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computed with $Q(x_i,x_j)$ becomes small. Finally, since

$$W(P^t) = \sum_1^n I(x_{m_i}, x_{m_{j(i)}})$$

$|W(P^t) - W(Q^t)|$ becomes small as

$$\max_x |P(x) - Q(x)|$$

becomes small. As a consequence of (5) it then follows that

$$\max_{t \in T} |W(P_s^t) - W(P^t)| \xrightarrow{s} 0 \text{ with probability 1.} \quad (6)$$

The implication of (6) is simply that for all s sufficiently large we will, with probability 1, always pick a tree in T' if we choose $t(s)$ such that $W(P_s^t)$ is maximum. Using the theorem of Chow and Liu quoted earlier with (6), (3) now follows.

The same ideas as outlined above also yield the following statement. If P is an arbitrary distribution and $P_s^{t(s)}$ is picked as before, then

$$W(P_s^{t(s)}) \xrightarrow{s} \max_{t \in T} W(P^t) \text{ with probability 1}$$

even though $P_s^{t(s)}$ itself may not converge.

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Short Convolutional Codes With Maximal Free Distance for Rates $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$

KNUD J. LARSEN

***Abstract*—This paper gives a tabulation of binary convolutional codes with maximum free distance for rates $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ for all constraint lengths (measured in information digits) ν up to and including $\nu = 14$. These codes should be of practical interest in connection with Viterbi decoders.**

A binary convolutional code of rate $R = 1/n$ and constraint length ν, measured in information digits, is specified by its code generating polynomials

$$G^{(i)}(D) = 1 + g_1^{(i)}D + g_2^{(i)}D^2 + \cdots + g_{\nu-1}^{(i)}D^{\nu-1}$$

for $1 \le i \le n$ where each $g_j^{(i)}$ is a binary digit. It is now well known that the Viterbi decoding algorithm is the maximum-likelihood decoding rule for the *trellis* defined by such a code [1] and that surprisingly good performance on memoryless channels such as the deep-space channel can be obtained for codes with small enough ν, say $\nu \le 10$, so that the Viterbi decoder could actually be implemented [2]. It is also well known [2]–[4] that the free distance d_{free} of the convolutional code is the appropriate criterion of goodness for the convolutional code used with Viterbi decoding.

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TABLE I
RATE $\frac{1}{2}$ CODES WITH MAXIMUM FREE DISTANCE

A. Noncatastrophic Codes

ν	N	generators(octal)		d_{free}	bound
3	6	5	7[1]	5	5
4	8	15	17[1]	6	6
5	10	23	35[1]	7	8
6	12	53	75[1]	8	8
7	14	133	171[1]	10	10
8	16	247	371[1]	10	11
9	18	561	753[1]	12	12
10	20	1167	1545	12	13
11	22	2335	3661	14	14
12	24	4335	5723	15	16
13	26	10533	17661	16	16
14	28	21675	27123	16	17

B. Catastrophic Codes

ν	N	generators(octal)		d_{free}	bound
5	10	27	35	8	8
12	24	5237	6731	16	16
14	28	21645	37133	17	17

[1] This code was found by Odenwalder [4] and is listed here for completeness.

The rates of most practical interest for Viterbi decoding on memoryless channels are $R = \frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$. $R = \frac{1}{2}$ codes with maximal d_{free} are already known for $\nu \le 9$ [4] and $R = \frac{1}{2}$ codes with maximal d_{free} are known for $\nu \le 24$ [5]. $R = \frac{1}{3}$ codes with maximal d_{free} are known for $\nu \le 8$ [4] and with nearly maximal d_{free} for $\nu \le 28$ [6]. The best $R = \frac{1}{4}$ codes reported are repetitions of the Bahl–Jelinek $R = \frac{1}{2}$ codes [5], i.e., $G^{(3)}(D) = G^{(1)}(D)$ and $G^{(4)}(D) = G^{(2)}(D)$. In this correspondence we report rate $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ codes with maximal d_{free} for $\nu \le 14$.

The newly found codes, together with some previously known codes with maximal d_{free} for rates $R = \frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ are listed in Tables I, II, and III, respectively, where we follow the usual practice of listing the generating polynomials by the octal form of the binary sequence $1, g_1^{(i)}, g_2^{(i)}, \cdots, g_{\nu-1}^{(i)}$, for $1 \le i \le n$. The number $N = \nu R^{-1}$ is the total constraint length. The optimality of d_{free} for these codes can be established from a simple upper bound, due to Heller [7],

$$d_{\text{free}} \le \min_{1 \le k} \left[\frac{n}{2} \frac{2^k}{2^k - 1} (\nu + k - 1) \right] \quad (1)$$

where [ ] denotes integer part of the enclosed expression. This bound can be improved [8] for some (n,ν) using the Griesmer bound for block codes [9]. The latter bound says that if d_0 is the minimum distance of an (N,k) binary linear code, and if $d_i = [(d_{i-1} + 1)/2]$, then $d_0 + d_1 + \cdots + d_{k-1} \le N$ [$= (\nu + k - 1)n$ in this case]. Thus by checking for every (n,ν) the bound (1) can in some cases be improved by one. The resulting upper bound is listed in the Tables I, II, and III.

From Tables II and III it is seen that optimal codes (in the sense of maximum d_{free}) achieving the bound for rates $R = \frac{1}{3}$ and $\frac{1}{4}$ were found for all constraint lengths (up to and including 14). These codes, all of which are noncatastrophic [10], were found by judicious choosing of the generating polynomials followed by a computer verification of their d_{free} using a corrected version of the algorithm given by Bahl *et al.* [11], [12].

TABLE II
RATE $\frac{1}{3}$ NONCATASTROPHIC CODES WITH MAXIMUM FREE DISTANCE

	N	generators (octal)			d_{free}	bound
3	9	5	7	7[1]	8	8
4	12	13	15	17[1]	10	10
5	15	25	33	37[1]	12	12
6	18	47	53	75[1]	13	13
7	21	133	145	175[2]	15	15
8	24	225	331	367[1]	16	16
9	27	557	663	711	18	18
10	30	1117	1365	1633	20	20
11	33	2353	2671	3175	22	22
12	36	4767	5723	6265	24	24
13	39	10533	10675	17661	24	24
14	42	21645	35661	37133	26	26

[1] This code was found by Odenwalder [4] and is listed here for completeness.

[2] This code was also found by Odenwalder [4], but was overlooked. The corresponding code in [4] has free distance only 14.

TABLE III
RATE $\frac{1}{4}$ NONCATASTROPHIC CODES WITH MAXIMUM FREE DISTANCE

ν	N	generators(octal)				d_{free}	bound
3	12	5	7	7	7	10	10
4	16	13	15	15	17	13	15
5	20	25	27	33	37	16	16
6	24	53	67	71	75	18	18
7	28	135	135	147	163	20	20
8	32	235	275	313	357	22	22
9	36	463	535	733	745	24	24
10	40	1117	1365	1633	1653	27	27
11	44	2387	2353	2671	3175	29	29
12	48	4767	5723	6265	7455	32	32
13	52	11145	12477	15573	16727	33	33
14	56	21113	23175	35527	35537	36	36

The noncatastrophic rate $\frac{1}{2}$ codes (Table I-A) are all optimal (i.e., maximum d_{free}) but some of them (ν = 5, 8, 10, 12, and 14) do not achieve the bound. The optimality is here established through a complete search covering all possibly optimal codes.

If we allow the codes to be catastrophic, which might be of interest in connection with framing of input data, we can find codes achieving the bound for ν = 5, 12, and 14, too, if the definition of d_{free} [3] is slightly modified: d_{free} is the weight of the minimum weight path in the trellis that diverges from the state **0** and later *reconverges* to this state; this reconvergence is not required in [3]. For a noncatastrophic code the two definitions are identical. The catastrophic codes for ν = 5, 12, and 14 are listed in Table I-B.

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