CHAPTER 17

SHOALING OF FINITE AMPLITUDE LONG WAVES ON A BEACH OF CONSTANT SLOPE

Yuichi Iwagaki
Professor
and
Tetsuo Sakai
Assistant Professor

Department of Civil Engineering
Kyoto University
Kyoto, Japan

ABSTRACT

A solution of finite amplitude long waves on constant sloping beaches is obtained by solving the equations of the shallow water theory of the lowest order. Non-linearity of this theory is taken into account, using the perturbation method. Bessel functions involved in the solution are approximated with trigonometric functions. The applicable range of this theory is determined from the two limit conditions caused by the hydrostatic pressure assumption and the trigonometric function approximation of Bessel functions.

The shoaling of this finite amplitude long waves on constant sloping beaches is discussed. Especially, the effects of the beach slope on the wave height change and the asymmetric wave profile near the breaking point are examined, which can not be explained by the concept of constancy of wave energy flux based on the theory of progressive waves in uniform depth. These theoretical results are presented graphically, and compared with curves of wave shoaling based on finite amplitude wave theories.

On the other hand, the experiments are conducted with respect to the transformation of waves progressing on beaches of three kinds of slopes (1/30, 1/20 and 1/10). The experimental results are compared with the theoretical curves to confirm the validity of the theory.

INTRODUCTION

As waves progress in shallow water, wave transformations occur due to the presence of the sea bottom. Especially, the changes of wave height, celerity and length in shoaling water are generally explained by using the assumption that the wave energy flux based on the theory of progressive waves in uniform depth is kept constant in shoaling water[1),2)]. On the other hand, some investigator[3),4)] have attempted to obtain a solution of wave transformation on the sloping beach considering the change of water depth as the bottom condition. The existing results of observations and experiments show that the change of wave height, celerity and length are well explained by the concept of constancy of energy flux based on theories of finite amplitude wave in uniform depth[2].

However, on sloping beaches, not only the wave height increases but also the wave profile becomes asymmetric[5]. Further, as Goda[6] pointed out recently, it is obvious that the beach slope affects the breaking wave height. Similarly, the beach slope will affect the wave transformations before breaking. These two problems, i.e. the asymmetry of wave profile and the effect of beach slope, can not be explained by using the approximate method of energy flux of waves in uniform depth. They will be explained by the solution of progressive waves on the beach mentioned above. However, none of existing theoretical investigations has given any solution to clarify these problems, except Biesel's investigation[7] which proposed a quantity representing the asymmetry of wave profile.

This paper treats analytically the two-dimensional wave transformation on constant sloping beaches in order to explain the asymmetry of wave profile and the beach slope effect on wave transformation. As the method to obtain the solution of progressive waves on the beach, two approximate methods exist, which are the small amplitude approximation and the shallow water approximation. In this paper, the shallow water theory of the lowest order[8] is used as the basic equations. These equations are non-linear, and the linear solution was already obtained[9]. It is seems that two problems mentioned above can be explained by taking this non-linearity into account. Carrier and Greenspan[10] and Ichiye[11] solved this non-linear shallow water theory already, but did not make clear these problems.

The non-linearity of this shallow water theory is taken into account herein by using the perturbation method as Ichiye did. Further, by using the asymptotic expansion of Bessel functions with trigonometric functions, the general solution of finite amplitude long waves progressing on the beach of a uniform slope is obtained. Based on this solution, the graphs showing the effects of beach slope on the wave height change and the asymmetry of wave profile are presented. On the other hand, the experiments with respect to wave shoaling on beaches of three kinds of slopes (1/30, 1/20 and 1/10) are conducted. The experimental results are compared with the theoretical ones in order to confirm the validity of this theory.

DERIVATION OF SOLUTION

BASIC EQUATIONS

The equations of two-dimensional shallow water theory of lowest order are as follows (see Fig.1) :

$$\left.\begin{aligned} & u_t + u \cdot u_x + g \cdot \eta_x = 0 , \\ & \eta_t + \{ u \cdot (\eta + h) \}_x = 0 , \end{aligned}\right\} \quad (1)$$

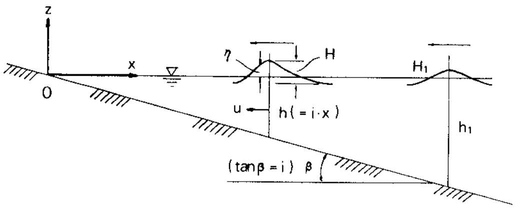

Fig.1 Sketch of waves on sloping beach

where $h = h(x)$ is the water depth, $\eta = \eta(x,t)$ the height of water surface above still water level and $u = u(x,t)$ the water particle velocity in the x-direction which is not dependent on the vertical coordinate. This implies that the pressure distribution is approximated with the hydrostatic distribution. Suffixes t and x denote the differentiations with t and x respectively. Eq.(1) is non-linear and frequently solved numerically with the method of characteristics. In this paper, the perturbation method is used in order to represent the non-linear effect analytically. That is, η and u are assumed to be expressed as the power series of a small quantity α as follows :

$$\eta=\alpha\cdot\eta^{(1)}+\alpha^2\cdot\eta^{(2)}+\cdots\cdots,\quad u=\alpha\cdot u^{(1)}+\alpha^2\cdot u^{(2)}+\cdots\cdots. \qquad (2)$$

Substituting Eq.(2) into Eq.(1) and rearranging with respect to α and α^2, the coefficients of α and α^2 lead respectively

$$u^{(1)}{}_t+g\cdot\eta^{(1)}{}_x=0,\quad \eta^{(1)}{}_t+\{u^{(1)}\cdot h\}_x=0, \qquad (3)$$

and

$$u^{(2)}{}_t+u^{(1)}\cdot u^{(1)}{}_x+g\cdot\eta^{(2)}{}_x=0,\quad \eta^{(2)}{}_t+\{u^{(1)}\cdot\eta^{(1)}+u^{(2)}\cdot h\}_x=0. \qquad (4)$$

SOLUTION OF $\eta^{(1)}$ AND $u^{(1)}$

Eliminating $u^{(1)}$ in Eq.(3), the following equation is derived :

$$\eta^{(1)}{}_{tt}-g\cdot\{\eta^{(1)}{}_x\cdot h\}_x=0. \qquad (5)$$

If the beach slope i is constant and the water depth h is $i{\times}x$ (see Fig.1), Eq.(5) is further reduced as follows :

$$\eta^{(1)}{}_{tt}-g\cdot\{\eta^{(1)}{}_{xx}\cdot i\cdot x+\eta^{(1)}{}_x\cdot i\}=0. \qquad (6)$$

$\eta^{(1)}$ is assumed to be expressed as

$$\eta^{(1)}(x,t)=\eta(x)\cdot\cos\sigma t. \qquad (7)$$

Therefore, from Eqs.(6) and (7), the equation of $\eta^{(1)}$ is obtained as follows :

$$x\cdot\eta_{xx}+\eta_x+(\sigma^2/g\,i)\cdot\eta=0. \qquad (8)$$

When the variable x is replaced with w through the relationship[12],

$$x=(g\,i/4\sigma^2)\cdot w^2, \qquad (9)$$

Eq.(8) is modified as

$$\eta_{ww}+(1/w)\cdot\eta_w+\eta=0. \qquad (10)$$

It is evident that Eq.(10) posesses a solution consisting of Bessel function $J_0(w)$ and Neumann function $N_0(w)$. A similar result is obtained when $\eta^{(1)}$ is assumed as

$$\eta^{(1)}(x,\ t)\ = \eta(x)\cdot\sin\sigma t\ . \qquad (11)$$

When x (therefore w) approaches infinite, Bessel and Neumann functions are expanded asymptotically as follows :

$$\left.\begin{aligned} J_\nu(w) &\sim \sqrt{2/\pi w}\cdot\cos(w-\nu\pi/2-\pi/4)\ ,\\ N_\nu(w) &\sim \sqrt{2/\pi w}\cdot\sin(w-\nu\pi/2-\pi/4)\ .\end{aligned}\right\} \qquad (12)$$

Accordingly the solution $\eta^{(1)}$ corresponding to the waves progressing in the negative x-direction (see Fig.1) is given as[9]

$$\eta^{(1)}(x,\ t)\ = a\cdot\{\cos\sigma t\cdot J_0(2\sigma\sqrt{\frac{x}{gi}}) - \sin\sigma t\cdot N_0(2\sigma\sqrt{\frac{x}{gi}})\}, \qquad (13)$$

where a is a constant related to the wave height. Using the relationships,

$$Z_0{}'(w) = -Z_1(w)\ ,\quad Z_1{}'(w) = Z_0(w) - w^{-1}\cdot Z_1(w)\ , \qquad (14)$$

and Eq.(3), $u^{(1)}$ is obtained as

$$u^{(1)}(x,\ t) = a\sqrt{\frac{g}{i}}\cdot x^{-\frac{1}{2}}\cdot\{\sin\sigma t\cdot J_1(2\sigma\sqrt{\frac{x}{gi}}) + \cos\sigma t\cdot N_1(2\sigma\sqrt{\frac{x}{gi}})\}\ . \qquad (15)$$

Eqs.(13) and (15) are the solutions of Eq.(1) when the non-linearity is neglected. Due to the natures of J_0 and N_0, the amplitude of $\eta^{(1)}$ increases with decrease in x. Therefore the solution $\eta^{(1)}$ can explain the fact of wave height increase in shoaling water. However, the wave profile of $\eta^{(1)}$ with time has the form of sine function, and can not explain the experimental fact of asymmetric and forward inclined wave profile on the sloping beach (see Fig.2[5]).

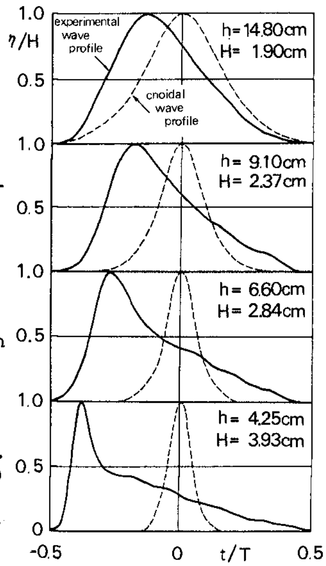

Fig.2 Asymmetric wave profile on sloping beach[5]

SOLUTION OF $\eta^{(2)}$ AND $u^{(2)}$

When $u^{(2)}$ is eliminated and the relation $h = i\times x$ is used in Eq.(4), the following equation is obtained.

$$\eta^{(2)}{}_{tt} - g\cdot\{\eta^{(2)}{}_{xx}\cdot i\cdot x + \eta^{(2)}{}_x\cdot i\}$$
$$= -\{u^{(1)}\cdot\eta^{(1)}\}_{xt} + \{u^{(1)}\cdot u^{(1)}{}_x\}_x\cdot i\cdot x + \{u^{(1)}\cdot u^{(1)}{}_x\}\cdot i\ . \qquad (16)$$

After substituting Eqs.(13) and (15) into Eq.(16), rearranging by using the relation of Eq.(14), the right side of Eq.(16) becomes as follows :

$$\cos 2\sigma t\cdot\{-\frac{3}{2}a^2\frac{\sigma^2}{i}x^{-1}\cdot(J_0^2 - J_1^2 - N_0^2 + N_1^2)$$

$$+\frac{5}{2}a^2\sqrt{\frac{g}{i}}\cdot\sigma\cdot x^{-3/2}\cdot(J_0J_1-N_0N_1)$$

$$-a^2\cdot g\cdot x^{-2}\cdot(J_1^2-N_1^2)\,\}$$

$$+\sin 2\sigma t\cdot\{\,3a^2\frac{\sigma^2}{i}x^{-1}\cdot(J_0N_0-J_1N_1)$$

$$-\frac{5}{2}a^2\sqrt{\frac{g}{i}}\cdot\sigma\cdot x^{-3/2}\cdot(J_1N_0+J_0N_1)$$

$$+2a^2\cdot g\cdot x^{-2}\cdot J_1N_1\,\}$$

$$+\{\,\frac{1}{2}a^2\frac{\sigma^2}{i}x^{-1}\cdot(J_0^2-J_1^2-N_0^2+N_1^2)$$

$$-\frac{3}{2}a^2\sqrt{\frac{g}{i}}\cdot\sigma\cdot x^{-3/2}\cdot(J_0J_1-N_0N_1)+a^2\cdot g\cdot x^{-2}\cdot(J_1^2-N_1^2)\,\}\,. \quad\text{.........................(17)}$$

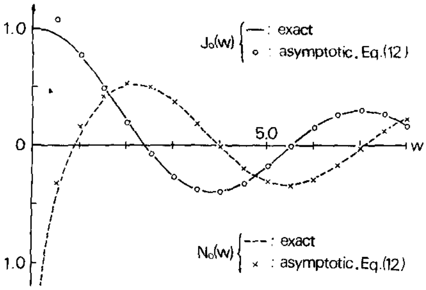

Fig.3 Approximate values by asymptotic expansion of Bessel and Neumann functions

In Fig.3, the approximate values of J_0 and N_0 based on Eq.(12) are compared with the exact ones. It is seen that the approximate values are accurate enough for the large value of w. When Bessel and Neumann functions in Eq.(17) are approximated with the trigonometric functions of Eq.(12), Eq.(17) becomes

$$\cos 2\sigma t\cdot[-3a^2\frac{\sigma}{\pi}\sqrt{\frac{g}{i}}\cdot x^{-3/2}\cdot\cos\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}$$

$$+\frac{5}{2}a^2\frac{g}{\pi}x^{-2}\cdot\sin\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}+a^2\frac{g}{\pi\sigma}\sqrt{gi}\cdot x^{-5/2}\cdot\cos\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}]$$

$$+\sin 2\sigma t\cdot[3a^2\frac{\sigma}{\pi}\sqrt{\frac{g}{i}}\cdot x^{-3/2}\cdot\sin\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}$$

$$+\frac{5}{2}a^2\frac{g}{\pi}x^{-2}\cdot\cos\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}-a^2\frac{g}{\pi\sigma}\sqrt{gi}\cdot x^{-5/2}\cdot\sin\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}]$$

$$+[a^2\frac{\sigma}{\pi}\sqrt{\frac{g}{i}}\cdot x^{-3/2}\cdot\cos\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}$$

$$-\frac{3}{2}a^2\frac{g}{\pi}\cdot x^{-2}\cdot\sin\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}-a^2\frac{g}{\pi\sigma}\sqrt{gi}\cdot x^{-5/2}\cdot\cos\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}]\,. \quad\text{..................(18)}$$

The solution of $\eta^{(2)}$ is assumed to be expressed as

$$\eta^{(2)}(x,t)=\cos 2\sigma t\cdot A(x)+\sin 2\sigma t\cdot B(x)+C(x)\,, \quad\text{..........................(19)}$$

Substituting Eq.(19) into the left side of Eq.(16), it becomes

$$\cos 2\sigma t\cdot\{-4\sigma^2\cdot A(x)-g\cdot i\cdot x\cdot A''(x)-g\cdot i\cdot A'(x)\}$$

$$+\sin 2\sigma t\cdot\{-4\sigma^2\cdot B(x)-g\cdot i\cdot x\cdot B''(x)-g\cdot i\cdot B'(x)\}$$

$$+\{-g\cdot i\cdot C''(x)-g\cdot i\cdot C'(x)\}\,. \qquad \cdots\cdots (20)$$

Comparing Eqs.(18) and (20), A(x), B(x) and C(x) are determined as follows :

$$A(x)=-a^2\frac{1}{\pi i}\cdot x^{-1}\cdot\sin\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}-\frac{3}{10}a^2\frac{1}{\pi\sigma}\sqrt{\frac{g}{i}}\cdot x^{-3/2}\cdot\cos\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}, \qquad \cdots\cdots (21)$$

$$B(x)=-a^2\frac{1}{\pi i}\cdot x^{-1}\cdot\cos\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}+\frac{3}{10}a^2\frac{1}{\pi\sigma}\sqrt{\frac{g}{i}}\cdot x^{-3/2}\cdot\sin\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}, \qquad \cdots\cdots (22)$$

$$C(x)=\frac{1}{4}a^2\frac{1}{\pi\sigma}\sqrt{\frac{g}{i}}\cdot x^{-3/2}\cdot\cos\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}-\frac{1}{16}a^2\frac{g}{\pi\sigma^2}\cdot x^{-2}\cdot\sin\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}. \qquad \cdots\cdots (23)$$

Substituting Eqs.(21), (22) and (23) into Eq.(20), the left side of Eq. (16) becomes

$$\begin{aligned}
&\cos 2\sigma t\cdot[-3a^2\frac{\sigma}{\pi}\sqrt{\frac{g}{i}}\cdot x^{-3/2}\cdot\cos\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}\\
&\quad+\frac{5}{2}a^2\frac{g}{\pi}x^{-2}\cdot\sin\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}+\frac{27}{40}a^2\frac{g}{\pi\sigma}\sqrt{gi}\cdot x^{-5/2}\cdot\cos\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}]\\
&+\sin 2\sigma t\cdot[3a^2\frac{\sigma}{\pi}\sqrt{\frac{g}{i}}\cdot x^{-3/2}\cdot\sin\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}\\
&\quad+\frac{5}{2}a^2\frac{g}{\pi}x^{-2}\cdot\cos\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}-\frac{27}{40}a^2\frac{g}{\pi\sigma}\sqrt{gi}\cdot x^{-5/2}\cdot\sin\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}]\\
&+[a^2\frac{\sigma}{\pi}\sqrt{\frac{g}{i}}\cdot x^{-3/2}\cdot\cos\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}-\frac{3}{2}a^2\frac{g}{\pi}\cdot x^{-2}\cdot\sin\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}\\
&\quad-a^2\frac{g}{\pi\sigma}\sqrt{gi}\cdot x^{-5/2}\cdot\cos\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}+\frac{1}{4}a^2\frac{g^2}{\pi\sigma^2}\cdot i\cdot x^{-3}\cdot\sin\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}].
\end{aligned} \qquad \cdots\cdots (24)$$

From the comparison between Eqs.(18) and (24), it is found that the difference exists between the constants in the third terms of coefficients of $\cos 2\sigma t$ and $\sin 2\sigma t$. Further, the forth term in the last part (independent on t) exists only in Eq.(24). Three terms in the coefficients of $\cos 2\sigma t$ and $\sin 2\sigma t$ involve $x^{-3/2}$, x^{-2} and $x^{-5/2}$ in turn, and the forth term in the last part of Eq.(24) involves x^{-3}. Using the relationships of $T = 2\pi/\sigma$ and $h = i\cdot x$, the ratios of the second, third and forth terms to the first term become :

2nd/1st term $\sim \{(gT/2\pi)/\sqrt{gh}\}\cdot i$, 3rd/1st term $\sim [\{(gT/2\pi)/\sqrt{gh}\}\cdot i]^2$,

and 4th/1st term $\sim [\{(gT/2\pi)/\sqrt{gh}\}\cdot i]^3$.

Therefore the higher term becomes smaller in proportion to i.

The third terms in the coefficients of $\cos 2\sigma t$ and $\sin 2\sigma t$ have $\cos\{2(2\sigma\sqrt{x/gi}-\pi/4)\}$ and $\sin\{2(2\sigma\sqrt{x/gi}-\pi/4)\}$ respectively, and are in the same phase as that of the first term. If $i = 1/10$, h = 20cm and T = 3sec, the ratio of third to first term $[\{(gT/2\pi)/\sqrt{gh}\}i]^2$ becomes smaller than 1/10, and the difference between constants of the third terms is negligible. Similarly, the forth term in the last part of Eq.(24), which has $\sin\{2(2\sigma\sqrt{x/gi}-\pi/4)\}$ is in the same phase as that of the second term, and also negligible.

Therefore, Eq.(19) with Eqs.(21), (22) and (23) is the solution of Eq. (16), and expressed as follows :

$$\begin{aligned}\eta^{(2)}(x,\ t) = {} & \cos 2\sigma t\Big[-a^2\frac{1}{\pi i}\cdot x^{-1}\cdot\sin\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\} \\ & -\frac{3}{10}a^2\frac{1}{\pi\sigma}\sqrt{\frac{g}{i}}\cdot x^{-3/2}\cdot\cos\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}\Big] \\ & +\sin 2\sigma t\cdot\Big[-a^2\frac{1}{\pi i}\,x^{-1}\cdot\cos\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\} \\ & +\frac{3}{10}a^2\frac{1}{\pi\sigma}\sqrt{\frac{g}{i}}\cdot x^{-3/2}\cdot\sin\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}\Big] \\ & +\Big[\frac{1}{4}a^2\frac{1}{\pi\sigma}\sqrt{\frac{g}{i}}\cdot x^{-3/2}\cdot\cos\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\} \\ & -\frac{1}{16}a^2\frac{g}{\pi\sigma^2}\cdot x^{-2}\cdot\sin\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}\Big]\,. \qquad \cdots\cdots(25)\end{aligned}$$

Using the relationship of Eq.(12), $\eta^{(1)}$ of Eq.(13) becomes

$$\eta^{(1)}(x,\ t) = a\cdot\left(\frac{\sqrt{gi}}{\pi\sigma}\right)^{1/2}\cdot x^{-1/4}\cdot\cos\left(\sigma t+2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4}\right). \qquad \cdots\cdots(26)$$

For the sake of comparison with Eq.(26), Eq.(25) is finally reduced as follows :

$$\begin{aligned}\eta^{(2)}(x,\ t) = {} & a^2\frac{1}{\pi i}\cdot x^{-1}\cdot\cos\{2(\sigma t+2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})+\frac{\pi}{2}+\tan^{-1}(\frac{3}{10}\frac{\sqrt{gi}}{\sigma}\cdot x^{-1/2})\} \\ & +\Big[\frac{1}{4}a^2\frac{1}{\pi\sigma}\sqrt{\frac{g}{i}}\cdot x^{-3/2}\cdot\cos\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}-\frac{1}{16}a^2\frac{g}{\pi\sigma^2}\cdot x^{-2}\cdot\sin\{2(2\sigma\sqrt{\frac{x}{gi}}-\frac{\pi}{4})\}\Big]. \qquad \cdots\cdots(27)\end{aligned}$$

APPLICABLE RANGE OF SOLUTION

The basic equation (1) is the shallow water theory of the lowest order[8], and the pressure distribution is assumed to be hydrostatic. Therefore these solutions are applicable only when the water depth is considerably smaller than the wave length. If the wave celerity is equal to that of long waves $\sqrt{gh}$,

$$h/L = 1/(T\sqrt{g/h})\,. \qquad \cdots\cdots(28)$$

The upper limit of h/L implies that the lower limit of $T\sqrt{g/h}$ exists and that the upper limit of the water depth h exists if the wave period is given. On the other hand, the lower limit of h also exists. As the solution of $\eta^{(1)}$, Eq.(26) is used in which Bessel and Neumann functions are approximated with trigonometric functions by Eq.(12), which is the asymptotic expansion when $|w| \to \infty$. Comparing between the exact and approximate values of J_0, N_0, J_1 and N_1, it is found that the approximate values are accurate enough when $|w| \geqq 1.0$. Using Eq.(9), this condition is rewritten as follows :

$$T\sqrt{g/h} \leq 4\pi/i \quad \text{.........} (29)$$

This means that the lower limit of h determined by the beach slope exists when the wave period is given. Two conditions mentioned above are shown in Fig.4. Two lines showing the lower limit of $T\sqrt{g/h}$ are given for the two limiting values of h/L.

DETERMINATION OF α

The second part of the right side of Eq.(27) consists of two terms involving $x^{-3/2}$ and x^{-2} respectively. The ratios of these to the first term in the first part of the right side are $\{(gT/2\pi)/\sqrt{gh}\}i$ and $\{\{(gT/2\pi)/\sqrt{gh}\}i\}^2$ respectively. Therefore the second part is smaller than the first part by the order of $\{(gT/2\pi)/\sqrt{gh}\}i$. Further, because of independence of t, the second part affects only the still water level and does not affect the wave height and wave profile. Becuase this paper deals with the beach slope effect on the wave height change and wave profile asymmetry, the second part of the right side of Eq.(27) is neglected.

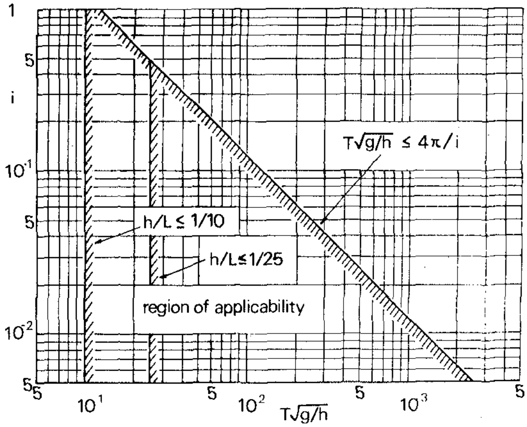

Fig.4 Applicable range

Substituting $\eta^{(1)}$ of Eq.(26) and $\eta^{(2)}$ of Eq.(27) into Eq.(2), the wave profile is given by

$$\eta(x,t) = \alpha\cdot a\cdot\left(\frac{\sqrt{gi}}{\pi\sigma}\right)^{1/2}\cdot x^{-1/4}\cdot\cos\left(\sigma t + 2\sigma\sqrt{\frac{x}{gi}} - \frac{\pi}{4}\right)$$

$$+\alpha^2\cdot\frac{a^2}{\pi i}\cdot x^{-1}\cdot\cos\left\{2\left(\sigma t + 2\sigma\sqrt{\frac{x}{gi}} - \frac{\pi}{4}\right) + \frac{\pi}{2} + \tan^{-1}\left(\frac{3}{10}\frac{\sqrt{gi}}{\sigma}\cdot x^{-1/2}\right)\right\} \quad . \quad (30)$$

Now, h_1/L_0 is applied as the small quantity α, where h_1 is the largest water depth in the applicable range of this solution (see Fig.4) and L_0 is the deep-water wave length of the small amplitude wave theory, $gT^2/2\pi$.

DETERMINATION OF a

Substituting $\alpha = h_1/L_0$ into Eq.(30) and rearranging, Eq.(30) is simply rewritten as follows :

$$\eta / h_1 = A^{(1)} \cdot \cos\theta + A^{(2)} \cdot \cos(2\theta + \delta), \qquad (31)$$

where $A^{(1)}$, $A^{(2)}$, θ and δ are are given by

$$\left.\begin{aligned} A^{(1)} &= 2^{-1/4} \cdot \pi^{-3/4} \cdot i^{1/2} \cdot (h_1 / L_0)^{3/4} \cdot (h_1 / h)^{1/4} \cdot (a / h_1), \\ A^{(2)} &= \pi^{-1} \cdot (h_1 / L_0)^2 \cdot (h_1 / h) \cdot (a / h_1)^2, \\ \theta &= 2\pi / T \cdot t + 2\pi\{\sqrt{2/\pi} \cdot (h / L_0)^{-1/2}\} \cdot x / L_0 - \pi / 4, \\ \delta &= \pi / 2 + \tan^{-1}\{3/10 \cdot \sqrt{2\pi}^{\,-1} \cdot i \cdot (h / L_0)^{-1/2}\}. \end{aligned}\right\} \qquad (32)$$

If $A^{(1)}{}_1$ and $A^{(2)}{}_1$ denote $A^{(1)}$ and $A^{(2)}$ at $h = h_1$ respectively, these are

$$\left.\begin{aligned} A^{(1)}{}_1 &= 2^{-1/4} \cdot \pi^{-3/4} \cdot i^{1/2} \cdot (h_1 / L_0)^{3/4} \cdot (a / h_1), \\ A^{(2)}{}_1 &= \pi^{-1} \cdot (h_1 / L_0)^2 \cdot (a / h_1)^2. \end{aligned}\right\} \qquad (33)$$

Because $\tan^{-1}\{\ \}$ in δ of Eq.(32) is smaller than $\pi/2$ at h_1, it is neglected. Therefore the wave profile η_1 at h_1 is given as follows :

$$\eta_1 / h_1 = A^{(1)}{}_1 \cdot f(\theta), \qquad (34)$$

where

$$f(\theta) = \cos\theta - b \cdot \sin 2\theta,$$

$$b = A^{(2)}{}_1 / A^{(1)}{}_1. \qquad (35)$$

Fig.5 shows the profile of $f(\theta)$ in the range of $-\pi \leqq \theta \leqq \pi$. It is evident that the relative wave height H_1/h_1 at h_1 is given

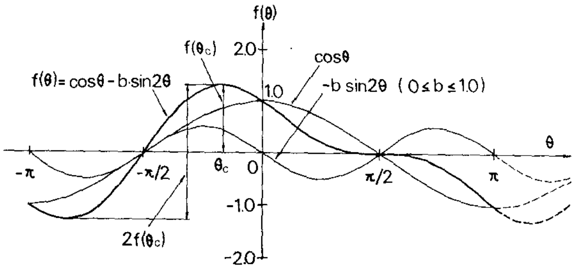

Fig.5 Profile of f(θ)

$$H_1 / h_1 = 2 \times A^{(1)}{}_1 \cdot f(\theta_c). \qquad (36)$$

θ_c is determined from $df/d\theta = 0$. After all, considering $-\pi/2 \leqq \theta_c \leqq 0$,

$$\sin\theta_c = (1/4b - \sqrt{1/16b^2 + 2})/2. \qquad (37)$$

In Eq.(36), $A^{(1)}{}_1$ is the function of a/h_1, and $f(\theta_c)$ is also the function of a/h_1 because b ($= A^{(2)}{}_1/A^{(1)}{}_1$) is dependent on a/h_1. Therefore a is determined from H_1 by using Eq.(36).

DISCUSSION OF SOLUTION

For simplicity, the second term of right side of Eq.(31) is assumed to be smaller than the first term at h_1 and negligible. Therefore the wave height H_1 is equal to twice of the amplitude of the first term. Using this relationship, Eq.(31) is reduced as follows :

$$\eta / h_1 = 1/2 \cdot (H_1 / h_1) \cdot (h_1 / h)^{1/4} \cdot \cos\theta + \sqrt{2\pi} \cdot (h_1 / L_0)^{1/2} \cdot i^{-1} \cdot (H_1 / 2h_1)^2 \cdot (h_1 / h) \cdot \cos(2\theta + \delta) . \quad (38)$$

As seen in Eq.(38), the amplitude of the first term (the linear solution) is inversely proportional to the 1/4th power of the water depth h, and independen on the beach slope i. This agrees with Green[13]'s law with respect to the long wave transformation on very gentle sloping beaches without wave reflection and energy loss. On the other hand, the amplitude of the second term representing the nonlinear effect is inversely proportional to h, and the rate of wave height increase with decrease in the water depth is much larger than that of the first term. Further it depends on h_1/L_0 and the beach slope i. Especially, the amplitude of the second term increases inversely proportionally to the beach slope i, the effect of which has not been made clear previously.

The authors[2),5)] presented previously the theoretical curves of wave shoaling based on the wave energy flux of the hyperbolic wave theory which is the approximate representation of Laitone's cnoidal waves of the second approximation[14)]. Before these investigations, as the preliminary step, Iwagaki[15)] obtained the theoretical equation of wave height change based on the first approximation. The characteristics of this equation agree with the relationship between the amplitude of the second term and the water depth that the amplitude is inversely proportional to the water depth h. Furthermore, the period of the second term is a half of that of the first term, and the phase of the second term is in advance of that of the first term by δ. As seen from Eq.(32), δ consists of $\pi/2$ and $\tan^{-1}\{(3/10)\cdot(2\pi)^{-1/2}\cdot i\cdot(h/L_0)^{-1/2}\}$, which increases with decrease in the water depth. As descrebed later, the difference of phase explains the asymmetry of wave profile.

NUMERICAL COMPUTATION

RESTRICTION ON NUMERICAL COMPUTATION

As shown in Fig.4, the applicable range of the water depth h exists when the wave period T and the beach slope i are given, which is determined by two limit conditions of the hydrostatic pressure assumption and the trigonometric function approximation of Bessel and Neumann functions. However, some arbitrarity exists in determination of the limiting value of h/L, which corresponds to the largest water depth h_1 in the range. In this paper, the following value is adapted :

$$h / L \leq 1/20 . \quad (39)$$

From Eqs.(28) and (39), the following equation is obtained :

$$T\sqrt{g/h} \geq 20 . \quad (40)$$

Using the deep-water wave length $L_0 = gT^2/2\pi$, h_1 is given by

$$h_1 / L_0 = 0.0157 . \quad (41)$$

Before the changes in the wave height and profile with decrease in the water depth on the beach of constant slope i are discussed numerically, the wave height H_1 at h_1 is calculated by using the theoretical curves of wave height change, which the authors[2),5)] obtained based on the wave energy flux of the hyperbolic wave theory. If the value of deep-water wave steepness H_0/L_0 (H_0 : deep-water wave height) is given, the value of H_1/H_0 is obtained from the theoretical curve when $h_1/L_0 = 0.0157$ and the value of $H_1/h_1 = (H_1/H_0) \times (H_0/L_0)/(h_1/L_0)$ is calculated. According to the theoretical curves of hyperbolic waves, when $H_0/L_0 \geqq 0.006$, waves already break at the water depth where $h_1/L_0 = 0.0157$. Therefore, the deep-water wave steepness to which this theory is applicable is restricted to considerably small values.

When the beach slope i is given, the value of a/h_1 can be calculated as described previously. However, the value of $A^{(2)}{}_1/A^{(1)}{}_1$ increases with decrease in i for a constant value of H_0/L_0 and with increase in H_0/L_0 for a constant value of i. The numerical computation is restricted in the range of $A^{(2)}/A^{(1)} \leqq 1$. Therefore, for the larger value of H_0/L_0, the value of i for which the computation can be made becomes larger.

RESULTS OF NUMERICAL COMPUTATION

The numerical computations were conducted for the cases that the deep-water wave steepness $H_0/L_0 = 0.004$, 0.002, 0.001, 0.0004, 0.0002 and 0.0001. As mentioned above, for each value of H_0/L_0, the value of H_1/H_0 was calculated using the theoretical curves by the energy flux method[2),5)], for the largest water depth $h_1/L_0 = 0.0157$, where the computation was started. Further, the values of a/h_1 were calculated for given values of i. The values of i in computations were 1/10, 1/20, 1/30, 1/50, 1/100 and 1/200. However, due to the restriction on the beach slope mentioned above, the computations were conducted for the larger values of i than 1/100 when $H_0/L_0 = 0.001$, 1/50 when $H_0/L_0 = 0.002$, and 1/20 when $H_0/L_0 = 0.004$.

For each value of H_0/L_0 and i, using Eqs.(31) and (32), η/h_1 was computed for many values of h/L_0 which are smaller than h_1/L_0. The numerical computations were performed with FACOM 230-60 at the DATA PROCESSING CENTER, KYOTO

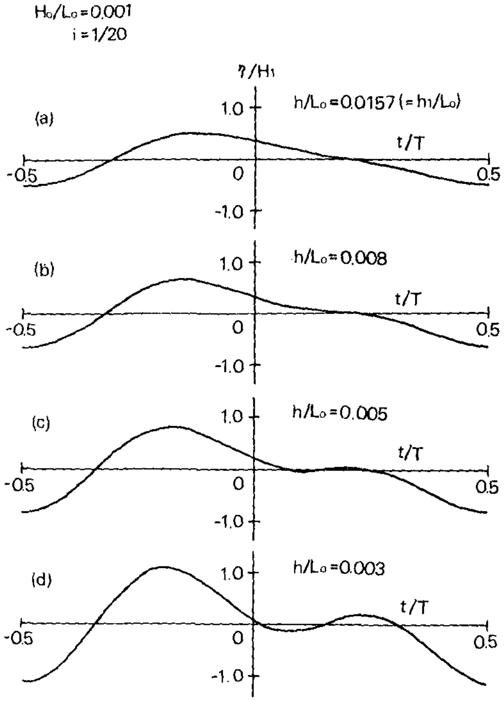

Fig.6 Wave profile change with decrease in water depth

UNIV.. One example is shown in Fig.6, for the case that $H_0/L_0 = 0.001$ and $i = 1/20$. The wave profile η is divided by the wave height H_1 at the water depth h_1. Therefore $(\eta/H_1)_{max} - (\eta/H_1)_{min} = 1.0$ in (a). This value increases with decrease in the water depth h (in order of (b), (c) and (d)), and it becomes larger than 2.0 in (d). Further, it is found that the slope of front face of the wave becomes steeper and that of back face becomes more gentle, so that the wave profile becomes more asymmetric and forward inclined. Thus, the asymmetric deformation of the wave profile in shoaling water can be explained by this solution.

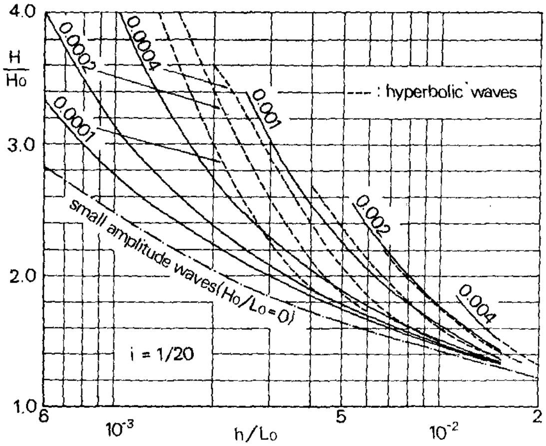

Fig.7 Effect of deep-water wave steepness on wave height change

Several examples of wave height change are shown in Figs.7 and 8. The beach slope i is constant (1/20) in Fig.7 and the deep-water wave steepness H_0/L_0 is constant (0.001) in Fig.8, which shows the effect of the beach slope on wave height change. The theoretical curves obtained under the assumption of constancy of energy flux of hyperbolic waves[2),5)] are also shown with broken lines. Similarly the curve of small amplitude waves is shown with chain line. At the upper end of each curve of this theory (full line) the value of $A^{(2)}/A^{(1)}$ approaches 1.0.

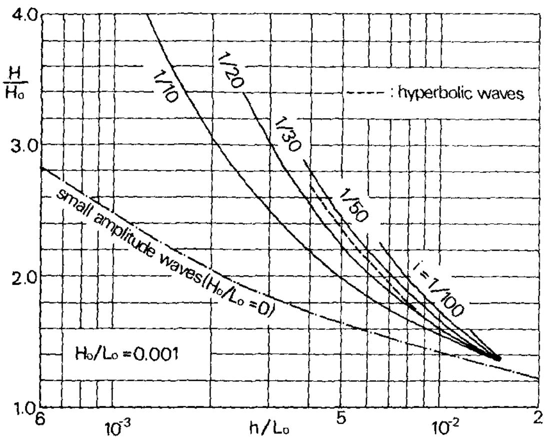

Fig.8 Effect of beach slope on wave height change

In general, the wave heights by this theory are always larger than those of small amplitude waves, and the rate of wave height increase with decrease in the water depth is larger than that of small amplitude waves. Further, as well as the curves of hyperbolic waves, the rate of wave height increase is larger for the larger value of the deep-water wave steepness. However, as the deep-water wave steepness becomes smaller, the value itself of this theory becomes smaller than that of hyperbolic waves. On the other hand, for a constant deep-water wave steepness as in Fig.8, when the beach slope i is small, tha rate of wave height increase becomes large. This effect of the beach slope is expected from the fact that i^{-1} is involved in the coefficient of the second term of Eq.(38). Thus, this theory clarifies the effect of beach slope on the wave height change, which has not been made clear theoretically before.

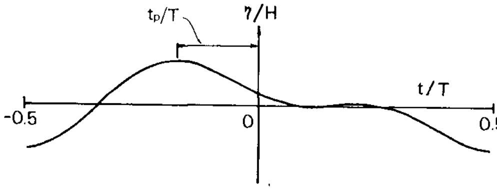

Fig.9 Parameter of asymmetry of wave profile

As a parameter representing the degree of asymmetry of the wave profile, t_p/T shown in Fig.9 is taken. Several examples of the change of t_p/T in shoaling water are shown in Fig.10 and 11. Fig.10 shows the case that the beach slope is constant (1/20) and Fig.11 is for the constant deep-water wave steepness (0.001). It is obvious that the value of t_p/T increases with decrease in the water depth and the wave profile becomes more asymmetric and forward inclined. It is found from the figures that the wave profile becomes more asymmetric for the larger deep-water wave steepness and for the smaller beach slope.

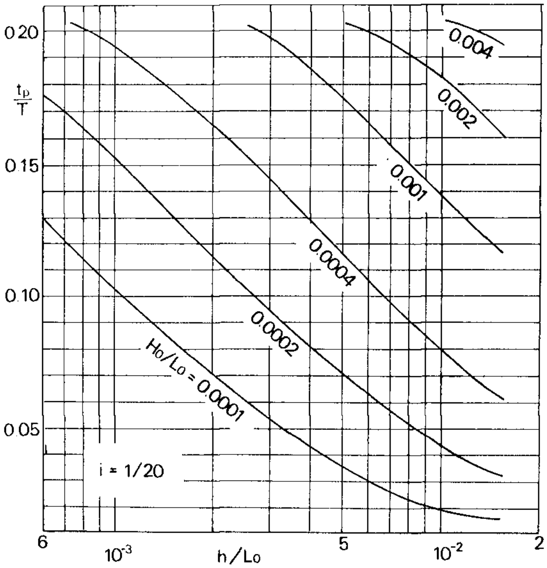

Fig.10 Effect of deep-water wave steepness on wave profile asymmetry

As mentioned above, Biesel[7] derived theoretically the average value of the slopes of front and back faces of the wave profile, while the parameter t_p/T is same as the " wave horizontal asymmetry " discussed experimentally by Adeyemo[16]. The wave profile asymmetry corresponds to the asymmetry of time variation of water particle velocity, and affects the time variations of wave forces on coastal structures and motions of bed materials. Therefore, the parameter t_p/T may has more important engineering significance than Biesel's parameter.

COMPARISON WITH EXPERIMENTAL RESULTS

The equipment and procedure of the experiments with respect to wave shoaling on beaches were same as that described by Iwagaki[2], in which the experiments for the beach slope of only 1/20 were conducted.

WAVE HEIGHT CHANGE

In Fig.12 (1) ∿ (3), the experimental results of wave height change for the beach slopes of 1/30, 1/20 and 1/10 are shown. In Fig. 12 (1) (i = 1/30), although the experimental values can not be directly compared with the theoretical values by this theory because of the lack of data for corresponding values of H_0/L_0, it seems that the trend of the experimental values agrees roughly with that by this theory and the values are larger than those by the hyperbolic wave theory[2),5)]. In Fig. 12 (2) (i = 1/20), the experimental values when H_0/L_0 = 0.0026 agree with those by this theory and all of the experimental values agree well with those by the hyperbolic wave theory. In Fig.12 (3) (i = 1/10), the experimental values are much smaller than those by this theory. Therefore, the experimental results confirm the prediction by this theory that the wave height becomes larger on the gentle slope beach than on the steep slope beach.

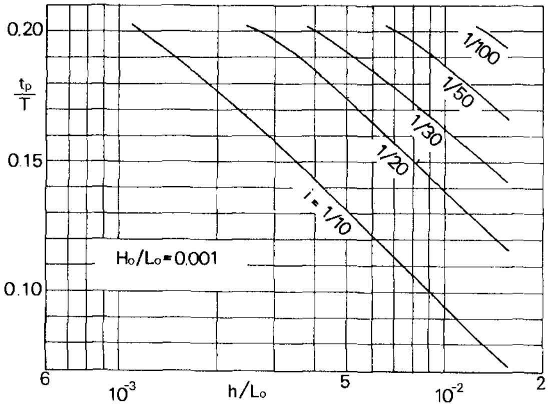

Fig.11 Effect of beach slope on wave profile asymmetry

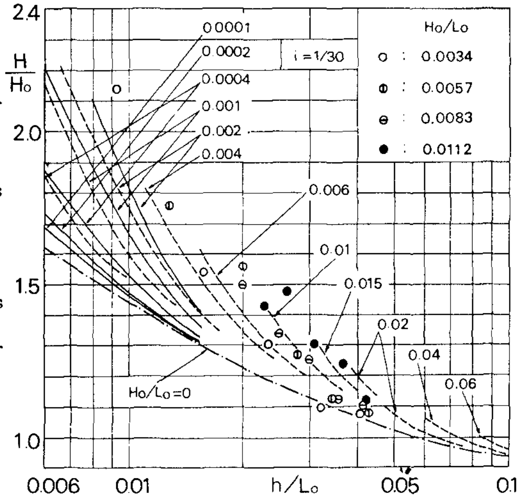

Fig.12 (1) Comparison with experimental results of wave height change (i = 1/30)

WAVE PROFILE ASYMMETRY

In Fig.13 (1) ∿ (3), the experimental results of wave profile asymmetry for the beach slopes of 1/30, 1/20 and 1/10 are shown. The experimental values of a parameter t_p/T increase with decrease in the water depth. This agrees with the behaviour of the curves by this theory. However, the experimental values themselves are considerably different from those of theoretical curves. Although the common range of values of h/L_0 and H_0/L_0 between the experimental results and the theoretical curves is limited, it is found that the experimental values begin to increase at the larger value of h/L_0 than the theoretical curves and the rates of increase are also larger than those of theoretical ones. It seems that the experimental values of t_p/T are larger for larger values of the deep-water wave steepness. This trend of the experimental values agrees with that of this theory. However, because of scatter of experimental values, the effect of the beach slope on the wave profile asymmetry can not be confirmed by the experimental results.

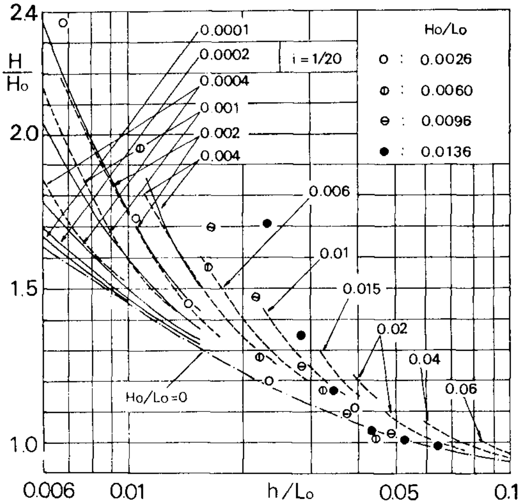

Fig.12 (2) Comparison with experimental results of wave height change (i = 1/20)

CONCLUSION

A solution of finite amplitude long waves on beaches of constant slope was obtained by solving the shallow water theory of the lowest order with the perturbation method. This solution was used to explain the beach slope effect on the wave height change and the wave profile asymmetry in shoaling water, which have not been made clear theoretically. The theoretical results were shown graphically. On the other hand, the experiments on wave shoaling were conducted in order to confirm the validity of this theory.

It was found theoretically that the rate of wave height increase is larger on

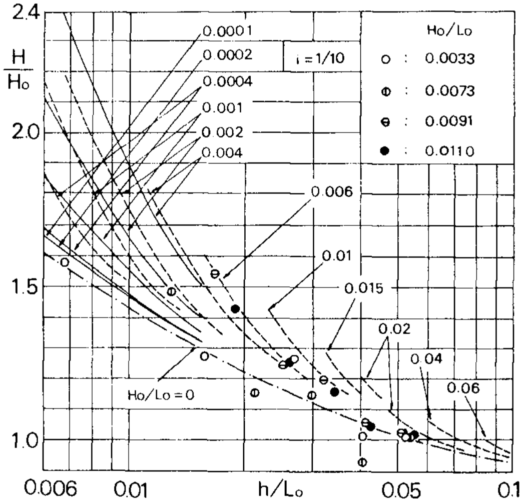

Fig.12 (3) Comparison with experimental results of wave height change (i = 1/10) the gentle slope beach than on the steep slope beach. The experimental results agreed qualitatively with this theoretical prediction.

It was found theoretically that the wave profile becomes asymmetric with decrease in the water depth and more asymmetric for the larger deep-water wave steepness and smaller beach slope. The experimental results confirmed the theoretical prediction on the effects of the water depth and the deep-water wave steepness.

This study is the part of the authors' investigations on the finite amplitude waves and their shoaling during recent several years. The authors wish that the results of this study contribute to advance of investigations on these problems and practical applications.

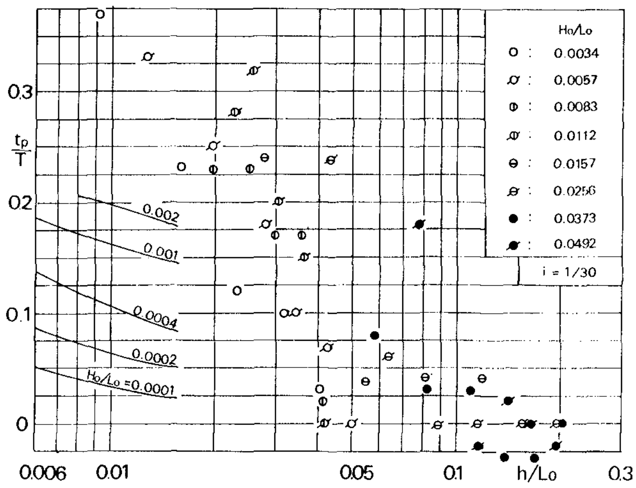

Fig.13 (1) Comparison with experimental results of wave profile asymmetry t_p/T ($i = 1/30$)

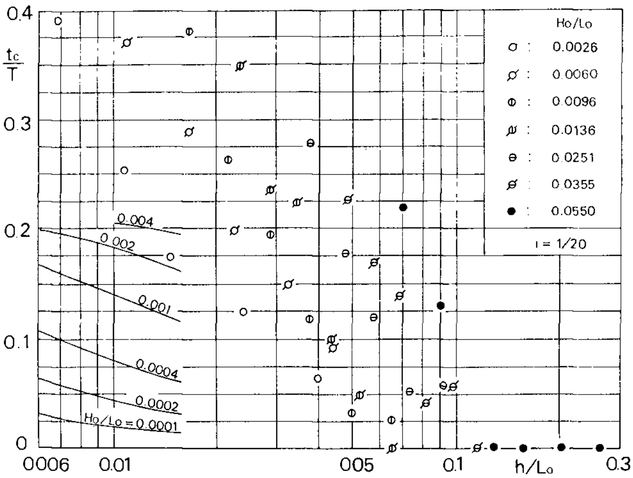

Fig.13 (2) Comparison with experimental results of wave profile asymmetry t_p/T (i = 1/20)

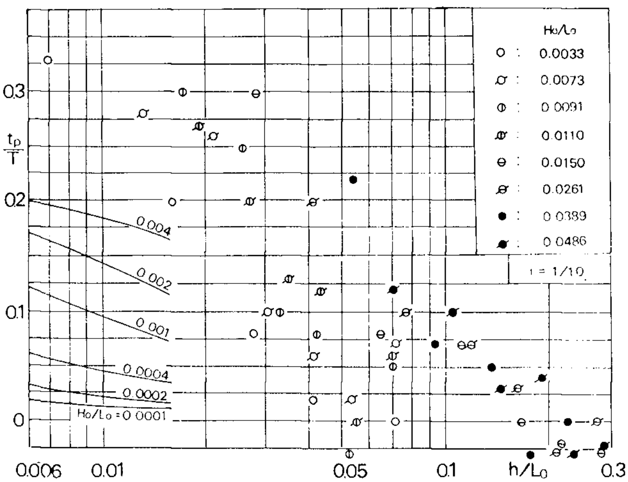

Fig.13 (3) Comparison with experimental results of wave profile asymmetry t_p/T (i = 1/10)

REFERENCES

1) Le Méhauté, B. and L.M. Webb : Periodic gravity waves over a gentle slope at a third order of approximation, Proc. 9th Conf. Coastal Eng., pp.23-40, 1964.
2) Iwagaki, Y. : Hyperbolic waves and their shoaling, Proc. 11th Conf. Coastal Eng., pp.124-144, 1968.
3) Friedrichs, K.O. : Water waves on a shallow sloping beach, Comm. Pure and Applied Math., Vol.1, pp.109-134, 1948.
4) Tlapa, G.A., C.C. Mei and P.S. Eagleson : An asymptotic theory for water waves on beaches of mild slope, M.I.T. Hydrodynamics Laboratory Report No. 90, 1966.
5) Iwagaki, Y. and T. Sakai : On the shoaling of finite amplitude waves (2), Proc. 15th Conf. Coastal Eng. in Japan, pp.10-15, 1968 (in Japanese).
6) Goda, Y. : A synthesis of breaker indices, Trans. JSCE, Vol.2, Part 2, pp. 227-230, 1970.
7) Biesel, F. : Study of wave propagation in water of gradually varying depth, Gravity Waves, Circular No.521, Nat. Bureau of Standard, Washington D.C., 1951.
8) Stoker, J.J. : Water Waves, Interscience Pub. Inc., New York, pp.22-32, 1957.
9) Homma, M. : Deformation of long waves, Jour. JSCE, Vol.19, No.9, 1933 (in Japanese).
10) Carrier, G.H. and H.P. Greenspan : Water waves of finite amplitude on a sloping beach, Jour. Fluid Mech., Vol.4, pp.97-109, 1958.
11) Ichiye, T. : Some remarks on the non-linear theory of shallow water waves on a sloping beach, The Oceanographical Magazine, Vol.4, No.4, pp.159-166, 1953.
12) Bourodimos, E.L. and A.T. Ippen : Wave transformation in an open channel transition, Proc. ASCE, Vol.94, HY.5, pp.1317-1329, 1968.
13) Green, G. : On the motion of waves in a variable canal of small depth and width, Camb. Trans. Vol.VI, 1837.
14) Laitone, E.V. : The second approximation to cnoidal and solitary waves, Jour. Fluid Mech., Vol.9, pp.430-444, 1961.
15) Iwagaki, Y : Wave transformation, Lecture note of the 1967 summer seminor on hydraulic engineering B, JSCE, 1967 (in Japanese).
16) Adeyemo, M.D. : Effect of beach slope and shoaling on wave asymmetry, Proc. 11th Conf. Coastal Eng., pp.145-172, 1968.