Short Cycles make W-hard problems hard: FPT algorithms for W-hard Problems in Graphs with no short Cycles *

Venkatesh Raman and Saket Saurabh

The Institute of Mathematical Sciences,
Chennai 600 113.
{vraman|saket}@imsc.res.in

Abstract. We show that several problems that are hard for various parameterized complexity classes on general graphs, become fixed parameter tractable on graphs with no small cycles.

More specifically, we give fixed parameter tractable algorithms for Dominating Set, t-Vertex Cover (where we need to cover at least t edges) and several of their variants on graphs with girth at least five. These problems are known to be $W[i]$-hard for some $i \geq 1$ in general graphs. We also show that the Dominating Set problem is $W[2]$-hard for bipartite graphs and hence for triangle free graphs.

In the case of Independent Set and several of its variants, we show these problems to be fixed parameter tractable even in triangle free graphs. In contrast, we show that the Dense Subgraph problem where one is interested in finding an induced subgraph on k vertices having at least l edges, paramaterized by k, is $W[1]$-hard even on graphs with girth at least six.

Finally, we give an $O(\log p)$ ratio approximation algorithm for the Dominating Set problem for graphs with girth at least 5, where p is the size of an optimum dominating set of the graph. This improves the previous $O(\log n)$ factor approximation algorithm for the problem, where n is the number of vertices of the input graph.

1 Introduction

Parameterized complexity is a practical approach to deal with intractable computational problems having some small parameters. For decision problems with input size n, and a parameter k (which typically, and in all the

* A preliminary version of this paper appeared as *Triangles, 4-cycles and Parameterized (In-)Tractability* in the proceedings of 10th Scandinavian Workshop on Algorithm Theory (SWAT 2006) [24].

problems we consider in this paper, is the solution size), the goal here is to design an algorithm with runtime $f(k)n^{O(1)}$ where f is a function of k alone, as contrasted with a trivial $n^{k+O(1)}$ algorithm. Problems having such an algorithm is said to be fixed parameter tractable (FPT), and such algorithms are practical when small parameters cover practical ranges. The book by Downey and Fellows [8] provides a good introduction to the topic of parameterized complexity. For recent developments see the books by Flum and Grohe [14] and Niedermeier [17].

There is a hierarchy of intractable parameterized problem classes above FPT, the main ones are:

$$FPT \subseteq M[1] \subseteq W[1] \subseteq M[2] \subseteq W[2] \subseteq \cdots \subseteq W[P] \subseteq XP.$$

The principal analogue of the classical intractability class NP is $W[1]$. A convenient source of $W[1]$-hardness reductions is provided by the result that Independent Set is complete for $W[1]$ [8]. Other highlights of the theory include that Dominating Set, by contrast, is complete for $W[2]$ [8]. Surprisingly we show that these problems and several of their variants that are known to be hard in the W-hierarchy, are fixed parameter tractable on graphs that have no short cycles – more specifically on graphs with girth at least five. These problems are known to be NP-complete on such graphs as well [4, 5]. We also look at the Set Cover problem where the size of the intersection of any pair of sets is bounded by a fixed constant. While the general version of Set Cover is known to be $W[2]$-complete, we prove this special version fixed parameter tractable.

Most of our algorithms are based on the method of *kernelization*. The main idea of *kernelization* is to replace a given instance (I, k) by a simpler instance (I', k') using some *data reduction rules* in polynomial time such that (I, k) is a *yes* instance if and only if (I', k') is a *yes* instance and $|I'|$ is bounded by a function of k alone. The reduced instance is called *kernel* for the problem. For most of our problems we give polynomial sized kernel in polynomial time.

1.1 Organization of the Rest of the Paper

In Section 2, we look at the Dominating Set problem and show that the problem is $W[2]$-complete even in bipartite graphs and split graphs (a graph in which the vertices can be partitioned into a clique and an independent set). Though variations of Dominating Set like Red-Blue Dominating Set [10] and Constrained Dominating Set [16] have been studied before and shown to be W[2]-complete, to the best of our knowledge the standard Dominating Set problem (which we consider here) in bipartite graphs has not been studied before. Our observation means that the dominating set problem is $W[2]$-complete in triangle free graphs. Then we show that the problem is FPT if the input graph has girth at least 5. It turns out that this result can be generalized to several variants of the Dominating Set problem on graphs with girth at least five.

In Section 3, we look at the Set Cover problem for which Dominating Set is a special instance. Set Cover problem is known to be $W[2]$-complete [8]. Here we show that if the set cover instance satisfies the property that the intersection of any pair of its sets is bounded by a fixed constant then the problem is fixed parameter tractable.

In Section 4, we look at t-Vertex Cover and t-Dominating Set problems. These are generalizations of Vertex Cover and Dominating Set problems. In the t-Vertex Cover problem, we are interested in finding a set of at most k vertices covering at least t edges and in the t-Dominating Set problem the objective is to find a set of at most k vertices that dominates at least t vertices. Both these problems have been parameterized in two different ways: by k alone and by both k and t. Both these problems are fixed parameter tractable when parameterized by both k and t. Bläser [6] gave $O(2^{O(t)}n^{O(1)})$ algorithm for both the problems using color coding technique. Guo et. al. [18] have shown that t-Vertex Cover is $W[1]$-complete when parameterized by k alone. It is easy to see that the t-Dominating Set is $W[2]$-complete by a reduction from Dominating set when parameterized by k alone. We show that both these problems are fixed parameter tractable in graphs with girth at least five, when parameterized by k alone.

In Section 5, we look at the Independent Set problem and several of its variants. We show that these problems are fixed parameter tractable in triangle free graphs while they are W[1]-complete in general graphs.

In contrast to our results in earlier sections, in Section 6, we exhibit a problem that is $W[1]$-hard in graphs with no small cycles. This is the Dense Subgraph problem [22]. Here, given a graph $G = (V, E)$ and positive integers k and l, the problem is to determine whether there exists a set of at most k vertices $C \subseteq V$ such that the induced subgraph on C has at least l edges; here k is the parameter.

In Section 7, we deviate and look at the approximability result of the Dominating Set problem. We conclude that the Dominating Set problem is as hard to approximate in bipartite graphs as in general undirected graphs. We also give an approximation algorithm of factor $O(\log p)$ for the Dominating Set problem if the input graph has girth at least 5, where p is the size of an optimum dominating set of the input graph. This improves the previously known approximation algorithm of factor $O(\log n)$, where n is the number of vertices in the input graph.

Section 8 gives some concluding remarks and open problems.

We assume that all our graphs are simple and undirected. Given a graph $G = (V, E)$, n represents number of vertices, and m represents the number of edges. For a subset $V' \subseteq V$, by $G[V']$ we mean the subgraph of G induced on V'. By $N(u)$ we represent all vertices (excluding u) that are adjacent to u, and by $N[u]$, we refer to $N(u) \cup \{u\}$. Similarly, for a subset $D \subseteq V$, we define $N[D] = \cup_{v \in D} N[v]$. By the girth of a graph, we mean the length of the shortest cycle in the graph. We say that a graph is a G_i graph if the girth of the graph is at least i. A vertex is said to dominate all its neighbors.

2 Dominating Set and its Variants

In this section we look at the Dominating Set problem and its variants.

> Dominating Set: Given a graph $G = (V, E)$ and an integer $k \geq 0$, determine whether there exists a set $D \subseteq V$, of size at most k, such that for every vertex $u \in V$, $N[u] \cap D \neq \emptyset$.

We say that the set D "*dominates*" the vertices of G. We first show that Dominating Set problem is W[2]-complete in bipartite graphs and split graphs by a reduction from the same problem in general undirected graphs. Then we give a fixed parameter tractable algorithm for the problem in graphs with girth at least 5.

2.1 Dominating Set in Bipartite and Split Graphs

Theorem 1. Dominating Set *problem is $W[2]$-complete in bipartite graphs.*

Proof. We prove this by giving a reduction from the Dominating Set problem in general undirected graphs. Given an instance $(G = (V, E), k)$ of Dominating Set, we construct a bipartite graph $H = (V', E')$. Let z_1 and z_2 be two new vertices (not in V). Now $V' = V_1 \cup V_2$ where $V_1 = \{u_1 \mid u \in V\} \cup \{z_1\}$ and $V_2 = \{u_2 \mid u \in V\} \cup \{z_2\}$. If there is an edge (u, v) in E then we draw the edges (u_1, v_2) and (v_1, u_2). We also draw edges of the form (u_1, u_2) for every $u \in V$. Finally, we add an edge from every vertex in V_1 to z_2. This completes the construction of H.

We show that G has a dominating set of size k if and only if H has a dominating set of size $k+1$. Let D be a dominating set of size k in G. Then clearly $D' = \{u_1 \mid u \in D\} \cup \{z_2\}$ is a dominating set of size $k+1$ in H. Conversely, let K be a dominating set in H of size $k+1$. Observe that either z_1 or z_2 must be part of K as z_2 is the unique neighbor of z_1. Without loss of generality, we can assume that $z_2 \in K$, as otherwise we could delete z_1 and include z_2 in K and still have a dominating set of size at most $k+1$ in H. Now take $D = \{u \mid u \in V,\ u_1 \textit{ or } u_2 \in K\}$. Clearly D is of size k. We show that D is a dominating set in G. For any $u \notin D$, $u_2 \notin K$ and hence there exists some $v_1 \in K$ such that v_1 dominates u_2 in H. But this implies $v \in D$ and $(v, u) \in E$, which shows that v dominates u. This proves that D is a dominating set of size k for G and establishes the theorem. □

Since every bipartite graph is also triangle free, we have the following corollary.

Corollary 1. Dominating Set *problem is* $W[2]$*-complete in triangle free graphs.*

Theorem 2. Dominating Set *problem is* $W[2]$*-complete in split graphs.*

Proof. We again prove this by giving a reduction from the Dominating Set problem in general undirected graphs. Given an instance $(G = (V, E), k)$ of Dominating Set, we construct a split graph $H = (V', E')$. We create two copies of V namely $V_1 = \{u_1 \mid u \in V\}$ and $V_2 = \{u_2 \mid u \in V\}$. If there is an edge (u, v) in E then we draw the edges (u_1, v_2) and (v_1, u_2). We also draw edges of the form (u_1, u_2) for every $u \in V$. Now we make $H[V_1]$ a complete graph by adding all arcs of the form (u_1, v_1) for every pair of vertex $u_1,\ v_1 \in V_1$. This completes the construction of H. It is easy to see that H is a split graph with $H[V_1]$ as a clique and $H[V_2]$ as an independent set.

As in the proof of Theorem 1, it is easy to see that G has a dominating set of size k if and only if H has a dominating set of size k. □

An undirected graph is chordal if every cycle of length greater than three possesses a chord, that is, an edge joining two nonconsecutive vertices of the cycle. It is well known that every split graph is also a chordal graph and hence Theorem 2 implies that the Dominating Set is $W[2]$-complete in chordal graphs. As a corollary of Theorem 2 we get the following.

Corollary 2. Dominating Set *problem is $W[2]$-complete in chordal graphs.*

2.2 FPT Algorithm for Dominating Set in G_5 Graphs

We give a fixed parameter tractable algorithm for the Dominating Set problem in graphs with girth at least 5 (G_5 graphs) and also observe that various other W-hard problems become tractable for G_5 graphs.

Our algorithm follows a branching strategy where at every iteration we find a vertex that needs to be included in the Dominating Set which we are trying to construct. Once a vertex is included, we can at best delete that vertex. Though the neighbors of the vertex are dominated, we can not remove these vertices from further consideration as they can be useful to dominate other vertices.

Hence we resort to a coloring scheme for the vertices, similar to the one suggested by Alber et al. in [2, 3]. At any point of time of the algorithm, the vertices are colored as below:

1. Red - The vertex is included in the dominating set D which we are trying to construct.
2. White - The vertex is not in the set D, but it is dominated by some vertex in D.
3. Black - The vertex is not dominated by any vertex of D.

Now we define the dominating set problem on the graph with vertices colored with White, Black or Red as above. We call a graph colored red, white and black as above, as a rwb-graph.

> RWB-Dominating Set: Let G be a G_5 graph (graph with girth at least 5) with vertices colored with Red, White or Black satisfying the following conditions, and let k be a positive integer parameter. Let R, W and B be the set of vertices colored red, white and black respectively.
> 1. Every white vertex is a neighbor of a red vertex.
> 2. Black vertices have no red neighbors.
> 3. $|R| \leq k$
>
> Does G have at most $k - |R|$ vertices that dominate all the black vertices?

It is easy to verify that if we start with a general G_5 graph with all vertices colored black, and color all vertices we want to include in the dominating set as red, and their neighbors as white, the graph we obtain at every intermediate step is a rwb-graph, and the problem we will have at the intermediate steps is the RWB-Dominating Set problem.

The following lemma essentially shows that if the rwb-graph has a black or white vertex dominating more than k black vertices, then such a vertex must be part of every solution of size at most k, if one exists.

Lemma 1. *Let* $(G = (R \cup W \cup B, E), k)$ *be an instance of the* RWB-Dominating Set *problem where* G *is a* G_5 *graph and* k *a positive integer. Let* v *be a black or white vertex with more than* $k - |R|$ *black neighbors. Then if* G *has a set of size at most* $k - |R|$ *that dominates all black vertices, then* v *must be part of every such set.*

Proof. Let D be a set of size $k - |R|$ that dominates all black vertices in G, and suppose $v \notin D$. Let X be the set of black neighbors of v which are not in D and Y be the set of black neighbors of v in D. So $|X| + |Y| > k - |R|$. Observe that for every $v_x \in X$ we have a neighbor $u_x \in D$ which is not in Y (otherwise v, v_x, u_x is a 3 length cycle). Similarly, for $x,\ y \in X,\ \ x \neq y \Rightarrow u_x \neq u_y$. Otherwise v, x, u_x, y will form a cycle of length 4. This means that $|D| \geq |X| + |Y| > k - |R|$ which is a contradiction. □

Given a rwb-graph, Lemma 1 suggests the following simple reduction rule.

(R1) If there is a white or a black vertex v having more than $k - |R|$ black neighbors, then color v red and color its neighbors white.

Our goal now is to pick enough white or black vertices to dominate the black vertices. So the following reduction rules are obviously justified.

(R2) If a white vertex is not adjacent to a black vertex, delete the white vertex.
(R3) If there is an edge between two white vertices, delete the edge.
(R4) If $|R| > k$, then stop and return NO.

The following lemma follows from Lemma 1.

Lemma 2. *Let* $G = (R \cup W \cup B, E)$ *be an instance of* RWB-Dominating Set *and let* $G' = (R' \cup W' \cup B', E')$ *be the reduced instance after applying rules* (R1) *to* (R4) *once. Let* k *be an integer parameter. Then* G *is a yes instance if and only if* G' *is a yes instance. That is* G *has a set of size at most* $k - |R|$ *dominating all vertices in* B *if and only* G' *has a set of size at most* $k - |R'|$ *dominating all vertices in* B'*.*

Let G be an instance of RWB-Dominating Set and let G' be the reduced instance after applying the reduction rules $(R1) - (R4)$ until no longer possible. Then we show that if G' is a yes instance (and hence G is a yes instance), the number of vertices in G' is bounded by polynomial in k. More precisely we show the following lemma.

Lemma 3. *Let (G, k) be a yes instance of* RWB-Dominating Set *and (G', k') be the reduced instance of (G, k) after applying the rules $(R1) - R(4)$ until no longer possible. Then, the number of vertices in G' is $O(k^3)$, that is, a kernel of size at most $O(k^3)$ can be obtained for* RWB-Dominating Set.

Proof. Let R', B' and W' be the set of vertices colored red, black and white respectively in G'. We argue that each of $|R'|$, $|B'|$ and $|W'|$ is bounded by a function of k.

Because of $(R4)$ (and the fact that G' is a yes instance), $|R'| \leq k$.

Because of $(R1)$, every vertex colored white or black has at most $k - |R'|$ black neighbors. Also we know that no red vertex has a black neighbor. Since G' is a yes instance, there are at most k ($k - |R'|$ to be more precise) black or white vertices dominating all black vertices. Since each of them can dominate at most k black vertices, we conclude that $|B'|$ can be at most k^2.

We argue that $|W'| \leq k^3$. Towards this end, we just show that every black vertex has at most k white neighbors. Since $|B'| \leq k^2$, and every white vertex is adjacent to some black neighbor (because of $(R2)$ and $(R3)$), the conclusion will follow.

Note that every white vertex has a red neighbor. Observe that the white neighbors of any black vertex (any vertex for that matter) will have all distinct red neighbors. I.e. if w_1 and w_2 are white neighbors of a black vertex b, then there is no overlap between the red neighbors of w_1 and the red neighbors of w_2. This is because if w_1 and w_2 have a common red neighbor r, then we will have a 4-cycle b, w_1, r, w_2, b. Since $|R'| \leq k$, it follows that a black vertex can have at most k white neighbors.

This proves the required claim. □

Thus we have the following theorem.

Theorem 3. *The* RWB-Dominating Set *problem can be solved in $O(k^{k+O(1)} + n^{O(1)})$ time for G_5 graphs.*

Proof. It is easy to see that the reduction rules $(R1)$ to $(R4)$ take polynomial time to execute. When none of these rules can be executed, by

Lemma 3, we have that the number of vertices in the resulting graph is $O(k^3)$, and each vertex has at most k black neighbors. We can just try all possible subsets of size at most k of the vertex set of the reduced graph, to see whether that subset dominates all the black vertices. If any of them does, then we say YES and NO otherwise. This will take $O(k^{3k+O(1)})$ time.

Alternatively, we can apply a branching technique on the black vertices, by selecting a black vertex or any of its neighbors in the dominating set. More precisely, let v be a black vertex. Then we branch on $N[v]$ by including $w \in N[v]$ in the possible dominating set D we are constructing and look for a solution of size $k-1$ in $G - \{w\}$ where w is colored red and all its neighbors are colored white for every $w \in N[v]$. This results in an $O((k+1)^{k+O(1)})$ time algorithm. □

Now to solve the general Dominating Set problem in G_5 graphs, simply color all vertices black and solve the resulting RWB-Dominating Set problem using Theorem 3. Thus we have

Theorem 4. *Parameterized* Dominating Set *problem can be solved in* $O(k^{k+O(1)} + n^{O(1)})$ *time for* G_5 *graphs.*

Parameterized version of Connected Dominating Set (where one is interested in dominating set which is connected) or Independent Dominating Set (where one is interested in dominating set which is independent) are also known to be W[2]-complete [8]. Since the reduction rules ($R1$)-($R4$) apply for any dominating set, using Lemma 3 we can obtain a kernel of size at most $O(k^3)$ for both these problems. For the Independent Dominating Set problem we also check that R remains an independent set when we add a vertex to it while applying reduction rule ($R1$), else we return NO. Furthermore in the proof of the Theorem 3, we try all possible subsets of size at most k and look for a connected or independent dominating set, as required. This results in the following corollary.

Corollary 3. *Parameterized* Connected Dominating Set *and* Independent Dominating Set *problems can be solved in* $O(k^{3k+O(1)}+n^{O(1)})$ *time for* G_5 *graphs.*

A number of other variants of dominating set problem which are W[2]-hard can be shown to be fixed parameter tractable in a similar way for G_5 graphs though not using kernelization. We give necessary details for a few of them in the next subsections.

2.3 Red-Blue Dominating Set and Constraint Bipartite Dominating Set

In this section we give FPT algorithms for Red-Blue Dominating Set and Constrained Bipartite Dominating Set problems for G_5 graphs. We first give an algorithm for Red-Blue Dominating Set problem which is defined as follows.

> Red-Blue Dominating Set [10]: Given a bipartite graph $G = (V, E)$ with V bipartitioned as $V_{red} \cup V_{blue}$ and a positive integer k. Does there exist a subset $D \subseteq V_{red}$ with $|D| \leq k$ and $V_{blue} \subseteq N(D)$.

Theorem 5. Red-Blue Dominating Set *is FPT for* G_5 *graphs.*

Proof. Any two vertices in V_{red} have at most one common neighbor in V_{blue} as otherwise there will be a four cycle in G. Hence, the following reduction rule is justified.

(R1′) if $x \in V_{red}$ has degree more than k then include $x \in D$.

The correctness of $(R1')$ follows from the fact that if we do not select x in D then we need more than k vertices from V_{red} to dominate $N(x)$ as any vertex $y \in V_{red}$, $y \neq x$, can dominate at most one vertex of $N(x)$. Hence after exhaustively applying reduction rule $(R1')$ if the size of D is more than k we answer NO.

Remove $N[D]$ from G, i. e., set $V_{red} = V_{red} \backslash D$ and $V_{blue} = V_{blue} \backslash N(D)$. Now the degree of every vertex in V_{red} is at most k and we are looking for a set of size at most $k - |D|$ in V_{red} such that it dominates all the vertices of V_{blue}. Since every vertex in V_{red} has degree at most k, the size of the set V_{blue} is bounded above by k^2 ($(k - |S|)k$ to be precise) else the answer is NO. We can not bound the size of the set V_{red} anymore, as we do not have any bound on the degree of the vertices in V_{blue}. So to find the desired dominating set in V_{red} (dominating all the vertices in V_{blue}) we do as follows:

- For all partitions $\mathscr{P}$ of V_{blue} into at most $k - |D|$ parts, say $\mathscr{P} = \{P_1, P_2, \cdots, P_j\}$, $1 \leq j \leq k - |D|$, for each P_i, $1 \leq i \leq j$ check whether there exists a vertex $u_i \in V_{red}$ such that $P_i \subseteq N(u_i)$. Call the partition $\mathscr{P}$ *valid* if for all $1 \leq i \leq j$, there exists $u_i \in V_{blue}$ such that $P_i \subseteq N(u_i)$ and the set $\{u_i \mid 1 \leq i \leq j\}$ is called the *witness set*. If any partition $\mathscr{P}$ is valid then return YES with the corresponding witness set else return NO.

It is easy to see that there exits a subset of V_{red} of size at most $k - |D|$ dominating all vertices of V_{blue} if and only if there exists a valid partition. Number of ways in which n indistinguishable objects can be partitioned into r ways is $\binom{n+r-1}{r-1}$ [20]. Hence the total number of partitions $\mathscr{P}$ considered for our case is upper bounded by

$$\sum_{i=1}^{k-|D|} \binom{k^2+i-1}{i-1}.$$

Since the total number of partitions is upper bounded by $O(k^{2k+O(1)})$, the result that Red-Blue Dominating Set is FPT for G_5 graphs follows. □

Next we study Constraint Bipartite Dominating Set problem which is defined as follows.

> Constraint Bipartite Dominating Set (CBDS) [16]: Given a bipartite graph $G = (V, E)$ with V partitioned as $V_1 \cup V_2$ and positive integers k_1 and k_2. Does there exist subsets $D_1 \subseteq V_1$ and $D_2 \subseteq V_2$ with $|D_1| \leq k_1$ and $|D_2| \leq k_2$ such that $V_2 \subseteq N(D_1)$ and $V_1 \subseteq N(D_2)$.

Theorem 6. *Parameterized* Constraint Bipartite Dominating Set *is FPT for G_5 graphs.*

Proof. To solve this problem we just need to solve two instances of Red-Blue Dominating Set problem. The instances of Red-Blue Dominating Set problem we solve are:

1. $V_{red} = V_1$, $V_{blue} = V_2$ and parameter is k_1; and
2. $V_{red} = V_2$, $V_{blue} = V_1$ and parameter is k_2.

We return YES for CBDS problem if both the instances return YES and as D_1 the red-blue dominating set returned by instance 1 and as D_2 the red-blue dominating set returned by instance 2. If either of the instances of Red-Blue Dominating Set problem returns NO, then we return NO for the CBDS problem. □

2.4 Threshold Dominating Set

This problem generalizes Dominating Set and is formally defined as follows.

Threshold Dominating Set (TDS) [7]: Given a graph $G = (V, E)$ and positive integers k and r. Is there a set of at most k vertices $V' \subseteq V$ such that for every vertex $u \in V$, $N[u]$ contains at least r elements of V'?

Theorem 7. Threshold Dominating Set *parameterized by k is FPT for G_5 graphs.*

Proof. First we observe that if $k < r$, then the answer is NO. We assume that $r \leq \log n$, as otherwise $k \geq \log n$ and we have a kernel of size at most 2^k. Now we can solve the problem by checking all subsets of size at most k for the desired threshold dominating set.

Our algorithm is again based on the following simple reduction rule whose correctness follows from Lemma 1.

(R1″) if $x \in V$ has degree more than k then include $x \in V'$.

So basically we select all the vertices of degree more than k of V in V' and hence if the size of V' is more than k then we answer NO.

Next we assign a color to all the vertices. We assign white color to all the vertices (including vertices in V') which have enough (at least r) neighbors in V' and black to the rest. Let B and W, as usual, represent the set of black and white vertices respectively and set $B' = B \setminus V'$ and $W' = W \setminus V'$. Apply reduction rules $(R2)$ and $(R3)$ of Lemma 2 exhaustively. The rule $(R3)$ makes $G[W]$ an independent set. Now the problem reduces to finding a set S' of size at most $k - |V'|$ in $V \setminus V'$ such that $V' \cup S'$ is a desired threshold dominating set for G, in particular for the vertices of B. Since every vertex in $V \setminus V'$ has degree at most k and we are looking for S' of size at most k in $V \setminus V'$, the size of $|B|$ is bounded above by k^2, as otherwise we answer NO.

Now what we have is a generalized version of Threshold Dominating Set problem where we have a set of $j \leq k^2$ black vertices $B = \{u_1, \cdots, u_j\}$, each with a positive integer r_i ($r_i = r - |N[v_i] \cap V'|$), $1 \leq i \leq j$. We are looking for a set $S' \subseteq (W' \cup B')$ of size at most $k - |V'|$ such that for every $u_i \in B$, $|N(u_i) \cap S'| \geq r_i$ in G' where the vertex set of G' is $V(G') = B \cup W'$ and the edge set of G' is $E(G') = \{(u, v) \in E \mid u \in W',\ v \in B \text{ or } u \in B, v \in B\}$.

To solve this generalized version of Threshold Dominating Set problem, we need to generalize our partition arguments used in the Theorem 5 suitably. The major differences are that G' is no more bipartite and that there are vertices which need more than 1 (possibly r) vertices in the desired threshold dominating set. To overcome this difficulty, we make a multiset M from B by having r_i copies for each vertex $u_i \in B$. Clearly the size of $|M|$ is bounded above by rk^2. Now if we apply the partition idea of Theorem 5 it is possible that the same vertex may dominate multiple copies of the same vertex. To deal with this call a partition $\mathscr{P} = \{P_1, P_2, \cdots, P_\alpha\}$ *valid* if (a) there exists a subset $S' \subseteq B' \cup W'$ forming a *system of distinct representatives*; that is for all $1 \leq i \leq \alpha$, there exists a *distinct* $u_i \in S'$ such that $P_i \subseteq N(u_i)$ and (b) each P_i contains at most one copy of any vertex of B. The set S' is a *witness set*. So to find the desired threshold dominating set in $B' \cup W'$ we proceed as follows.

– For all partitions $\mathscr{P}$ of M in at most $k - |V'|$ parts, say $\mathscr{P} = \{P_1, P_2, \cdots, P_\alpha\}$, $1 \leq \alpha \leq k - |V'|$, we check whether $\mathscr{P}$ is a valid partition. If any partition $\mathscr{P}$ is valid then return YES with the corresponding witness set else return NO.

For a fixed partition $\mathscr{P} = \{P_1, P_2, \cdots, P_\alpha\}$, we can do the validity testing and find a corresponding witness set in polynomial time as follows. Testing for duplicate copies in P_i's are easy. For the other part we first define the set

$$I_i = \{u \in (B' \cup W') \mid P_i \subseteq N_{G'}(u)\},$$

where $N_{G'}(u)$ denotes the neighbors of u in G'. Now we make the bipartite incidence graph for the sets $\{I_1, \cdots, I_\alpha\}$, that is a bipartite graph $G^* = (X \cup Y, E'')$, where X has a vertex x_i for every set I_i and $Y = \cup_{l=1}^{\alpha} I_l$ and there is an edge between (x_i, u) if $u \in I_i$. Now finding a "valid" system of distinct representatives reduces to finding a maximum bipartite matching in G^* saturating X, for which there is a classical polynomial time algorithm of Edmonds [12].

The total number of partitions $\mathscr{P}$ considered for our case is upper bounded by

$$\sum_{i=1}^{k-|V'|} \binom{rk^2 + i - 1}{i - 1},$$

which is at most $O((rk^2)^{k+O(1)})$. Since $r \leq \log n$ and $(\log n)^k \leq n + (2k \log k)^k$ for all n and $k \leq n$, we have the desired result that Threshold Dominating Set problem is FPT for G_5 graphs. □

3 Set Cover with Bounded Intersection among Sets

Dominating Set problem is well known to be a special instance of the Set Cover problem defined below.

SET COVER: Given a base set (or universe) $\mathcal{U} = \{s_1, s_2, \cdots, s_n\}$, a collection $\mathcal{S} = \{S_1, S_2, \cdots, S_m\}$ of subsets of $\mathcal{U}$ ($S_i \subseteq \mathcal{U}, 1 \le i \le m$) such that $\cup_{i=1}^{m} S_i = \mathcal{U}$ and a positive integers k, does there exist a sub-collection $\mathcal{S}'$ of $\mathcal{S}$ of size at most k such that $\cup_{S_j \in \mathcal{S}'} S_j = \mathcal{U}$.

Given an instance $(G = (V, E), k)$ of the DOMINATING SET problem, we can formulate it as an instance of the SET COVER problem by taking $\mathcal{U} = V$ and $\mathcal{S} = \{S_v = N[v] \mid v \in V\}$. It is easy to verify that G has a dominating set of size k if and only if $(\mathcal{U}, \mathcal{S})$ has a set cover of size at most k. Hence the parameterized version of the SET COVER problem is $W[2]$-complete [8].

Here, we show that a special case of the SET COVER problem, that generalizes the DOMINATING SET problem for G_5 graphs to be fixed parameter tractable. More specifically, we show if the SET COVER instance $(\mathcal{U}, \mathcal{S})$ satisfies the property that for any pair of sets S_i and S_j in $\mathcal{S}$, $|S_i \cap S_j| \le c$, for a fixed constant c, then the problem is fixed parameter tractable. We call this variant of the SET COVER problem, where every pair of sets in the given family intersect in at most c elements, as BOUNDED INTERSECTION SET COVER (BISC) problem.

Observe that if the input graph G of the dominating set problem is a G_5 graph then the sets in its corresponding set cover instance satisfies a property that for any pair of sets S_u and S_v in $\mathcal{S}$, $|S_u \cap S_v| \le 2$.

Theorem 8. *The* BISC *problem is fixed parameter tractable.*

Proof. If there is a set $S_i \in \mathcal{S}$ such that $|S_i| > ck$ then S_i must be in every k-sized set cover. Otherwise, we need more than k sets to cover all the elements of $\mathcal{U}$ since every other set can cover at most c elements of S_i. So this gives us a following simple reduction rule:

Rule 1: Given a set cover instance, $(\mathcal{U}, \mathcal{S}, k)$, if there exists $S_i \in \mathcal{S}$ such that $|S_i| > ck$ then obtain a new reduced instance of set cover as following:
- $\mathcal{U} \leftarrow \mathcal{U} - S_i$.
- $\mathcal{S} \leftarrow \{S - S_i \mid S \in \mathcal{S}\}$. If there are multiple copies of some set, then remove all but one copy of the same.
- $k \leftarrow k - 1$

If k becomes 0 and $\mathcal{U}$ is non-empty then this is a no instance for the problem and we stop. We apply the **Rule 1** until all the sets in $\mathcal{S}$ is of size at most ck', where $k' \le k$. As k' sets of size ck' can only cover at most $ck'^2 \le ck^2$ elements of $\mathcal{U}$, if $|\mathcal{U}| > ck^2$ then it is a no instance of the problem. The reduction rule also ensures that every set in $\mathcal{S}$ is distinct. But then the number of distinct sets of size at most ck in $\mathcal{S}$ can be at most the number of distinct subsets of $\mathcal{U}$. This gives us that if $|\mathcal{U}| \geq 2ck$ then

$$|\mathcal{S}| = \sum_{i=1}^{ck} \binom{|\mathcal{U}|}{i} \leq ck \binom{|\mathcal{U}|}{ck} \leq ck \left(\frac{cek^2}{ck} \right)^{ck} = ce^{ck} k^{ck+1}$$

and if $|\mathcal{U}| < 2ck$ then

$$|\mathcal{S}| = \sum_{i=1}^{ck} \binom{|\mathcal{U}|}{i} \leq 2^{2ck} = 4^{ck}.$$

Now it suffices to try each sub-collection $\mathcal{S}' \subseteq \mathcal{S}$ of size k and return YES if any of them covers the set $\mathcal{U}$ and NO otherwise. This has following time complexity:

$$\binom{ce^{ck} k^{ck+1}}{k} \leq \left(\frac{ce^{ck} k^{ck+1} e}{k} \right)^k = (ce)^k (ek)^{ck^2}.$$

Since c is a fixed constant, it follows that the running time results in a fixed parameter tractable algorithm. ◻

4 *t*-Vertex Cover and *t*-Dominating Set Problems

t-Vertex Cover and t-Dominating Set problems are respectively, generalizations of classical Vertex Cover and Dominating Set problems. Here the objective is not to cover all the edges or to dominate all the vertices but to cover at least t edges or to dominate at least t vertices with at most k vertices. More precisely they are defined as follows:

> t-Vertex Cover: Given a graph $G = (V, E)$ and positive integers k and t, does there exist a set of at most k vertices $C \subseteq V$ such that $|\{e = (u, v) \in E \mid C \cap \{u, v\} \neq \emptyset\}| \geq t$.
>
> t-Dominating Set: Given a graph $G = (V, E)$ and positive integers k and t, does there exist a set of at most k vertices $D \subseteq V$ such that $|N[D]| \geq t$.

The t-Vertex Cover and t-Dominating Set problems have been parameterized in two ways. They are either parameterized by k or by t and k. Both these problems are FPT when parameterized by both k and

t [6] and are hard for different level of W-hierarchy when parameterized by k alone. t-Vertex Cover is $W[1]$-complete [18] and t-Dominating Set is $W[2]$-complete when parameterized by k alone.

Here, we first give a simple algorithm for the t-Vertex Cover when parameterized by both t and k and then show that this problem is FPT even when parameterized by k alone in G_5 graphs. We then extend this result to the t-Dominating Set problem for G_5 graphs when parameterized by k alone.

Our algorithms for the t-Vertex Cover depend on the following lemma.

Lemma 4. *Let $(G = (V, E), k, t)$ be a yes instance of the t-*Vertex Cover *and v be a vertex of maximum degree in G. Then there exists a t-vertex cover C whose intersection with $N[v]$ is nonempty, i.e. $N[v] \cap C \neq \emptyset$.*

Proof. Since G is a yes instance of the problem there exists a t-vertex cover C of size at most k and covering at least t edges. If $N[v] \cap C = \emptyset$ then choose $C' = C - \{u\} + \{v\}$ where u is any vertex in C. Since v is a vertex of highest degree and none of its neighbors is in C, C' also covers at least t edges and is of size at most k. □

Suppose that the given graph has maximum degree bounded by d. Since there exists a t-vertex cover containing either a maximum degree vertex u or one of the neighbors of u, we can branch on u and on each of the (at most) d neighbors of u giving rise to a $(d + 1)$-way branching. The following theorem is immediate from this.

Theorem 9. *Let G be a graph with maximum degree d. Then t-*Vertex Cover *can be solved in $O((d + 1)^k n)$ time.*

Given a graph $G = (V, E)$ and positive integer parameters t and k, if there exists a vertex of degree at least t then we get a t-vertex cover by choosing the vertex. So without loss of generality, we can assume that every vertex has degree at most $t - 1$. Then from Theorem 9, we have

Corollary 4. *t-*Vertex Cover *can be solved in $O(t^k n)$ in general graphs.*

Suppose, instead of trying to cover at least t edges, we want to cover all but t edges (where t is a parameter) using at most k vertices. That is, we want an induced subgraph on $n - k$ vertices with at most t edges. We call it the $(m - t)$-Vertex Cover problem. Such a parameterization is known as dual parameterization and dual problems are, in general, natural and equally interesting [8, 21]. For example Vertex Cover is fixed parameter tractable whereas the dual of Vertex Cover is the Independent Set problem (which is the same as choosing $n-k$ vertices to cover all edges) and is W[1] complete.

The $(m-t)$-Vertex Cover problem can also be parameterized in two ways, by k alone and by k and t. When we have both t and k as parameters then we solve this problem by branching on an edge $e = (u, v)$. Here we branch by choosing either the vertex u or the vertex v or e which means that we are looking for a solution which contains either u or v or does not cover e. So we get the following branching recurrence:

$$T(k, t) \leq 2T(k-1, t) + T(k, t-1).$$

This immediately gives us the following theorem.

Theorem 10. $(m-t)$-Vertex Cover *can be solved in* $O(3^{t+k}(n+m))$ *time. Thus* $(m-t)$ Vertex Cover *is fixed parameter tractable if parameterized by* t *and* k.

When $(m-t)$-Vertex Cover problem is parameterized by k alone, we can show the following theorem.

Theorem 11. *The* $(m-t)$-Vertex Cover *problem is* $W[1]$*-hard when parameterized by* k *alone.*

Proof. We give a reduction from W[1]-complete t-Vertex Cover problem where we need at most k vertices to cover at least t edges. Given $(G = (V, E), k, t_1)$, an instance of t-Vertex Cover problem, we map it to $(G = (V, E), k, t_2)$ where $t_2 = |E| - t_1$ as an instance of $(m-t)$-Vertex Cover problem. Now it is easy to see that $(G = (V, E), k, t_1)$ is a yes instance of t-Vertex Cover problem if and only if $(G = (V, E), k, t_2)$ is a yes instance of $(m-t)$-Vertex Cover problem. □

Now we show that the t-Vertex Cover problem is FPT for G_5 graphs when parameterized by k alone. We will see that this result also applies to $(m-t)$-Vertex Cover problem when parameterized by k alone.

Theorem 12. t-Vertex Cover *is fixed parameter tractable for* G_5 *graphs when parameterized by* k *alone. The algorithm runs in* $O((k+1)^k(n+m))$ *time.*

Proof. Without loss of generality we can assume that the maximum degree of this graph is not bounded by a function of k, otherwise the problem is fixed parameter tractable by Theorem 9. Let v_0 be a vertex of highest degree and let $v_1, v_2, \ldots, v_r$ be its neighbors. Further assume that

$$deg(v_1) \geq deg(v_2) \geq \cdots deg(v_k) \geq \cdots \geq deg(v_r).$$

Let $A = \{v_0, v_1, \cdots, v_k\}$. We show that if there exists any t-vertex cover then there is one which contains either v_0 or one of its k highest degree neighbors. More precisely, we prove the following claim:

Claim : There exists a t-vertex cover C such that $A \cap C \neq \emptyset$, if one exists.

The claim says that if there exists any t-vertex cover then there exists a t-vertex cover C containing at least one vertex of A. We then branch on the vertices of the set A, and look for a solution of size $k-1$, covering t-deg(v_i) edges in $G-\{v_i\}$, where $0 \leq i \leq k$ and recursively use this claim on the respective subgraphs. Hence the claim proves that t-vertex cover is fixed parameter tractable.

Now we are left with proving the claim. We show the claim by contradiction. Assume to the contrary that no t-vertex cover intersects A. By Lemma 4 we know that there exists a t-vertex cover C containing one of v_0's neighbors. Let v_l be a neighbor of v_0 in C. Because of our assumption $l > k$. Suppose for some $v_i,\ 1 \leq i \leq k,\ N(v_i) \cap C = \emptyset$. Then we can obtain a t-vertex cover $C' = C - \{v_l\} + \{v_i\}$ of size at most k and covering at least t edges as deg(v_i)$\geq$ deg(v_l). So we now assume that for each $v_i,\ 1 \leq i \leq k,\ N(v_i) \cap C \neq \emptyset$. Let $B_i = N(v_i) \cap C$. Observe that for each i, B_i does not contain v_l otherwise that will imply v_0, v_i, v_l is a triangle. Suppose for some $i \neq j$, $u \in B_i \cap B_j$ then v_0, v_i, u, v_j is a cycle of length 4. Hence $B_i \cap B_j = \emptyset$ for all i, j such that $i \neq j$. So this implies that

$$\sum_{i=1}^{k} |B_i| \geq k.$$

So we have $B_i \neq \emptyset$, $B_i \subseteq C - \{v_l\}$ and their pairwise intersections are empty. But this implies

$$\sum_{i=1}^{k} |B_i| \leq |C - \{v_l\}| \leq k-1$$

which contradicts that $\sum_{i=1}^{k} |B_i| \geq k$. This in turn proves the claim.

Since we branch on the vertices of A whose size is bounded by $k+1$, we get an algorithm of time complexity $O((k+1)^k n)$. $\square$

Since the runtime in Theorem 12 was independent of t, we get

Theorem 13. $(m-t)$-Vertex Cover *can be solved in* $O((k+1)^k(n+m))$ *time for* G_5 *graphs when parameterized by* k *only.*

By arguments similar to those used in Theorem 12, we can show the following.

Theorem 14. t-Dominating Set *can be solved in* $O((k+1)^k n^{O(1)})$ *time for* G_5 *graphs when parameterized by* k *only.*

5 Independent Set and its Variants in G_4 Graphs

Independent Set problem asks for an induced subgraph on k vertices which only contains isolated vertices. More precisely:

> Independent Set : Given a graph $G = (V, E)$ and an integer $k \geq 0$, determine whether there exists a set of at most k vertices $I \subseteq V$ such that the subgraph induced by I does not contain any edges.

Independent Set problem is W[1]-complete for general graphs. We show that the Independent Set and some of its variants are fixed parameter tractable for *triangle free* graphs. We use *Ramsey theory* to get a kernel of size $O(k^2)$ for these problems.

Theorem 15. *Parameterized* Independent Set *problem can be solved in* $O(kn + k^{O(k)})$ *in* G_4 *graphs (triangle free graphs).*

Proof. Given any two integers p and q, there exists a number $R(p, q)$ such that any graph on at least $R(p, q)$ vertices contains an independent set of size p or a clique of size q. $R(p, q)$, for various values of p and q are known as *Ramsey Numbers*. It is well known that $R(p, q) \leq \binom{p+q-2}{q-1}$ [20]. And if $n > R(p, q)$ then either an independent set of size p or a clique of size q can be found in $O((p + q)n)$ time by transforming the inductive arguments used in the proof of Theorem 27.3 in [20] for the upper bound of $R(p, q)$ to a constructive algorithm.

If $k \leq 2$, then we can check in linear time whether the graph has an independent set of size 2 or not. So let us assume that $k \geq 3$. If the number of vertices $n > k^2 \geq R(k, 3)$ then we know that this graph has either an independent set of size k or a clique of size 3. But since the input graph is triangle free, we know that it must have an independent set of size k and can be found in $O(kn)$ time. Otherwise we know that $n \leq k^2$. In this case, we try all possible subsets of size at most k to see whether the graph has an independent set of size k or not. If any of them does, then we answer YES and answer NO otherwise. This will take $O(k^{O(k)})$ time. This completes the proof. □

Theorem 15 can be extended to a larger class of problems where one is interested in finding a subset inducing a "hereditary property". A graph property Π is a collection of graphs. A graph property Π is non-trivial if Π has at least one graph and does not include all graphs. A non-trivial property is said to be *hereditary* if a graph G is in property Π implies that every *induced subgraph* of G is also in Π. Given any property Π, let $P(G, k, \Pi)$ be the problem defined below:

> P(G, k, Π): Given a graph $G = (V, E)$ and a positive integer k, determine whether there exists a set of k vertices $V' \subseteq V$ such that $G[V']$ is in Π.

Khot and Raman [21] studied this problem and showed the following theorem.

Theorem 16. *(Khot and Raman [21]) Let Π be a hereditary property that includes all independent sets but not all cliques (or vice versa). Then the problem $P(G, k, \Pi)$ is $W[1]$ hard.*

The proof of the following theorem is exactly as in the proof of Theorem 15, by considering the Ramsey numbers $R(k, c)$.

Theorem 17. *Let Π be a hereditary property that includes all independent sets. Then the problem $P(G, k, \Pi)$ restricted to G_c graphs for any fixed constant $c \geq 3$ is fixed parameter tractable and can be solved in $O(kn + k^{O(k)} n^{O(1)})$ time.*

Given a graph $G = (V, E)$ and a positive integer $k \geq 0$, Acyclic Subgraph, Bipartite Subgraph and Planar Subgraph problems ask whether there exists a subset $V' \subseteq V$, such that $|V'| \geq k$ and $G[V']$ is acyclic, bipartite or planar respectively. All these problems are known to be W[1]-hard [8, 21] in general graphs. As a corollary to Theorem 17 we have following:

Corollary 5. Acyclic, Bipartite *and* Planar Subgraph *problems are fixed parameter tractable with time complexity $O(kn + k^{O(k)} n^{O(1)})$ for G_c graphs for any fixed constant $c \geq 3$.*

Corollary 5 shows that Acyclic and Planar Subgraph problems are fixed parameter tractable for bipartite graphs. In fact we can easily obtain much improved FPT algorithms for these problems for bipartite graphs. Observe that a bipartite graph has an independent set (and hence planar or acyclic induced subgraph) on $n/2$ vertices. So, if $k \leq n/2$ then for both these problems the answer is YES and otherwise $k > n/2$ or $n < 2k$ and hence we get a kernel of size at most $2k$ for both the ACYCLIC and PLANAR SUBGRAPH problems for bipartite graphs. Now we check all k sized subsets of the vertex set to see whether the subset induces an acyclic subgraph or planar subgraph. Since $\binom{n}{k} \leq \binom{2k}{k} \leq 2^{2k} = 4^k$, we get an $O(4^k n^{O(1)})$ time algorithm for both these problems for bipartite graphs.

Minimum feedback vertex set, which is a subset of vertices whose removal makes the graph acyclic, is a complement of the vertex set of the maximum ACYCLIC SUBGRAPH problem. Fomin et al. [15] have shown that the minimum feedback vertex set can be found in $O(1.7548^n)$ time in undirected graphs. So together with this result and the kernel of size $2k$ we get $O(1.7548^{2k} n^{O(1)}) = O(3.0793^k n^{O(1)})$ time algorithm for the ACYCLIC SUBGRAPH problem. Putting together everything we get the following theorem.

Theorem 18. *The parameterized* ACYCLIC SUBGRAPH *and* PLANAR SUBGRAPH *problems can be solved in* $O(3.08^k k^{O(1)} + n^{O(1)})$ *and* $O(4^k k^{O(1)} + n^{O(1)})$ *time, respectively, for bipartite graphs.*

Another problem which can be shown to be FPT for G_c graphs for any fixed constant $c \geq 3$ is the IRREDUNDANT SET problem, which is known to be W[1]-complete [9] in general graphs.

> IRREDUNDANT SET: Given a graph $G = (V, E)$ and a positive integer k. Is there a set $V' \subseteq V$ of cardinality at least k having the property that each vertex $u \in V'$ has a private neighbor ? A private neighbor of a vertex $u \in V'$ is a vertex $u' \in N[u]$ (possibly $u' = u$) with the property that for every vertex $v \in V' \setminus \{u\}$, $u' \notin N[v]$.

This follows from a simple observation that every independent set is also an irredundant set. Then the following theorem can be proved on the lines of Theorem 15, by considering the Ramsey Numbers $R(k, c)$.

Theorem 19. IRREDUNDANT SET *is FPT for* G_c *graphs for any fixed constant* $c \geq 3$.

6 Is everything easy on graphs with no small cycles ?

In contrast to the results presented in the previous sections, here we show a problem to be $W[1]$-hard even in bipartite graphs with girth at least 6 (G_6 graphs). Observe that in graphs with large girth the Clique problem is trivial. We look at Dense Subgraph problem [22] which is a generalization of the Clique problem.

> Dense Subgraph: Given a graph $G = (V, E)$ and positive integers k and l, determine whether there exists a set of at most k vertices $C \subseteq V$ such that $G[C]$ has at least l edges, i.e. the induced subgraph on C has at least l edges. (Note that l is at most $\binom{k}{2}$.)

It is easy to observe that Dense Subgraph problem is W[1]-hard when parameterized by k, by a simple reduction from Clique. But we give a reduction from Clique to Dense Subgraph problem parameterized by k which shows that the problem is W[1]-hard even in bipartite graphs with girth at least 6.

Theorem 20. Dense Subgraph *is $W[1]$-hard for bipartite graphs with girth at least* 6 *when parameterized by k.*

Proof. We give a reduction from Clique. Let (G, k) be an instance of Clique with $k \geq 3$. We make the graph $G = (V, E)$ bipartite by subdividing every edge. Let $G' = (V', E')$ be the resulting subgraph. Here, $V' = V \cup W$ where $W = \{w_{uv} \mid (u, v) \in E\}$ and E', the set of edges, consists of (u, w_{uv}) and (v, w_{uv}) for every $w_{uv} \in W$. Take $k' = k + \binom{k}{2}$ and $l = 2\binom{k}{2}$.

Observe that G' is a bipartite graph as every cycle is of even length and the girth is at least 6 as the girth of G is at least 3. We claim that G has a clique of size k if and only if G' has a subgraph on k' vertices with at least l edges. Also note that every vertex in W has degree 2 as they represent edges in the original graph. Now suppose G has a clique of size k on vertex set $C = \{v_1, v_2, \cdots, v_k\}$. Then $C' = C \cup \{w_{uv} \mid u, v \in C\}$ is a vertex set of dense subgraph in G' having k' vertices and l edges as $G[C]$ has at least $\binom{k}{2}$ edges.

Conversely, let C' be a set of k' vertices such that $G'[C']$ has at least l edges. Let $O = V \cap C'$. Clearly $G'[C']$ is bipartite with O and $N = C' - O$ as the two parts of the vertex set, and every vertex in N has degree at most 2. Since the number of edges in $G'[C'] = l = 2\binom{k}{2}$, and since every vertex in N has degree at most 2, $|N| \geq \binom{k}{2}$ and hence $|O| \leq k$. Let $t = |O|$. We claim that $t = k$. Suppose not. Then $t \leq k - 1$. Also, since $k \geq 3$, $t \geq 1$. Let n_1 and n_2 be the degree 1 and degree 2 vertices in N respectively. Since G has no multiple edges, no pair of vertices in N with degree 2 can be adjacent to the same pair of vertices in O and hence

$n_2 \le \binom{t}{2}$. Then the number of edges in $G[C']$ is:

$$\begin{aligned} 2\binom{k}{2} \le |E(G[C'])| &= 2n_2 + n_1 \\ &= k' - t + n_2 \\ &= k + \binom{k}{2} - t + n_2 \\ &\le k + \binom{k}{2} - t + \binom{t}{2} \qquad (\text{ since } n_2 \le \tbinom{t}{2}). \end{aligned}$$

From the above it implies that

$$t + \binom{k}{2} \le k + \binom{t}{2}. \tag{1}$$

If $t = 1$ then

$$k + \binom{1}{2} = k < 1 + \binom{k}{2},$$

a contradiction to inequality (1). So let $2 \le t \le k-1$. But then

$$k + \binom{t}{2} \le k + \binom{k-1}{2} = \binom{k}{2} + 1 < \binom{k}{2} + t,$$

again a contradiction to inequality (1). This implies that $|O| = k$. As a result of this, $|N| = \binom{k}{2}$ and every vertex in N has degree 2. Every vertex of degree 2 in N represents an edge in $G[O]$. This shows that the vertices of O form a clique in the original graph. □

7 Approximation of Dominating Set

In this section we give some results concerning approximation of the Dominating Set problem for bipartite and G_5 graphs. We refer to [25] for all the basic definitions regarding approximation algorithms.

Feige [13] showed that (1-$o(1)$) $\ln n$ is a threshold below which the Dominating Set problem cannot be approximated efficiently unless NP has slightly super-polynomial time algorithm. Here, $\ln n$ represents natural logarithm. Under the same hypothesis $\frac{(1-o(1))\ln n}{2}$ is a threshold below which the Dominating Set problem can not be approximated for bipartite graphs. This result follows from the reduction in Theorem 1. Towards this end, we just show that if we have factor α approximation algorithm for the dominating set problem in bipartite graphs then it implies 2α factor approximation algorithm for the dominating set problem in general undirected graphs. Given a graph G, we apply Theorem 1 to obtain the bipartite graph H and apply the factor α approximation algorithm for dominating set problem in bipartite graphs to get a dominating set D for H. We obtain a dominating set D' for G from the dominating set D for H as in the proof of Theorem 1. Let OPT_G denote the size of an optimum dominating set for the graph G. Now note that

$$
\begin{aligned}
|D'| &< |D| \\
&\leq \alpha \cdot OPT_H \\
&\leq \alpha \cdot (OPT_H - 1) + \alpha \\
&\leq \alpha \cdot OPT_G + \alpha \cdot OPT_G \qquad \text{(since } OPT_H - 1 = OPT_G\text{)} \\
&= (2\alpha) \cdot OPT_G.
\end{aligned}
$$

Furthermore the result of Feige [13] together with Theorem 2 imply that the approxmability of Dominating Set problem has the same threshold of (1-$o(1)$) $\ln n$ even for split graphs. The above discussion results in the following theorem.

Theorem 21. Dominating Set *problem can not be approximated efficiently below* (1-$o(1)$) $\ln n$ *in bipartite and split graphs unless* $NP \subset DTIME(n^{O(\log\log n)})$.

An approximation algorithm of factor $O(\log n)$ is known for the Dominating Set problem using the reduction to the Set Cover problem (see discussion before Theorem 8 in Section 3) and the following proposition.

Proposition 1 ([19, 23]). *Let* $(\mathcal{U}, \mathcal{S})$ *be a set cover instance such that* $|U| = n$*. Then we can find a set cover* $\mathcal{S}' \subseteq \mathcal{S}$ *of size at most* $\mathcal{H}_n \cdot (OPT)$ *where* $\mathcal{H}_n = \sum_{i=1}^{n} 1/i$ *and* OPT *is the size of the optimum solution of the set cover instance.* $\mathcal{H}_n \leq \ln n + 1$.

Here we outline a slightly improved approximation algorithm for Dominating Set problem in G_5 graphs. This approximation algorithm has a factor $O(\log l)$ where l is the size of the optimum dominating set. The idea of the algorithm is to use the reduction rules developed in Section 2.2 and obtain an instance of size $O(l^3)$ with the property that maximum degree of the graph is bounded by l and then use the following proposition on the corresponding set cover instance of the problem.

Proposition 2 ([11]). *Let* $(\mathcal{U}, \mathcal{S})$ *be a set cover instance such that* $|U| = n$ *and the size of each set* $S_i \in \mathcal{S}$ *is bounded by* q*, that is* $|S_i| \leq q$*. Then we can find a set cover* $\mathcal{S}' \subseteq \mathcal{S}$ *of size at most* $(\mathcal{H}_q - 1/2) \cdot (OPT)$ *where*

$\mathcal{H}_q = \sum_{i=1}^{q} 1/i$ and OPT is the size of the optimum solution of the set cover instance.

Observe that the reduction rules $(R1) - (R4)$ depend on k whereras here we have an optimization problem. Hence apply reduction rules for all values for k between 1 and n and if the reduced instance as viewed as the Set Cover problem instance satisfies the hypothesis of Proposition 2 then we obtain a dominating set for G by applying Proposition 2. Finally we return the dominating set of smallest size among the ones obtained for different k. Our detailed algorithm is described below. We outline our algorithm in terms of *rwb-graphs* described in Section 2.2.

`Algo-Dom-SET(G=(V,E))`
(***Input:*** A G_5 graph. ***Output:*** A dominating set of G.)

Step 1: Given an undirected graph $G = (V, E)$. Make it a rwb-graph by coloring all vertices of V black; that is $R = \emptyset, W = \emptyset$ and $B = V$. $\mathcal{I} = \emptyset$.

Step 2: for $(j = 1$ to $n)$ do as follows:

Step 2a: Apply reduction rules $(R1) - (R4)$ on $(G = (R \cup W \cup B, E), j)$ until no longer possible and obtain an instance $(G_j = (R^j \cup W^j \cup B^j, E^j), j - |R^j|)$.

Step 2b: If $(|W^j| + |B^j| \leq 2j^3)$ and the maximum degree of G_j is at most j then

$$\mathcal{I} = \mathcal{I} \cup \{(G_j = (R^j \cup W^j \cup B^j, E^j), j - |R^j|)\}$$

(In this step we obtain a set of instances which could possibly lead to an optimum dominating set. So we have

$$\mathcal{I} = \{(G_k = (R^k \cup W^k \cup B^k, E^k), k - |R^k|) \;\mid\; |W^k| + |B^k| \leq 2k^3 \text{ and maximum degree of } G_k \text{ is at most } k\})$$

Step 3: We obtain a set cover instance $(\mathcal{U}_k, \mathcal{S}_k)$ from the reduced graph G^k by taking $\mathcal{U} = B^k$ and having sets S_u for $u \in (W^k \cup B^k)$. $S_u = N(u) \cap B^k$ if $u \in W^k$ and $S_u = N[u] \cap B^k$ if $u \in B^k$. Obtain $\mathcal{P}$, the set of instances for the set cover problem, by changing every instance in $\mathcal{I}$ to the set cover instance. That is:

$$\mathcal{P} = \{(\mathcal{U}_k, \mathcal{S}_k) \;\mid\; (G_k = (R^k \cup W^k \cup B^k, E^k), k - |R^k|) \in \mathcal{I}\}.$$

Step 4: Apply Proposition 2 to every instance of the set cover problem in $\mathcal{P}$ and obtain the following set of solutions

$$\mathcal{SOL} = \{\mathcal{S}'_k \mid \mathcal{S}'_k \subseteq \mathcal{S}_k, (\mathcal{U}_k, \mathcal{S}_k) \in \mathcal{P}\}.$$

Let $\mathcal{V}(\mathcal{S}'_k)$ represent the set of vertices in G_k corresponding to the sets in the collection $\mathcal{S}'_k$. Obtain the following set

$$\mathcal{DOM} = \{\mathcal{V}(\mathcal{S}'_k) \cup R_k \mid \mathcal{S}'_k \in \mathcal{SOL} \text{ \& } R^k \text{ the red vertices of } G_k\}.$$

of possible dominating sets for G and return the one with the minimum size in $\mathcal{DOM}$ as a dominating set for G.

Theorem 22. *Let $G = (V, E)$ be a G_5 graph on n vertices. Then the algorithm* `Algo-Dom-SET` *outputs a dominating set of size at most $(\mathcal{H}_{p+1} - 1/2) \cdot p$ in polynomial time where $H_p = \sum_{i=1}^{p} 1/i$ and p is the size of the optimum solution of a dominating set of G. That is,* `Algo-Dom-SET` *is an approximation algorithm with performance ratio of $\ln(p+2) + 1/2$ for the dominating set problem for G_5 graphs.*

Proof. It is clear that the algorithm `Algo-Dom-SET` takes polynomial time. Proposition 2 ensures that the algorithm returns a dominating set for G. Now we show that the algorithm is a factor of $\mathcal{H}_{p+1} - 1/2$ approximation algorithm for the dominating set problem for G_5 graphs which will complete the proof of the theorem.

Let l be the smallest positive integer in Step 2 of the algorithm such that $(G_l = (R^l \cup W^l \cup B^l, E^l), l - |R^l|) \in \mathcal{I}$. The reduction rules ensures that $(G = (R \cup W \cup B, E), k)$ is a *no* instance for $1 \leq k \leq l-1$ and hence we have $p \geq l$.

Consider the instance $(G_p = (R^p \cup W^p \cup B^p, E^p), p - |R^p|) \in \mathcal{I}$. Observe that the instance G_p has an optimum dominating set of size $p - |R^p|$ and the maximum degree of the graph is bounded by p. When we apply the factor $(\mathcal{H}_q - 1/2)$ set cover approximation algorithm in Step 4 on the instance $(\mathcal{U}_p, \mathcal{S}_p)$, where each set in $\mathcal{S}_p$ is bounded by $p+1$, we obtain $\mathcal{S}'_p \subseteq \mathcal{S}_p$ of size at most $|\mathcal{S}'_p| \leq (\mathcal{H}_{p+1} - 1/2)(p - |R^p|)$. Now the size of the dominating set $R_p \cup \mathcal{V}(\mathcal{S}'_p)$ corresponding to this instance for G is :

$$\begin{aligned} |R_p| + |\mathcal{V}(\mathcal{S}'_p)| &= |R_p| + |\mathcal{S}'_p| \\ &\leq |R_p| + (\mathcal{H}_{p+1} - 1/2)(p - |R^p|) \\ &\leq |R_p|(\mathcal{H}_{p+1} - 1/2) + (\mathcal{H}_{p+1} - 1/2)(p - |R^p|) \\ &= (\mathcal{H}_{p+1} - 1/2)p. \end{aligned}$$

Since we return a dominating set of minimum size among the sets in $\mathcal{DOM}$ as a dominating set for G it is clear that its size is also bounded by $(\mathcal{H}_{p+1} - 1/2)p$. This completes the proof. □

8 Conclusion and Discussions

In this paper we showed that if the input graphs do not possess short cycles then the neighborhood problems such as DOMINATING SET, Independent Set and several of their variants are fixed parameter tractable. We have also shown that the restriction on girth is optimal if we do not put further restriction on the graph classes. This is the first time, to our knowledge, that the parameterized complexity of graph problems are classified by girth.

Most of the algorithms given here are just parameterized complexity classification algorithms. We believe that more efficient FPT algorithms should be possible. Obtaining a $O(c^k n^{O(1)})$, c a constant, algorithm for all these problems remains an open problem.

We also gave an improved approximation algorithm for DOMINATING SET problem in graphs with girth at least 5. It would be interesting to explore the possibility of improved approximation algorithms for other problems on graphs with no small cycles.

Furthermore, it is worth exploring excluding structures as subgraphs other than cycles to see whether some W-hard problems become FPT.

Acknowledgments. We thank the referees for several useful comments. We would also like to thank T. Erlebach and R. E. Tarjan for suggesting an investigation of approximation algorithms for the DOMINATING SET problem for graphs with no small cycles.

References

1. J. Alber, H. L. Bodlaender, H. Fernau, T. Kloks and R. Niedermeier. *Fixed parameter algorithms for Dominating Set and related problems on Planar Graph.* Algorithmica **33** (2002) 461-493.
2. J. Alber, H. Fan, M. R. Fellows, H. Fernau, R. Niedermeier, F. Rosamond and U. Stege. *A refined search tree technique for dominating set on planar graphs.* Journal of Computer and System Sciences **71** (2005) 385-405.
3. J. Alber, M. R. Fellows and R. Niedermeier. *Polynomial time data reduction for dominating Set.* Journal of the ACM **51(3)** (2004) 363-384.
4. V. E. Alekseev. *On easy and hard hereditary classes of graphs with respect to the Independent Set problem.* Discrete Applied Mathematics **132(1-3)** (2003) 17-26.
5. V. E. Alekseev, D. V. Korobitsyn and V. V. Lozin. *Boundary classes of graphs for the Dominating Set problem.* Discrete Mathematics **285(1-3)** (2004) 1-6.
6. M. Bläser. *Computing small partial coverings.* Information Processing Letters **85(6)** (2003) 327-331.
7. R. G. Downey and M. R. Fellows. *Threshold Dominating Sets and an improved characterization of W[2].* Theoretical Computer Science **209(1-2)** (1998) 123-140.

8. R. G. Downey and M. R. Fellows. *Parameterized Complexity*. Springer Verlag, (1999).
9. R. G. Downey, M. R. Fellows and V. Raman. *The complexity of irredundant sets parameterized by size*. Discrete Applied Mathematics **100(3)** (2000) 155-167.
10. R. G. Downey, M. R. Fellows, A. Vardy and G. Whittle. *The parametrized complexity of some fundamental problems in coding theory.* SIAM Journal on Computing **29(2)** (1999) 545-570.
11. R. Duh and M. Fürer. *Approximation of k-set cover by semi-local optimization.* In the Proceedings of the 29th Annual ACM Symposium on Theory of Computing (STOC), (1997) 256-264.
12. J. Edmonds. *Paths, trees and flowers*. Canadian Journal of Mathematics **17** (1965) 449-467.
13. U. Feige. *A threshold of ln n for approximating set cover*. Journal of the ACM **45(4)** (1998) 634-652.
14. J. Flum and M. Grohe. *Parameterized Complexity Theory*. Springer-Verlag, (2006).
15. F. V. Fomin, S. Gaspers and A. V. Pyatkin. *Finding a Minimum Feedback Vertex Set in time* $O(1.7548^n)$. In the Proceedings of the 2nd International Workshop on Parameterized and Exact Computation (IWPEC), LNCS **4169** (2006) 184-191.
16. H. Fernau. *Parameterized algorithms: A graph-theoretic approach*. Habilitationsschrift (Universität Tübingen, Tübingen, Germany) (2005).
17. R. Niedermeier. *Invitation to Fixed-Parameter Algorithms*. Oxford Lecture Series in Mathematics and Its Applications, Oxford University Press, (2006).
18. J. Guo, R. Niedermeier and S. Wernicke. *Parameterized complexity of generalized Vertex Cover problems*. In the Proceeding of the 9th International Workshop Algorithms and Data Structures (WADS), LNCS **3608** (2005) 36-48.
19. David S. Johnson. *Approximation algorithms for combinatorial problems.* Journal of Computer and System Sciences **9(3)** (1974) 256-278.
20. S. Jukna. *Extremal Combinatorics*. Springer-Verlag, (2001).
21. S. Khot and V. Raman. *Parameterized complexity of finding Subgraphs with hereditary properties*. Theoretical Computer Science **289(2)** (2002) 997-1008.
22. G. Kortsarz and D. Peleg. *On choosing a dense subgraph*. In the Proceeding of the 34th Annual Symposium on Foundations of Computer Science (FOCS), (1993) 692-701.
23. L. Lovàsz. *On the ratio of optimal fractional and integral covers*. Discrete Mathematics **13** (1975) 383-390.
24. V. Raman and S. Saurabh. *Triangles, 4-Cycles and parameterized (in-) tractability*. In the Proceeding of the 10th Scandinavian Workshop on Algorithm Theory (SWAT), LNCS **4059** (2006) 304-315.
25. V. V. Vazirani. *Approximation Algorithms*. Springer-Verlag, (2001).