J. DIFFERENTIAL GEOMETRY
67 (2004) 229-255

SHOKUROV'S BOUNDARY PROPERTY

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Abstract

For a birational analogue of minimal elliptic surfaces $f \colon X \to Y$, the singularities of the fibers allow us to define a log structure (Y, B_Y) in codimension one on Y. Via base change, we have a log structure $(Y', B_{Y'})$ in codimension one on Y', for any birational model Y' of Y. We show that these codimension one log structures glue to a unique log structure, defined on some birational model of Y (Shokurov's BP Conjecture). As applications, we obtain Inversion of Adjunction for the above mentioned fiber spaces, and the invariance of Shokurov's FGA-algebras under adjunction.

0. Introduction

Our aim is an extension to the category of log pairs of Kodaira's canonical bundle formula for elliptic surfaces [**18**]. We recall Kodaira's formula in a generalized form due to Ueno [**23**], Kawamata [**12**] and Fujita [**9**]. Let $f \colon X \to Y$ be a fibration of non-singular proper varieties whose general fibre is an elliptic curve, and assume that the J-invariant function extends to a morphism $J \colon Y \to \mathbb{P}^1$. Then,

$$K_X + B_X = f^*(K_Y + B_Y + M_Y).$$

Here, K_X, K_Y are suitable canonical divisors on X and Y, respectively. The *moduli part* M_Y is a $\mathbb{Q}$-Weil divisor such that $12M_Y$ is a Cartier divisor and $\mathcal{O}_Y(12M_Y) \simeq J^*\mathcal{O}_{\mathbb{P}^1}(1)$. The *discriminant* $B_Y = \sum_P b_P P$ is supported by the codimension one points P of Y for which the geometric fibre $X_P = X \times_Y \operatorname{Spec}(\overline{k(P)})$ is singular. In terms of Kodaira's classification of degenerate fibers ([**17**], Theorem 6.2), its coefficients are ($m\mathrm{I}_b$ is a multiple fibre of multiplicity m, and $b \geq 0$)

This work was supported through a European Community Marie Curie Fellowship.
Received 12/10/2002.

X_P	$m\mathrm{I}_b$	I_b^*	II	II^*	III	III^*	IV	IV^*
b_P	$1-m^{-1}$	$1/2$	$1/6$	$5/6$	$1/4$	$3/4$	$1/3$	$2/3$

Finally, $B_X = E^+ - E^-$, where E^+, E^- are effective $\mathbb{Q}$-Weil divisors on X such that E^- supports no fibers over codimension one points of Y, and $\operatorname{codim}(f(E^+), Y) \geq 2$.
Recall that a *log pair* (X, B) is a normal variety X endowed with a $\mathbb{Q}$-Weil divisor B such that the log canonical divisor $K + B$ is $\mathbb{Q}$-Cartier. Here, K is the Weil divisor of zeros and poles of a top rational differential form on X; it is called the canonical divisor of X and is unique up to linear equivalence.

The term B_X was invisible in Kodaira's original formula, since f was minimal and $\dim(Y) = 1$. However, Kawamata [**14**, **15**] pointed out that B_Y, which was originally computed in terms of the local monodromies around the classified degenerate fibers, is uniquely determined by the log pair (X, B_X): $1 - b_P$ is the largest real number t such that the log pair $(X, B_X + tf^*(P))$ has log canonical singularities over the generic point of P. In particular, the moduli part is also determined by the log pair (X, B_X) and the above adjunction formula.

The objects we are interested in are *lc-trivial fibrations* $f\colon (X, B) \to Y$. They consist of a contraction $f\colon X \to Y$ of proper normal varieties and a log pair structure (X, B) such that the log canonical divisor $K+B$ is $\mathbb{Q}$-linearly trivial over Y. More precisely:

(1) the log pair (X, B) has at most Kawamata log terminal singularities over the generic point of Y;
(2) $\operatorname{rank} f_*\mathcal{O}_X(\lceil \mathbf{A}(X, B)\rceil) = 1$;
(3) there exist a positive integer r, a rational function φ on X and a $\mathbb{Q}$-Cartier divisor D on Y such that

$$K + B + \frac{1}{r}(\varphi) = f^*D.$$

This type of fibrations appear naturally in higher codimensional adjunction [**14**, **15**, **1**] and in the study of parabolic fiber spaces [**6**, **5**]. We refer the reader to Remark 2.2 for examples where the technical assumption (2) is satisfied, and to Section 1 for the definition of Shokurov's discrepancy b-divisor $\mathbf{A}(X, B)$. See also Example 2.3 for the classification of the generic fibre in case f has relative dimension one.

Kawamata's formula defines the *discriminant* B_Y of (X, B) on Y, and the *moduli part* is the unique $\mathbb{Q}$-Weil divisor M_Y on Y satisfying the adjunction formula

$$K + B + \frac{1}{r}(\varphi) = f^*(K_Y + B_Y + M_Y).$$

The following properties are desirable for applications.

- *Inversion of Adjunction*: (Y, B_Y) is a log pair having the same type of singularities as (X, B).
- *Semi-ampleness*: M_Y is semi-ample, that is, there exists a positive integer m such that mM_Y is a Cartier divisor and the linear system $|mM_Y|$ is base point free.
- *Boundedness*: the minimal value of m is bounded in terms of the log pair structure induced on the geometric general fibre $(X, B) \times_Y \operatorname{Spec}(\overline{k(Y)})$.

If Y is a curve, Inversion of Adjunction holds by the very definition of B_Y, and Semi-ampleness can be reduced to a result of Fujita [8].

Theorem 0.1. *Let $f\colon (X, B) \to Y$ be an lc-trivial fibration such that $\dim(Y) = 1$. Then, the moduli $\mathbb{Q}$-divisor M_Y is semi-ample.*

If Y has dimension at least two, Prokhorov found an example where Inversion of Adjunction fails. The reason is that the linear system $|mM_Y|$ might have base points in codimension at least two, for every large and divisible integer m. Thus, we expect that the desired properties hold only after a suitable birational base change: if $f'\colon X' \to Y'$ is a fiber space induced via a birational base change $\sigma\colon Y' \to Y$, we have an induced lc-trivial fibration $f'\colon (X', B_{X'}) \to Y'$, where $B_{X'}$ is defined by $\mu^*(K + B) = K_{X'} + B_{X'}$:

$$\begin{array}{ccc} (X, B) & \xleftarrow{\ \mu\ } & (X', B_{X'}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xleftarrow{\ \sigma\ } & Y' \end{array}$$

We denote by $B_{Y'}$ and $M_{Y'}$ the discriminant and moduli part of the lc-trivial fibration $f'\colon (X', B_{X'}) \to Y'$, respectively. Our main result is the stabilization of the induced structure on Y, after a suitable blow-up:

Theorem 0.2. *Let $f\colon (X, B) \to Y$ be an lc-trivial fibration. Then, there exists a proper birational morphism $Y' \to Y$ such that*

(i) *$K_{Y'} + B_{Y'}$ is a $\mathbb{Q}$-Cartier divisor, and $\nu^*(K_{Y'} + B_{Y'}) = K_{Y''} + B_{Y''}$ for every proper birational morphism $\nu\colon Y'' \to Y'$.*

(ii) *$M_{Y'}$ is a nef $\mathbb{Q}$-Cartier divisor and $\nu^*(M_{Y'}) = M_{Y''}$ for every proper birational morphism $\nu\colon Y'' \to Y'$.*

The first part is the positive answer to Shokurov's BP Conjecture [**22**], p. 92. Prokhorov and Shokurov [**20**] proved (i), by a different method, in a special case when X is a 3-fold and Y is a surface (they also obtain an explicit description of Y'). Modulo the first part, (ii) is a result of Kawamata [**15**]. Furthermore, Theorems 0.1 and Theorem 0.2(ii) generalize similar results of Fujino [**5**] for parabolic fiber spaces. We have three applications:

(A) Inversion of Adjunction holds for the induced lc-trivial fibration $f''\colon (X'', B_{X''}) \to Y''$, for every birational model Y'' which dominates Y' (Theorem 3.1).

(B) Shokurov has reduced the existence of flips to the finite generatedness of certain (FGA) algebras which are asymptotically saturated with respect to a Fano variety ([**22**], Conjecture 4.39). We obtain a descent property for asymptotic saturation of algebras (Proposition 6.3). In particular, FGA algebras are invariant under restriction to exceptional log canonical centers (Theorem 6.5).

(C) Kawamata–Shokurov's Adjunction Conjecture [**1**] follows from inversion of adjunction and semi-ampleness (for exceptional lc centers). Adjunction to lc centers of codimension two is due to Kawamata [**14**]. As a corollary of Theorem 0.1, adjunction to 1-dimensional exceptional lc centres holds as well.

Besides Theorem 0.1, the semi-ampleness of the moduli part is known in the following cases (denote the general fibre by F and let $B_F = B|_F$):

(i) $\dim(F) = 1$ and B_F is effective (Kawamata [**14**]).

(ii) F is an abelian variety and $B_F = 0$, or F is a surface of Kodaira dimension zero and $-B_F$ is effective (Ueno [**24**], Fujino [**5**]).

The proofs of Theorems 0.1 and 0.2 are based on techniques developed for the proof of Iitaka's Addition Conjecture, especially [**12**] (see [**19**] for an excellent survey). Some of the applications in (B) are explained conjecturally in [**22**].

1. Preliminary

A *variety* is a reduced and irreducible scheme of finite type, defined over an algebraically closed field of characteristic zero. An open subset U of a variety X is called *big* if $X \setminus U \subset X$ has codimension at least two. A *contraction* is a proper morphism $f\colon X \to Y$ such that $\mathcal{O}_Y = f_*\mathcal{O}_X$.

Let $\pi\colon X \to S$ be a proper morphism from a normal variety X, and let $L \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$.

1.1. Divisors. An L-Weil divisor is an element of $Z^1(X) \otimes_{\mathbb{Z}} L$. The *round up* (*down*) divisor $\lceil D \rceil$ ($\lfloor D \rfloor$) is defined componentwise. Two $\mathbb{R}$-Weil divisors D_1, D_2 are L- *linearly equivalent*, denoted $D_1 \sim_L D_2$, if there exist $q_i \in L$ and rational functions $\varphi_i \in k(X)^\times$ such that $D_1 - D_2 = \sum_i q_i(\varphi_i)$. An $\mathbb{R}$-Weil divisor D is called

(i) *L-Cartier* if $D \sim_L 0$ in a neighborhood of each point of X.
(ii) *Relatively nef* if D is $\mathbb{R}$-Cartier and $D \cdot C \geq 0$ for every proper curve C contracted by π.
(iii) *Relatively free* if D is a Cartier divisor and the natural map
$$\pi^*\pi_*\mathcal{O}_X(D) \to \mathcal{O}_X(D)$$
is surjective.
(iv) *Relatively ample* if π is a projective morphism and the numerical class of D belongs to the real cone generated by relatively ample Cartier divisors.
(v) *Relatively semi-ample* if there exists a contraction $\Phi\colon X \to Y/S$ and a relatively ample $\mathbb{R}$-divisor H on Y such that $D \sim_{\mathbb{R}} \Phi^*H$. If D is rational, this is equivalent to mD being relatively free for sufficiently large and divisible positive integers m.
(vi) *Relatively big* if there exists $C > 0$ such that $\operatorname{rank} \pi_*\mathcal{O}_X(mD) \geq Cm^d$ for m sufficiently large and divisible, where d is the dimension of the generic fibre of π.

A divisor D has *simple normal crossings* if it is reduced and its components are non-singular divisors intersecting transversely, in the smooth ambient space X.

1.2. B-divisors. (V.V. Shokurov [21, 22]) An *L-b-divisor* $\mathbf{D}$ of X is a family $\{\mathbf{D}_{X'}\}_{X'}$ of L-Weil divisors indexed by all birational models X' of X, such that $\mu_*(\mathbf{D}_{X''}) = \mathbf{D}_{X'}$ if $\mu\colon X'' \to X'$ is a birational contraction.

Equivalently, $\mathbf{D} = \sum_E \operatorname{mult}_E(\mathbf{D})E$ is an L-valued function on the set of all geometric valuations of the field of rational functions $k(X)$, having finite support on some (hence any) birational model of X.

Example 1.1.

(1) Let ω be a top rational differential form of X. The associated family of divisors $\mathbf{K} = \{(\omega)_{X'}\}_{X'}$ is called the *canonical b-divisor* of X.

(2) A rational function $\varphi \in k(X)^\times$ defines a b-divisor $\overline{(\varphi)} = \{(\varphi)_{X'}\}_{X'}$.
(3) An $\mathbb{R}$-Cartier divisor D on a birational model X' of X defines an $\mathbb{R}$-b-divisor $\overline{D}$ such that $(\overline{D})_{X''} = \mu^* D$ for every birational contraction $\mu\colon X'' \to X'$.
(4) For an $\mathbb{R}$-b-divisor $\mathbf{D}$, the round up (down) b-divisor $\lceil \mathbf{D} \rceil$ ($\lfloor \mathbf{D} \rfloor$) is defined componentwise.

An $\mathbb{R}$-b-divisor $\mathbf{D}$ is called L-b-*Cartier* if there exists a birational model X' of X such that $\mathbf{D}_{X'}$ is L-Cartier and $\mathbf{D} = \overline{\mathbf{D}_{X'}}$. In this case, we say that $\mathbf{D}$ *descends* to X'. The *relative Iitaka dimension* $\kappa(X/S, \mathbf{D})$ of a L-Cartier b-divisor $\mathbf{D}$ is defined as the relative Kodaira dimension of $\mathbf{D}_{X'}$, where X'/S is a model where $\mathbf{D}$ descends.

An $\mathbb{R}$-b-divisor $\mathbf{D}$ is *b-nef/S* (*b-free/S, b-semi-ample/S, b-big/S*) if there exists a birational contraction $X' \to X$ such that $\mathbf{D} = \overline{\mathbf{D}_{X'}}$, and $\mathbf{D}_{X'}$ is nef (free, semi-ample, big) relative to the induced morphism $X' \to S$.

To any $\mathbb{R}$-b-divisor $\mathbf{D}$ of X, there is an associated *b-divisorial sheaf* $\mathcal{O}_X(\mathbf{D})$. If $U \subset X$ is an open subset, then $\Gamma(U, \mathcal{O}_X(\mathbf{D}))$ is the set of rational functions $\varphi \in k(X)$ (including 0) such that $\mathrm{mult}_E(\overline{(\varphi)} + \mathbf{D}) \geq 0$ for every geometric valuation E with $c_X(E) \cap U \neq \emptyset$. Here, $c_X(E)$ is the center on X of the geometric valuation E.

1.3. Log pairs. A *log pair* (X, B) is a normal variety X endowed with a $\mathbb{Q}$-Weil divisor B such that $K + B$ is $\mathbb{Q}$-Cartier. A *log variety* is a log pair (X, B) such that B is effective. A *relative log pair* (*variety*) $(X/S, B)$ consists of a proper morphism $\pi\colon X \to S$ and a log pair (variety) structure (X, B). The *discrepancy b-divisor* of a log pair (X, B) is the $\mathbb{Q}$-b-divisor of X defined by the following formula:

$$\mathbf{A}(X, B) = \mathbf{K} - \overline{K + B}.$$

A birational map of log pairs $f\colon (X, B) \dashrightarrow (X', B_{X'})$ is called (*log*) *crepant* if $\mathbf{A}(X, B) = \mathbf{A}(X', B_{X'})$. For a geometric valuation E of $k(X)$, the *log discrepancy* of E with respect to (X, B) is

$$a(E; X, B) := 1 + \mathrm{mult}_E(\mathbf{A}(X, B)).$$

The *minimal log discrepancy* of (X, B) in a proper closed subset $W \subset X$, is

$$a(W; X, B) := \inf_{c_X(E) \subseteq W} a(E; X, B).$$

The log pair (X, B) has *log canonical* (*Kawamata log terminal*) *singularities* if $a(E; X, B) \geq 0$ ($a(E; X, B) > 0$) for every valuation E. The *non-klt locus* $\mathrm{LCS}(X,B)$ (*non-log canonical locus* $(X,B)_{-\infty}$) is the union of all centers $c_X(E)$ of geometric valuations E with $a(E;X,B) \le 0$ ($a(E;X,B) < 0$). An *lc place* is a geometric valuation E such that $a(E;X,B) = 0$ and $c_X(E) \not\subseteq (X,B)_{-\infty}$, and its center $c_X(E)$ on X is called an *lc center*. An lc centre C is *exceptional* if there exists a unique lc place E with $c_X(E) = C$. We also denote

$$\mathbf{A}^*(X,B) = \mathbf{A}(X,B) + \sum_{a(E;X,B)=0} E.$$

A *relative generalized log Fano* variety is a relative log variety $(X/S,B)$ such that $-(K+B)$ is ample$/S$.

2. The discriminant and moduli b-divisors

Definition 2.1. An *lc-trivial fibration* $f\colon (X,B) \to Y$ consists of contraction of normal varieties $f\colon X \to Y$ and a log pair (X,B), satisfying the following properties:

(1) (X,B) has Kawamata log terminal singularities over the generic point of Y;
(2) rank $f_*\mathcal{O}_X(\lceil \mathbf{A}(X,B) \rceil) = 1$;
(3) There exists a positive integer r, a rational function $\varphi \in k(X)^\times$ and a $\mathbb{Q}$-Cartier divisor D on Y such that

$$K + B + \frac{1}{r}(\varphi) = f^*D.$$

Remark 2.2. The property rank $f_*\mathcal{O}_X(\lceil \mathbf{A}(X,B) \rceil) = 1$ holds in the following examples:

(a) f is birational to the Iitaka fibration of a functional algebra $\mathcal{L}$ which is asymptotically $\mathbf{A}(X,B)$-saturated (Lemma 6.2);
(b) The log pair (F,B_F) has Kawamata log terminal singularities and B_F is effective (F is a general fiber of f and $B_F = B|_F$);
(c) Let W be the normalization of an exceptional log canonical centre of a log variety (X,B), and let $h\colon E \to W$ be the unique lc place over W. By adjunction, there exists a $\mathbb{Q}$-divisor B_E such that $h\colon (E,B_E) \to W$ is an lc-trivial fibration (see [**3**]).

Example 2.3. Assume that the geometric generic fibre $F = X_{\bar\eta}$ of an lc-trivial fibration has dimension one. Then, the induced log pair structure (F,B_F) is classified as follows (g is the genus of the curve F):

(i) $g = 0$ and $B_F = \sum_{i=1}^l b_i P_i$, with $b_i \in \mathbb{Q} \cap [0,1)$ and $\sum_i b_i = 2$.

(ii) $g = 1$, $B_F = -b_1 P_1 + \sum_{i=2}^{l} b_i P_i$ is a torsion $\mathbb{Q}$-divisor, where $0 \le b_i \le 1$.
(iii) $g = 2$ and $B_F = -P_1 - P_2$, where P_1, P_2 are (possibly equal) points of F such that $P_1 + P_2 \sim_{\mathbb{Q}} K_F$ and $\dim |P_1 + P_2| = 0$.

Define $B_Y = \sum_{P \subset Y} b_P P$, where the sum runs after all prime divisors of Y, and

$$1 - b_P = \sup\{t \in \mathbb{R}; ^{\exists} U \ni \eta_P, (X, B + tf^*(P)) \text{ lc sing}/U\}.$$

The coefficients b_P are well defined, since (X, B) has at most log canonical singularities over the general point of Y, and each prime divisor is Cartier in a neighborhood of its general point. It is easy to see that the sum has finite support, so B_Y is a well defined $\mathbb{Q}$-Weil divisor on Y. By (3), there exists a unique $\mathbb{Q}$-Weil divisor M_Y such that the following *adjunction formula* holds:

$$K + B + \frac{1}{r}(\varphi) = f^*(K_Y + B_Y + M_Y).$$

Definition 2.4 ([1]). The $\mathbb{Q}$-Weil divisors B_Y and M_Y are called the *discriminant* and *moduli part* of the lc-trivial fibration $f\colon (X, B) \to Y$. Note that $K_Y + B_Y + M_Y$ is $\mathbb{Q}$-Cartier.

Remark 2.5. The above adjunction formula gives a one-to-one correspondence between the choices of M_Y and rational functions with $\mathbb{Q}$-coefficients $(1/r)\varphi$ such that $K_F + B_F + (1/r)(\varphi|_F) = 0$, where F is the general fibre of f.

If M_Y and M'_Y correspond to $(1/r)\varphi$ and $(1/r)\varphi'$, respectively, then there exists a rational function $\theta \in k(Y)^\times$ such that $\varphi' = \varphi f^*\theta$ and $rM'_Y = (\theta) + rM_Y$. The smallest possible value of r is the positive integer $b(F, B_F)$, uniquely defined by

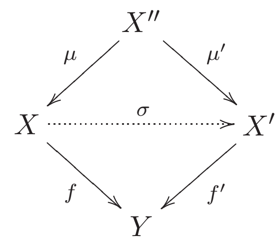
$$\{m \in \mathbb{N}; m(K_F + B_F) \sim 0\} = b(F, B_F)\mathbb{N}.$$

Thus, we may assume $r = b(F, B_F)$, up to a $\mathbb{Q}$-linear equivalence of M_Y.

According to the following lemma, B_Y and M_Y are independent of the choice of a crepant model of (X, B) over Y.

Lemma 2.6. *Let $\sigma\colon X \dashrightarrow X'$ be a birational map defined over Y, and let $f'\colon X' \to Y$ be the induced morphism. Then, there exists a unique $\mathbb{Q}$-Weil divisor $B_{X'}$ such that $\sigma\colon (X, B) \dashrightarrow (X', B_{X'})$ is a crepant birational map. Moreover, (X, B) and $(X', B_{X'})$ induce the same discriminant on Y.*

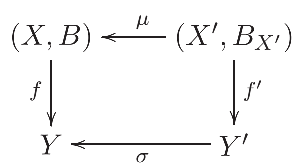
Proof. There exists a common normal birational model of X and X' which makes the following diagram commute



Let $K_{X''} + B_{X''} = \mu^*(K + B)$ be the log pullback. Since $K_{X''} + B_{X''} + \frac{1}{r}(\varphi) = \mu'^*(f'^* D)$ and μ' is birational, we have $K_{X''} + B_{X''} = \mu'^*(K_{X'} + B_{X'})$, where $B_{X'} := \mu'_*(B_{X''})$. Therefore, there exists a crepant log structure on X'. The uniqueness of $B_{X'}$ is clear.

Finally, note that $\mu^*(K + B + tf^*(P)) = K_{X''} + B_{X''} + t(f \circ \mu)^*(P) = \mu'^*(K_{X'} + B_{X'} + tf'^*(P))$. Therefore, the thresholds $1 - b_P$ induced by $K + B$ and $K_{X'} + B_{X'}$ coincide. q.e.d.

Let $\sigma \colon Y' \to Y$ be a birational contraction from a normal variety Y'. Let X' be a resolution of the main component of $X \times_Y Y'$ which dominates Y'. The induced morphism $\mu \colon X' \to X$ is birational, and let $(X', B_{X'})$ be the crepant log structure on X', i.e., $\mu^*(K + B) = K_{X'} + B_{X'}$



We say that the lc-trivial fibration $f' \colon (X', B_{X'}) \to Y'$ is induced by base change. Let $B_{Y'}$ be the discriminant of $K_{X'} + B_{X'}$ on Y'. Since the definition of the discriminant is divisorial and σ is an isomorphism over codimension one points of Y, we have $B_Y = \sigma_*(B_{Y'})$. This means that there exists a unique $\mathbb{Q}$-b-divisor $\mathbf{B}$ of Y such that $\mathbf{B}_{Y'}$ is the discriminant on Y' of the induced fibre space $f' \colon (X', B_{X'}) \to Y'$, for every birational model Y' of Y. We call $\mathbf{B}$ the *discriminant* $\mathbb{Q}$-*b-divisor* induced by (X, B) on the birational class of Y. Accordingly, there exists a unique $\mathbb{Q}$-*b-divisor* $\mathbf{M}$ of Y such that

$$K_{X'} + B_{X'} + \frac{1}{r}(\varphi) = f^*(K_{Y'} + \mathbf{B}_{Y'} + \mathbf{M}_{Y'})$$

for every lc-trivial fibration $f' \colon (X', B_{X'}) \to Y'$ induced by base change on a birational model Y' of Y. We call $\mathbf{M}$ the *moduli* $\mathbb{Q}$*-b-divisor* of Y, induced by the lc-trivial fibration $f \colon (X, B) \to Y$. We restate Theorem 0.2 in terms of b-divisors.

Theorem 2.7. *Let $f \colon (X, B) \to Y$ be a lc-trivial fibration, and let $\pi \colon Y \to S$ be a proper morphism. Let $\mathbf{B}$ and $\mathbf{M}$ be the induced discriminant and moduli $\mathbb{Q}$-b-divisors of Y. Then,*

(1) $\mathbf{K} + \mathbf{B}$ *is $\mathbb{Q}$-b-Cartier,*
(2) $\mathbf{M}$ *is b-nef/S.*

We expect Theorem 2.7 to hold if we allow $\mathbb{R}$-boundaries and $\mathbb{R}$-linear equivalence instead of $\mathbb{Q}$-boundaries and $\mathbb{Q}$-linear equivalence in Definition 2.1, or if (X, B) has log canonical singularities over the generic point of Y. In the latter case, the assumption rank $f_*\mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil) = 1$ should be replaced by rank $f_*\mathcal{O}_X(\lceil \mathbf{A}^*(X, B) \rceil) = 1$.

3. Inversion of Adjunction

Let $f \colon (X, B) \to Y$ be an lc-trivial fibration. The $\mathbb{Q}$-b-divisor of Y

$$\mathbf{A}_{\mathrm{div}} := -\mathbf{B}$$

is called the *divisorial discrepancy b-divisor* ([**22**], p. 92). Theorem 2.7 (1) is equivalent to the following property: there exists a birational model Y' of Y such that $\mathbf{A}_{\mathrm{div}} = \mathbf{A}(Y'', \mathbf{B}_{Y''})$ for every birational model Y'' which dominates Y'. As a corollary, Inversion of Adjunction holds for the induced morphism of log pairs $f \colon (X, B) \to (Y, \mathbf{B}_Y)$, after a sufficiently high birational base change:

Theorem 3.1 (Inversion of Adjunction)**.** *Let $f \colon (X, B) \to Y$ be an lc-trivial fibration such that $\mathbf{A}_{\mathrm{div}} = \mathbf{A}(Y, \mathbf{B}_Y)$. Then, there exists a positive integer N such that*

$$\frac{1}{N} a(f^{-1}(Z); X, B) \le a(Z; Y, B_Y) \le a(f^{-1}(Z); X, B)$$

for every closed subset $Z \subset Y$, where $a(Z; Y, B_Y)$ and $a(f^{-1}(Z); X, B)$ are the minimal log discrepancies of (Y, B_Y) in Z, and (X, B) in $f^{-1}(Z)$ respectively.

In particular, $(Y, \mathbf{B}_Y)$ has Kawamata log terminal (log canonical) singularities in a neighborhood of a point $y \in Y$ if and only if (X, B) has Kawamata log terminal (log canonical) singularities in a neighborhood of $f^{-1}(y)$.

Proof. The assumption $\mathbf{A}_{\mathrm{div}} = \mathbf{A}(Y, \mathbf{B}_Y)$ means that the Base Change Conjecture ([1], Section 3) holds for $f\colon (X,B) \to Y$. The claim is proved in [1], Proposition 3.4, but with N depending on Z. The possible values for minimal log discrepancies of a fixed log pair are finite ([2], Theorem 2.3), hence a maximal value $N = \max_{Z \subset Y} N(Z)$ exists. q.e.d.

Lemma 3.2. *Let $f\colon (X,B) \to Y$ be an lc-trivial fibration such that $\mathbf{A}_{\mathrm{div}} = \mathbf{A}(Y, \mathbf{B}_Y)$. Assume, moreover, that X, Y are non-singular varieties, and the divisors $B, \mathbf{B}_Y$ have simple normal crossings support. Then, $f_*\mathcal{O}_X(\lceil -B \rceil) = \mathcal{O}_Y(\lceil -\mathbf{B}_Y \rceil)$.*

Proof. By the simple normal crossings assumption, $\mathcal{O}_X(\lceil \mathbf{A}(X,B) \rceil) = \mathcal{O}_X(\lceil -B \rceil)$. Since (X,B) has Kawamata log terminal singularities over the generic point of Y, we have a natural inclusion

$$\mathcal{O}_Y|_V \subseteq f_*\mathcal{O}_X(\lceil -B \rceil)|_V$$

for some open subset $V \subset Y$. Since $\operatorname{rank} f_*\mathcal{O}_X(\lceil \mathbf{A}(X,B) \rceil) = 1$, the above inclusion is an equality, after possibly shrinking V. Thus, we may identify $f_*\mathcal{O}_X(\lceil -B \rceil)$ with a subsheaf of the constant sheaf $k(Y)$. We first show that $f_*\mathcal{O}_X(\lceil -B \rceil) \subseteq \mathcal{O}_Y(\lceil -\mathbf{B}_Y \rceil)$. Let φ be a rational function of Y such that $(f^*\varphi) + \lceil -B \rceil \geq 0$, and let P be a prime divisor of Y. We may replace X by some resolution, so that there exists a prime divisor Q of X with $f(Q) = P$ and

$$1 - \operatorname{mult}_P(\mathbf{B}_Y) = \frac{1 - \operatorname{mult}_Q(B)}{m_{Q/P}},$$

where $m_{Q/P}$ is the multiplicity of $f^*(P)$ at Q. By assumption, we have $\operatorname{mult}_Q(f^*\varphi) + 1 - \operatorname{mult}_Q(B) > 0$. But $\operatorname{mult}_Q(f^*\varphi) = m_{Q/P} \cdot \operatorname{mult}_P(\varphi)$. Hence, $\operatorname{mult}_P(\varphi) + 1 - \operatorname{mult}_P(\mathbf{B}_Y) > 0$. Therefore, $(\varphi) + \lceil -\mathbf{B}_Y \rceil$ is effective at P.

Conversely, assume $(\varphi) + \lceil -\mathbf{B}_Y \rceil$ is effective, and fix a prime divisor Q of X. There exists a birational base change

$$\begin{array}{ccc} (X,B) & \xleftarrow{\ \mu\ } & (X', B_{X'}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xleftarrow[\ \sigma\ ]{} & Y' \end{array}$$

such that $P := f(Q)$ is a prime divisor of Y'. We have $\sigma^*(K_Y + \mathbf{B}_Y) = K_{Y'} + \mathbf{B}_{Y'}$ by $\mathbf{A}_{\mathrm{div}} = \mathbf{A}(Y, \mathbf{B}_Y)$. Furthermore, the simple normal crossings assumption implies $\sigma_*\mathcal{O}_{Y'}(\lceil -\mathbf{B}_{Y'}\rceil) = \mathcal{O}_Y(\lceil -\mathbf{B}_Y\rceil)$. Therefore, $(\varphi) + \lceil -\mathbf{B}_{Y'}\rceil \geq 0$, and hence, $\mathrm{mult}_P(\varphi) + 1 - \mathrm{mult}_P(\mathbf{B}_{Y'}) > 0$. Since

$$1 - \mathrm{mult}_P(\mathbf{B}_Y) \leq \frac{1 - \mathrm{mult}_Q(B_{X'})}{m_{Q/P}},$$

we infer $\mathrm{mult}_Q(f^*\varphi) + 1 - \mathrm{mult}_Q(B_{X'}) > 0$, i.e., $(f^*\varphi) + \lceil -B\rceil$ is effective at Q. q.e.d.

Remark 3.3. Let $f\colon (X, B) \to Y$ be an lc-trivial fibration. Let L be a $\mathbb{Q}$-Cartier divisor on Y, and set $B' = B + f^*L$. Then, $f\colon (X, B') \to Y$ is an lc-trivial fibration, with moduli $\mathbb{Q}$-b-divisor $\mathbf{M}' = \mathbf{M}$, and discriminant $\mathbb{Q}$-b-divisor $\mathbf{B}' = \mathbf{B} + \overline{L}$.

4. Covering tricks and base change

Theorem 4.1 ([10]). *Let X be a non-singular quasi-projective variety endowed with a divisor D with simple normal crossings singularities, and let N be a positive integer. Then, there exists a finite Galois covering $\tau\colon \tilde{X} \to X$ satisfying the following conditions:*

(1) *$\tilde{X}$ is a non-singular quasi-projective variety, and there exists a simple normal crossings divisor Σ_X such that τ is étale over $X \setminus \Sigma_X$, and $\tau^{-1}(\Sigma_X)$ is a divisor with simple normal crossings;*
(2) *The ramification indices of τ over the prime components of D are divisible by N.*

Sketch of proof. We may assume that X is projective (by Hironaka's resolution of singularities, we can compactify as a complement of simple normal crossings divisor in a projective variety, construct the cover, and then restrict back to the original variety). Let A be a very ample divisor such that $NA - D_i$ is very ample for each component D_i of D. Let $n = \dim(X)$. There exists $H_1^{(i)}, \ldots, H_n^{(i)} \in |NA - D_i|$ for every D_i, such that $\Sigma_X := D + \sum_{i,j} H_j^{(i)}$ is a divisor with simple normal crossings. Let $X = \cup U_\alpha$ be an affine cover, and let $D_i + H_j^{(i)} = (\varphi_{j\alpha}^{(i)})$ on U_α. The field extension $L := k(X)[\sqrt[N]{\varphi_{j\alpha}^{(i)}}; i, j]$ is independent of the choice of α. Let $\tilde{X}$ be the normalization on X in L. Then $\tilde{X}$ is non-singular and τ is a Kummer cover which is étale outside Σ_X, and $\tau^{-1}(\Sigma_X)$ has simple normal crossings. q.e.d.

Remark 4.2. In the above notations, assume that $\varrho\colon Y \to X$ is a surjective morphism from a non-singular quasi-projective variety Y such that $\varrho^{-1}(D)$ has simple normal crossings. Then, we may assume that $\tau\colon \tilde{X} \to X$ fits into a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xleftarrow{\;g\;} & \tilde{Y} \\ {\scriptstyle\tau}\big\downarrow & & \big\downarrow{\scriptstyle\nu} \\ X & \xleftarrow{\;\varrho\;} & Y \end{array}$$

satisfying the following properties:

(1) ν is a finite covering and g is a projective morphism;
(2) $\tilde{Y}$ is non-singular quasi-projective;
(3) there exists a simple normal crossings divisor Σ_Y such that ν is étale over $Y \setminus \Sigma_Y$, $\nu^{-1}(\Sigma_Y)$ has simple normal crossings, and $\varrho^{-1}(\Sigma_X) \subseteq \Sigma_Y$.

Proof. In the proof of Theorem 4.1, we may choose the divisors $H_j^{(i)}$ so that $\varrho^{-1}(D + \sum_{i,j} H_j^{(i)})$ is a divisor with simple normal crossings on Y. Let $\tau_1\colon \bar{Y} \to Y$ be the normalization of the main component of the pull back of τ to Y.

$$\begin{array}{ccccc} \tilde{X} & \longleftarrow & \bar{Y} & \longleftarrow & \tilde{Y} \\ {\scriptstyle\tau}\big\downarrow & & {\scriptstyle\tau_1}\big\downarrow & & \big\downarrow \\ X & \xleftarrow{\;\varrho\;} & Y & \xleftarrow{\;\pi\;} & Y_1 \end{array}$$

Then τ_1 is a finite cover whose ramification locus is contained in the simple normal crossings divisor $\varrho^{-1}(\Sigma_X)$. Let N' be the least common multiple of its ramification indices, and construct by Theorem 4.1 a finite cover $\pi\colon Y_1 \to Y$ with respect to $\varrho^{-1}(\Sigma_X)$ and N'. Let $\tilde{Y}/Y_1$ be the normalization of the main component of the pull back of τ_1 to Y_1. The induced map $\nu\colon \tilde{Y} \to Y$ is a finite cover. By construction, $\tilde{Y}/Y_1$ is étale, hence, $\tilde{Y}$ is non-singular. There exists a simple normal crossings divisor Σ_Y containing $\varrho^{-1}(\Sigma_X)$ such that π is étale over $Y \setminus \Sigma_Y$ and $\pi^{-1}(\Sigma_Y)$ has simple normal crossings. Therefore, $\nu^{-1}(\Sigma_Y)$ has simple normal crossings. q.e.d.

Theorem 4.3 (Semi-stable reduction in codimension one [**16**, **25**])**.** *Let $f\colon X \to Y$ be a surjective morphism of non-singular varieties. Assume Σ_X, Σ_Y are simple normal crossings divisors on X and Y respectively, such that $f^{-1}(\Sigma_Y) \subseteq \Sigma_X$ and f is smooth over $Y \setminus \Sigma_Y$. Then, there exists a positive integer N such that the following hold:*

Let $\pi\colon Y' \to Y$ be a finite covering from a non-singular variety Y' such that $\Sigma_{Y'} := \pi^{-1}(\Sigma_Y)$ has simple normal crossings and N divides the ramification indices of π over the prime components of $\Sigma_{Y'}$. Then, there exists a commutative diagram

$$\begin{array}{ccccc} X & \longleftarrow & X \times_Y Y' & \xleftarrow{\ p\ } & X' \\ {\scriptstyle f}\downarrow & & \downarrow & \swarrow {\scriptstyle f'} & \\ Y & \xleftarrow[\pi]{} & Y' & & \end{array}$$

with the following properties:

(a) *X' is non-singular and $\Sigma_{X'} := {\pi'}^{-1}(\Sigma_X)$ has simple normal crossings, where $\pi'\colon X' \to X$ is the induced projective morphism;*
(b) *p is a projective morphism which is an isomorphism above $Y' \setminus \Sigma_{Y'}$. In particular, f' is smooth over $Y' \setminus \Sigma_{Y'}$;*
(c) *f' is* semi-stable in codimension one*: the fibers over* (*generic*) *codimension one points of Y' have simple normal crossings singularities.*

Sketch of proof. Let $f^*(\Sigma_X) = \sum n_i E_i$, and let N be the least common multiple of the n_i's corresponding to components E_i which dominate some component of Σ_Y. Consider a finite base change $Y' \to Y$ as above. Over the generic point of each prime component Q of $\Sigma_{Y'}$, $X \times_Y Y'$ admits a resolution with the desired properties [**16**]. Therefore, there exists a closed subscheme $B \subset X \times_Y Y'$, supported over $\Sigma_{Y'}$, and a closed subset $Z \subset \Sigma_{Y'}$ with $\operatorname{codim}(Z, Y') \geq 2$, such that the blow-up of $X \times_Y Y'$ in B has the desired properties over $Y' \setminus Z$. Then, we may take X' to be any resolution of the blow-up, which is an isomorphism outside its singular locus, and such that (a) holds. q.e.d.

Theorem 4.4 ([**7**, **10**])**.** *Let $f\colon X \to Y$ be a projective morphism of non-singular algebraic varieties. Assume f is semi-stable in codimension one, and there exists a simple normal crossings divisor Σ_Y such that f is smooth over $Y \setminus \Sigma_Y$. Then, the following properties hold:*

(1) *$f_*\omega_{X/Y}$ is a locally free sheaf on Y;*

(2) $f_*\omega_{X/Y}$ *is* semi-positive*: let* $\nu\colon C \to Y$ *be a proper morphism from a non-singular projective curve* C*, and let* $\mathcal{L}$ *be an invertible quotient of* $\nu^*(f_*\omega_{X/Y})$*. Then,* $\deg(\mathcal{L}) \geq 0$*;*

(3) *Let* $\varrho\colon Y' \to Y$ *be a projective morphism from a non-singular variety* Y' *such that* $\varrho^{-1}(\Sigma_Y)$ *is a simple normal crossings divisor. Let* $X' \to (X \times_Y Y')_{\mathrm{main}}$ *be a resolution of the component of* $X \times_Y Y'$ *which dominates* Y'*, and let* $h\colon X' \to Y'$ *be the induced fibre space:*

$$\begin{array}{ccc} X & \longleftarrow & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \overset{\varrho}{\longleftarrow} & Y' \end{array}$$

Then, there exists a natural isomorphism $\varrho^*(f_*\omega_{X/Y}) \xrightarrow{\sim} f'_*\omega_{X'/Y'}$ *which extends the base change isomorphism over* $Y \setminus \Sigma_Y$*.*

Sketch of proof. By the Lefschetz principle and flat base change, we may assume $k = \mathbb{C}$. Let $Y_0 = Y \setminus \Sigma_Y, X_0 = f^{-1}(Y_0)$, and let $d = \dim(X/Y)$. The locally free sheaf $H_0 := R^d f_*\mathbb{Q}_{X_0} \otimes_{\mathbb{Q}_{Y_0}} \mathcal{O}_{Y_0}$ is endowed with the integrable Gauss–Manin connection and is the underlying space of a variation of Hodge structure of weight d on Y_0, with $F^d H_0 = f_*\omega_{X_0/Y_0}$. Since f is semi-stable in codimension one, H_0 has unipotent local monodromies around the components of Σ_Y. Let H be the *canonical extension* [**4**] of H_0. By Schmid's asymptotic behaviour of variations of Hodge structure, the natural inclusion

$$f_*\omega_{X/Y} \to j_*(F^d H_0) \cap H$$

is an isomorphism and $f_*\omega_{X/Y}$ is locally free [**10**]. The semi-positivity follows from unipotence and Griffiths' semi-positivity of the curvature of the last piece of a variation of Hodge structure [**7**, **10**].

For base change, the sheaf $f'_*\omega_{X'/Y'}$ is independent of birational changes in X' over Y'. Thus, we may assume that $X' \to X \times_Y Y'$ is an isomorphism above $Y' \setminus \Sigma_{Y'}$, where $\Sigma_{Y'} = \varrho^{-1}(\Sigma_Y)$. Let H'_0 be the variation of Hodge structure on $Y' \setminus \Sigma_{Y'}$ induced by f', and let H' be its canonical extension to Y'. Since H_0 has unipotent local monodromies around the components of Σ_Y, the canonical extension is compatible with base change [**12**]:

$$H' \xrightarrow{\sim} \varrho^* H.$$

This isomorphism preserves the extensions of the Hodge filtration, hence it induces an isomorphism $\varrho^*(f_*\omega_{X/Y}) \xrightarrow{\sim} f'_*\omega_{X'/Y'}$. q.e.d.

Theorem 4.5 ([8, 11]). *Let $f\colon X \to Y$ be a contraction from a non-singular projective variety X to a projective curve Y, and let E be a quotient locally free sheaf of $f_*\omega_{X/Y}$. If $\deg(\det(E)) = 0$, then $\det(E)^{\otimes m} \simeq \mathcal{O}_Y$ for some positive integer m.*

Sketch of proof. By [8], E is a local system which is a direct summand of $f_*\omega_{X/Y}$. Since $E|_{Y_0}$ is a local subsystem of the variation of Hodge structure H_0, $\det(E|_{Y_0})^{\otimes m} \simeq \mathcal{O}_{Y_0}$ for some positive integer m [4]. By flatness, $\det(E)^{\otimes m} \simeq \mathcal{O}_Y$. q.e.d.

5. An auxiliary relative 0-log pair

We prove Theorems 0.1 and 2.7 in this section. The following finite base change formula ([1], Theorem 3.2) is essential:

Lemma 5.1. *Consider a commutative diagram of normal varieties*

$$\begin{array}{ccc} (X,B) & \xleftarrow{\;\nu\;} & (X',B_{X'}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xleftarrow[\;\tau\;]{} & Y' \end{array}$$

with the following properties:

(1) *(X,B) is a log pair with log canonical singularities over the generic point of Y;*
(2) *τ is a finite morphism, ν is generically finite, and f, f' are proper surjective;*
(3) $\nu^*(K+B) = K_{X'} + B_{X'}$.

Let B_Y and $B_{Y'}$ be the discriminants of $K+B$ and $K_{X'}+B_{X'}$ on Y and Y' respectively. Then, $\tau^*(K_Y+B_Y) = K_{Y'}+B_{Y'}$ (pull back of $\mathbb{Q}$-Weil divisors under a finite morphism).

The category of lc-trivial fibrations is closed under generically finite base changes. In order to *normalize* the discriminant B_Y and the moduli part M_Y, we have to replace the generic fibre of X/Y by a generically finite cover. The property rank $f_*\mathcal{O}_X(\lceil \mathbf{A}(X,B) \rceil) = 1$ is not invariant under this operation. Thus, we will consider an auxiliary fibre space (cf. [**12**, **19**]). Throughout this section, we consider the following *set-up*:

$$\begin{array}{ccccc} (X,B) & \xleftarrow{\pi} & \tilde{X} & \overset{d}{\dashleftarrow} & (V,B_V) \\ {\scriptstyle f}\downarrow & \swarrow_{\tilde{f}} & & \swarrow_{h} & \\ Y & & & & \end{array}$$

where $f\colon (X,B)\to Y$ is an lc-trivial fibration, $b = b(F,B_F)$ and

$$K + B + \frac{1}{b}(\varphi) = f^*(K_Y + B_Y + M_Y)$$

$\pi\colon \tilde{X} \to X$ is the normalization of X in $k(X)(\sqrt[b]{\varphi})$ and $d\colon V \dashrightarrow \tilde{X}$ is a proper birational map from a non-singular variety V. The induced rational map $g\colon V \dashrightarrow X$ is generically finite, so there exists a unique log structure (V,B_V) such that $g\colon (V,B_V) \dashrightarrow (X,B)$ is crepant. We assume the following properties hold:

(i) X, V, Y are non-singular quasi-projective varieties endowed with simple normal crossings divisors $\Sigma_X, \Sigma_V, \Sigma_Y$ on X, V and Y, respectively;
(ii) f and h are projective morphisms;
(iii) f and h are smooth over $Y \setminus \Sigma_Y$, and Σ_X^h/Y and Σ_V^h/Y have relative simple normal crossings over $Y \setminus \Sigma_Y$;
(iv) $f^{-1}(\Sigma_Y) \subseteq \Sigma_X$, $f(\Sigma_X^v) \subseteq \Sigma_Y$ and $h^{-1}(\Sigma_Y) \subseteq \Sigma_V$, $h(\Sigma_V^v) \subseteq \Sigma_Y$;
(v) B, B_V and B_Y, M_Y are supported by Σ_X, Σ_V and Σ_Y, respectively.

In this context, the properties (1) and (2) in the definition of the lc-trivial fibration $f\colon (X,B)\to Y$ are equivalent to

(vi) $\lceil -B_F \rceil$ is an effective divisor and $\dim_k H^0(F, \lceil -B_F \rceil) = 1$.

Lemma 5.2. *The following properties hold for the above set-up:*

(1) *The extension $k(V)/k(X)$ is Galois and its Galois group G is cyclic of order b. There exists $\psi \in k(V)^\times$ such that $\psi^b = \varphi$ and a generator of G acts by $\psi \mapsto \zeta\psi$, where $\zeta \in k$ is a fixed primitive b^{th}-root of unity;*
(2) *The relative log pair $h\colon (V,B_V)\to Y$ satisfies all properties of an lc-trivial fibration, except that* $\operatorname{rank} f_*\mathcal{O}_X(\lceil \mathbf{A}(V,B_V) \rceil)$ *might be bigger than one;*
(3) *Both $f\colon (X,B)\to Y$ and $h\colon (V,B_V)\to Y$ induce the same discriminant and moduli part on Y;*

(4) *The group G acts naturally on $h_*\mathcal{O}_V(K_{V/Y})$. The eigensheaf corresponding to the eigenvalue ζ is $\mathcal{L} := f_*\mathcal{O}_X(\lceil -B + f^*B_Y + f^*M_Y\rceil)\cdot\psi$.*

(5) *Assume that $h\colon V \to Y$ is semi-stable in codimension one. Then M_Y is an integral divisor, $\mathcal{L}$ is semi-positive and $\mathcal{L} = \mathcal{O}_Y(M_Y)\cdot\psi$.*

Proof.

(2) We have $K_V + B_V + (\psi) = h^*(K_Y + B_Y + M_Y)$, and clearly (V, B_V) has Kawamata log terminal singularities over the generic point of Y. The generic fibre H of h is a non-singular birational model of the normalization of $k(F)$ in $k(F)(\sqrt[b]{\varphi|_F})$. Since b is minimal with $b(K_F + B_F) \sim 0$, H is connected. Therefore, $\mathcal{O}_Y = h_*\mathcal{O}_V$, i.e., h is a contraction.

(3) It follows from (2) and Lemma 5.1. Note that the assumption $\operatorname{rank} f_*\mathcal{O}_X(\lceil \mathbf{A}(V, B_V)\rceil) = 1$ is not required in the definition of the discriminant and moduli part.

(4) The group G acts on $\tilde{f}_*\omega_{\tilde{X}/Y}$, with eigensheaf decomposition

$$\tilde{f}_*\omega_{\tilde{X}/Y} = \bigoplus_{i=0}^{b-1} f_*\mathcal{O}_X(\lceil (1-i)K_{X/Y} - iB + if^*B_Y + if^*M_Y\rceil)\cdot\psi^i$$

Since $B - f^*(B_Y + M_Y)$ is supported by the simple normal crossings divisor Σ_X, $\tilde{X}$ has rational singularities. In particular, $h_*\omega_{V/Y} = \tilde{f}_*\omega_{\tilde{X}/Y}$ is independent of the choice of V.

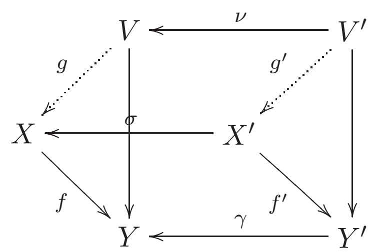
(5) By the semi-stable assumption, there exists a big open subset $Y^\dagger \subseteq Y$ such that $(-B_V + h^*B_Y)|_{h^{-1}(Y^\dagger)}$ is effective and supports no fibres of h. Since $(\psi|_H) + K_H = -B_H \geq 0$, ψ is a rational section of $h_*\mathcal{O}_V(K_{V/Y})$. Furthermore, $\psi \mapsto \zeta\psi$ implies that ψ is a rational section of $\mathcal{L}$. Therefore, $\mathcal{L} \subseteq k(Y)\psi$, since $\mathcal{L}$ has rank one by (vi) and (4). We have $(h^*a \cdot \psi) + K_{V/Y} = h^*((a) + M_Y) + (-B_V + h^*B_Y)$.

Since $-B_V + h^*B_Y$ is effective over $Y^\dagger$, we infer that $\mathcal{O}_Y(M_Y)\psi|_{Y^\dagger} \subseteq h_*\mathcal{O}_V(K_{V/Y})|_{Y^\dagger}$. Therefore, $\mathcal{O}_Y(M_Y)\psi|_{Y^\dagger} \subseteq \mathcal{L}|_{Y^\dagger}$. Conversely, let $h^*a \cdot \psi$ be a section of $\mathcal{L}$. Then, $h^*a \cdot \psi$ is a section of $h_*\mathcal{O}_V(K_{V/Y})$, i.e., $(h^*a \cdot \psi) + K_{V/Y} \geq 0$. Since $-B_V + h^*B_Y$ contains no fibres over codimension one points of Y, this implies $(a) + M_Y \geq 0$. In particular, $\mathcal{L} \subseteq \mathcal{O}_Y(M_Y)\psi$. Therefore, $\mathcal{O}_Y(M_Y)\psi|_{Y^\dagger} = \mathcal{L}|_{Y^\dagger}$. Since $Y^\dagger \subseteq Y$ is a big open subset, this implies $\mathcal{L}^{**} = \mathcal{O}_Y(M_Y)\psi$. By Theorem 4.4,

$h_*\mathcal{O}_V(K_{V/Y})$ is locally free and semi-positive. Its direct summand $\mathcal{L}$ is locally free and semi-positive as well, hence the conclusion.

Finally, for each prime divisor P of Y there exists a prime divisor Q of X such that $h(Q) = P$ and $\operatorname{mult}_Q(-B_V + h^*B_Y) = 0$. We infer from (2) that $\operatorname{mult}_Q h^*(M_Y) = 1$. But $\operatorname{mult}_Q h^*(P) = 1$, hence M_Y is an integral Weil divisor. q.e.d.

Remark 5.3. Let $\gamma\colon Y' \to Y$ be a generically finite morphism from a non-singular quasi-projective variety Y'. Assume there exists a simple normal crossings divisor $\Sigma_{Y'}$ which contains $\gamma^{-1}(\Sigma_Y)$ and the locus where γ is not étale. By base change, there exists a commutative diagram



such that $(V', B_{V'}) \dashrightarrow (X', B_{X'}) \to Y'$ satisfies the same properties (i)–(v). Here, $B_{X'}, B_{V'}$ are induced by crepant pull back, $\Sigma_{X'} \supseteq \sigma^{-1}(\Sigma_X), \Sigma_{V'} \supseteq \nu^{-1}(\Sigma_V)$ and $\varphi' = \sigma^*\varphi \in k(X')^\times$. We say that the set-up $(V', B_{V'}) \dashrightarrow (X', B_{X'}) \to Y'$ *is induced by* $(V, B_V) \to (X, B) \to Y$ *via the base change* $\gamma\colon Y' \to Y$.

Proposition 5.4. *There exists a finite Galois cover* $\tau\colon Y' \to Y$ *from a non-singular variety* Y' *which admits a simple normal crossings divisor supporting* $\tau^{-1}(\Sigma_Y)$ *and the locus where* τ *is not étale, and such that* $h'\colon V' \to Y'$ *is semi-stable in codimension one for some set-up* $(V', B_{V'}) \dashrightarrow (X', B_{X'}) \to Y'$ *induced by base change.*

Proof. Let N be the positive integer associated to $V \to Y$ by Theorem 4.3. By Theorem 4.1, there exists a finite Galois cover $\tau\colon Y' \to Y$ such that $\tau^*(\Sigma_Y)$ is divisible by N and there exists a simple normal crossings divisor $\Sigma_{Y'}$ containing $\tau^{-1}(\Sigma_Y)$ and the locus where τ is not étale. By Theorem 4.3, there exists an induced set-up $(V', B_{V'}) \dashrightarrow (X', B_{X'}) \to Y'$ induced by base change, so that $h'\colon V' \to Y'$ is semi-stable in codimension one. q.e.d.

Proposition 5.5. *Let $\gamma\colon Y' \to Y$ be a generically finite projective morphism from a non-singular variety Y'. Assume there exists a simple normal crossings divisor $\Sigma_{Y'}$ on Y' which contains $\gamma^{-1}(\Sigma_Y)$, and the locus where γ is not étale. Let $M_{Y'}$ be the moduli part of the induced set-up $(V', B_{V'}) \dashrightarrow (X', B_{X'}) \to Y'$. Then, $\gamma^*(M_Y) = M_{Y'}$.*

Proof.

Step 1: Assume that V/Y and V'/Y' are semi-stable in codimension one. In particular, M_Y and $M_{Y'}$ are integral divisors. Since h is semi-stable in codimension one, Theorem 4.4 implies

$$h'_*\mathcal{O}_{V'}(K_{V'/Y'}) \xrightarrow{\sim} \gamma^*(h_*\mathcal{O}_V(K_{V/Y})).$$

This isomorphism is natural, hence compatible with the action of the Galois group G. We have an induced isomorphism of eigensheaves corresponding to ζ: $\gamma^*\mathcal{O}_Y(M_Y) \simeq \mathcal{O}_{Y'}(M_{Y'})$. Therefore, $\gamma^* M_Y - M_{Y'}$ is linearly trivial, and is exceptional over Y. Thus, $\gamma^* M_Y = M_{Y'}$.

Step 2: By Theorems 4.3 and 4.1, we can construct a commutative diagram

$$\begin{array}{ccc} \bar{Y} & \xleftarrow{\gamma'} & \bar{Y}' \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau'} \\ Y & \xleftarrow{\gamma} & Y' \end{array}$$

as in Remark 4.2, so that $\bar{V}/\bar{Y}$ is semi-stable in codimension one for an induced set-up $(\bar{V}, B_{\bar{V}}) \dashrightarrow (\bar{X}, B_{\bar{X}}) \to \bar{Y}$.

By Theorems 4.3 and 4.1, we replace $\bar{Y}'$ by a finite covering so that $\bar{V}'/\bar{Y}'$ is semi-stable in codimension one for an induced set-up $(\bar{V}', B_{\bar{V}'}) \dashrightarrow (\bar{X}', B_{\bar{X}'}) \to \bar{Y}'$. By Step 1, we have $M_{\bar{Y}'} = \gamma'^*(M_{\bar{Y}})$. Since τ and τ' are finite coverings, Lemma 5.1 implies $\tau^*(M_Y) = M_{\bar{Y}}$ and $\tau'^*(M_{Y'}) = M_{\bar{Y}'}$. Therefore, $\tau'^*(M_{Y'} - \gamma^*(M_Y)) = 0$, which implies $M_{Y'} = \gamma^*(M_Y)$. q.e.d.

Proof of Theorem 2.7. Let $f\colon (X, B) \to Y$ be an lc-trivial fibration with $b = b(F, B_F)$ and

$$K + B + \frac{1}{b}(\varphi) = f^*(K_Y + B_Y + M_Y).$$

We replace X by a resolution, so that X is non-singular, quasi-projective, and $B - f^*(B_Y + M_Y)$ is supported by a simple normal crossings divisor Σ_X. Let V be a resolution of the normalization of X in

$k(X)(\varphi^{1/b})$ such that B_V has simple normal crossings support. We may assume that f, h are projective morphisms, after a birational base change. Then, there exists a closed subvariety $\Sigma_f \subsetneq Y$ such that $(V, B_V) \dashrightarrow (X, B) \to Y$ satisfies the assumptions of the set-up in the beginning of this section, except that Σ_f may not be the support of a simple normal crossings divisor. Let $\sigma\colon Y' \to Y$ be an embedded resolution so that $\Sigma_{Y'} := \sigma^{-1}(\Sigma_f)$ is a divisor with simple normal crossings. There exists an induced set-up $(V', B_{V'}) \dashrightarrow (X', B') \to Y'$.

We claim that $\sigma^*(\mathbf{M}_{Y'}) = \mathbf{M}_{Y''}$ and $\sigma^*(K_{Y'} + \mathbf{B}_{Y'}) = K_{Y''} + \mathbf{B}_{Y''}$ for every birational contraction $\sigma\colon Y'' \to Y'$. By Hironaka's resolution of singularities, there exists a diagram of birational morphisms

$$\begin{array}{ccc} Y'' & \longleftarrow & Y''' \\ \sigma \big\downarrow & \swarrow \sigma' & \\ Y' & & \end{array}$$

such that Y''' is a non-singular quasi-projective variety admitting a simple normal crossings divisor which supports ${\sigma'}^{-1}(\Sigma_{Y'})$ and the exceptional locus of Y'''/Y'. By Proposition 5.5, ${\sigma'}^*(\mathbf{M}_{Y'}) = \mathbf{M}_{Y'''}$ and consequently, ${\sigma'}^*(K_{Y'} + \mathbf{B}_{Y'}) = K_{Y'''} + \mathbf{B}_{Y'''}$. Since Y'''/Y'' is a birational morphism, the claim follows.

Let $\tau : \bar{Y}' \to Y'$ be a covering given by Proposition 5.4. By Lemma 5.2, $M_{\bar{Y}'}$ is a Cartier divisor and $\mathcal{O}_{\bar{Y}'}(M_{\bar{Y}'})$ is a semi-positive invertible sheaf. In particular, $M_{\bar{Y}'}$ is nef$/S$, but $\tau^*(M_{Y'}) = M_{\bar{Y}'}$ according to Lemma 5.1. Hence, $M_{Y'}$ is nef$/S$. q.e.d.

Proof of Theorem 0.1. By the Lefschetz principle, we may assume $k = \mathbb{C}$. After a finite base change (Lemma 5.1), we may assume that the induced root fiber space $h\colon V \to Y$ is *semi-stable*. By construction, the invertible sheaf $\mathcal{L} := \mathcal{O}_Y(M_Y) \subset h_*\omega_{V/Y}$ is a direct summand.

We know that M_Y is a nef Cartier divisor on the curve Y. If $\deg(M_Y) > 0$, then M_Y is ample, in particular, semi-ample. If $\deg(M_Y) = 0$, Theorem 4.5 implies $\mathcal{L}^{\otimes m} \simeq \mathcal{O}_Y$. Therefore, M_Y is semi-ample. q.e.d.

6. Asymptotically saturated algebras

We first recall some terminology from [**22**]. Let $\pi\colon X \to S$ be a proper morphism. A *normal functional algebra* of X/S is an $\mathcal{O}_S$-algebra of the form

$$\mathcal{L} = \mathcal{R}_{X/S}(\mathbf{M}_\bullet) = \bigoplus_{i=0}^{\infty} \pi_* \mathcal{O}_X(\mathbf{M}_i),$$

where $\{\mathbf{M}_i\}$ is a sequence of b-free/S b-divisors of X such that $\mathbf{M}_i + \mathbf{M}_j \le \mathbf{M}_{i+j}$ for every i and j. The sequence of $\mathbb{Q}$-b-divisors $\mathbf{D}_i = \frac{1}{i}\mathbf{M}_i$ is called the *characteristic sequence* of $\mathcal{L}$. The algebra $\mathcal{L}$ is *bounded* if there exists an $\mathbb{R}$-b-divisor $\mathbf{D}$ of X such that $\mathbf{D}_i \le \mathbf{D}$ for every i. The $\mathcal{O}_S$-algebra $\mathcal{L}$ is finitely generated if and only if the characteristic sequence $\mathbf{D}_\bullet$ is constant up to a truncation ([**22**], Theorem 4.28). For an $\mathbb{R}$-b-divisor $\mathbf{A}$, the algebra $\mathcal{L}$ is *asymptotically* $\mathbf{A}$*-saturated*, if there exists a positive integer I such that

$$\pi_* \mathcal{O}_X(\lceil \mathbf{A} + j\mathbf{D}_i \rceil) \subseteq \pi_* \mathcal{O}_X(\mathbf{M}_j) \text{ for } I|i, j.$$

The *Kodaira dimension* of $\mathcal{L}$ is $\kappa(\mathcal{L}) := \max_i \kappa(X/S, \mathbf{M}_i)$. We say that $\mathcal{L}$ is a *big algebra* if $\kappa(\mathcal{L}) = \dim(X/S)$.

Definition 6.1. Let $\mathcal{L}$ be a normal functional algebra of X/S. There exists a unique rational map with connected fibers $f \colon X \dashrightarrow Y/S$ and a big normal functional algebra $\mathcal{L}'$ of Y/S, such that $f^* \colon k(Y) \to k(X)$ induces a quasi-isomorphism of $\mathcal{O}_S$-algebras

$$f^* \colon \mathcal{L}' \to \mathcal{L}.$$

We say that $(f, \mathcal{L}')$ is the *Iitaka fibration* of $\mathcal{L}$.

Proof. (cf. [**22**], Lemma 6.22) Let $\mathcal{L} = \mathcal{R}_{X/S}(\mathbf{M}_\bullet)$. Since $\mathbf{M}_i + \mathbf{M}_j \le \mathbf{M}_{i+j}$ and the $\mathbf{M}_i$'s are b-free, there exists $I \in \mathbb{N}$ and a rational map $f \colon X \dashrightarrow Y/S$ which is the Iitaka contraction of $\mathbf{M}_i$ for every i divisible by I. Up to a quasi-isomorphism, we may assume that the b-free b-divisors $\mathbf{M}_i$ are effective. Since f has connected fibers, there exists a convex sequence $\mathbf{M}'_\bullet$ such that $\mathbf{M}_i = f^*(\mathbf{M}'_i)$ for every $I|i$. In particular, $\mathcal{L}$ is quasi-isomorphic to the big algebra $\mathcal{L}' := \mathcal{R}_{Y/S}(\mathbf{M}'_\bullet)$. q.e.d.

Lemma 6.2. *Let* $(f \colon X \dashrightarrow Y/S, \mathcal{L}')$ *be the Iitaka fibration of a normal functional algebra* $\mathcal{L}$*. If* $\mathcal{L}$ *is asymptotically* $\mathbf{A}$*-saturated, then* $\operatorname{rank} f'_* \mathcal{O}_{X'}(\lceil \mathbf{A} \rceil) \le 1$*, where* $f' \colon X' \to Y$ *is a regular representative of the rational map* f*.*

Proof. We may assume that $f' = f$ and $\mathcal{L} = \mathcal{R}_{X/S}(f^* \mathbf{M}'_\bullet)$, where $\mathcal{L}' = \mathcal{R}_{Y/S}(\mathbf{M}'_\bullet)$ is the induced big algebra. By assumption, there exists i such that $\mathbf{D}_i$ is b-big/S. After passing to higher models, we may assume that $\mathbf{D}_i$ descends to Y. There exists a birational contraction $\mu\colon Y \to Z/S$ and an ample/S $\mathbb{Q}$-divisor H on Z such that $(\mathbf{D}_i)_Y \sim_{\mathbb{Q}} \mu^*H$. For j sufficiently large and divisible, the $\mathcal{O}_Z$-sheaf

$$\mu_*(f_*\mathcal{O}_X(\lceil \mathbf{A} + jf^*\mathbf{D}_i \rceil)) = \mu_* f_*\mathcal{O}_X(\lceil \mathbf{A} \rceil) \otimes \mathcal{O}_Z(jH)$$

is π-generated. Therefore, $f_*\mathcal{O}_X(\lceil \mathbf{A}+jf^*\mathbf{D}_i \rceil)$ is generically π-generated. Asymptotic saturation implies that $f_*\mathcal{O}_X(\lceil \mathbf{A} + jf^*\mathbf{D}_i \rceil)$ is contained in the b-divisorial sheaf $\mathcal{O}_Y(\mathbf{M}_j)$ on an open subset of Y. The latter has rank one, and hence, $f_*\mathcal{O}_X(\lceil \mathbf{A} \rceil)$ has rank at most one. q.e.d.

Proposition 6.3 (cf. [**22**], Proposition 4.50)**.** *Consider a commutative diagram*

$$\begin{array}{ccc} (X, B) & \xrightarrow{\;f\;} & Y \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\sigma} & \\ & S & \end{array}$$

and a normal functional algebra $\mathcal{L} = \mathcal{R}_{X/S}(\mathbf{M}_\bullet)$ *with the following properties:*

(a) *$f\colon (X, B) \to Y$ is an lc-trivial fibration;*
(b) *$\mathcal{L}$ is bounded and asymptotically $\mathbf{A}(X, B)$-saturated;*
(c) *There exist b-divisors $\mathbf{M}'_i$ of Y such that $\mathbf{M}_i = f^*(\mathbf{M}'_i)$ for all i.*

Then, $\mathcal{L}' := \mathcal{R}_{Y/S}(\mathbf{M}'_\bullet)$ is a normal bounded functional algebra of Y/S, which is asymptotically $\mathbf{A}_{\mathrm{div}}$-saturated. Moreover, the natural map $f^*\colon \mathcal{L}' \to \mathcal{L}$ is an isomorphism of $\mathcal{O}_S$-algebras.

Proof. It is clear that $\mathcal{L}'$ is a functional algebra of Y/S, and $f^*\colon \mathcal{L}' \to \mathcal{L}$ is an isomorphism of $\mathcal{O}_S$-algebras. The algebra is normal since each $\mathbf{M}'_i$ is b-free. Let $\mathbf{D}'_i = \frac{1}{i}\mathbf{M}'_i$ be the characteristic sequence of $\mathcal{L}'$.

We first check that $\mathcal{L}'$ is bounded. After passing to higher models, we may assume that there exists an effective Cartier divisor E on X such that $f^*\mathbf{D}'_i = \mathbf{D}_i \le \overline{E}$. Let E' be the divisorial support of $f(\mathrm{Supp}(E^v)) \subset Y$. For each i, we can find a birational model X'/Y' of X/Y, fitting in the commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{\;h\;} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \longleftarrow & Y' \end{array}$$

such that $\mathbf{D}'_i$ and $\mathbf{D}_i$ descend on Y' and X', respectively. In particular, $f'^*((\mathbf{D}'_i)_{Y'}) \le h^*E$. Since $\mathbf{D}'_i$ is effective and Y'/Y is an isomorphism over a big open subset of Y, we conclude that $(\mathbf{D}'_i)_Y$ is supported by E'. This holds for every i, and hence, $\mathbf{D}'_\bullet$ is bounded.

It remains to check asymptotic $\mathbf{A}_{\mathrm{div}}$-saturation. Fix two positive integers i, j which are divisible by I. By Theorem 2.7, we may assume the following properties hold (after a birational base change):

(i) X, Y are non-singular;
(ii) $\mathbf{D}'_i, \mathbf{D}'_j$ descend to Y (in particular $\mathbf{D}_i, \mathbf{D}_j$ descend to X). Denote $D'_i = (\mathbf{D}'_i)_Y$ and $D'_j = (\mathbf{D}'_j)_Y$;
(iii) $\mathrm{Supp}(B) \cup \mathrm{Supp}(f^*D'_i)$ and $\mathrm{Supp}(\mathbf{B}_Y) \cup \mathrm{Supp}(D'_i)$ are simple normal crossings divisors on X and Y, respectively;
(iv) $\mathbf{A}_{\mathrm{div}} = \mathbf{A}(Y, \mathbf{B}_Y)$.

Under these assumptions, the asymptotic saturation for i, j means

$$\pi_*\mathcal{O}_X(\lceil -B + jf^*D'_i \rceil) \subseteq \pi_*\mathcal{O}_X(jf^*D'_j).$$

By Lemma 3.2 and Remark 3.3, we have

$$f_*\mathcal{O}_X(\lceil -B + jf^*D'_i \rceil) = \mathcal{O}_Y(\lceil -\mathbf{B}_Y + jD'_i \rceil).$$

Since $\pi_*\mathcal{O}_X(jf^*D'_j) = \sigma_*\mathcal{O}_Y(\mathbf{M}'_j)$, we infer

$$\sigma_*\mathcal{O}_Y(\lceil -\mathbf{B}_Y + jD'_i \rceil) \subseteq \sigma_*\mathcal{O}_Y(jD'_j).$$

Therefore, $\sigma_*\mathcal{O}_Y(\lceil \mathbf{A}_{\mathrm{div}} + j\mathbf{D}'_i \rceil) \subseteq \sigma_*\mathcal{O}_{Y'}(\mathbf{M}'_j)$, by (i)–(iv) again. q.e.d.

Example 6.4 (Reduction to big algebras)**.** Let $(X/S, B)$ be a relative log pair, and let $\mathcal{L}$ be a normal bounded functional algebra with Iitaka fibration $(f\colon X \dashrightarrow Y/S, \mathcal{L}')$, satisfying the following properties:

(i) $K_{X'} + B_{X'} \sim_{\mathbb{Q}} f'^*D$, where $f'\colon X' \to Y$ is a regular model of the rational map f and $B_{X'}$ is a crepant boundary ($\mathbf{A}(X, B) = \mathbf{A}(X', B_{X'})$);
(ii) $(X', B_{X'})$ has klt singularities over the generic point of Y;
(iii) $\mathcal{L}$ is asymptotically $\mathbf{A}(X, B)$-saturated.

By Lemma 6.2, $f'\colon (X', B_{X'}) \to Y$ is an lc-trivial fibration. Let $\mathbf{B}, \mathbf{M}$ be the induced boundary and moduli $\mathbb{Q}$-b-divisors of Y, respectively. By Theorem 2.7 and Proposition 6.3, we may replace Y by a higher birational model so that the following properties hold:

(a) $\mathbf{A}_{\mathrm{div}} = \mathbf{A}(Y, \mathbf{B}_Y)$;
(b) $K_{X'} + B_{X'} \sim_{\mathbb{Q}} f'^*(K_Y + \mathbf{B}_Y + \mathbf{M}_Y)$;

(c) $\mathcal{L}'$ is normal, bounded and asymptotically $\mathbf{A}(Y, \mathbf{B}_Y)$-saturated;

The above example is a first step towards a reduction of 0LP algebras ([22], Remark 4.40) to the big case. To complete the reduction, we need to know that the moduli b-divisor $\mathbf{M}$ is b-semi-ample. However, the b-nef property of the moduli b-divisor is enough for some applications to the Fano case. We show that the restriction of an FGA algebra to an exceptional log canonical centre is again an FGA algebra (cf. [22], Proposition 4.50 for lc centers of codimension one)

Theorem 6.5. *Let $(X/S, B)$ be a relative generalized Fano log variety, let $\nu\colon W \to X$ be the normalization of an exceptional lc centre of (X, B) and let $\mathcal{L} = \mathcal{R}_{X/S}(\mathbf{M}_\bullet)$ be a normal bounded functional algebra of X/S such that the following hold:*

(i) *$\mathcal{L}$ is asymptotically $(\mathbf{A}(X, B) + E)$-saturated, where E is the unique lc place over W;*
(ii) *There exists an open subset $U \subseteq X$ such that $U \cap \nu(W) \neq \emptyset$, $\mathbf{D}_i|_U = \overline{D}|_U$ $\forall i$ for some $\mathbb{Q}$-Cartier divisor D on X;*
(ii*) *U contains $(X, B)_{-\infty} \cap \nu(W)$ and $C \cap \nu(W)$, for every lc centre $C \neq \nu(W)$ of (X, B).*

Then, there exists a well defined restricted algebra *$\mathcal{L}_{\restriction W}$ of W/S, with the following properties:*

(1) *$\mathcal{L}_{\restriction W} = \mathcal{R}_{W/S}(\mathbf{M}'_i)$ is a normal, bounded functional algebra.*
(2) *$\mathcal{L}_{\restriction W}$ is $\mathbf{A}(W, B_W)$-saturated, where $(W/S, B_W)$ is a relative generalized Fano log variety;*
(3) *$\mathrm{LCS}(W, B_W) \subset U' := U|_W$ and $\mathbf{D}'_i|_{U'} = \overline{D|_{U'}}$ for every i;*
(4) *The $\mathcal{O}_S$-algebras $\mathcal{L}_{\restriction W}$ and $\mathcal{L}_{\restriction E}$ are quasi-isomorphic.*

Proof. Let H be an ample/S $\mathbb{Q}$-divisor on X such that $-(K+B+H)$ is ample/S. As in [3], Theorem 4.9, we construct an effective $\mathbb{Q}$-divisor B_W on W such that $(W/S, B_W)$ is a relative generalized Fano log variety with $(K + B + H)|_W \sim_{\mathbb{Q}} K_W + B_W$ and $\mathrm{LCS}(W, B_W)$ is contained in the union of $(X, B)_{-\infty}$ and all lc centres of (X, B) different than $\nu(W)$. In particular, $\mathrm{LCS}(W, B_W) \subset U'$. Consider the induced diagram

$$\begin{array}{ccc} (E, B_E) & \subset & (X', B_{X'}) \\ \downarrow & & \downarrow \\ W & \to & (X, B) \end{array}$$

By adjunction and Kawamata–Viehweg vanishing, $\mathcal{L}_{\restriction E}$ is asymptotically $\mathbf{A}(E, B_E)$-saturated ([22], Proposition 4.50). By (ii), there exist b-free/S b-divisors $\mathbf{M}'_i$ of W such that $\mathbf{M}_{i}{}_{\restriction E} = h^*(\mathbf{M}'_i)$ for every i. By construction, $(E, B_E) \to W/S$ is an lc-trivial fibration for which Proposition 6.3 applies. Therefore, $\mathcal{L}_{\restriction W} := \mathcal{R}_{W/S}(\mathbf{M}'_i)$ is quasi-isomorphic to $\mathcal{L}_{\restriction E}$, it is normal, bounded and asymptotically $\mathbf{A}_{\mathrm{div}}$-saturated. From the construction of B_W (choosing W' high enough so that $\mathbf{A}_{\mathrm{div}} = \mathbf{A}(W', \mathbf{B}_{W'})$, in the proof of [3], Theorem 4.9), we have $\mathbf{A}(W, B_W) \le \mathbf{A}_{\mathrm{div}}$. Therefore, $\mathcal{L}_{\restriction W}$ is asymptotically $\mathbf{A}(W, B_W)$-saturated.

Finally, $\mathbf{D}_i|_U = \overline{D|_U}$ implies $\mathbf{D}'_i|_{U'} = \overline{D|_{U'}}$. q.e.d.

7. Acknowledgments

I would like to thank Professors Yujiro Kawamata, Miles Reid and Vyacheslav V. Shokurov for useful discussions.

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