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SHOCK MODELS AND WEAR PROCESSES

BY J. D. ESARY, A. W. MARSHALL AND F. PROSCHAN[1]

U.S. Naval Postgraduate School, University of Rochester and Florida State University

The life distribution $H(t)$ of a device subject to shocks governed by a Poisson process is considered as a function of the probabilities P_k of not surviving the first k shocks. Various properties of the discrete failure distribution P_k are shown to be reflected in corresponding properties of the continuous life distribution $H(t)$. As an example, if P_k has discrete increasing hazard rate, then $H(t)$ has continuous increasing hazard rate. Properties of P_k are obtained from various physically motivated models, including that in which damage resulting from shocks accumulates until exceedance of a threshold results in failure. We extend our results to continuous wear processes. Applications of interest in renewal theory are obtained. Total positivity theory is used in deriving many of the results.

1. Introduction. In this paper we study some models for the life distribution of a device subjected to a sequence of shocks occurring randomly in time as events in a Poisson process. If the device has a probability $\bar{P}_k$ of surviving the first k shocks, $k = 0, 1, 2, \cdots$, then the probability $\bar{H}(t)$ that the device survives beyond time t can be represented in the form

$$\bar{H}(t) = \sum_{k=0}^{\infty} \bar{P}_k e^{-\lambda t}(\lambda t)^k/k! \,, \qquad t \geqq 0 \,, \tag{1.1}$$

for some $\lambda > 0$. Shock models of this kind have been considered by a number of authors, e.g., by Esary (1957), Epstein (1958) and by Gaver (1963).

Survival functions with the form of $\bar{H}$ have a number of pleasant properties which are noted in Section 2 for later reference. In Section 3 we see how some properties of the shock survival probabilities $\bar{P}_k$ natural in reliability models are reflected as properties of $\bar{H}$. These results are useful in conjunction with the results of Sections 4 and 6, where we obtain properties of the $\bar{P}_k$ from physically motivated models. Thus, our overall aim is to obtain properties of the survival function of a device from models for the stochastic mechanism leading to failure.

The simplest model for the shock survival probabilities $\bar{P}_k$ assumes that each shock causes a random damage, that damages on successive shocks are independent

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and identically distributed, and that failure occurs when the accumulated damage exceeds a specified threshold x. Thus, $\bar{P}_k = F^{(k)}(x)$ where $F^{(k)}$ is the kth convolution of the distrbution F of damage sustained from any given shock. References to several earlier treatments of this model are given in Section 4. Our principal tool is the result that $[F^{(k)}(x)]^{1/k}$ is decreasing in $k = 1, 2, \cdots$ so long as $F(z) = 0$ for $z < 0$ (damages are never negative). From results in Section 3, this yields the conclusion that $\bar{H}$ has an increasing hazard rate average, a property of interest in the reliability context. (See Birnbaum, Esary, and Marshall (1966).) The monotonicity of $[F^{(k)}(x)]^{1/k}$ also has some applications in renewal theory. In addition to wear or damage that is accumulated in positive increments at isolated time points, we consider some hypotheses for continuous wear processes. The results there may be of some independent interest; we show that a certain kind of Markov process has first passage time distributions with increasing hazard rate averages.

There are several obvious directions for extending the models of this paper, some of which have already received attention in the literature. Although we have limited ourselves to the case that shocks are governed by a homogeneous Poisson process, interesting new results can be obtained by dropping the condition of homogeneity, or by replacing the Poisson process with some other renewal process.

In what follows we use the term "increasing" to mean "non-decreasing" and "decreasing" to mean "non-increasing."

2. Preliminary definitions and calculations. We collect here some basic facts which are required in later sections. The reader may proceed to Section 3 and refer to these results as needed.

Let us suppose that $\bar{H}$ is a survival function of the form

$$\bar{H}(t) = \sum_{k=0}^{\infty} \bar{P}_k e^{-\lambda t}(\lambda t)^k/k! \,, \qquad t \geqq 0 \,, \tag{2.1}$$
$$= 1 \,, \qquad t < 0 \,.$$

If $\bar{P}_k$ is interpreted as the probability of "surviving k shocks," then

$$1 \geqq \bar{P}_0 \geqq \bar{P}_1 \geqq \cdots \,, \tag{2.2}$$

and the probability of "failure on the kth shock" is given by

$$p_0 = 1 - \bar{P}_0 \,,$$
$$p_k = \bar{P}_{k-1} - \bar{P}_k \,, \qquad k = 1, 2, \cdots \,.$$

Since $\bar{H}$ is a survival function, i.e., $1 - \bar{H} = H$ is a distribution function, it must be that $\bar{P}_k \to 0$ as $k \to \infty$. It may happen that (2.1) is a survival function even though (2.2) is violated, but we assume that (2.2) holds.

Except for the nonnegative mass $1 - \bar{P}_0$ at the origin, we easily compute that H has a density h given by

$$h(t) = \lambda \sum_{k=1}^{\infty} p_k e^{-\lambda t}(\lambda t)^{k-1}/(k-1)! \,, \qquad t > 0 \,. \tag{2.3}$$

Moreover, the hazard rate r is given by

$$(2.4)\qquad \begin{aligned} r(t) &= h(t)/\bar{H}(t) \\ &= \lambda\{1 - [\sum_{k=0}^{\infty} \bar{P}_{k+1}(\lambda t)^k/k!]/[\sum_{k=0}^{\infty} \bar{P}_k(\lambda t)^k/k!]\}\,, \qquad t > 0\,. \end{aligned}$$

Notice from (2.4) that

$$(2.5)\qquad r(t) \leqq \lambda\,, \qquad t > 0\,.$$

Furthermore, $r(t) = \lambda$ for *some* $t > 0$ if and only if $\bar{P}_k = 0$ for all $k > 0$ and then $r(t) = \lambda$ for *all* $t > 0$. (If $\bar{P}_0 = 1$, this means $\bar{H}$ is exponential.) Since $\bar{H}(t) = \bar{H}(0)\exp\{-\int_0^t r(x)\,dx\}$, it follows from (2.5) that

$$(2.6)\qquad \bar{H}(t) \geqq \bar{H}(0)e^{-\lambda t} \qquad \text{for all } t \geqq 0\,.$$

When a survival function can be written in the form (2.1), the $\bar{P}_k$ can be calculated from successive derivatives of $\bar{H}$ evaluated at 0: Denoting the jth derivative of $\bar{H}$ by $\bar{H}_j$, we have

$$(2.7)\qquad \bar{P}_k = \sum_{j=0}^{k} \binom{k}{j}\bar{H}_j(0)/\lambda^j \qquad \text{and} \qquad p_k = -\sum_{j=1}^{k} \binom{k-1}{j-1}\bar{H}_j(0)/\lambda^j\,.$$

A survival function which can be written in the form (2.1) can, for any $\nu > 0$, also be written in the form

$$\bar{H}(t) = \sum_{k=0}^{\infty} \bar{Q}_k e^{-\nu t}(\nu t)^k/k!\,, \qquad t \geqq 0\,,$$

where

$$(2.8)\qquad \bar{Q}_k = \nu^{-k} \sum_{j=0}^{k} \binom{k}{j}(\nu - \lambda)^{k-j}\lambda^j\bar{P}_j\,.$$

Trivially, $\bar{P}_0 = 1$ implies $\bar{Q}_0 = 1$. Furthermore, if $\nu > \lambda$, then (2.2) implies that $\bar{Q}_k$ is decreasing in k.

It is important to realize that *any* survival function $\bar{H}$ on $[0, \infty)$ can be approximated by survival probabilities of the form (2.1) with λ sufficiently large. If

$$\bar{H}_\lambda(t) = \sum_{k=0}^{\infty} \bar{H}(k/\lambda)e^{-\lambda t}(\lambda t)^k/k!\,, \qquad t \geqq 0\,,$$

then

$$(2.9)\qquad \lim_{\lambda\to\infty} \bar{H}_\lambda(t) = \bar{H}(t)$$

at continuity points of $\bar{H}$. See (1.5) of Feller (1966) page 219.

The jth moment μ_j of the distribution given by (2.1) can be obtained as

$$(2.10)\qquad \mu_j = \frac{j!}{\lambda^j} \sum_{k=0}^{\infty} \binom{k+j-1}{k}\bar{P}_k\,, \qquad j = 1, 2, \cdots\,.$$

Of course these moments are finite only if the relevant series converge.

EXAMPLE 2.1. A particularly interesting example of (2.1) occurs with $\bar{P}_k = \theta^k$. In this case $\bar{H}(t) = e^{-\lambda(1-\theta)t}$ is exponential. One special case is $\theta = 0$, i.e., $\bar{P}_0 = 1$, $\bar{P}_k = 0$ for $k \geqq 1$. It is not difficult to see from (2.7) that $\bar{H}$ is exponential only if $\bar{P}_k = \theta^k$ for some $\theta < 1$.

EXAMPLE 2.2. An interesting extension of Example 2.1 is obtained if we begin with $\bar{P}_k = 1$, $k = 0, 1, \cdots, n$, $\bar{P}_k = 0$, $k > n$. Then (2.3) becomes the density of a gamma distribution of order $n + 1$, i.e.,

$$h(t) = \lambda(\lambda t)^n e^{-\lambda t}/n! \,. \tag{2.11}$$

Using (2.8), we see more generally that H given by (2.1) has a gamma density of the form (2.11) if

$$\bar{P}_k = \theta^k \sum_{j=0}^{k} \binom{k}{j} \left(\frac{1-\theta}{\theta}\right)^j , \qquad k \leqq n ,$$
$$= \theta^k \sum_{j=0}^{n} \binom{k}{j} \left(\frac{1-\theta}{\theta}\right)^j , \qquad k > n ,$$

where $\theta < 1$.

The ideas behind (2.1)—a Poisson source of shocks and probabilities $\bar{P}_k$ of surviving k shocks—can be extended in the following way. Suppose that there are in fact several independent Poisson sources of shocks, and shocks from various sources may have different effects when combined in different ways. Thus, the probabilities $\bar{P}_k$ are replaced by probabilities $\bar{P}_{i_1, i_2, \cdots, i_l}$ of surviving i_1 shocks of the first kind, i_2 shocks of the second kind, etc. Here, (2.1) is replaced by the survival probability

$$\bar{H}(t) = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_l=0}^{\infty} \prod_{j=1}^{l} e^{-\lambda_j t} \frac{(\lambda_j t)^{i_j}}{i_j!} \bar{P}_{i_1, \cdots, i_l} \,, \qquad t \geqq 0 \,. \tag{2.12}$$

If we let $k = i_1 + \cdots + i_l$, $\lambda = \sum_{j=1}^{l} \lambda_j$, $q_j = \lambda_j/\lambda$, $j = 1, 2, \cdots, l$, (2.12) can be written as (2.1) where

$$\bar{P}_k = \sum_{i_j \geqq 0,\, \sum i_j = k} \binom{k}{i_1, \cdots, i_l} \prod_{j=1}^{l} q_j^{\,i_j} \bar{P}_{i_1, \cdots, i_l} \,. \tag{2.13}$$

Thus, the simple model first introduced does in fact include the apparently more general one.

We make constant use of the methods of total positivivity in studying distributions of the form (2.1). Consequently, we record here some relevant definitions.

DEFINITION 2.3. Let A and B be subsets of the real line. A function K on $A \times B$ is said to be *sign consistent of order n* (SC_n) with sign $\varepsilon = \pm 1$ if $\varepsilon \det |K(x_i, y_j)|_{i,j=1,2,\cdots,n} \geqq 0$ whenever each $x_i \in A$, $y_j \in B$, and $x_1 < x_2 < \cdots < x_n$, $y_1 < y_2 < \cdots < y_n$. K is said to be *totally positive* of order n (TP_n) if it is SC_m for $m = 1, 2, \cdots, n$ with $\varepsilon = 1$. A function ϕ defined on the real line (the integers) is said to be a *Pólya frequency function* (*sequence*) of order n (PF_n) if $\phi(x - y)$ is TP_n in real (integer) x and y.

The function $K_1(x, y) = e^{xy}$ is totally positive of all finite orders (TP_∞) in $x, y \in (-\infty, \infty)$ (Karlin (1968) page 15), so that $K_2(r, t) = t^r$ is TP_∞ in $t \in (0, \infty)$ and $r \in (-\infty, \infty)$.

3. Properties of $\bar{H}$ from properties of the $\bar{P}_k$. Certain kinds of properties, when imposed on the $\bar{P}_k$ in (2.1), are reflected as analogous properties of $\bar{H}$. For example, we have already seen that if $\bar{P}_k$ is decreasing in k, then $\bar{H}(t)$ is decreasing in t. Below, various other properties of the $\bar{P}_k$ of interest in reliability are shown to carry over to $\bar{H}$; some definitions are convenient for stating these results. In these definitions, we assume that the distribution F satisfies $F(z) = 0$ for $z < 0$ because we are concerned in this paper with life distributions. It is also tacitly assumed in these definitions that variables are restricted to avoid zero denominators.

A distribution F or survival function $\bar{F}$ is said to be or to have:

(i) *a PF_2 density* if F has a density f such that $f(x + t)/f(t)$ is decreasing in t whenever $x > 0$.

One can easily check that this definition is equivalent to the one given in Definition 2.3. The conditions that $\log f$ is concave and that f is strongly unimodal are also equivalent.

(ii) *increasing hazard rate* (IHR) if $\bar{F}(x + t)/\bar{F}(t)$ is decreasing in t whenever $x > 0$.

When F has a density, this is equivalent to the condition that for some version f of the density, the hazard rate $r(t) = f(t)/\bar{F}(t)$ is increasing in t. Also, F is IHR if and only if $\log \bar{F}$ is concave, and F is IHR if and only if $\bar{F}$ is PF_2. To say that the life distribution F of an item is IHR is to say that the residual life length of an unfailed item of age t is stochastically decreasing in t.

(iii) *decreasing mean residual life* (DMRL) if $\int_0^\infty \bar{F}(x + t)\, dx/\bar{F}(t)$ is decreasing in t.

To say that the life distribution F of an item is DMRL is equivalent to saying that the residual life of an unfailed item of age t has a mean that is decreasing in t.

(iv) *increasing hazard rate average* (IHRA) if $[\bar{F}(t)]^{1/t}$ is decreasing in $t > 0$.

When a hazard rate r exists, this is equivalent to the condition that the hazard rate average $(1/t) \int_0^t r(u)\, du$ is increaing in t. In another formulation, this condition says that $(1/t)[-\log \bar{F}(t)]$ is increasing in $t > 0$ ($-\log \bar{F}(t)$ is a starshaped function).

(v) *new better than used* (NBU) if $\bar{F}(x) \geqq \bar{F}(t + x)/\bar{F}(t)$ for all x, $t \geqq 0$.

To say that the life distribution F of an item is NBU is equivalent to saying that the life length of a new item is stochastically greater than the residual life length of an unfailed item of age t, $t \geqq 0$.

(vi) *new better than used in expectation* (NBUE) if $\int_0^\infty \bar{F}(x)\, dx \geqq \int_0^\infty \bar{F}(t + x) dx/\bar{F}(t)$ for all $t \geqq 0$.

To say that the life distribution F of an item is NBUE is equivalent to saying that the expected life length of a new item is greater than the expected residual life length of an unfailed item of age $t > 0$.

The strongest of these properties, that f is PF_2, is possessed by many of the commonly encountered densities, such as uniform densities and gamma densities of order $\alpha \geqq 1$. Such densities have found diverse applications in reliability theory, inventory theory, statistical decision theory, etc., and have been discussed extensively by Karlin (1968).

The IHR property is basic to a considerable amount of reliability theory, and is discussed by Barlow and Proschan (1965). Statistical procedures based on a generalization of this property have been developed by Barlow (1968) and by Barlow and van Zwet (1970), who cite a series of earlier papers on the subject.

The class of IHRA distributions was characterized by Birnbaum, Esary and Marshall (1966) as the smallest class of distributions containing the exponential distributions which is closed under both formation of coherent systems and limits in distribution. These distributions have been subsequently studied, e.g., by Barlow and Marshall (1967), Barlow (1968), and Doksum (1969a, b).

The NBU and NBUE distributions have recently been encountered by Marshall and Proschan (1970) in conjection with studies of replacement policies. They show that the NBU property must be present if replacement policies are to be beneficial in a certain sense.

There is a corresponding set of analogous properties, obtained by reversing the direction of monotonicity or the direction of the inequality in these definitions. These properties are (i′) logarithmically convex density, (ii′) decreasing hazard rate (DHR), (iii′) *increasing* mean residual life (IMRL), (iv′) decreasing hazard rate average (DHRA), (v′) new worse than used (NWU), and (vi′) new worse than used in expectation (NWUE).

Among these properties, only the first two have received much attention. Densities which are logarithmically convex arise in consideration of passage times in birth-death processes (see Keilson (1971) who also cites earlier references). The DHR distributions are discussed in reliability contexts by Barlow and Proschan (1965).

With the above definitions, we are in a position to state the main result of this section.

THEOREM 3.1. *Suppose that*

$$\bar{H}(t) = \textstyle\sum_{k=0}^{\infty} \bar{P}_k e^{-\lambda t}(\lambda t)^k/k! \,, \qquad t \geqq 0$$

where $1 = \bar{P}_0 \geqq \bar{P}_1 \geqq \cdots$. *Then*

(3.1) *H has a PF_2 density if p_{k+1}/p_k is decreasing in $k = 1, 2, \cdots$, i.e., if $\{p_k, k \geqq 1\}$ is a PF_2 sequence;*

(3.2) *H is* IHR *if $\theta_k = \bar{P}_k/\bar{P}_{k-1}$ is decreasing in $k = 1, 2, \cdots$, i.e., if $\{\bar{P}_k, k \geqq 0\}$ is a PF_2 sequence;*

(3.3) *H is* DMRL *if $\sum_{j=k}^{\infty} \bar{P}_j/\bar{P}_k$ is decreasing in $k = 0, 1, \cdots$;*

(3.4) *H is* IHRA *if $\bar{P}_k^{1/k}$ is decreasing in $k = 1, 2, \cdots$;*

(3.5) *H is* NBU *if $\bar{P}_j \bar{P}_k \geqq \bar{P}_{j+k}$, $j, k = 0, 1, \cdots$;*

(3.6) *H is* NBUE *if $\bar{P}_k \sum_{j=0}^{\infty} \bar{P}_j \geqq \sum_{j=k}^{\infty} \bar{P}_j$, $k = 0, 1, \cdots$.*

Observe that the conditions of (3.1)—(3.6) on the $\bar{P}_k$ are in every case discrete analogues of the conditions concluded to hold for $\bar{H}$.

To prove (3.1)—(3.4), we use the variation diminishing property of the totally positive kernel $K(k, t) = e^{-\lambda t}(\lambda t)^k/k!$. See Karlin (1968) for a discussion of this property.

PROOF OF (3.1). Since p_{k+1}/p_k is decreasing in $k = 1, 2, \cdots$, the sequence $\{\log p_k, k = 1, 2, \cdots\}$ is concave. Thus, for any $a > 0$, $\zeta > 0$, $p_k - a\zeta^k$ changes sign at most twice, in the order $-, +, -$ if two sign changes occur. Because

$$h(t) - a\lambda\zeta e^{-(1-\zeta)\lambda t} = \lambda \sum_{k=0}^{\infty} (p_{k+1} - a\zeta^{k+1})e^{-\lambda t}(\lambda t)^k/k!\ ,$$

we conclude from the variation diminishing property that for any $c > 0$ and any $\theta < \lambda$, $h(t) - ce^{-\theta t}$ has at most two sign changes, in the order $-, +, -$ if two sign changes occur.

Next, consider the case that $\theta \geqq \lambda$; we shall show that here, $h(t) - ce^{-\theta t}$ has at most one sign change. To do this, differentiate $h(t)$ in (2.3) to verify that

$$h'(t) \geqq -\lambda h(t)\ .$$

Suppose that for some t_0, $h(t_0) = ce^{-\theta t_0}$. Then

$$\begin{aligned} h'(t_0) &\geqq -\lambda ce^{-\theta t_0} \\ &\geqq -ce^{-\theta t_0} = \frac{d}{dt} ce^{-\theta t}|_{t=t_0}\ , \end{aligned}$$

i.e., $ce^{-\theta t}$ crosses $h(t)$ only from above, so that there can be at most one sign change.

Combining these results, we see that for all $c, \theta > 0$, $h(t) - ce^{-\theta t}$ has at most two sign changes, in the order $-, +, -$ if two sign changes occur. This means that $\log h(t)$ is concave in t, i.e., h is PF_2. □

PROOF OF (3.2). One can prove (3.2) in the same way as (3.1) is proved above but with $\bar{P}_k$ in place of p_k. Here, (2.5) is used in place of $h'(t) \geqq -\lambda h(t)$. □

A result similar to (3.1) and (3.2) but more general in several respects has been given by Karlin (1968) page 107.

PROOF OF (3.3). Here, we have the hypothesis that for any $c \geqq 0$, $\lambda^{-1} \sum_{i=k}^{\infty} \bar{P}_i - c\bar{P}_k$ has at most one sign change, from $+$ to $-$ if one occurs. Hence

$$\int_t^\infty \bar{H}(x)\, dx - c\bar{H}(t) = \sum_{k=0}^{\infty} (\lambda^{-1} \sum_{i=k}^{\infty} \bar{P}_i - c\bar{P}_k)e^{-\lambda t}(\lambda t)^k/k!$$

has at most one sign change, from $+$ to $-$ if one occurs. This means that $\int_t^\infty \bar{H}(x)\, dx/\bar{H}(t)$ is decreasing in t, i.e., H is DMRL. □

PROOF OF (3.4). If $\bar{P}_k^{1/k}$ is decreasing in k, $\bar{P}_k - \zeta^k$ $(0 \leqq \zeta \leqq 1)$ has at most one sign change, from $+$ to $-$ if one occurs. This means that $\bar{H}(t) - e^{-(1-\zeta)\lambda t} = \sum_{k=0}^{\infty} (\bar{P}_k - \zeta^k)e^{-\lambda t}(\lambda t)^k/k!$ has in t the same sign change property. By (2.6) and $\bar{H}(0) = 1$, this means that for any $\theta > 0$, $\bar{H}(t) - e^{-\theta t}$ has at most one sign change, from $+$ to $-$ if one occurs. Consequently $[\bar{H}(t)]^{1/t}$ is decreasing in t, i.e., H is IHRA. □

PROOF OF (3.5). Under the conditions of (3.5),

$$\bar{H}(x)\bar{H}(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \bar{P}_k \bar{P}_l e^{-\lambda(x+t)} \frac{(\lambda x)^k}{k!} \frac{(\lambda t)^l}{l!}$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{j} \bar{P}_k \bar{P}_{j-k} e^{-\lambda(x+t)} \frac{(\lambda x)^k}{k!} \frac{(\lambda t)^{j-k}}{(j-k)!}$$

$$\geqq \sum_{j=0}^{\infty} \frac{\bar{P}_j}{j!} e^{-\lambda(x+t)} \sum_{k=0}^{j} \binom{j}{k} (\lambda x)^k (\lambda t)^{j-k} = \bar{H}(x+t) . \ \Box$$

PROOF OF (3.6). From the hypothesis of (3.6) it follows that

$$\sum_{k=0}^{\infty} [\bar{P}_k \sum_{j=0}^{\infty} \bar{P}_j - \sum_{j=k}^{\infty} \bar{P}_j] (\lambda t)^k / k! \geqq 0 ,$$

which can be rewritten as

$$\sum_{j=0}^{\infty} \bar{P}_j \geqq [\sum_{k=0}^{\infty} (\sum_{j=k}^{\infty} \bar{P}_j)(\lambda t)^k / k!] / [\sum_{k=0}^{\infty} \bar{P}_k (\lambda t)^k / k!] .$$

It is not difficult to see that this is just the condition that H is NBUE. $\Box$

The following companion to Theorem 3.1 can be proved by modifying the proof of Theorem 3.1 in obvious ways.

THEOREM 3.2. *Suppose that* $\bar{H}(t) = \sum_{k=0}^{\infty} \bar{P}_k e^{-\lambda t} (\lambda t)^k / k!$, *where* $1 \geqq \bar{P}_0 \geqq \bar{P}_1 \geqq \cdots$. *Then*

(3.7) *H has a density logarithmically convex on* $(0, \infty)$ *if* p_{k+1}/p_k *is increasing in* $k = 1, 2, \cdots$;

(3.8) *H is* DHR *if* $\bar{P}_k / \bar{P}_{k-1}$ *is increasing in* $k = 1, 2, \cdots$;

(3.9) *H is* IMRL *if* $\sum_{j=k}^{\infty} \bar{P}_j / \bar{P}_k$ *is increasing in* $k = 0, 1, \cdots$;

(3.10) *H is* DHRA *if* $\bar{P}_k^{1/k}$ *is increasing in* $k = 1, 2, \cdots$;

(3.11) *H is* NWU *if* $\bar{P}_j \bar{P}_k \leqq \bar{P}_{k+j}$, $j, k = 0, 1, \cdots$;

(3.12) *H is* NWUE *if* $\bar{P}_k \sum_{j=0}^{\infty} \bar{P}_j \leqq \sum_{j=k}^{\infty} \bar{P}_j$, $k = 0, 1, \cdots$.

The conditions of Theorem 3.1 are *not* necessary conditions. In this connection, (2.9) has an interesting consequence. If $\bar{H}$ has the form (2.1) and is, e.g., IHR, then even though $\bar{P}_{k+1}/\bar{P}_k$ is not decreasing in k, $\bar{H}$ can be approximated by the survival functions $\bar{H}_\lambda$ which are IHR and have the form (2.1) with $\bar{P}_{k+1}/\bar{P}_k = \bar{H}((k+1)/\lambda)/\bar{H}(k/\lambda)$ which *is* decreasing in k. Similar statements can be made concerning the conditions of (3.1), (3.3), (3.4), (3.5) and (3.6).

Conditions weaker than those of (3.2) which also insure that $\bar{H}$ given by (1.1) is IHR have been given by Murthy and Lientz (1968). Their results are contained in the following theorem.

THEOREM 3.3. *If* $\bar{P}_0 = 1$ *and*

(3.13) $$S_k = \sum_{j=0}^{k} \binom{k}{j} [\bar{P}_{j+1} \bar{P}_{k-j+1} - \bar{P}_j \bar{P}_{k-j+2}] \geqq 0 , \qquad k = 0, 1, \cdots ,$$

then $\bar{H}$, *given by* (1.1), *is* IHR.

PROOF. By using (2.4) and differentiating $\lambda - r(t)$, it is easily seen that $r(t)$ is increasing in t if and only if

$$\sum_{k=0}^{\infty} [(\lambda t)^k/k!] \sum_{j=0}^{k} \binom{k}{j}[\bar{P}_{j+1}\bar{P}_{k-j+1} - \bar{P}_j\bar{P}_{k-j+2}] \geqq 0, \qquad t \geqq 0, \tag{3.14}$$

from which the theorem is immediate. □

The condition of (3.2) that $\bar{P}_k/\bar{P}_{k-1}$ is decreasing in $k = 1, 2, \cdots$ can be seen to imply (3.13) if S_k is written in the form

$$S_k = \bar{P}_{k+1}\bar{P}_1 - \bar{P}_0\bar{P}_{k+2} + \sum_{j=1}^{[k/2]} [\binom{k}{j} - \binom{k}{j-1}][\bar{P}_{j+1}\bar{P}_{k-j+1} - \bar{P}_j\bar{P}_{k-j+2}],$$

and if (3.2) is used with the fact that $\binom{k}{j} - \binom{k}{j-1} \geqq 0$, $j = 1, 2, \cdots, [k/2]$.

If $\bar{P}_k = \zeta^k$ when k is even and $\bar{P}_k = \zeta^{k-1}$ when k is odd, $0 < \zeta < 1$, use (3.13) and the fact that $(1-1)^k = 0$ to check that $S_0 = 1 - \zeta^2 > 0$ and $S_k = 0$ for $k > 0$. Here, (3.13) is satisfied, but the condition of (3.2) is violated. Thus (3.13) is strictly weaker than (3.2). If $\bar{P}_0 = 1$, $\bar{P}_1 = \frac{1}{2}$, $\bar{P}_2 = \bar{P}_3 = \frac{1}{8}$, and $\bar{P}_j = 0$ for $j > 3$, then (3.13) is violated for $k = 1$ but (3.14) is satisfied so that $\bar{H}$ given by (1.1) is IHR. Thus (3.13), like (3.2), is not a necessary condition.

A condition similar to (3.13) can be given for F to be DMRL. In fact, H is DMRL if and only if

$$\bar{H}_1(t) = \frac{1}{\mu} \int_t^\infty \bar{H}(x)\,dx = \frac{1}{\mu\lambda} \sum_{k=0}^{\infty} (\sum_{j=k}^{\infty} \bar{P}_j) e^{-\lambda t}(\lambda t)^k/k!$$

is IHR, where $\mu = \int_0^\infty \bar{H}(x)\,dx$. Thus a sufficient condition for DMRL can be obtained by replacing $\bar{P}_i$ with $\sum_{j=i}^{\infty} \bar{P}_j$ in (3.13). It is easily verified that the resulting condition is weaker than the more complicated condition obtained by Murthy and Lientz (1968).

One can generalize (3.1), (3.2), (3.7) and (3.8) from PF_2 to PF_n (see Definition 2.3). This is done in the following theorem, which is closely related to Lemma 2.3, page 109, of Karlin (1968). Karlin's lemma has stronger hypotheses and stronger conclusions. Our proof is more elementary.

THEOREM 3.4. *Suppose that*

$$\bar{H}(t) = \sum_{k=0}^{\infty} \bar{P}_k e^{-\lambda t}(\lambda t)^k/k!, \qquad t \geqq 0$$

where $1 = \bar{P}_0 \geqq \bar{P}_1 \geqq \cdots$. *Then*

(3.15) $h(s+t)$ *is* SC_n *in* $s, t > 0$ *with sign* ε *if* p_{k+j} *is* SC_n *in* $j, k \geqq 1$ *with sign* ε;

(3.16) $\bar{H}(s+t)$ *is* SC_n *in* $s, t > 0$ *with sign* ε *if* $\bar{P}_{j+k}$ *is* SC_n *in* $j, k \geqq 0$ *with sign* ε.

PROOF. Since the proofs of (3.15) and (3.16) are essentially the same, we prove only (3.16). We compute

$$\begin{aligned}\bar{H}(s+t) = e^{-\lambda(s+t)} \sum_{k=0}^{\infty} \bar{P}_k \frac{[\lambda(s+t)]^k}{k!} &= e^{-\lambda(s+t)} \sum_{k=0}^{\infty} \bar{P}_k \sum_{j=0}^{k} \frac{(\lambda s)^j}{j!}\frac{(\lambda t)^{k-j}}{(k-j)!} \\ &= e^{-\lambda(s+t)} \sum_{j=0}^{\infty}\sum_{k=j}^{\infty} \bar{P}_k \frac{(\lambda s)^j}{j!}\frac{(\lambda t)^{k-j}}{(k-j)!} \\ &= e^{-\lambda(s+t)} \sum_{j=0}^{\infty} \frac{(\lambda s)^j}{j!} \sum_{l=0}^{\infty} \bar{P}_{j+l} \frac{(\lambda t)^l}{l!}.\end{aligned}$$

If $\bar{P}_{j+l}$ is SC_n in j, $l = 0, 1, \cdots$ with sign ε, then by the basic composition formula (Karlin, page 17) and the total positivity of t^l, $Q(t, j) = \sum_{l=0}^{\infty} \bar{P}_{j+l}(\lambda t)^l/l!$ is SC_n in $j = 0, 1, \cdots$ and $t \geqq 0$ with sign ε. By a repetition of this argument, $\bar{H}(s + t) = e^{-\lambda(s+t)} \sum_{j=0}^{\infty} Q(t, s)(\lambda s)^j/j!$ is SC_n in $s, t \geqq 0$ with sign ε. □

4. Cumulative damage threshold models for the $\bar{P}_k$. The model considered here has been described in our context by Cox (1962) page 91, and it has been discussed, e.g., by Morey (1965) and by Murthy and Lientz (1968). Closely related models involving compound Poisson processes have been considered by many other authors; see, e.g., Feller (1966) page 179.

Suppose that the ith shock to an item causes a random damage X_i. Damages accumulate additively, and the kth shock is survived by the item if $X_1 + \cdots + X_k$ does not exceed the capacity or threshold x of the item.

The case that $X_1, X_2, \cdots$ are independent with common distribution F is particularly simple and interesting. Here

$$\bar{P}_k = F^{(k)}(x), \qquad k = 0, 1, \cdots, \tag{4.1}$$

where $F^{(k)}$ denotes the kth convolution of F, $k = 1, 2, \cdots$, and $F^{(0)}$ is degenerate at 0.

Alternatively it may happen that successive shocks become increasingly effective in causing wear or damage, even though they are independent. This means that $F_i(z)$, the distribution function of the ith damage, is decreasing in $i = 1, 2, \cdots$ for each z. Here

$$\bar{P}_0 = 1 \quad \text{and} \quad \bar{P}_k = F_1 * F_2 * \cdots * F_k(x), \qquad k = 1, 2, \cdots, \tag{4.2}$$

where $*$ denotes convolution.

Often, successive damages are neither independent nor identically distributed. A primary reason for this is that an accumulation of damage may result in a loss of resistance to further damage. In this case the magnitudes of successive damages are dependent, so that (4.2) does not apply. Rather, it may be reasonable to assume that

$$P\{X_k \leqq u \mid X_1, \cdots, X_{k-1}\} \text{ depends on } X_1, \cdots, X_{k-1} \text{ only via } Z_{k-1} = X_1 + \cdots + X_{k-1}, \tag{4.3}$$

$$P\{X_k \leqq u \mid Z_{k-1} = z\} \text{ is decreasing in } z \geqq 0, \tag{4.4}$$

$$P\{X_k \leqq u \mid Z_{k-1} = z\} \geqq P\{X_{k+1} \leqq u \mid Z_k = z\}, \quad z \geqq 0,\ k = 1, 2, \cdots, \text{ where } Z_0 = 0. \tag{4.5}$$

Condition (4.4) restates the assumption that an accumulation of damage lowers resistance to further damage. Condition (4.5), says that for any given accumulation of damage, later shocks are apt to be more severe. Probably the case of equality in (4.5) is more important than inequality. With conditions (4.3), (4.4) and (4.5),

$$\bar{P}_0 = 1 \quad \text{and} \quad \bar{P}_k = F^{[k]}(x), \qquad k = 1, 2, \cdots, \tag{4.6}$$

where $F^{[k]}(x) = P\{X_1 + \cdots + X_k \leqq x\} = \int_0^x P\{X_k \leqq x - z \mid Z_{k-1} = z\}\, dF^{[k-1]}(z)$. Of course, (4.1) and (4.2) are special cases.

For the case that $\bar{P}_k$ is given by (4.1) we have the following lemma.

LEMMA 4.1. *If F is a distribution function satisfying $F(z) = 0$ for all $z < 0$, then*

$$[F^{(k)}(x)]^{1/k} \quad \textit{is decreasing in} \quad k = 1, 2, \cdots .$$

PROOF. $F^{(2)}(x) = \int_0^x F(x - z)\, dF(z) \leqq \int_0^x F(x)\, dF(z) = [F(x)]^2$, so that $F(x) \geqq [F^{(2)}(x)]^{\frac{1}{2}}$. Now suppose that $[F^{(k-1)}(x)]^{1/(k-1)} \geqq [F^{(k)}(x)]^{1/k}$, i.e., $F^{(k-1)}(x) \geqq [F^{(k)}(x)]^{(k-1)/k}$ for all x. Then

$$\begin{aligned}
[F^{(k)}(x)]^{k+1} &= F^{(k)}(x)[\int F^{(k-1)}(x - z)\, dF(z)]^k \\
&\geqq F^{(k)}(x)[\int [F^{(k)}(x - z)]^{(k-1)/k}\, dF(z)]^k \\
&= \{\int [F^{(k)}(x)]^{1/k}[F^{(k)}(x - z)]^{1-(1/k)}\, dF(z)\}^k \\
&\geqq \{\int F^{(k)}(x - z)\, dF(z)\}^k = [F^{(k+1)}(x)]^k . \qquad \square
\end{aligned}$$

Next, suppose that $\bar{P}_k$ is given by (4.2).

LEMMA 4.1a. *If F_i are distribution functions satisfying $F_i(z) = 0$ for $z < 0$, $i = 1, 2, \cdots$, and if $F_i(z)$ is decreasing in i for all z, then*

$$[F_1 * F_2 * \cdots * F_k(x)]^{1/k} \quad \textit{is decreasing in} \quad k = 1, 2, \cdots .$$

PROOF. The inductive proof of Lemma 4.1 applies here, with the added step that

$$\int F_1 * \cdots * F_k(x - z)\, dF_k(z) \geqq \int F_1 * \cdots * F_k(x - z)\, dF_{k+1}(z) . \qquad \square$$

A similar generalization of Lemma 4.1 holds for (4.6).

LEMMA 4.1b. *If $X_1, X_2, \cdots$ are nonnegative random variables with a joint distribution which satisfies* (4.3), (4.4), *and* (4.5), *then*

$$[P\{X_1 + \cdots + X_k \leqq x\}]^{1/k} \quad \textit{is decreasing in} \quad k = 1, 2, \cdots .$$

PROOF. Using (4.4), then (4.5) with $z = 0$ and $k = 1$, we obtain

$$\begin{aligned}
F^{[2]}(x) &= \int_0^x P\{X_2 \leqq x - z \mid X_1 = z\}\, dF^{[1]}(z) \\
&\leqq \int_0^x P\{X_2 \leqq x - z \mid X_1 = 0\}\, dF^{[1]}(z) \\
&\leqq \int_0^x P\{X_1 \leqq x - z\}\, dF^{[1]}(z) \leqq [F^{[1]}(x)]^2 .
\end{aligned}$$

To complete an induction, suppose that $[F^{[k]}(z)]^{1/k} \leqq [F^{[k-1]}(z)]^{1/(k-1)}$. Then for $z \leqq x$, $F^{[k]}(z) = [F^{[k]}(z)]^{1/k}[F^{[k]}(z)]^{(k-1)/k} \leqq [F^{[k]}(x)]^{1/k}F^{[k-1]}(z)$. Using this and (4.4), and then using (4.5), we obtain

$$\begin{aligned}
[F^{[k+1]}(x)]^k &= [\int_0^x P\{X_{k+1} \leqq x - z \mid Z_k = z\}\, dF^{[k]}(z)]^k \\
&\leqq [\int_0^x P\{X_{k+1} \leqq x - z \mid Z_k = z\}\, dF^{[k-1]}(z)]^k F^{[k]}(x) \\
&\leqq [\int_0^x P\{X_k \leqq x - z \mid Z_{k-1} = z\}\, dF^{[k-1]}(z)]^k F^{[k]}(x) \\
&= [F^{[k]}(x)]^{k+1} . \qquad \square
\end{aligned}$$

COROLLARY 4.2. *If F is a distribution function such that $F(z) = 0$ for all $z < 0$,* *then the survival function*

$$(4.7) \qquad \bar{H}(t) = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} F^{(k)}(x)$$

is IHRA. *If* $F_i(z) = 0$ *for* $z < 0$ *and* $F_i(z)$ *is decreasing in* $i = 1, 2, \cdots$, *then*

$$(4.7\text{a}) \qquad \bar{H}(t) = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} F_1 * \cdots * F_k(x)$$

is IHRA. *Still more generally, if* $X_1, X_2, \cdots$, *are nonnegative random variables satisfying* (4.3), (4.4), *and* (4.5), *then*

$$(4.7\text{b}) \qquad \bar{H}(t) = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} P\{X_1 + \cdots + X_k \leqq x\}$$

is IHRA.

PROOF. This is an immediate application of (3.4) of Theorem 3.1, and of Lemmas 4.1, 4.1a and 4.1b. □

We wish to emphasize that the IHRA property has been obtained in Corollary 4.2 as an implication of a natural physical model. The only hypothesis imposed upon F is that it be the distribution of a nonnegative random variable.

We remark that the survival functions (4.7), (4.7a), and (4.7b) are, for fixed t, distribution functions in the suppressed variable x. From this viewpoint, (4.7) is a compound Poisson distribution. The stochastic process $\{Z(t), t \geqq 0\}$ which represents in our model the amount of damage accumulated by time t is a compound Poisson process, and for each t, (4.7) as a function of x is the distribution function of $Z(t)$.

The following corollary considers a special case of (2.11), where there are several independent sources of shocks.

COROLLARY 4.3. *If* $F_1, F_2, \cdots, F_l$ *are distribution functions such that* $F_j(z) = 0$ *for all* $z < 0$, $j = 1, 2, \cdots, l$, *then the survival function*

$$\bar{H}(t) = \sum_{i_1=0}^{\infty} \cdots \sum_{i_l=0}^{\infty} \left[\prod_{j=1}^{l} e^{-\lambda_j t} \frac{(\lambda_j t)^{i_j}}{i_j!} \right] F_1^{(i_1)} * \cdots * F_l^{(i_l)}(x)$$

is IHRA.

PROOF. Since this is a special case of (2.12) we may rewrite $\bar{H}$ in the form of (2.1) where

$$\bar{P}_k = \sum_{i_j \geqq 0, \sum i_j = k} \binom{k}{i_1, \cdots, i_j} \prod_{j=1}^{l} q_j^{i_j} F_1^{(i_1)} * \cdots * F_l^{(i_l)}(x)$$

and $q_j = \lambda_j / \sum_{i=1}^{l} \lambda_i$. But then we see that $\bar{P}_k = F^{(k)}(x)$ where $F(x) = \sum_{j=1}^{l} q_j F_j(x)$, so that the first part of Corollary 4.2 applies. □

Let us return briefly to Lemma 4.1, to identify those distribution functions F for which $[F^{(k)}(x)]^{1/k}$ is constant in k.

THEOREM 4.4. *Let* $F(z) = 0$ *for* $z < 0$. *Then* $[F^{(k)}(x)]^{1/k}$ *is independent of* $k = 1, 2, \cdots$, *if and only if* F *has no mass in the interval* $(0, x]$.

PROOF. First note that $[F^{(k)}(x)]^{1/k}$ is independent of k if and only if $F^{(k)}(x) = [F(x)]^k$. If $X_1, X_2, \cdots$ are independent random variables with distribution function F, this can be equivalently written as

$$P\{X_1 + \cdots + X_k \leqq x\} = [P\{X_1 \leqq x\}]^k$$
$$= P\{X_1 \leqq x, X_2 \leqq x, \cdots, X_k \leqq x\}, \qquad k = 1, 2, \cdots.$$

Clearly this holds if $P\{0 < X_i \leqq x\} = 0$. On the other hand, this relation means that $P\{x/k < X_i \leqq x\} = 0$, $k = 1, 2, \cdots$. □

COROLLARY 4.5. *The survival function* (4.7) *is exponential if and only if F has no mass in the interval* $(0, x]$.

PROOF. This is immediate from Theorem 4.4 and Example 2.1. □

Having obtained the results of Lemma 4.1, it is natural to ask if one can reach there the stronger conclusion that $F^{(k)}(x)/F^{(k-1)}(x)$ is decreasing in $k = 1, 2, \cdots$, without strengthening the hypotheses. The following example shows that this is not possible.

EXAMPLE 4.6. Suppose that F places mass p at 1 and mass $1 - p$ at 3. Take $x = 3.5$. Then $F(x) = 1$, $F^{(2)}(x) = p^2$, $F^{(3)}(x) = p^3$, and $F^{(i)}(x) = 0$, $i \geqq 4$.

By comparing with $k = 2$ and $k = 3$, we see that $F^{(k)}(x)/F^{(k-1)}(x)$ is not decreasing in k. Moreover, by comparing with $k = 1$ and $k = 2$ and taking $p = \frac{1}{2}$, we see that $\sum_{j=k}^{\infty} F^{(j)}(x)/F^{(k)}(x)$ is not decreasing in k. □

From Example 4.6, it follows that (3.2) cannot be used to show that H given by (4.7) is IHR. Moreover, (3.2) connot be used to obtain the weaker conclusion that H is DMRL. On the other hand with $\bar{P}_k = F^{(k)}(x)$ and F as in Example 4.6, the condition of Theorem 3.3 is satisfied, so that H is in this case IHR. Thus, we must still consider the possibility that H given by (4.7) is IHR.

EXAMPLE 4.7. For $j = 1, 2, \cdots$, suppose that F_j places mass p at 1 and mass $1 - p$ at j. Take $x_j = j + \frac{1}{2}$. Then $F_j(x_j) = 1$, $F_j^{(i)}(x_j) = p^i$ for $i = 2, 3, \cdots, j$, and $F_j^{(i)}(x_j) = 0$ for $i \geqq j + 1$. With $F = F_j$ and $x = x_j$ in (4.7), we obtain

$$\bar{H}_j(t) = e^{-t}\left[1 + t + \frac{t^2p^2}{2!} + \cdots + \frac{t^jp^j}{j!}\right].$$

The algebra involved in checking monotónicity of the hazard rate of H_j is cumbersome for large j. Consider therefore $H^* = \lim_j H_j$, which is given by

$$\bar{H}^*(t) = e^{-t}[t - pt + e^{pt}].$$

It is easy to check that for sufficiently large t, the hazard rate of H^* is actually decreasing. Because the class of IHR distributions is closed under limits in distribution, this means that for sufficiently large j, H_j is not IHR. Moreover, H^* is not DMRL either, so that for sufficiently large j, H_j is not DMRL. □

Example 4.7 show that although $\bar{H}$ given by (4.7) is IHRA, it need not be

IHR or even DMRL. But if we are willing to impose some hypotheses on the distribution function F, stronger conclusions concerning H are obtainable.

THEOREM 4.8. *If F is a distribution function such that $F(z) = 0$ for all $z < 0$, and if F has a density f that is PF_2, then $[F^{(k)}(x) - F^{(k+1)}(x)]/[F^{(k-1)}(x) - F^{(k)}(x)]$ is decreasing in $k = 1, 2, \cdots$, so that H given by* (4.7) *has a PF_2 density.*

PROOF. The monotonicity conclusion of this theorem is due to Karlin and Proschan (1960). The remainder of the theorem is immediate from (3.1). □

This result is also true for convolutions of unlike F_i, and similar results hold for higher order total positivity.

Now, let us consider conditions on F in order that H be IHR. To apply (3.2), we would like $F^{(k)}(x)/F^{(k-1)}(x)$ to be decreasing in $k = 1, 2, \cdots$. That this need not hold when F is IHR can be seen by taking $F(t) = t/2$, $0 \leqq t < 1$, $F(t) = 1$, $t \geqq 1$ (an equal mixture of a uniform and a degenerate distribution), by taking $x = 1$, and by comparing the ratios for $k = 2$ and $k = 3$. However, if we assume that F is PF_2 rather than that $\bar{F}$ is PF_2 (F is IHR), we obtain the desired result.

THEOREM 4.9. *If $F(x) = 0$ for $x < 0$, the following chain of implications holds: $F(x)$ is PF_2 in $x \Rightarrow F^{(k)}(x)$ is TP_2 in k and $x \Rightarrow F^{(k)}(x)$ is PF_2 in k (for each x) $\Rightarrow H$ given by* (4.7) *is* IHR.

PROOF. For $x_1 \leqq x_2$, let $D(x_1, x_2) = F^{(n)}(x_1)F^{(n+1)}(x_2) - F^{(n)}(x_2)F^{(n+1)}(x_1)$. If $x_1 < 0$, then $D = 0$. If $x_1 \geqq 0$ and $n \geqq 1$, then $D(x_1, x_2) = \int_0^\infty [F^{(n)}(x_1)F^{(n)}(x_2 - \theta) - F^{(n)}(x_2)F^{(n)}(x_1 - \theta)]\, dF(\theta)$. Since the PF_2 property of F is preserved under convolutions (Barlow and Proschan (1965) Theorem 5.3, page 38), $F^{(n)}(x)$ is PF_2 in x. Thus the integrand, and hence $D(x_1, x_2)$, is nonnegative. This proves the first implication. The second implication is similarly obtained because $\int_0^\infty D(x - \theta, x)\, dF(\theta) \geqq 0$. The third implication follows from (3.2). □

The first implication is true for convolutions of unlike F_i; the second and third implications similarly generalize if $F_i(z)$ is decreasing in i for all z.

Continuous wear: first passage times. Throughout this section, we have considered models for wear (damage) that accumulates in discrete amounts at isolated points in time. Let us forgo the assumption of discreteness, and denote the wear accumulated in the time interval $[0, t]$ by $Z(t)$, $t \geqq 0$. The stochastic process $\{Z(t), t \geqq 0\}$ will in practice normally satisfy

(4.8) $Z(0) = 0$, and $Z(t + \Delta) - Z(t) \geqq 0$ for all $t, \Delta \geqq 0$ with probability one. This condition simply says that a device enters service (at time 0) with no accumulated wear, and that wear is always nonnegative. A model of this kind has been considered by Morey (1965) who additionally assumes that $P\{Z(t + \Delta) > x \mid Z(t) \leqq x\}$ is decreasing in $x \geqq 0$ whenever $t, \Delta \geqq 0$. Rather than this, we shall supplement condition (4.8) with

(4.9) $\{Z(t), t \geqq 0\}$ is a Markov process, and

(4.10) $P\{Z(t+\Delta)-Z(t)\leqq u \mid Z(t)=z\}$ is decreasing in both z and t in the region $t\geqq 0$, $z\geqq 0$ and $\Delta\geqq 0$. Loosely speaking (4.9) says that given the amount of wear accumulated by time t, details of earlier history are not relevant in predicting the amount of wear to be accumulated by some future time $t+\Delta$. Condition (4.10) says that, for fixed time t, an accumulation to wear can only weaken a device and make it more prone to further wear. Also, (4.10) says that given equal amounts of accumulated wear, an older device is more prone to additional wear than a young one. Of course, these conditions are not universally satisfied in practice, but they often can be verified from physical considerations.

If a device has a known capacity for wear, it will fail when the accumulated wear first exceeds this capacity, say x. Thus failure times are first passage times.

THEOREM 4.10. *If the process* $\{Z(t),\, t\geqq 0\}$ *satisfies* (4.8), (4.9), *and* (4.10), *then the first passage time* $T_x=\inf\{t\colon Z(t)>x\}$ *has an* IHRA *distribution.*

PROOF. Let H_x be the distribution of T_x, and let F_t be the distribution of $Z(t)$, $t\geqq 0$. Choose $\Delta>0$ and consider the random variables $X_i=Z(i\Delta)-Z((i-1)\Delta)$, $i=1,2,\cdots$. Then $X_1, X_2,\cdots$ satisfy the conditions of Lemma 4.1b, so that

$$[F_{k\Delta}(x)]^{1/k\Delta} \quad \text{is decreasing in} \quad k=1,2,\cdots .$$

This means that $[F_s(x)]^{1/s}\geqq[F_t(x)]^{1/t}$ whenever $s\leqq t$ and s/t is rational. In case $s<t$ but s/t is not rational, the inequality still holds, as can be seen by approximating t from below by rational multiples of s. Since $T_x>t$ if and only if $Z(t+\varepsilon)\leqq x$ for some $\varepsilon>0$, it follows that $\bar{H}_x(t)=\lim_{\varepsilon\to 0}F_{t+\varepsilon}(x)$. Thus if $s\leqq t$,

$$[\bar{H}_x(s)]^{1/s}-[\bar{H}_x(t)]^{1/t}=\lim_{\varepsilon\to 0}\{[F_{s+\varepsilon}(x)]^{1/(s+\varepsilon)}-[F_{t+\varepsilon}(x)]^{1/(t+\varepsilon)}\}\geqq 0 . \qquad \square$$

There are a number of stochastic processes which satisfy the conditions (4.8), (4.9), and (4.10) of Theorem 4.10. For example, processes satisfy the conditions if sample functions start at the origin and the increments of the process are non-negative, stationary, and independent. This, of course, includes the compound Poisson processes, and the infinitesimal renewal processes.

Applications in renewal theory. The lemmas in the early part of this section have immediate applications to renewal processes, and to certain Markov processes. The most obvious of these amounts to little more than a restatement of Lemma 4.1.

COROLLARY 4.11. *Let* $N(x)$ *be the number of renewals in* $[0, x]$ *for an ordinary renewal process* (not *automatically counting the origin as a renewal point*). *Then*

$$(4.11) \qquad [P\{N(x)\geqq k\}]^{1/k} \quad \textit{is decreasing in} \quad k=1,2,\cdots .$$

Lemma 4.1b has a similar application.

COROLLARY 4.12. *Let* $\hat{N}(x)$ *be the number of renewals in* $[0, x]$ *for a stationary renewal process* (not *automatically counting the origin as a renewal point*). *If the underlying recurrence time distribution* F *is* NBUE, *then* (4.11) *holds with* $\hat{N}$ *in place of* N.

The condition that F is NBUE is just the condition that for the stationary renewal process $\{X_1, X_2, X_3, \cdots\}$, the distribution F_1 of X_1 and the common distribution F of $X_2, X_3, \cdots$ satisfy $F_1(x) \geqq F(x)$ for all x. This condition was encountered by Barlow and Proschan (1964) and by Marshall and Proschan (1970) in a renewal theory context. They show that for an ordinary renewal process with a recurrence distribution F that is NBUE with mean μ,

$$M(t) = EN(t) \leqq t/\mu \qquad \text{for all } t \geqq 0 .$$

Discrete versions of the IHRA property, of which (4.11) is one, have apparently received little attention. We mention one application to illustrate the utility of (4.11).

Let $\bar{P}_k$, $k = 0, 1, \cdots$ be a decreasing sequence of nonnegative numbers such that $\bar{P}_0 = 1$, let $p_k = \bar{P}_{k-1} - \bar{P}_k$, $k = 1, 2, \cdots$, and let

$$B_j = \sum_{k=0}^{\infty} \binom{j+k}{j} p_k = \sum_{k=0}^{\infty} \binom{j+k-1}{k} \bar{P}_k .$$

If $\bar{P}_k^{1/k}$ is decreasing in $k = 1, 2, \cdots$ and $\bar{P}_k < 1$ for some $k > 0$, then $B_j < \infty$ for all j, and

$$B_j^{1/j} \text{ is decreasing in } j = 1, 2, \cdots . \tag{4.12}$$

This can be proved using the variation diminishing property of $\binom{j+k-1}{k}$ which is totally positive in k and j (Karlin (1968) page 137).

If $\bar{P}_k = P\{N(x) \geqq k\}$ (or $\bar{P}_k = P\{\hat{N}(x) \geqq k\}$ and F is NBUE), then from Corollary 4.11 (or Corollary 4.12) together with (4.12), it follows that $B_j < \infty$ for all j and $B_j^{1/j}$ is decreasing in $j = 1, 2, \cdots$. Using $B_1^2 \geqq B_2$, it follows that $\operatorname{Var} N(x) \leqq [EN(x)]^2 + EN(x)$. This inequality, while rather crude for large x, may be useful when x is small.

5. Random threshold for cumulative damage. Often one is interested in an item for which there is a significant individual variation in ability to withstand shocks. Moreover, there may be no practical way to inspect an individual item to determine its threshold x. In this case the threshold must be regarded as a random variable.

Let us now turn our attention to the case that the threshold x is random with distribution G such that $G(0) = 0$. We shall assume that the damages $X_1, X_2, \cdots$ from successive shocks are mutually independent with common distribution F, and in addition we assume that $X_1, X_2, \cdots$ are independent of the threshold. Then the shock survival probabilities are given by

$$\bar{P}_k = \int_0^\infty F^{(k)}(x)\, dG(x) , \qquad k = 0, 1, \cdots , \tag{5.1}$$

and $\bar{H}$ is given by

$$\bar{H}(t) = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \int_0^\infty F^{(k)}(x)\, dG(x) . \tag{5.2}$$

For technical reasons, it is convenient to assume in what follows that $F^{(k)}$ and G have no common discontinuities. Then there is no problem in writing $\bar{P}_k$ as

$$\bar{P}_k = E\bar{G}(X_1 + \cdots + X_k) .$$

The conclusion of Corollary 4.2 that $\bar{H}$ given by (4.7) is IHRA is particularly satisfying because the only condition required of F (that $F(z) = 0$ for $z < 0$) is physically reasonable in terms of the model we have introduced. Similar hypothesis-free results about (5.2) would be even more useful. The results that we obtain below again do not depend on F, but we cannot expect them at the same time to be free of hypotheses on G. For if we allow F to be degenerate, say at a, then $\bar{P}_k$ given by (5.1) takes the form $\bar{P}_k = \bar{G}(ka)$. Thus any conditions obtained on the $\bar{P}_k$ are reflected as conditions on $\bar{G}$.

THEOREM 5.1. *The survival function* (5.2) *is exponential for all* F *such that* $F(z) = 0$ *for* $z < 0$ *if and only if* G *is exponential.*

POOF. From Example 2.1, we know that $\bar{H}$ is exponential if and only if

$$\int_0^\infty F^{(k)}(x)\,dG(x) = \left[\int_0^\infty F(x)\,dG(x)\right]^k . \tag{5.3}$$

If G is exponential, then $\int_0^\infty F(x)\,dG(x) = \int_0^\infty \bar{G}(x)\,dF(x)$ is the Laplace transform of F, so that (5.3) holds.

Now, suppose $\bar{H}$ is exponential, i.e., suppose (5.3) for all F. Take F degenerate at x_0. Then we have from (5.3) that $\bar{G}(kx_0) = [\bar{G}(x_0)]^k$, $k = 1, 2, \cdots$. Since $\lim_{x_0\to 0} x_0[u/x_0] = u$, it follows that if G is continuous at u, then

$$\begin{aligned}\bar{G}(u) &= \lim_{x_0\to 0} \bar{G}(x_0[u/x_0]) = \lim_{x_0\to 0} \{[\bar{G}(x_0)]^{1/x_0}\}^{x_0[u/x_0]} \\ &= \{\lim_{x_0\to 0} [\bar{G}(x_0)^{1/x_0}]\}^u ;\end{aligned}$$

Using the monotonicity of $\bar{G}$, we conclude that for all u, $\bar{G}(u) = e^{-\lambda u}$ for some λ. $0 < \lambda \leqq \infty$. □

This theorem give conditions for equality in inequalities or monotonicity results in several of the theorems which follow.

THEOREM 5.2. *Let* $\bar{P}_k = \int_0^\infty F^{(k)}(x)\,dG(x)$, $k = 0, 1, \cdots$, *where* $F(z) = 0$ *for* $z < 0$ *and* $G(0) = 0$.

(a) *If* G *is* IHRA, *then* H, *given by* (5.2) *is* IHRA.
(b) *If* G *is* IHR, *then* $\bar{P}_k^{1/k}$ *is decreasing in* $k = 1, 2, \cdots$ *for all* F.
(c) *If* $\bar{P}_k^{1/k}$ *is decreasing in* $k = 1, 2, \cdots$ *for all* F, *then* G *is* IHRA.

An unresolved question remains: Is $\bar{P}_k^{1/k}$ decreasing in $k = 1, 2, \cdots$ if G is IHRA? If this is true, (a) would follow from (3.4). We have been able to prove that $\bar{P}_k^{1/k}$ is decreasing only with the stronger hypothesis that G is IHR. Consequently, we cannot make use of (3.4) to prove that H is IHRA whenever G is IHRA. This result is obtained only by bringing to bear a body of theory that otherwise plays no role in this paper. References for required facts and definitions are given.

PROOF THAT (a) G IS IHRA IMPLIES H IS IHRA. Let $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ be mutually independent random variables, the X_i having the common distribution function F, and the Y_i being exponentially distributed. Let $\{N(t), t \geqq 0\}$

be a Poisson process with parameter λ and almost surely right continuous sample paths. Define

$$T_i = \inf\{t : X_1 + \cdots + X_{N(t)} > Y_i\}, \quad i = 1, 2, \cdots,$$

i.e., $T_i > t$ if and only if $X_1 + \cdots + X_{N(t)} \leqq Y_i$. For any finite subset I of positive integers, $\min_{i \in I} Y_i$ is exponentially distributed, so that from Theorem 5.1 we conclude that

$$\min_{i \in I} T_i = \inf\{t : X_1 + \cdots + X_{N(t)} > \min_{i \in I} Y_i\}$$

is also exponentially distributed (although the T_i are of course dependent). Now let τ be the life function of a coherent structure ϕ of order n (see Esary and Marshall (1970a)). Then it follows (Esary and Marshall (1970b) Application 5.3) that $\tau(T_1, \cdots, T_n)$ has an IHRA distribution.

Denoting the characteristic (indicator) function of a set A by χ_A, we compute that

$$\begin{aligned} P\{\tau(\mathbf{T}) > t\} &= \textstyle\sum_{k=0}^{\infty} P\{\tau(\mathbf{T}) > t \mid N(t) = k\} e^{-\lambda t}(\lambda t)^k/k! \\ &= \textstyle\sum_{k=0}^{\infty} E\phi(\chi_{\{X_1 + \cdots + X_k \leq Y_1\}}, \cdots, \chi_{\{X_1 + \cdots + X_k \leq Y_n\}}) e^{-\lambda t}(\lambda t)^k/k! \\ &= \textstyle\sum_{k=0}^{\infty} P\{\tau(Y_1, \cdots, Y_n) > X_1 + \cdots + X_k\} e^{-\lambda t}(\lambda t)^k/k! \\ &= \textstyle\sum_{k=0}^{\infty} e^{-\lambda t} \dfrac{(\lambda t)^k}{k!} \int F^{(k)}(x)\, dG_\tau(x), \end{aligned}$$

where G_τ is the distribution function of $\tau(Y_1, \cdots, Y_n)$.

Since G is IHRA, $\bar{G}$ can be approximated by an increasing sequence of survival functions each having, for some n, the form of $\bar{G}_\tau$ (Birnbaum, Esary and Marshall (1966) page 822). It follows that if $\bar{H}$ is given by (5.2), then

$$\bar{H}(t) = \lim_{n \to \infty} P\{\tau_n(T_1, \cdots, T_n) > t\}$$

where for each n, $\tau_n(T_1, \cdots, T_n)$ has an IHRA distribution. Hence H is IHRA. □

Multivariate questions which arise to a small degree in the above proof are subjected to further investigation in a paper by Esary and Marshall (in preparation).

PROOF THAT (b) G IS IHR IMPLIES $\bar{P}_k^{1/k}$ IS DECREASING. By writing $\bar{P}_k = E\bar{G}(X_1 + \cdots + X_k)$ where the X_i are independent and have distribution function F, and by using the fact that G is NBU, we have

$$E\bar{G}(X_1 + X_2) \leqq E\bar{G}(X_1)\bar{G}(X_2) = [E\bar{G}(X_1)]^2$$

i.e., $\bar{P}_1 \geqq \bar{P}_2^{\frac{1}{2}}$.

Now, suppose that for all IHR distributions K

$$\int F^{(n-1)}(x)\, dK(x) \geqq [\int F^{(n)}(x)\, dK(x)]^{(n-1)/n}.$$

We shall apply this in the induction which follows, with

$$K(z) = G_y(z) = [G(y + z) - G(y)]/\bar{G}(y).$$

G_y is IHR because G is IHR (G_y would not necessarily be IHRA just because G

is IHRA). Note in advance that the calculations below are followed by explanations of the steps involved.

$$\begin{aligned}\{\textstyle\int F^{(n)}(x)\,dG(x)\}^{n+1} &= \bar{P}_n[\textstyle\int\int F^{(n-1)}(x-y)\,dF(y)\,dG(x)]^n \\ &= \bar{P}_n[\textstyle\int\int F^{(n-1)}(x-y)\,dG(x)\,dF(y)]^n \\ &= \bar{P}_n[\textstyle\int\int F^{(n-1)}(z)\,dG_y(z)\bar{G}(y)\,dF(y)]^n \\ &\geqq \{\bar{P}_n^{1/n}\textstyle\int [F^{(n)}(z)\,dG_y(z)]^{(n-1)/n}\bar{G}(y)\,dF(y)\}^n \\ &\geqq [\textstyle\int\int F^{(n)}(z)\,dG_y(z)\bar{G}(y)\,dF(y)]^n = [\textstyle\int F^{(n+1)}(x)\,dG(x)]^n .\end{aligned}$$

Here the first inequality follows from the inductive hypothesis. The second inequality is an application of $\bar{P}_n \geqq \int F^{(n)}(z)\,dG_y(z)$; this follows from the monotonicity of $F^{(n)}$, because G is IHR so that $G(z) \leqq G_y(z)$ for all z. The last equality is a reapplication of the first three steps of the calculations. □

PROOF THAT (c) $\bar{P}_k^{1/k}$ IS DECREASING IMPLIES G IS IHRA. Take F to be degenerate at x_0. Then $\bar{P}_k^{1/k} = [\bar{G}(kx_0-)]^{1/k}$ is decreasing in k, e.g., $[\bar{G}(kx_0)]^{1/kx_0}$ is decreasing in k, $k = 1, 2, \cdots$. This means that $[\bar{G}(\alpha t-)]^{1/\alpha t} \geqq [\bar{G}(t-)]^{1/t}$ whenever $\alpha < 1$ is rational. Thus we can approximate $s < t$ from above by rational multiples of t to obtain $[\bar{G}(s)]^{1/s} \geqq [\bar{G}(t-)]^{1/t} \geqq [\bar{G}(t)]^{1/t}$. □

THEOREM 5.3. *Let* $\bar{P}_k = \int_0^\infty F^{(k)}(x)\,dG(x)$, $k = 0, 1, \cdots$, *where* $F(z) = G(z) = 0$ *for* $z < 0$. *Then* $\bar{P}_{j+k} \leqq \bar{P}_j\bar{P}_k$ *for* $j, k = 0, 1, \cdots$, *for all* F *if and only if* G *is* NBU. *If* G *is* NBU, *then* H *given by* (5.2) *is* NBU.

PROOF. Let $X_1, X_2, \cdots$ be independent random variables with distribution F, and suppose that G is NBU. Then

$$\begin{aligned}\bar{P}_{j+k} &= E\bar{G}(X_1 + \cdots + X_{j+k}) \leqq E\bar{G}(X_1 + \cdots + X_j)\bar{G}(X_{j+1} + \cdots + X_{j+k}) \\ &= E\bar{G}(X_1 + \cdots + X_j)E\bar{G}(X_{j+1} + \cdots + X_k) = \bar{P}_j\bar{P}_k .\end{aligned}$$

Now suppose that $\bar{P}_{j+k} \leqq \bar{P}_j\bar{P}_k$ for $j, k \geqq 0$. Choose $s, t > 0$ and take F to be degenerate at $x_k = s/k$. From $\bar{P}_{j+k} \leqq \bar{P}_j\bar{P}_k$ we obtain $\bar{G}((j+k)x_k) \leqq \bar{G}(jx_k)\bar{G}(kx_k)$, i.e.

$$\bar{G}(s + jx_k) \leqq \bar{G}(s)\bar{G}(jx_k) , \qquad j = 0, 1, \cdots .$$

Choose $j = j_k = [t/x_k] + 1$. Then $t + x_k \geqq j_k x_k > t$. Since $x_k \to 0$ as $k \to \infty$, and since $\bar{G}$ is right continuous, we have by taking limits that $\bar{G}(s + t) \leqq \bar{G}(s)\bar{G}(t)$, i.e., G is NBU.

The last part of the theorem is a consequence of the first part and of (3.5). □

This theorem remains true if $F^{(k)}(x)$ is replaced by $F_1 * F_2 * \cdots * F_k(x)$ where $F_i(z)$ is decreasing in i for all z.

In view of Theorem 5.1, 5.2, and 5.3, it is interesting to note that with $\bar{P}_k$ given by (5.1)

(i) $\{p_k = \bar{P}_{k-1} - \bar{P}_k, k = 1, 2, \cdots\}$ need not be a PF_2 sequence when G has a PF_2 density,

(ii) $\bar{P}_k/\bar{P}_{k-1}$ need not be decreasing in $k = 1, 2, \cdots$ when G is IHR,

(iii) $\sum_{j=k}^{\infty} \bar{P}_j/\bar{P}_k$ need not be decreasing in $k = 1, 2, \cdots$ when G is DMRL,
(iv) It need not be that $\bar{P}_k \sum_{j=0}^{\infty} \bar{P}_j \geqq \sum_{j=k}^{\infty} \bar{P}_j$, $k = 0, 1, \cdots$ when G is NBUE.

We have already seen that the first three of these negative results are true because the conclusions do not hold for all F even when G is degenerate. To see the fourth result, take

$$\bar{G}(x) = 1 \quad \text{for} \quad x < 3\,, \qquad \bar{G}(x) = \tfrac{1}{4} \quad \text{for} \quad 3 \leqq x < 7\,,$$
$$\bar{G}(x) = 0 \quad \text{for} \quad x \geqq 7\,.$$

Then G is NBUE. Take F degenerate at 13/8. Then $\bar{P}_k = \bar{G}(13k/8)$, i.e.,

$$\bar{P}_0 = \bar{P}_1 = 1\,, \qquad \bar{P}_2 = \bar{P}_3 = \bar{P}_4 = \tfrac{1}{4}\,, \qquad \bar{P}_j = 0 \quad \text{for} \quad j > 4\,.$$

The conclusion of (iv) is violated with $k = 2$.

For the case of a nonrandom threshold, we used conditions on F in Theorems 4.8 and 4.9 to obtain stronger conclusions on H than would otherwise be possible. These results have analogs in the case of a random threshold.

THEOREM 5.4. *Let F and G be distribution functions such that $F(z) = 0$ for $z < 0$ and $G(0) = 0$, and suppose F and G have densities f and g that are PF_2. Then*

$$[\textstyle\int F^{(k)}(x)\,dG(x) - \int F^{(k+1)}(x)\,dG(x)]/[\int F^{(k-1)}(x)\,dG(x) - \int F^{(k)}(x)\,dG(x)]$$

is decreasing in $k = 1, 2, \cdots$, so that H given by (5.2) has a PF_2 density.

PROOF. Let $w(k) = \int F^{(k)}(x)\,dG(x) - \int F^{(k+1)}(x)\,dG(x)$. Then

$$\begin{aligned} w(k + l) &= \textstyle\iint [F^{(k)}(x - z) - F^{(k+1)}(x - z)]g(x)f^{(l)}(z)\,dx\,dz \\ &= \textstyle\iint [F^{(k)}(u) - F^{(k+1)}(u)]g(u + z)f^{l}(z)\,du\,dz\,. \end{aligned}$$

Since $F^{(k)}(u) - F^{(k+1)}(u)$ is TP_2 in $k = 1, 2, \cdots$ and $u \geqq 0$ (see Theorem 4.8 or Karlin and Proschan (1960) Theorem 2), it follows from the basic composition formula for totally positive functions (Karlin (1968) page 17) that $\int [F^{(k)}(u) - F^{(k+1)}(u)]g(u + z)\,du$ is TP_2 in $k = 1, 2, \cdots$ and $-z$. Since $f^{l}(z)$ is TP_2 in l and z (Karlin and Proschan (1960)) it follows by the same argument that $w(k + l)$ is TP_2 in $k = 1, 2, \cdots$ and $-l$, $l = 1, 2, \cdots$. Hence w is PF_2. ▯

We remark that Theorem 2. (2) of Morey (1966) is a stronger result than Theorem 5.4. However, his omission of any condition on G is an apparent oversight.

THEOREM 5.5. *Let F and G be distributions such that $F(z) = 0$ for $z < 0$ and $G(0) = 0$. If F is PF_2, and if G has a density g that is PF_2, then the $\bar{P}_k$ given by (5.1) satisfy $1 = \bar{P}_0 \geqq \bar{P}_1 \geqq \cdots$ and $\bar{P}_k/\bar{P}_{k-1}$ is decreasing in $k = 1, 2, \cdots$, so H given by (5.2) is* IHR.

PROOF. This is a special case of Theorem 5.4 of Karlin (1968) page 130. ▯

6. Maximum shock threshold models for $\bar{P}_k$. Here we consider the case that shocks to a device do no damage unless they exceed a critical threshold x. If the threshold is exceeded, the device fails; otherwise it is "as good as new."

This model may be an appropriate description, e.g., for the fracture of brittle materials such as glass.

Although the concept here appears quite different from that of Section 4, we hasten to point out that this model can be regarded as a special case of the model treated there. One needs only to take in Section 4 damage distributions F that place no mass on $(0, x]$. Then, the damages caused by shocks are either 0, or they exceed x and cause failure.

If we assume that the magnitudes $X_1, X_2, \cdots$ of successive shocks are independent and have corresponding distributions $F_1, F_2, \cdots$, then for the model here, the probability $\bar{P}_k$ of surviving k shocks is given by

$$\bar{P}_k = \prod_{i=1}^k F_i(x) , \qquad k = 1, 2, \cdots , \tag{6.1}$$

and the corresponding survival functions is

$$\bar{H}(t) = \sum_{k=0}^\infty e^{-\lambda t} \frac{(\lambda t)^k}{k!} \prod_{i=1}^k F_i(x) , \qquad t \geqq 0 . \tag{6.2}$$

THEOREM 6.1. *If $F_i(x) = F_1(x)$, $i = 2, 3, \cdots$, then $\bar{H}$ given by* (6.2) *is exponential. If $F_i(x)$ is decreasing in $i = 1, 2, \cdots$, then $\bar{H}$ is* IHR. *If $F_i(x)$ is increasing in $i = 1, 2, \cdots$, then $\bar{H}$ is* DHR.

PROOF. This is a trivial consequence of Theorem 3.1.

Of course one can use Theorem 3.1 to obtain conditions on the F_i in order that $\bar{H}$ have various other properties. However, the cases mentioned in Theorem 6.1 have some practical interest.

Suppose that the shock magnitudes are independent and identically distributed with distribution F, but the occurrence of each shock causes a change in the threshold level that does *not* depend on shock magnitudes. Denote the successive threshold levels by $x_1, x_2, \cdots$. Then

$$\bar{P}_k = \prod_{i=1}^k F(x_i) , \qquad k = 1, 2, \cdots .$$

This form is similar to that of (6.1), and Theorem 6.1 applies with $F(x_i)$ in place of $F_i(x)$. Consequently, if successive shocks each cause a lowering (raising) of the threshold, i.e., x_i is decreasing (increasing) in i, then $\bar{H}$ is IHR (DHR).

Let us now consider the modified case of (6.1), where the threshold level x is random, but independent of the shock magnitudes. Let us further suppose that $F_i(x) = F(x)$, $i = 1, 2, \cdots$. Then

$$\bar{P}_k = \int_0^\infty [F(x)]^k \, dG(x) , \qquad k = 0, 1, \cdots . \tag{6.3}$$

THEOREM 6.2. *If $\bar{P}_k$ is given by* (6.3), *then $\bar{H}$ given by* (2.1) *has a density that is logarithmically convex on* $(0, \infty)$.

PROOF. From (6.3) and (2.3) we compute that

$$h(t) = \int_0^\infty e^{-\lambda t \bar{F}(x)} \bar{F}(x) \, dG(x) .$$

Thus, h is a mixture of exponential functions, and concequently h is logarithmically convex (see, e.g., Artin (1931)).

Alternatively, one can use the fact that the $\bar{P}_k$ of (6.3) form the moment sequence of a random variable taking values in [0, 1]. Because of this, properties of the $\bar{P}_k$ are well known, see, e.g., Karlin and Shapley (1953) page 55). In particular, if $p_k = \bar{P}_{k-1} - \bar{P}_k$, $k = 1, 2, \cdots$, then $p_i p_{i+2} - p_{i+1}^2 \geqq 0$, $i = 1, 2, \cdots$, i.e., $\{p_k\}$ is a logarithmically convex sequence. The theorem follows from this and (3.7). □

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A. W. Marshall
Department of Statistics
University of Rochester
Rochester, New York 14627

J. D. Esary
Department of Operations Research
and Administrative Sciences
Naval Postgraduate School
Monterey, California 93940

F. Proschan
Department of Statistics
Florida State University
Tallahassee, Florida 32306