Omega^{-1}; P).$$

Then $E^*(g; s; f, \Omega^{-1}; P)$ is continued to an entire function of s and satisfies the functional equation $E^*(g; s; f, \Omega^{-1}; P) = E^*({}^t g^{-1}; -s; \tilde{f}, \Omega; P)$. Next set

(9.9)
$$E^*(g; s; \mathbf{1}; Q) = \xi_E\left(s + \frac{n}{2}\right) E(g; s; \mathbf{1}; Q).$$

Then $E^*(g; s; \mathbf{1}; Q)$ is continued to a meromorphic function of s on C, holomorphic except at simple poles $s = n/2$ and $-n/2$ with residues $|d_E|$ and $-|d_E|$, respectively, and satisfies the functional equation $E^*(g; s; \mathbf{1}; Q) = E^*({}^t g^{-1}; -s; \mathbf{1}; Q)$. The normalized Eisenstein series (9.8) and (9.9) are slowly increasing functions of g on $G_E \backslash G_A$ with central characters Ω^{-1} and $\mathbf{1}$ (the trivial character), respectively. We now define a convolution attached to (f, F) of Rankin-Selberg type by

(9.10)
$$Z^*_{f,F}(s_1, s_2) = \int_{Z_A G_E \backslash G_A} F(g) E^*(g; s_1; f, \Omega^{-1}; P) E^*(g; s_2; \mathbf{1}; Q) dg.$$

The integral (9.10) is absolutely convergent if $\mathrm{Re}(s_1)$ and $\mathrm{Re}(s_2)$ are sufficiently large. By the properties of the Eisenstein series stated above, $Z^*_{f,F}(s_1, s_2)$ is continued to an entire function of (s_1, s_2) on C^2 (note that the Eisenstein series is orthogonal to any cusp forms). The main result of this section is stated as follows:

9.4. THEOREM. *We have*
$$Z^*_{f,F}(s_1, s_2) = |d_E|^{n-1} \xi\left(\Omega\omega \otimes \tilde{F}; \frac{n-1}{n} s_1 + \frac{1}{n} s_2 + \frac{1}{2}\right) W_{f,F}\left(1; \frac{s_1 - s_2}{n}\right).$$

9.5. To prove the theorem, we need some preparation. For i, j ($1 \le i, j \le n$, $i \ne j$), put $U_{ij}=\{1_n+a\cdot E_{ij}\mid a\in E\}$, where $E_{ij}=(\delta_{ik}\delta_{jl})_{1\le k,l\le n}\in M_n(E)$. For j $(1\le j\le n)$, let $w_j\in G_E$ be the permutation matrix corresponding to the transposition $(1\,j)$ and put $U_j=\prod_{1\le i<j}U_{ij}$. Note that U_j is a subgroup of G_E and $U_n=N_{P,E}$ (cf. §9.3). The Bruhat decomposition for G implies

$$E(g;s;\mathbf{1};Q)=\sum_{j=1}^{n}\sum_{u\in U_j}\phi(w_jug;s;\mathbf{1};Q)\,. \tag{9.11}$$

9.6. The following result is elementary and we omit its proof.

LEMMA. *For $X\in A^{n-1}$ and $s\in \mathbf{C}$, we have*

$$\phi\left(w_n\begin{bmatrix}1_{n-1} & X\\ 0 & 1\end{bmatrix};s;\mathbf{1};Q\right)=v_{s+n/2}(X)\,.$$

9.7. THE PROOF OF THEOREM 9.4. Set

$$Z_{f,F}(s_1,s_2)=\int_{Z_AG_E\backslash G_A}F(g)E(g;s_1;f,\Omega^{-1};P)E(g;s_2;\mathbf{1};Q)dg\,.$$

Note that $Z^*_{f,F}(s_1,s_2)=\xi_E(s_2+n/2)\xi(\Omega\omega\otimes f;s_1+1)Z_{f,F}(s_1,s_2)$. Unwinding the Eisenstein series $E(g;s_1;f,\Omega^{-1};P)$, we have

$$\begin{aligned}Z_{f,F}(s_1,s_2)&=\int_{Z_AP_E\backslash P_A}F(p)\phi(p;s_1;f,\Omega^{-1};P)E(p;s_2;\mathbf{1};Q)d_lp\\&=\int_{G_{o,E}\backslash G_{o,A}}dg_o\int_{E^{n-1}\backslash A^{n-1}}dX\,F\left(\begin{bmatrix}1_{n-1} & X\\ 0 & 1\end{bmatrix}\begin{bmatrix}g_o & 0\\ 0 & 1\end{bmatrix}\right)f(g_o)|\det g_o|_A^{s_1/n-1/2}\\&\quad\times E\left(\begin{bmatrix}1_{n-1} & X\\ 0 & 1\end{bmatrix}\begin{bmatrix}g_o & 0\\ 0 & 1\end{bmatrix};s_2;\mathbf{1};Q\right),\end{aligned}$$

where

$$d_lp=|\det g_o|^{-1}dg_odX\left(p=\begin{bmatrix}1_{n-1} & X\\ 0 & 1\end{bmatrix}\begin{bmatrix}g_o & 0\\ 0 & 1\end{bmatrix}\right)$$

is a left invariant measure on $Z_A\backslash P_A$. By (9.11), $Z_{f,F}(s_1,s_2)$ equals

$$\begin{aligned}&\sum_{j=1}^{n}\int_{G_{o,E}\backslash G_{o,A}}dg_o\int_{E^{n-1}\backslash A^{n-1}}dX\,F\left(\begin{bmatrix}1_{n-1} & X\\ 0 & 1\end{bmatrix}\begin{bmatrix}g_o & 0\\ 0 & 1\end{bmatrix}\right)f(g_o)|\det g_o|_A^{s_1/n-1/2}\\&\quad\times\sum_{u\in U_j}\phi\left(w_ju\begin{bmatrix}1_{n-1} & X\\ 0 & 1\end{bmatrix}\begin{bmatrix}g_o & 0\\ 0 & 1\end{bmatrix};s_2;\mathbf{1};Q\right).\end{aligned}$$

We claim that the term for j with $1\le j\le n-1$ vanishes. Observe that every w_ju $(u\in U_j)$ normalizes $N_{P,A}$ and that $\phi(g;s;\mathbf{1};Q)$ is a left $N_{P,A}$-invariant function of g. Then we have

$$\int_{E^{n-1}\backslash A^{n-1}} F\left(\begin{bmatrix} 1_{n-1} & X \\ 0 & 1 \end{bmatrix}\begin{bmatrix} g_o & 0 \\ 0 & 1 \end{bmatrix}\right)\sum_{u\in U_j}\phi\left(w_j u\begin{bmatrix} 1_{n-1} & X \\ 0 & 1 \end{bmatrix}\begin{bmatrix} g_o & 0 \\ 0 & 1 \end{bmatrix}; s_2; \mathbf{1}; Q\right)dX$$
$$=\sum_{u\in U_j}\phi\left(w_j u\begin{bmatrix} g_o & 0 \\ 0 & 1 \end{bmatrix}; s_2; \mathbf{1}; Q\right)\int_{E^{n-1}\backslash A^{n-1}} F\left(\begin{bmatrix} 1_{n-1} & X \\ 0 & 1 \end{bmatrix}\begin{bmatrix} g_o & 0 \\ 0 & 1 \end{bmatrix}\right)dX=0$$

by the cuspidality of F, which proves our claim. Thus $Z_{f,F}(s_1, s_2)$ equals

$$\int_{G_{o,E}\backslash G_{o,A}} dg_o \int_{A^{n-1}} dX F\left(\begin{bmatrix} 1_{n-1} & X \\ 0 & 1 \end{bmatrix}\begin{bmatrix} g_o & 0 \\ 0 & 1 \end{bmatrix}\right) f(g_o)|\det g_o|_A^{s_1/n-1/2}$$
$$\times\phi\left(w_n\begin{bmatrix} 1_{n-1} & X \\ 0 & 1 \end{bmatrix}\begin{bmatrix} g_o & 0 \\ 0 & 1 \end{bmatrix}; s_2; \mathbf{1}; Q\right).$$

Since

$$\phi\left(w_n\begin{bmatrix} 1_{n-1} & X \\ 0 & 1 \end{bmatrix}\begin{bmatrix} g_o & 0 \\ 0 & 1 \end{bmatrix}; s_2; \mathbf{1}; Q\right)=|\det g_o|_A^{-(s_2/n+1/2)} v_{s_2+n/2}(g_o^{-1}X)$$

by Lemma 9.6, $Z_{f,F}(s_1, s_2)$ equals

$$\int_{A^{n-1}}\left\{\int_{G_{o,E}\backslash G_{o,A}} F\left(\begin{bmatrix} g_o & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1_{n-1} & X \\ 0 & 1 \end{bmatrix}\right) f(g_o)|\det g_o|_A^{(s_1-s_2)/n} dg_o\right\} v_{s_2+n/2}(X)dX$$
$$=\int_{A^{n-1}} W_{f,F}\left(\begin{bmatrix} 1_{n-1} & X \\ 0 & 1 \end{bmatrix}; \frac{s_1-s_2}{n}\right) v_{s_2+n/2}(X)dX.$$

The theorem now follows from Proposition 9.2. q.e.d

9.8. REMARK. In view of Theorem 9.4 and §8.8 (iii), we have proved that the product $E(g; s_1; f, \Omega^{-1}; P)E(g; s_2; \mathbf{1}; Q)$ has no *cuspidal component* unless f is generic.

10. Orbit decomposition.

10.1. Let $G_1=GL(n+1)$. We often regard $G=GL(n)$ (and $G_o=GL(n-1)$) as a subgroup of G_1 via the embedding

$$\imath(g)=\begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix} \quad (g\in G).$$

Let P_1 be the standard parabolic subgroup of G_1 corresponding to the partition $n+1=1+(n-1)+1$ as in §7.1. Recall that $P_1=N_1M_1$, where

$$N_1=\left\{\begin{bmatrix} 1 & * & * \\ 0 & 1_{n-1} & * \\ 0 & 0 & 1 \end{bmatrix}\in G_1\right\} \quad\text{and}\quad M_1=\left\{\begin{bmatrix} t & & \\ & g_o & \\ & & t' \end{bmatrix}\middle| t, t'\in GL(1), g_o\in G_o\right\}.$$

For i $(1\le i\le n+1)$, let $e_i={}^t(0, \dots, 0, 1, 0, \dots, 0)\in E^{n+1}$ be the vector with the i-th component 1 and the others 0. Then $P_1=\{g_1\in G_1\,|\,g_1\cdot e_1=\lambda e_1, {}^tg^{-1}\cdot e_{n+1}=\lambda' e_{n+1}$ $(\lambda, \lambda'\neq 0)\}$.

10.2. Let $\mathscr{X}=\{(x, y)\in E^{n+1}\times E^{n+1}\,|\,\langle x, y\rangle:={}^txy=1\}$. The group $G_{1,E}$ acts on $\mathscr{X}$ transitively by $g_1\cdot(x, y)=(g_1x, {}^tg_1^{-1}y)$. Then G_E is the isotropy subgroup of $(e_1, e_1)\in\mathscr{X}$ in $G_{1,E}$.

10.3. Define the elements Υ_i $(0\leq i\leq 5)$ of $G_{1,E}$ as follows:

$$(10.1)\quad \Upsilon_o=\begin{bmatrix}1&0&0\\0&1_{n-1}&0\\1&0&1\end{bmatrix},\quad \Upsilon_1=\begin{bmatrix}1&0&0\\e&1_{n-1}&0\\0&0&1\end{bmatrix},\quad \Upsilon_2=1_{n+1},\quad \Upsilon_3=\begin{bmatrix}0&0&-1\\0&1_{n-1}&0\\1&-{}^te&0\end{bmatrix},$$

$$\Upsilon_4=\begin{bmatrix}0&0&-1\\0&1_{n-1}&0\\1&0&0\end{bmatrix},\quad \Upsilon_5=\begin{bmatrix}1&-{}^te&0\\e&(1_{n-1}-e\cdot{}^te)&0\\0&0&1\end{bmatrix},$$

where $e={}^t(1, 0, \dots, 0)\in E^{n-1}$ ($e=1$ if $n=2$). We put $\mathscr{Q}_i=\Upsilon_i^{-1}P_1\Upsilon_i\cap G$. Then, viewed as subgroups of G, the $\mathscr{Q}_i$'s are given as follows:

$$(10.2)\quad \mathscr{Q}_o=\left\{\begin{bmatrix}g_o&0\\0&1\end{bmatrix}\middle|\, g_o\in G_o\right\},\quad \mathscr{Q}_1=\left\{\begin{bmatrix}1&*&*\\0&g'&*\\0&0&t\end{bmatrix}\middle|\, g'\in GL(n-2), t\neq 0\right\},$$

$$\mathscr{Q}_2=\left\{\begin{bmatrix}g_o&*\\0&t\end{bmatrix}\middle|\, g_o\in G_o, t\neq 0\right\},\quad \mathscr{Q}_3=\left\{\begin{bmatrix}1&0&0\\ *&g'&0\\ *&*&t\end{bmatrix}\middle|\, g'\in GL(n-2), t\neq 0\right\},$$

$$\mathscr{Q}_4=\left\{\begin{bmatrix}g_o&0\\ *&t\end{bmatrix}\middle|\, g_o\in G_o, t\neq 0\right\},\quad \mathscr{Q}_5=\left\{\begin{bmatrix}t&*&*\\0&g'&*\\0&0&t'\end{bmatrix}\middle|\, g'\in GL(n-2), t, t'\neq 0\right\}.$$

10.4. PROPOSITION. (i) $G_{1,E}=\coprod_{i=0}^{5} P_{1,E}\Upsilon_i\iota(G_E)$ (*disjoint union*).

(ii) *If* $i>0$, *there exists a normal subgroup* U_i *of* $\mathscr{Q}_i$ *such that* U_i *is the unipotent radical of a proper parabolic subgroup of* G *and that* $\Upsilon_iU_i\Upsilon_i^{-1}\subset N_1$.

PROOF. Let $(x, y)\in\mathscr{X}$ and write

$$x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\begin{matrix}1\\n-1\\1\end{matrix},\qquad y=\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}\begin{matrix}1\\n-1\\1\end{matrix}.$$

Put

$$\mathscr{X}_o=\{(x, y)\in\mathscr{X}\mid x_3\neq 0, y_1\neq 0\}\,,\quad \mathscr{X}_1=\{(x, y)\in\mathscr{X}\mid x_2\neq 0, x_3=0, y_1\neq 0\}$$
$$\mathscr{X}_2=\{(x, y)\in\mathscr{X}\mid x_2=0, x_3=0, y_1\neq 0\}\,,\quad \mathscr{X}_3=\{(x, y)\in\mathscr{X}\mid x_3\neq 0, y_1=0, y_2\neq 0\}$$
$$\mathscr{X}_4=\{(x, y)\in\mathscr{X}\mid x_3\neq 0, y_1=0, y_2=0\}\,,\quad \mathscr{X}_5=\{(x, y)\in\mathscr{X}\mid x_3=y_1=0\}\,.$$

We have $\mathscr{X}=\coprod_{i=0}^{5}\mathscr{X}_i$ (disjoint union) and $\mathscr{X}_i=P_{1,E}\Upsilon_i\cdot(\mathrm{e}_1, \mathrm{e}_1)$ $(0\leq i\leq 5)$, which proves (i). Put

$$U_1=U_2=U_5=\left\{\begin{bmatrix} 1_{n-1} & * \\ 0 & 1\end{bmatrix}\in G\right\}\quad\text{and}\quad U_3=U_4=\left\{\begin{bmatrix} 1_{n-1} & 0 \\ * & 1\end{bmatrix}\in G\right\}.$$

Then it is easily checked that each U_i satisfies the condition of (ii). q.e.d.

11. Rankin-Selberg convolution (II).

11.1. We keep the notation of §8 and §10. Let $F\in\mathscr{A}_{\mathrm{cusp}}(G_E\backslash G_A/K_A;\Omega;\Xi)$ and $f\in\mathscr{A}_{L^2}(G_{o,E}\backslash G_{o,A}/K_{o,A};\omega;\xi)$. Put $K_{1,A}=\prod_{v\in\mathscr{P}}K_{1,v}$, where $K_{1,v}$ is defined in the same way as that for K_v in §8.2. For $s, s'\in\boldsymbol{C}$, we define a function $\phi(*; s, s'; f)$ on $G_{1,A}$ by

$$\phi\left(n_1\begin{bmatrix} t & & \\ & g_o & \\ & & t'\end{bmatrix}k_1; s, s'; f\right)=f(g_o)|t|_A^{s+n/2}|t'|_A^{-(s'+n/2)}\,, \tag{11.1}$$

where $n_1\in N_{1,A}$, $t, t'\in A^\times$, $g_o\in G_{o,A}$ and $k_1\in K_{1,A}$. If Re(s) and Re(s') are sufficiently large, the Eisenstein series

$$\mathscr{E}(g_1; s, s'; f)=\sum_{\gamma_1\in P_{1,E}\backslash G_{1,E}}\phi(\gamma_1 g_1; s, s'; f) \tag{11.2}$$

is absolutely convergent. Put

$$\mathscr{E}^*(g_1; s, s'; f)=\xi_E(s+s'+1)\xi(f; s'+1)\xi(\tilde{f}; s+1)\mathscr{E}(g_1; s, s'; f)\,, \tag{11.3}$$

where $\tilde{f}(x)=f({}^t x^{-1})\in\mathscr{A}_{L^2}(G_{o,E}\backslash G_{o,A}/K_{o,A};\omega^{-1};\xi^{-1})$ and $\xi(f;s)$ is the completed standard L-function of f (see §8.5). Then $\mathscr{E}^*(g_1; s, s'; f)$ is continued to a meromorphic function of (s, s') on $\boldsymbol{C}^2$ and satisfies the functional equation

$$\mathscr{E}^*(g_1; -s', -s; f)=\mathscr{E}^*(g_1; s, s'; f)\,.$$

Moreover, $\mathscr{E}^*(g_1; s, s'; f)$ is a slowly increasing function of g_1 on $G_{1,E}\backslash G_{1,A}$.

11.2. Let $C_c^\infty(\boldsymbol{R}_+^\times)$ be the space of compactly supported smooth functions on $\boldsymbol{R}_+^\times$ (the set of positive real numbers). For $\upsilon\in C_c^\infty(\boldsymbol{R}_+^\times)$, we set

$$\mathscr{Z}_{f,F}^*(s, s'; \upsilon)=\int_{G_E\backslash G_A}F(g)\mathscr{E}^*(\iota(g); s, s'; f)\upsilon(|\det g|_A)dg\,. \tag{11.4}$$

Since $F(g)$ is rapidly decreasing and $\mathscr{E}^*(\iota(g); s, s'; f)$ is slowly increasing on $G_E\backslash G_A$, the integral (11.4) is absolutely convergent and defines a meromorphic function of (s, s') on

C^2. Note that the integral

$$\int_{G_E\backslash G_A} F(g)\mathscr{E}^*(\iota(g); s, s'; f)dg$$

is not necessarily absolutely convergent.

11.3. Take a sequence $\{\upsilon_j\}$ in $C_c^\infty(\boldsymbol{R}_+^\times)$ satisfying

$$0<\upsilon_1(x)\le\upsilon_2(x)\le\cdots\le 1 \tag{11.5}$$

$$\lim_{j\to\infty}\upsilon_j(x)=1 \tag{11.6}$$

for every $x>0$. The aim of this section is to show the following:

11.4. THEOREM. *Assume that* Re(s), Re(s') *are sufficiently large. Then we have*

$$\lim_{j\to\infty}\mathscr{Z}^*_{f,F}(s, s'; \upsilon_j)=|d_E|^{n-1}\xi\left(F; s+\frac{1}{2}\right)\xi\left(\tilde{F}; s'+\frac{1}{2}\right)\cdot W_{f,F}(1).$$

(Note that the limit is independent of the choice of $\{\upsilon_j\}$.)

11.5. For $g\in G_A$, we take $\alpha(g)$, $\alpha'(g)\in \boldsymbol{A}^\times$ and $\beta(g)\in C_{o,A}$ so that

$$\Upsilon_o\cdot\iota(g)=n_1\begin{bmatrix}\alpha(g)&&\\&\beta(g)&\\&&\alpha'(g)\end{bmatrix}k_1 \tag{11.7}$$

($n_1\in N_{1,A}$, $k_1\in K_{1,A}$). We put

$$\mathscr{Z}_{f,F}(s, s'; \upsilon)=\int_{G_E\backslash G_A}F(g)\mathscr{E}(\iota(g); s, s'; f)\upsilon(|\det g|_A)dg. \tag{11.8}$$

Note that

$$\mathscr{Z}^*_{f,F}(s, s'; \upsilon)=\xi_E(s+s'+1)\xi(f; s'+1)\xi(\tilde{f}; s+1)\cdot\mathscr{Z}_{f,F}(s, s'; \upsilon). \tag{11.9}$$

11.6. PROPOSITION (Basic identity). *For* $\upsilon\in C_c^\infty(\boldsymbol{R}_+^\times)$, *we have*

$$\mathscr{Z}_{f,F}(s, s'; \upsilon)=\int_{G_{o,A}\backslash G_A}W^\upsilon_{f,F}(\beta(g)^{-1}g)|\alpha(g)|_A^{s+n/2}|\alpha'(g)|_A^{-(s'+n/2)}dg,$$

where we put

$$W^\upsilon_{f,F}(g)=\int_{G_{o,E}\backslash G_{o,A}}f(x)F(xg)\upsilon(|\det x\cdot\det g|_A)dx \qquad (g\in G_A). \tag{11.10}$$

PROOF. Unwinding the Eisenstein series in (11.8) and using Proposition 10.4 (i), we obtain

$$\mathscr{Z}_{f,F}(s, s'; \upsilon)=\int_{G_E\backslash G_A} F(g)\sum_{i=0}^{5}\sum_{\gamma\in\mathscr{Q}_{i,E}\backslash G_E}\phi(\Upsilon_i\gamma g; s, s'; f)\upsilon(|\det g|_A)dg$$
$$=\sum_{i=0}^{5}\int_{\mathscr{Q}_{i,A}\backslash G_A} dg\int_{\mathscr{Q}_{i,E}\backslash\mathscr{Q}_{i,A}} F(qg)\phi(\Upsilon_i qg; s, s'; f)\upsilon(|\det(qg)|_A)d_l q\,,$$

where $d_l q$ is a left invariant measure of $\mathscr{Q}_{i,A}$. Since F is cuspidal, the integral over $\mathscr{Q}_{i,E}\backslash\mathscr{Q}_{i,A}$ vanishes for $i\geq 1$ in view of Proposition 10.4 (ii). It follows that $\mathscr{Z}_{i,F}(s, s'; \upsilon)$ is equal to

$$\int_{G_{o,A}\backslash G_A}\left\{\int_{G_{o,E}\backslash G_{o,A}} F(xg)\phi(\Upsilon_o xg; s, s'; f)\upsilon(|\det x\cdot\det g|_A)dx\right\}dg\,.$$

Since

$$\phi(\Upsilon_o xg; s, s'; f)=\phi(x\Upsilon_o g; s, s'; f)=f(x\beta(g))|\alpha(g)|_A^{s+n/2}|\alpha'(g)|_A^{-(s'+n/2)}$$

for $x\in G_o(A)$ and $g\in G(A)$, the integral over $G_{o,E}\backslash G_{o,A}$ is equal to $|\alpha(g)|_A^{s+n/2}\cdot|\alpha'(g)|_A^{-(s'+n/2)}W_{f,F}^{\upsilon}(\beta(g)^{-1}g)$. This completes the proof of the proposition. q.e.d.

11.7. For $g=\prod_{v\in\mathscr{P}}g_v\in G_A$ and $s\in\boldsymbol{C}$, we put

(11.11) $$N_{G_A,s}(g)=\prod_{v\in\mathscr{P}}N_{G_v,s}(g_v)$$

(for the definition of $N_{G_v,s}$, see (7.2)). For $\upsilon\in C_c^{\infty}(\boldsymbol{R}_+^{\times})$ and $s, s'\in\boldsymbol{C}$, we set

(11.12) $$J_{f,F}(s, s'; \upsilon)=\int_{G_A} W_{f,F}^{\upsilon}(g)N_{G_A,s'+n/2}(g)N_{G_A,s+n/2}({}^t g^{-1})dg\,.$$

11.8. Lemma. *For* $\upsilon\in C_c^{\infty}(\boldsymbol{R}_+^{\times})$ *and* $s, s'\in\boldsymbol{C}$ *with* Re(s), Re(s') *sufficiently large, we have*

$$J_{f,F}(s, s'; \upsilon)=|d_E|^{-(n-1)+n(n-1)/4}\frac{\xi(\tilde{f}; s+1)\xi(f; s'+1)}{\xi_E^{(n-1)}(s+s'+n)}\mathscr{Z}_{f,F}(s, s'; \upsilon)\,.$$

Proof. Observe that $\varphi * W_{f,F}^{\upsilon}=\xi_v^{\wedge}(\varphi)\cdot W_{f,F}^{\upsilon}$ for $\varphi\in\mathscr{H}_{G_{o,v}}$ $(v\in\mathscr{P}_f)$ and that $L_z W_{f,F}^{\upsilon}=\xi_v^{\wedge}(z)\cdot W_{f,F}^{\upsilon}$ for $z\in\mathscr{Z}_{G_{o,v}}$ $(v\in\mathscr{P}_{\infty})$. It follows that for any bi-$K_{o,A}$-invariant function φ we have

$$\int_{G_{o,A}} W_{f,F}^{\upsilon}\left(\begin{bmatrix}x & 0\\ 0 & 1\end{bmatrix}g\right)\varphi(x)dx=W_{f,F}^{\upsilon}(g)\int_{G_{o,A}}\phi_{\xi^{-1}}(x)\varphi(x)dx$$

if the integral is absolutely convergent. Here we put $\phi_{\xi^{-1}}(x)=\prod_{v\in\mathscr{P}}\phi_{\xi_v^{-1}}(x_v)$ for $x=\prod_{v\in\mathscr{P}}x_v\in G_{o,A}$. On the other hand, by Proposition 7.6 we have

$$N_{G_A,s'+n/2}\left(\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} g\right) N_{G_A,s+n/2}\left(\begin{bmatrix} {}^t x^{-1} & 0 \\ 0 & 1 \end{bmatrix} {}^t g^{-1}\right)$$
$$= N_{G_o,A,s'+n/2}(x\cdot\beta(g)) N_{G_o,A,s+n/2}({}^t(x\cdot\beta(g))^{-1}) \times |\alpha(g)|_A^{s+n/2} |\alpha'(g)|_A^{-(s'+n/2)}.$$

The lemma is then proved by an argument similar to that of §7.7. q.e.d.

11.9 Let $\{\upsilon_j\}$ be a sequence in $C_c^\infty(\boldsymbol{R}_+^\times)$ satisfying (11.5) and (11.6). In view of Lemma 11.8 and (11.9), it remains to show the following result to complete the proof of Theorem 11.4:

LEMMA. *Assume that* Re(s) *and* Re(s') *are sufficiently large. Then*

$$\lim_{j\to\infty} J_{f,F}(s, s'; \upsilon_j) = |d_E|^{n(n-1)/4} \frac{\xi\left(F; s+\frac{1}{2}\right)\xi\left(\tilde{F}; s'+\frac{1}{2}\right)}{\xi_E^{(n)}(s+s'+n)} \times W_{f,F}(1).$$

PROOF. By the definition of $W_{f,F}^{\upsilon}$ and the left K_A-invariance of $N_{G_A,s+s'+n}(g)$, the integral $J_{f,F}(s, s'; \upsilon_j)$ is equal to

$$\int_{G_A} dg \int_{G_{o,E}\backslash G_{o,A}} dx\, f(x) N_{G_A,s+s'+n}(g) |\det g|_A^{s+n/2} \upsilon_j(|\det x\cdot \det g|_A) \int_{K_A} F(xkg)dk.$$

By an argument similar to that in the proof of Lemma 6.8, we have

$$\int_{K_A} F(xkg)dk = F(x)\int_{K_A} \phi_\Xi(kg)dk,$$

where $\phi_\Xi(g) = \prod_{v\in\mathscr{P}} \phi_{\Xi_v}(g_v)$ for $g = \prod_{v\in\mathscr{P}} g_v \in G_A$. It follows that $J_{f,F}(s, s'; \upsilon_j)$ equals

$$\int_{G_{o,E}\backslash G_{o,A}} f(x)F(x)\left\{\int_{G_A} \phi_\Xi(g) N_{G_A,s+s'+n}(g) |\det g|_A^{s+n/2} \upsilon_j(|\det x\cdot \det g|_A) dg\right\} dx.$$

By (11.5), we have

$$\left|\int_{G_A} \phi_\Xi(g) N_{G_A,s+s'+n}(g) |\det g|_A^{s+n/2} \upsilon_j(|\det x\cdot\det g|_A)dg\right|$$
$$\le \int_{G_A} \phi_{\mathrm{Re}(\Xi)}(g) N_{G_A,\mathrm{Re}(s+s'+n)}(g) |\det g|_A^{\mathrm{Re}(s)+n/2} dg.$$

Observe that the last integral is absolutely convergent if Re(s) and Re(s') are sufficiently large and that the value is independent of j and $x\in G_{o,A}$. Since the integral $\int_{G_{o,E}\backslash G_{o,A}} f(x)F(x)dx$ is absolutely convergent (cf. Lemma 8.6), we may applying Fubini's theorem and Lebesgue's convergence theorem to obtain

$$\lim_{j\to\infty} J_{f,F}(s, s'; v_j) = \int_{G_{o,E}\backslash G_{o,A}} f(x)F(x)\left\{\int_{G_A} \phi_\Xi(g) N_{G_A,s+s'+n}(g)|\det g|_A^{s+n/2} \times \lim_{j\to\infty} v_j(|\det x\cdot \det g|_A)dg\right\}dx$$

$$= \int_{G_{o,E}\backslash G_{o,A}} f(x)F(x)dx \int_{G_A} \phi_\Xi(g) N_{G_A,s+s'+n}(g)|\det g|_A^{s+n/2} dg\ .$$

Since the integral over G_A is equal to

$$|d_E|^{n(n-1)/4}\frac{\xi\left(F; s+\frac{1}{2}\right)\xi\left(\tilde{F}; s'+\frac{1}{2}\right)}{\xi_E^{(n)}(s+s'+n)}$$

by Lemma 7.5, we have completed the proof of the lemma. q.e.d.

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