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Short-Range Order for the Triangular and Honeycomb Ising Nets in Ferro- and Antiferromagnetic Cases*)

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The short-range order parameters are evaluated for the triangular and honeycomb Ising nets in ferro- and antiferromagnetic cases by the method of Kaufman and Onsager.[1] For the antiferromagnetic triangular net we also evaluate the probabilities for the various spin configurations. When the three spins are located at the vertices of the smallest regular triangle, the probability that three neighbouring spins are all parallel is zero at $T=0$. However, it is 0.035120 or 0.202998 respectively at $T=0$ according as the vertical angle of the three neighbouring spins is 120° or 180°. Making use of these values, the configurational probabilities corresponding to four or five neighbouring spins are also evaluated at zero temperature.

§ 1. Introduction

The spin correlation functions for the various Ising nets have been studied by Kaufman and Onsager,[1] Sekiya and Naya,[2] Montroll, Potts and Ward,[3] and Hurst and Green.[4] Even in the present age the spin configurations for the antiferromagnetic triangular net are an interesting problem at low temperature, and spin correlations for rather small distances between the lattice sites will tell us several characteristic properties[5] for the net. In the present paper, we derive the spin pair correlations among a site and the next, $\cdots$, the fifth nearest neighbouring sites for the ferro- and antiferromagnetic honeycomb and triangular net by the method of Kaufman and Onsager.[1] The critical values for $T=T_c$ or $T=0$ are exactly solved by the elliptic substitutions. Our resulting curves are checked by the high-temperature expansions. Finally we find the family of curves for the probabilities corresponding to three neighbouring spin configurations and four or five neigbouring spin configurations at zero temperature.

§ 2. The calculation of the correlation functions

We will follow Kaufman and Onsager's[1] notation through this section. Now let us consider a lattice which is a triangular or honeycomb lattice according as a parameter H' tends to infinity or zero. (See Fig. 1a.) For the eigenvalue problem of this lattice, " Kaufman-Onsager's[1] characteristic operator **V** " is written as

*) Based partly on a dissertation submitted in 1961 to Osaka University. Quite recently the spin correlations on the triangular lattice have been published by J. Stephenson using the Paffian method, and their results are consistent with this paper.[6]

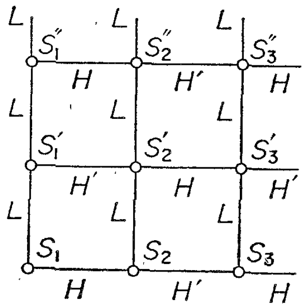

Fig. 1a. The lattice is reduced to a triangular or honeycomb lattice according as a parameter H' tends to infinity or zero.

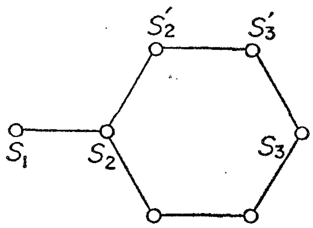

Fig. 1b. The isotropic honeycomb net, $H'=0$ in Fig. 1a.

$$\mathbf{V}=\mathbf{V}_1\cdot\mathbf{V}_2,$$

where

$$\mathbf{V}_1=\exp\Big(-\sum_{r=\text{odd}} iH\cdot P_{r+1}Q_r-\sum_{r=\text{even}} iH'\cdot P_{r+1}Q_r\Big)\cdot\exp\Big(\sum_{r=\text{all}} iL^*\cdot P_rQ_r\Big),$$

$$\mathbf{V}_2=\exp\Big(-\sum_{r=\text{odd}} iH'\cdot P_{r+1}Q_r-\sum_{r=\text{even}} iH\cdot P_{r+1}Q_r\Big)\cdot\exp\Big(\sum_{r=\text{all}} iL^*\cdot P_rQ_r\Big),\tag{1}$$

where L^* is a "dual" parameter in Onsager's paper.[1] The operator $\mathbf{V}$ is represented by a $2n$-dimensional cyclic matrix. In general a cyclic matrix can be diagonalized easily. In order to facilitate the calculation it is convenient to deal with the problem in subalgebra in $(1-\boldsymbol{U})/2$, where $\boldsymbol{U}\equiv(iP_1Q_1)\cdot(iP_2Q_2)\cdots(iP_nQ_n)$. According to Kaufman-Onsager,[1] the two spin correlations are connected to the average values of the 2nd rank spinor or 2-spinor quantity as follows:

$$\begin{aligned}
&\langle S_1S_2\rangle=\langle S_2'S_3'\rangle=\langle -iP_2Q_1\rangle,\\
&\langle S_2S_3\rangle=\langle S_1'S_2'\rangle=\langle -iP_3Q_2\rangle,\\
&\langle S_1S_1'\rangle=\langle S_2S_2'\rangle=\cosh 2H^*-\sinh 2H^*\cdot\langle iP_1Q_1\rangle,\\
&\langle S_1S_2'\rangle=\langle S_2S_3'\rangle=\cosh 2H^*\cdot\langle -iP_2Q_2\rangle+\sinh 2H^*\cdot\langle P_3P_2\rangle,\\
&\langle S_2S_3'\rangle=\langle -iP_3Q_2\rangle\cdot\cosh 2H^*+\sinh 2H^*\cdot\langle P_3P_1\rangle,\\
&\langle S_1S_3\rangle=\langle S_1'S_3'\rangle=\langle -iP_2Q_1\rangle\cdot\langle -iP_3Q_2\rangle+\langle P_3P_2\rangle\cdot\langle Q_2Q_1\rangle-\langle iP_2Q_2\rangle\cdot\langle iP_3Q_1\rangle.
\end{aligned}\tag{2}$$

The next step is to evaluate the average values of the 2-spinor quantities $P_k\cdot P_l$, $P_k\cdot Q_l$, $Q_k\cdot Q_l$ etc. The average value of $P_k\cdot P_l$ is given by the formula

$$\langle P_kP_l\rangle=\frac{2}{n^2\cdot\text{trace }\mathbf{V}^n}\lim_{\beta\to 0}\frac{\partial}{\partial\beta}\text{trace}\{\exp(\beta\sum_{i=\text{even}}P_{k+i}\cdot P_{l+i})\cdot\mathbf{V}\}^n,\tag{3}$$

where β is a parameter and n is the number of the parallel chains for the lattice. The translational symmetries are available, i.e.

$$\langle P_kP_l\rangle=\langle P_{k+2}\cdot P_{l+2}\rangle=\langle P_{k+4}\cdot P_{l+4}\rangle=\cdots.$$

The corresponding equations are applicable to the average values of P_kQ_l, Q_kQ_l.

It is feasible to carry out the calculation of trace $[\exp(\beta \sum_{i=\text{even}} P_{k+i}\cdot P_{l+i})\cdot \mathbf{V}]^n$ when the eigenvalue problem of the operator $\exp(\beta \sum_{i=\text{even}} P_{k+i}P_{l+i})\cdot \mathbf{V}$ is solvable. These eigen value problems are in general reduced to $2n$-dimensional eigenvalue problems. For our lattice, the eigenvalue problem of the operator $\exp(\beta \sum_{i=\text{even}} P_{k+i} \cdot P_{l+i})\cdot \mathbf{V}$ is decomposed into the eigenvalue problem based on the Kano and Naya method[7] which uses the shift operator.

For example we show the formulae for the case $\beta=0$. Then the four-dimensional matrix is written as

$$\begin{pmatrix} -S'S^*\varepsilon^{-1}, & -iS'C^*\varepsilon^{-1}, & C'C^*\varepsilon^{-1}, & iC'S^*\varepsilon^{-1}, \\ iSC^*\varepsilon, & -SS^*\varepsilon, & -iCS^*\varepsilon^{-1}, & CC^*\varepsilon^{-1} \\ iC^*\varepsilon, & iCS^*\varepsilon, & -SS^*\varepsilon^{-1} & -iCS^*\varepsilon^{-1}, \\ -iC'S^*\varepsilon, & C'C^*\varepsilon, & iS'C^*\varepsilon, & -S'S^*\varepsilon, \end{pmatrix} \tag{4}$$

where $\varepsilon=\exp i\omega=\exp 2\pi ir/n$, $(r=1, 2, 3, \cdots, n)$ and C, S, C', S', C^*, S^* are defined as follows:

$$\begin{aligned} &C=\cosh 2H,\ C'=\cosh 2H',\ C^*=\cosh 2L^*, \\ &S=\sinh 2H,\ S'=\sinh 2H',\ S^*=\sinh 2L^*. \end{aligned} \tag{5}$$

The eigenvalue λ_r of the matrix (4) is determined by

$$\begin{aligned} \frac{1}{2}(\lambda_r+\lambda_r^{-1}) &= -\frac{1}{2}S^*(S+S')\cdot\cos\omega+\frac{1}{2}\{(S^*)^2\cdot(S-S')^2\cdot\cos^2\omega+\mu\}^{1/2} \\ &=\cosh\gamma, \\ \mu&=2(C^*)^2\cdot(1+CC'+SS'),\ (0\leqq\omega\leqq 2\pi). \end{aligned} \tag{6}$$

Now we consider the case $\beta\neq 0$, and let the largest eigenvalue be $\exp(\sum_{j=1}^{n}\gamma_j)$. Then Eq. (3) becomes

$$\langle P_kP_l\rangle=\lim_{\beta\to 0}\frac{2}{n}\sum_{j=1}^{n}\frac{\partial\gamma_j}{\partial\beta}=\lim_{\beta\to 0}\frac{1}{\pi}\int_0^{\pi}(d\omega/\sinh\gamma_j)\cdot\left(\frac{\partial}{\partial\beta}\cosh\gamma_j\right). \tag{7}$$

With the aid of Eq. (7) we can get the average value of the operator $-iP_1Q_1$, $-iP_2Q_2$, $-iP_3Q_3$, etc. After some lengthy algebra we have

$$\begin{aligned} &\langle -iP_2Q_1\rangle=(C^*)^2\cdot(C'S+CS')\ \Sigma_0-2(S^*)^2\ S'C\Sigma_1-2S^*C\Sigma_2, \\ &\langle -iP_3Q_2\rangle=(C^*)^2\cdot(C'S+CS')\ \Sigma_0-2(S^*)^2\ SC'\Sigma_1-2S^*C'\Sigma_2, \\ &\langle iP_1Q_1\rangle=\langle iP_2Q_2\rangle \\ &\qquad =S^*C^*(1+CC'+SS')\Sigma_0-2SS'S^*C^*\Sigma_1-C^*(S+S')\Sigma_2, \end{aligned} \tag{8}$$

$$\langle P_2P_1\rangle = -S^*C^*(C'S+CS')\Sigma_0+2S'CS^*C^*\Sigma_1-C^*(C'-C)\Sigma_2,$$
$$\langle P_3P_2\rangle = -S^*C^*(C'S+CS')\Sigma_0+2SC'S^*C^*\Sigma_1+C^*(C'-C)\Sigma_2,$$
$$\langle Q_2Q_1\rangle = S^*C^*(C'S+CS')\Sigma_0-2S'CS^*C^*\Sigma_1-C^*(C'-C)\Sigma_2,$$
$$\langle Q_3Q_2\rangle = S^*C^*(C'S+CS')\Sigma_0-2SC'S^*C^*\Sigma_1+C^*(C'-C)\Sigma_2,$$
$$\langle iP_4P_2\rangle = \langle iP_3Q_1\rangle,$$

where Σ_0, Σ_1, Σ_2 are defined by the following integrals:

$$\Sigma_0 = \frac{1}{\pi}\int_0^\pi 1/D\cdot d\omega, \quad \Sigma_1 = \frac{1}{\pi}\int_0^\pi \cos^2\omega/D\cdot d\omega,$$

$$\Sigma_2 = \frac{1}{\pi}\int_0^\pi \cos\omega\cdot\cosh\gamma(\omega)/D\cdot d\omega, \tag{9}$$

$$D = \sinh\gamma(\omega)\{2\cosh\gamma(\omega)+S^*(S+S')\cdot\cos\omega\}.$$

An elliptic substitution is available for computing the integrals Σ_0, Σ_1, Σ_2. Let us choose for the elliptic modulus k and the complementary modulus k'. The expressions for k and k' are as follows:

$$k = a\{a^2+\mu/4\}^{-1/2}, \ k' = \mu^{1/2}\cdot\{a^2+\mu/4\}^{-1/2}/2,$$

where

$$a = |s^*(s-s')|/2. \tag{10}$$

We will define the Jacobian elliptic functions by

$$\cos\omega = \mathrm{cn}(u,k), \ \sin\omega = \mathrm{sn}(u,k),$$
$$\{1-k^2\cdot\sin^2\omega\}^{1/2} = \mathrm{dn}(u,k). \tag{10'}$$

Then in terms of the Jacobian elliptic functions we have

$$\cosh\gamma = \frac{a}{k}\cdot\mathrm{dn}(u,k)-b\cdot\mathrm{cn}(u,k),$$

where

$$b = S^*(S+S')/2$$

and

$$d\omega/D = kdu/2a\sinh\gamma,$$

$$\Sigma_0 = k/2\pi a\cdot\int_0^{2K} du/\sinh\gamma, \tag{11}$$

$$\Sigma_1 = k/2\pi a\cdot\int_0^{2K} \mathrm{cn}^2(u,k)\,du/\sinh\gamma,$$

$$\Sigma_2 = -b\Sigma_1 + \frac{1}{2\pi}\int_0^{2K} \mathrm{cn}(u, k)\,\mathrm{dn}(u, k)\,du/\sinh\gamma,$$

$$K = F\left(\frac{\pi}{2}, k\right).$$

The elliptic modulus k is equal to zero, unity or finite ($k \leqq 1/3$) according as the lattice is rectangular, triangular or honeycomb. From formulae (10), (10′) and (11) we have

$$\sinh\gamma = \{A\cdot\mathrm{cn}^2(u, k) + B\cdot\mathrm{cn}(u, k)\cdot\mathrm{dn}(u, k) + E\}^{1/2}. \tag{12}$$

where

$$A = a^2 + b^2, \quad B = -2ab/k, \quad E = a^2\cdot(k'/k)^2 - 1.$$

We will write Σ_0 by these elliptic functions as;

$$\Sigma_0 = k/2\pi a\cdot\int_0^{2K} du/\sinh\gamma$$

$$= k/2\pi a\cdot\int_0^{2K} \{A\,\mathrm{cn}^2(u, k)/\mathrm{dn}^2(u, k) + B\,\mathrm{cn}(u, k)/\mathrm{dn}(u, k)$$

$$+ E/\mathrm{dn}^2(u, k)\}^{-1/2}\cdot du/\mathrm{dn}(u, k). \tag{12′}$$

Then we can make use of the following formulae:

$$\mathrm{sn}(u+K, k) = \mathrm{cn}(u, k)/\mathrm{dn}(u, k),$$

$$\mathrm{dn}(u+K, k) = k'/\mathrm{dn}(u, k). \tag{13}$$

Then introducing a variable x defined by $x = \mathrm{sn}(u+K, k)$, we have

$$\Sigma_0 = -k/2\pi ak'\cdot\int_{-1}^{1} \varDelta^{-1}\cdot dx, \tag{14}$$

where

$$\mathrm{dn}(u+K, k)\cdot d(u+K) = -(1-x^2)^{1/2}\cdot dx,$$

$$\varDelta = \{(1-x^2)[\{b^2 + (k/k')^2\}x^2 - 2ab/k\cdot x + \{(a/k)^2 - (1/k')^2\}]\}^{1/2},$$

$$K \leqq x \leqq 3K.$$

By a similar way we can replace Σ_1, Σ_2 as follows:

$$\Sigma_1 = -(k'/k)^2\cdot\Sigma_0 + k'/2\pi ak\cdot\int_{-1}^{1}(1-k^2x^2)^{-1}\cdot\varDelta^{-1}\cdot dx, \tag{15}$$

$$\Sigma_2 = -b\Sigma_1 + 1/2\pi\cdot\int_{-1}^{1}(1-k^2x^2)^{-1}\cdot\varDelta^{-1}\cdot dx.$$

These formulae are also written by the Legendre elliptic integrals, and we shall deal with these integrations in a compact way or in a good approximation.

§ 3. Isotropic honeycomb net

In this chapter we consider a case $|H|=|L|$ with $H'=0$ in Fig. 1a. (See Fig. 1b.) For this case the calculations of the foregoing integrations are greatly simplified by virtue of the following relations:

$$SS^*=1,\ S^*C=C^*,\ SC^*=C,$$
$$k^2=\{1+2C^{2*}(1+C)\}^{-1}\leqq 1/9,$$
$$\cosh\gamma=1/2k\cdot\mathrm{dn}(u,k)-1/2\cdot\mathrm{cn}(u,k).$$

Especially at the critical temperature T_c, the integrals (14) and (15) can be expressed in a closed form, and k and $\sinh\gamma$ become

$$k=1/3,\ C=2,\ S=\sqrt{3},\ C^*=2/\sqrt{3},\ S^*=1/\sqrt{3},$$
$$\sinh\gamma=\sqrt{3}/2\sqrt{2}\cdot\{\mathrm{dn}(u,k)-\mathrm{cn}(u,k)\}^{1/2}\cdot\{3\,\mathrm{dn}(u,k)-\mathrm{cn}(u,k)\}^{1/2}.$$

For this case the integrals are easily evaluated as follows:

$$\Sigma_0-\Sigma_1=2\sqrt{3}/\pi\cdot\int_0^{2K}\{1-\mathrm{cn}^2(u,k)\}du/\{\mathrm{dn}(u,k)-\mathrm{cn}(u,k)\}^{1/2}$$
$$\times\{3\mathrm{dn}(u,k)-\mathrm{cn}(u,k)\}^{1/2}$$
$$=2\sqrt{3}/\pi\cdot\int_{-1}^{1}dx/(3-x)(1+x)^{1/2}(3-x)^{1/2}-\sqrt{3}/\pi\cdot\int_{-1}^{1}dx/(3+x)(1+x)^{1/2}(3-x)^{1/2}$$
$$=\sqrt{3}/\pi-1/3,$$
$$\Sigma_0-\Sigma_2=2\sqrt{3}/\pi\cdot\int_0^{2K}\{1-\mathrm{cn}(u,k)\cdot\cosh\gamma\}du/\{\mathrm{dn}(u,k)-\mathrm{cn}(u,k)\}^{1/2}$$
$$\times\{3\mathrm{dn}(u,k)-\mathrm{cn}(u,k)\}^{1/2}$$
$$=\sqrt{3}/\pi\cdot\int_{-1}^{1}dx/(3+x)(1+x)^{1/2}\cdot(3-x)^{1/2}$$
$$=1/3.$$

Thus we have the critical data

$$\langle P_3P_2\rangle=-4/\sqrt{3}\cdot(\Sigma_0-\Sigma_1)+2/\sqrt{3}\cdot(\Sigma_0-\Sigma_2)=-0.118\ 538,$$
$$\langle iP_3Q_1\rangle=-\sqrt{3}\langle P_3P_2\rangle=0.205\ 314,$$
$$\langle iP_1Q_1\rangle=2(\Sigma_0-\Sigma_2)=2/3,$$

$$\langle Q_2Q_1\rangle = 2/\sqrt{3}\cdot(\Sigma_0 - \Sigma_2) = 0.384\ 900,$$

$$\langle S_1S_2\rangle = 4/\sqrt{3}\cdot(\Sigma_0 - \Sigma_2) = 0.769\ 800,$$

$$\langle S_2S_3\rangle = 2/\sqrt{3}\cdot(\Sigma_0 - \Sigma_1) + 2/\sqrt{3}\cdot(\Sigma_0 - \Sigma_2) = 0.636\ 619,$$

$$\langle S_1S_2'\rangle = \langle S_2S_3'\rangle = 2/3,$$

$$\langle S_1S_3\rangle = 0.581\ 321.$$

The dependence of the spin pair correlation upon temperature is shown in Fig. 2.

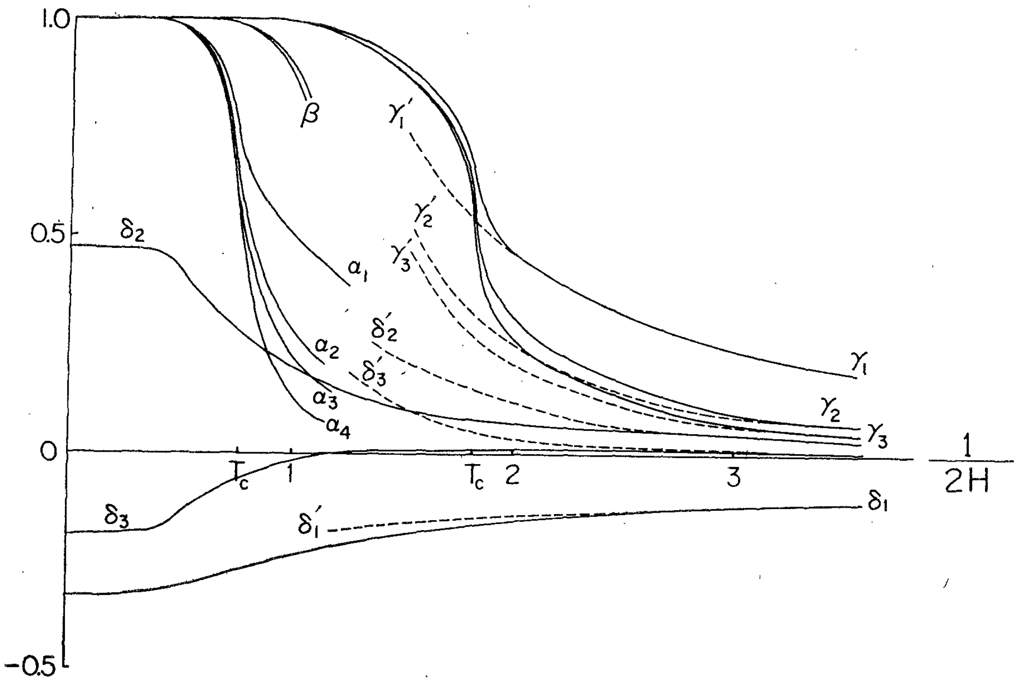

Fig. 2.
Short-range order of the plane Ising net
(α), Honeycomb net $\alpha_1:\langle S_1\ S_2\rangle$, $\alpha_2:\langle S_1\ S_2'\rangle$, $\alpha_3:\langle S_2\ S_3\rangle$, $\alpha_4:\langle S_1\ S_3\rangle$,
(β), Square net
(γ), Triangular net (Ferromagnetic case)
exact $\gamma_1:\langle S_1\ S_2\rangle$, $\gamma_2:\langle S_1\ S_3\rangle$, $\gamma_3:\langle S_1\ S_3'\rangle$,
high temperature approximation $\gamma_1':\langle S_1\ S_2\rangle$, $\gamma_2':\langle S_1\ S_3\rangle$, $\gamma_3':\langle S_1\ S_3'\rangle$,
(δ), Triangular net (Antiferrimagnetic case)
exact $\delta_1:\langle S_1\ S_2\rangle$, $\delta_2:\langle S_1\ S_3\rangle$, $\delta_3:\langle S_1\ S_3'\rangle$,
high temperature approximation $\delta_1':\langle S_1\ S_2\rangle$, $\delta_2':\langle S_1\ S_3\rangle$, $\delta_3':\langle S_1\ S_3'\rangle$.

§ 4. Isotropic triangular net (ferromagnetic net)

On this lattice it is simpler to discuss the lattice in Fig. 3 than in Fig. 1a for the calculation. (See Fig. 3a.) We can treat the problem on the analogy of the procedure of the foregoing section. Instead of formulae (2), we have for this lattice

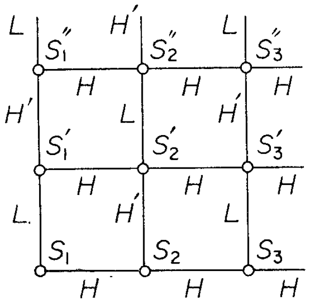

Fig. 3a.
The lattice is reduced to a triangular lattice as a parameter H' tends to infinity.

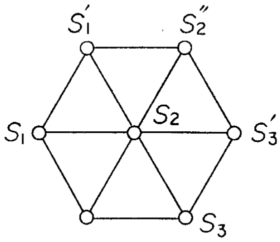

Fig. 3b.
For $H'\to\infty$, S_2, S_2' can be regarded as the same spin.

$$\begin{aligned}
\langle S_1S_2\rangle &= C^*-S^*\langle iP_2Q_2\rangle,\\
\langle S_1S_3\rangle &= \langle S_1S_2\rangle\langle S_2S_3\rangle+\langle P_3P_2\rangle\langle Q_2Q_1\rangle-\langle iP_2Q_2\rangle\langle iP_3Q_1\rangle,\\
\langle S_1S_3'\rangle &= C^*\langle S_1S_3\rangle+S^*\{\langle iP_1Q_1\rangle\langle P_4P_2\rangle-\langle P_2P_1\rangle\langle iP_2Q_3\rangle\\
&\qquad +\langle P_3P_2\rangle\langle -iP_2Q_1\rangle\}.
\end{aligned}\tag{16}$$

Moreover we must change formulae (8) for the following formulae:

$$\begin{aligned}
\langle -iP_2Q_1\rangle &= C(C+S)\Sigma_0^*-C^*\Sigma_2^*,\\
\langle iP_1Q_1\rangle &= CC^*\Sigma_0^*-2S\Sigma_1^*-2S\Sigma_2^*,\\
\langle iP_2Q_2\rangle &= CC^*\Sigma_0^*-2C\Sigma_2^*,\\
\langle P_2P_1\rangle &= -\langle Q_3Q_1\rangle\\
&= -C\Sigma_0^*+2C\Sigma_1^*-C(C^*-1)\Sigma_2^*,\\
\langle iP_2Q_3\rangle &= C(C+S)\Sigma_0^*-2C(S+C)\Sigma_1^*+C^*\Sigma_2^*,\\
\langle P_4P_2\rangle &= 4\Sigma_3^*,\\
\langle P_3P_2\rangle &= -\langle Q_2Q_1\rangle=-C\Sigma_0^*+C(C^*-1)\Sigma_2^*,\\
\cosh\gamma^* &= 1/2k^*\cdot\mathrm{dn}(u,k)+1/2\cdot\mathrm{cn}(u,k),
\end{aligned}\tag{17}$$

where k^*, Σ_0^*, Σ_1^*, Σ_2^*, Σ_3^* are as follows:

$$\begin{aligned}
&k^*=\{1+2C^2(1+C^*)\},\quad \Sigma_0^*=\frac{1}{\pi}\int_0^\pi d\omega/D^*,\\
&\Sigma_1^*=1/\pi\int_0^\pi \cos^2\omega/D^*\cdot d\omega,\\
&\Sigma_2^*=1/\pi\int_0^\pi \cosh\gamma^*\cdot\cos\omega/D^*\cdot d\omega,\\
&\Sigma_3^*=1/\pi\int_0^\pi \cosh\gamma^*\cdot\sin^2\omega\cdot\cos\omega/D^*\cdot d\omega,
\end{aligned}\tag{18}$$

$$D^* = \sinh \gamma^* \cdot (2 \cosh \gamma^* + \cos \omega).$$

The integration of Σ_0^*, Σ_1^* and Σ_2^* is similar to the cases of honeycomb net. In the previous section k^2 is equal to $\{1+2C^{2*}(1+C)\}^{-1}$ but in this case $(k^*)^2$ is equal to $\{1+2C^2(1+C^*)\}^{-1}$. Now C and C^*, S and S^* are replaced with each other, respectively. The so-called dual relation appears in this replacement and it is easy to evaluate Σ_0^*, Σ_1^*, Σ_2^* from the values of Σ_0, Σ_1, Σ_2. (Cf. Figs. 1a and 3a.) The dependence of the spin correlation upon temperature is shown in Fig. 2. Then we consider the correlation at the critical temperature in a closed form. The integration for $\Sigma_0^*-\Sigma_1^*$, $\Sigma_0^*-\Sigma_2^*$ is known from the values $\Sigma_0-\Sigma_1$, $\Sigma_0-\Sigma_2$ and the integration for Σ_3^* is reduced to the Legendre elliptic integrals of the three kinds. The elliptic modulus k^* is equal to 1/3 at $T=T_c$, and we have

$$\Sigma_0^*-\Sigma_1^* = \sqrt{3}/\pi - 1/3, \qquad \Sigma_0^*-\Sigma_2^* = 1/3,$$

$$\Sigma_3^* = 1/\pi \cdot \int_0^\pi \sin^2\omega \cdot \cos\omega \cdot \cosh\gamma \cdot d\omega / \sinh\gamma^* \cdot (2\cosh\gamma^* + \cos\omega)$$

$$= 40/3\sqrt{3} \cdot \int_{-1}^{1} x[(1+x)(3-x)]^{1/2} \cdot dx/(3+x)^2 \cdot (3-x)^2$$

$$= 4/3 - 5\sqrt{3}/2\pi$$

$$C = 2/\sqrt{3}, \ S = 1/\sqrt{3}, \ C^* = 2, \ S^* = \sqrt{3}.$$

Thus we have the critical values for the ferromagnetic triangular lattice:

$$\langle S_1S_2\rangle = 2(\Sigma_0^*-\Sigma_2^*) = 2/3,$$

$$\langle S_1S_3\rangle = 8(\Sigma_0^*-\Sigma_1^*) \cdot (\Sigma_0^*-\Sigma_2^*) = 8/3 \cdot (\sqrt{3}/\pi - 1/3) = 0.581\ 321,$$

$$\langle S_1S_3'\rangle = 8(\Sigma_0^*-\Sigma_1^*) \cdot \Sigma_3^* + 8(\Sigma_0^*-\Sigma_2^*) \cdot \Sigma_3^* + 16(\Sigma_0^*-\Sigma_1^*)^2$$

$$= -12/\pi^2 - 16/9 = 0.561\ 924.$$

§ 5. An isotropic triangular net (antiferromagnetic net)

We consider the lattice, $H=L$, $H<0$, $H'=\infty$ in Fig. 3a; and in this case we shall change L and H for $-L$ and $-H$ respectively. This transformation corresponds to

$$C \to C, \ S \to -S, \ C^* \to -C^*, \ S^* \to -S^*,$$

$$\gamma^* \to \gamma^\dagger, \ \Sigma_0^* \to \Sigma_0^\dagger, \ \Sigma_1^* \to \Sigma_1^\dagger, \ \Sigma_2^* \to \Sigma_2^\dagger, \ \Sigma_3^* \to \Sigma_3^\dagger,$$

where $\gamma^\dagger$, $\Sigma_0^\dagger$, $\Sigma_1^\dagger$, $\Sigma_2^\dagger$, $\Sigma_3^\dagger$ are similar to the case of ferromagnetic triangular net. In general, $\gamma^\dagger$ becomes a complex number and we shall evaluate the integral in a somewhat different way. We have

$$\Sigma_0^\dagger = 1/2\pi \cdot \int_0^\pi \{1/D_1^\dagger - 1/D_2^\dagger\}\, d\omega\,,$$

$$\Sigma_1^\dagger = 1/2\pi \cdot \int_0^\pi \{1/D_1^\dagger + 1/D_2^\dagger\} \cos \omega d\omega\,,$$

$$\Sigma_2^\dagger = 1/2\pi \cdot \int_0^\pi \{\cosh \gamma_1^\dagger / D_1^\dagger + \cosh \gamma_2^\dagger / D_2^\dagger\} \cos \omega d\omega\,, \tag{19}$$

$$\Sigma_3^\dagger = 1/2\pi \cdot \int_0^\pi \{\cosh \gamma_1^\dagger / D_1^\dagger + \cosh \gamma_2^\dagger / D_2^\dagger\} \sin^2\omega \cdot \cos \omega \cdot d\omega\,,$$

where

$$D_1^\dagger = \sinh \gamma_1^\dagger \cdot \{2\cosh \gamma_1^\dagger + \cos \omega\}\,,$$
$$D_2^\dagger = \sinh \gamma_2^\dagger \cdot \{2\cosh \gamma_2^\dagger + \cos \omega\}\,, \tag{20}$$
$$\cosh \gamma_1^\dagger = -1/2 \cdot \cos \omega + i/2 \cdot \{\mu - \cos^2\omega\}^{1/2},$$
$$\cosh \gamma_2^\dagger = -1/2 \cdot \cos \omega - i/2 \cdot \{\mu - \cos^2\omega\}^{1/2},$$
$$\mu = 2C^2 \cdot (C^* - 1)\,, \qquad 0 \leqq \omega \leqq \pi\,.$$

We define $\mathrm{cn}(u, k^\dagger)$, $\mathrm{sn}(u, k^\dagger)$ and $\mathrm{dn}(u, k^\dagger)$ by

$$\cos \omega = \mathrm{sn}(u, k^\dagger)\,, \ \sin \omega = \mathrm{cn}(u, k^\dagger)\,,$$
$$\{1 - (k^\dagger)^2\}^{1/2} = \mathrm{dn}(u, k^\dagger)\,, \ k^\dagger = (1 + \mu/4)^{-1} = \{1 + 1/2 \cdot C^2 (C^* - 1)\}^{-1}.$$

Now let us put

$$\sinh \gamma_1^\dagger = A + iB,\ A < 0,\ B > 0,\ 0 < \omega < \pi/2\,;$$
$$\sinh \gamma_2^\dagger = A - iB,\ A > 0,\ B > 0,\ \pi/2 < \omega < \pi\,, \tag{21}$$

where A and B are real. Then we can get the following relations:

$$\{\mu - \cos^2\omega\}^{1/2} = 1/k^\dagger \cdot \{2k^\dagger - 1 + \mathrm{dn}(u, k^\dagger)\}^{1/2} \cdot \{1 - 2k + \mathrm{dn}(u, k^\dagger)\}^{1/2},$$
$$2A = \pm 1/k^\dagger \cdot \{1 - \mathrm{dn}(u, k^\dagger)\}^{1/2} \cdot \{1 - 2k^\dagger + \mathrm{dn}(u, k^\dagger)\}^{1/2},$$
$$2B = 1/k^\dagger \cdot \{1 + \mathrm{dn}(u, k^\dagger)\}^{1/2} \cdot \{2k^\dagger - 1 + \mathrm{dn}(u, k^\dagger)\}^{1/2}, \tag{22}$$
$$A^2 + B^2 = 1/k^\dagger \cdot \mathrm{dn}(u, k^\dagger)\,.$$

Using the relations (21) and (22), $\Sigma_0^\dagger$, $\Sigma_1^\dagger$, $\Sigma_2^\dagger$, $\Sigma_3^\dagger$ can be reduced to the Legendre elliptic integral of three kinds:

$$\Sigma_0^\dagger = k^\dagger/\pi \cdot \int_0^{K\dagger} \{1 + \mathrm{dn}(u, k^\dagger)\}^{1/2} \cdot \{1 - 2k^\dagger + \mathrm{dn}(u, k^\dagger)\}^{-1/2} du$$

$$=k^{\dagger}/\pi\cdot\int_{(k^{\dagger})'}^{1}\{[x+(k^{\dagger})'][x-(k^{\dagger})'][1-x][1-2k^{\dagger}+x]\}^{-1/2}\cdot du,$$

where

$$K^{\dagger}=F(\pi/2,\,k^{\dagger}),\ \ (k^{\dagger})'=\{1-(k^{\dagger})^2\}^{1/2},$$

$$\Sigma_1{}^{\dagger}=1/\pi\,(k^{\dagger})^4\cdot\int_{(k^{\dagger})'}^{1}(1+x)^{1/2}\{[x^2-(k^{\dagger})'^2][1-2k^{\dagger}+x]\}^{-1/2}\cdot dx,$$

$$\Sigma_2{}^{\dagger}=-1/2\cdot\Sigma_1{}^{\dagger}-k^{\dagger}/2\pi\cdot\int_{(k^{\dagger})'}^{1}\{x^2-(k^{\dagger})'^2\}^{1/2}\{(1-x)(1-2k^{\dagger}+x)\}^{1/2}dx,$$

$$\Sigma_3{}^{\dagger}=\Sigma_2{}^{\dagger}+1/2\pi\,(k^{\dagger})^3\cdot\int_{(k^{\dagger})'}^{1}(1-x^2)^2[x^2-(k^{\dagger})'^2]^{-1/2}[(1-x)(1-2k^{\dagger}+x)]^{-1/2}\cdot dx,$$

$$+1/2\pi\,(k^{\dagger})^3\cdot\int_{(k^{\dagger})'}^{1}(1-x^2)\,[x^2-(k^{\dagger})'^2]^{-1/2}[(1-x)(1-x-2k^{\dagger})]^{1/2}\cdot dx,$$

where $x=\mathrm{dn}(u,\,k^{\dagger})$, and $(k^{\dagger})'$ is a complementally modulus for $k^{\dagger}$. Especially, near the zero temperature, we have the following approximate formulae:

$$\begin{aligned}&\langle S_1S_2\rangle\simeq-1/2\cdot(k^{\dagger}\sigma_1+\tau_1),\\&\langle S_1S_3\rangle\simeq\langle S_1S_2\rangle^2+(k^{\dagger}\sigma_1+\tau_1)(k^{\dagger}\sigma_1-\tau_1),\\&\langle S_1S_3'\rangle\simeq-\langle S_1S_3\rangle+2\langle S_1S_2\rangle^2\\&\qquad+(k^{\dagger}\sigma_1-\tau_1)\{2\cdot(k^{\dagger}\sigma_2-\tau_2)-1/2(k^{\dagger}\sigma_1-\tau_1)\},\end{aligned}\tag{23}$$

where

$$k^{\dagger}\sigma_1=1/\pi k^{\dagger}\cdot\int_{(k^{\dagger})'}^{1}[x^2-(k^{\dagger})'^2]\{[x^2-(k^{\dagger})'^2][1-x][1-2k^{\dagger}+x]\}^{-1/2}\cdot dx,$$

$$k^{\dagger}\sigma_2=1/\pi\,(k^{\dagger})^3\cdot\int_{(k^{\dagger})'}^{1}[x^2-(k^{\dagger})'^2]^2\cdot\{[x^2-(k^{\dagger})'^2][1-x][1-2k^{\dagger}+x]\}^{-1/2}\cdot dx,$$

$$\tau_1=k^{\dagger}/\pi\cdot\int_{(k^{\dagger})'}^{1}\{[x^2-(k^{\dagger})'^2]^{-1/2}[1-x]^{1/2}[1-2k^{\dagger}+x]^{1/2}\}\,dx,$$

$$\tau_2=1/\pi k^{\dagger}\cdot\int_{(k^{\dagger})'}^{1}[x^2-(k^{\dagger})^2]^{1/2}\{[1-x][1-2k^{\dagger}+x]\}^{1/2}\,dx.$$

At zero temperature,

$$k^{\dagger}=4/5,\ \ (k^{\dagger})'=3/5,\ \ -1+2k=(k^{\dagger})',$$

$$k^{\dagger}\sigma_1+\tau_1=1/3,\ k^{\dagger}\sigma_1-\tau_1=\sqrt{3}/\pi,$$
$$k^{\dagger}\sigma_2-\tau_2=1/3-\sqrt{3}/4\pi.$$

The approximations (23) are exact at $T=0$, and we have

$$\langle S_1S_2\rangle=-1/3,\ \langle S_1S_3\rangle=1/9+2/\sqrt{3}\pi=0.478\ 664,$$
$$\langle S_1S_3'\rangle=1/9-3/\pi^2=-0.192\ 852.$$

The curves for the antiferromagnetic triangular net are plotted in Fig. 2. Lastly the family of the curves of the short-range order for the ferromagnetic and antiferromagnetic triangular nets has been checked by the high-temperature expansion series:

$$\langle S_1S_2\rangle=k+2k^2+4k^3+9k^4+\cdots,$$
$$\langle S_1S_3\rangle=\quad 2k^2+5k^3+16k^4+\cdots,$$
$$\langle S_1S_3'\rangle=\quad k^2+6k^3+18k^4+\cdots,$$
$$k=\pm e^{-2H^*}\qquad (H=L).$$

These curves are shown in Fig. 2, and the comparison with the exact curve is quite satisfactory at high temperatures.

§ 6. The probabilities of the spin configuration between three, four and five neighbouring sites (an antiferromagnetic triangular net)

First we consider the probability p that the neighbouring two spins are parallel; it is written as

$$p=(1+\langle S_1S_2\rangle)/2. \tag{24}$$

The probability that the neighbouring two spins are anti-parallel is denoted by q; it is given by

$$q=1-p. \tag{25}$$

Farther we consider the probability a on condition that three neighbouring spins located at the vertices of the smallest regular triangle are parallel to each other. (See Fig. 4.) We also consider the probability b on condition that the neighbouring two spins are parallel but the third spin is anti-parallel to others. (See Fig. 4.) Then we easily get the following relations:

$$a+1/3\cdot b=p,\qquad 2/3\cdot b=q,$$
$$a=(1+3\langle S_1S_2\rangle)/4,\qquad b=3(1-\langle S_1S_2\rangle)/4. \tag{26}$$

Now let us introduce more complicated probabilities by the similar definitions. According to the schematic representation in Fig. 4, we can define the probabilities c, c', d, d', e and e', and their definitions are exactly similar

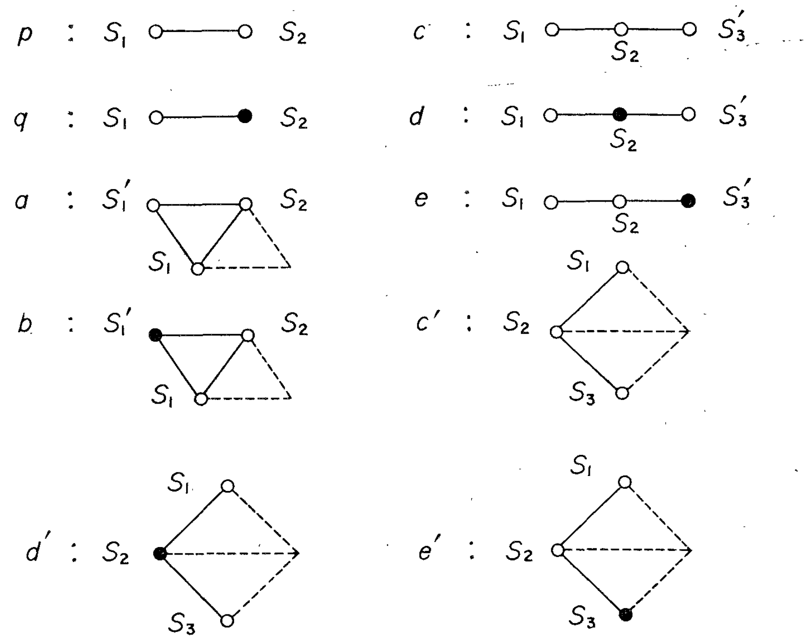

Fig. 4. The schematic representations of the two-spin probabilities p, q and three-spin probabilities a, b, c, d, e, c', d', e'.

to that of a and b. It is easily shown that these probabilities are satisfied by the following relations:

$$\begin{aligned}
&c+d+e=1, \qquad c'+d'+e'=1,\\
&c+d=(1+\langle S_1S_3'\rangle)/2,\ c'+d'=(1+\langle S_1S_3\rangle)/2,\\
&c+e/2=(1+\langle S_1S_2\rangle)/2,\ c'+e'/2=(1+\langle S_1S_2\rangle)/2,\\
&c=(1+2\langle S_1S_2\rangle+\langle S_1S_3'\rangle)/4,\\
&d=(1-2\langle S_1S_2\rangle+\langle S_1S_3'\rangle)/4,\\
&e=(1-\langle S_1S_3'\rangle)/2,\\
&c'=(1+2\langle S_1S_2\rangle+\langle S_1S_3\rangle)/4,\\
&d'=(1-2\langle S_1S_2\rangle+\langle S_1S_3\rangle)/4,\\
&e'=(1-\langle S_1S_3\rangle)/2.
\end{aligned} \tag{27}$$

The dependence of these probabilities upon temperature is shown in Fig. 5. The probability a is zero at $T=0$. However, c and c' are finite even at zero temperature and 0.202998 and 0.035120 respectively. At high temperature the curves for c and c', d and d', e and e' are almost equal to each other, respectively, but this is not so for a and b. But they are considerably different from each other at low temperaure, and it is seen that $e'>d>d'>e>c>c'$.

We will consider the probabilities of the four-spins configurations f, g_1, g_2, h_1 and h_2 at $T=0$. (See Fig. 6.) At zero temperature the probability that

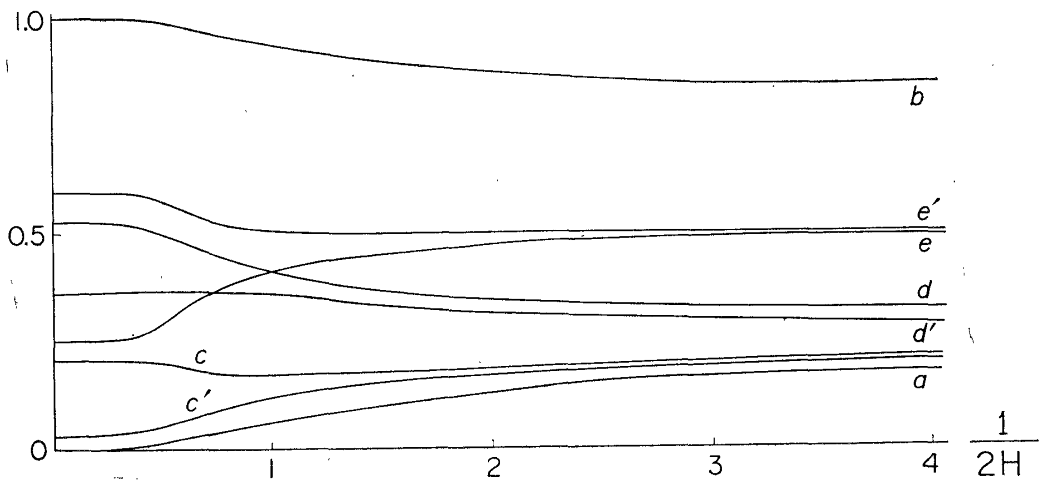

Fig. 5. The probabilities a, b, c, d, e, c', d', e' versus $1/2H$.

three neighbouring spins are located at the vertices of the smallest regular triangle are zero. By this fact it is easily shown that the probabilities f and g_1 are zero at $T=0$, and g_2, h_1, h_2 are equals to 0.07024, 0.59642, 0,33333, respectively, by the following relations:

$$f+g_1/2=a_0=0,\ f\geqq 0,\ g_1\geqq 0,$$

$$f+g_2/2=c_0',\ g_2/2+h_1/2=b_0/3,\ h_2=b_0/3,$$

where a_0, b_0 and c_0' are the values of a, b and c' at $T=0$.

Lastly we will consider the probabilities of the five-spins configurations i, j, k_1, k_2, k_3, l and m at $T=0$. (See Fig. 6.) It is known that $i=j=k_1=a_0=0$, $k_2=0.40599$, $k_3=0.07024$, $l=0.38085$, $m=0.14290$ by the following relations:

$$i+j/2+k_1/2=a_0=0,\ k_1/2+l/2+m=b_0/3,$$

$$k_1/2+k_3/2+j/4+l/4=e_0/2,$$

$$i+j/2+k_3/2=c_0',\ k_2/2+l/2+m=d_0,$$

where e_0 and d_0 are the values of e and d at $T=0$. We can show that $n>0.60899$ at $T=0$, and this probability is largest among the probabilities of the seven-spins. Through these values of probabilities the antiferromagnetic charactor of this lattice is indicated fairly well.

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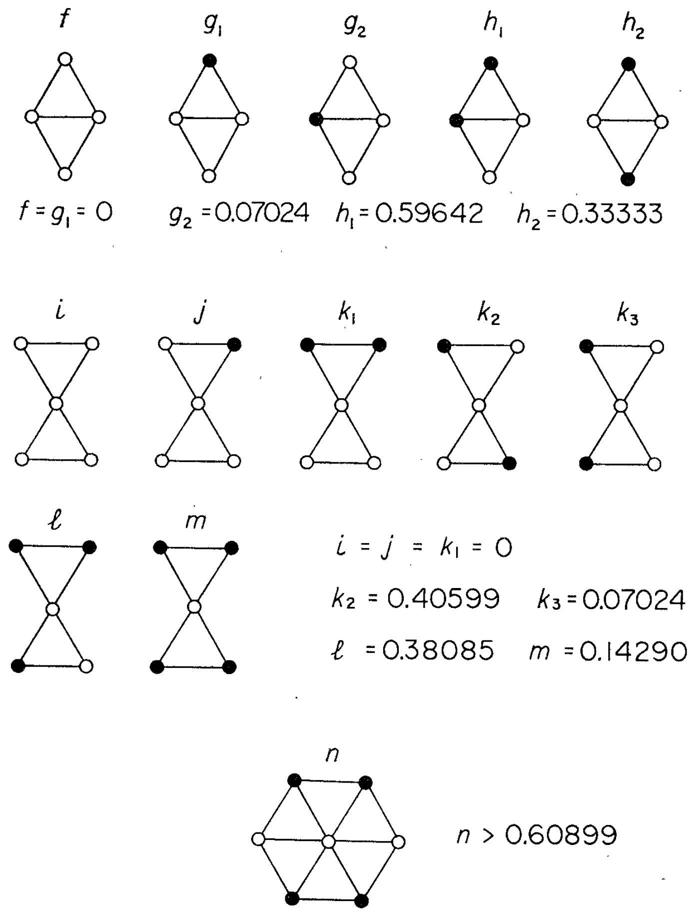

Fig. 6. The schematic representation of the four-spins probabilities f, g_1, g_2 h_1, h_2 and the five or seven-spins probabilities i, j, k_1, k_2, k_3, l, m or n at zero temperature. The values of these probabilities at $T=0$ are given. We can show that $n>0.60899$ at $T=0$ and this probability is largest among the probabilities of the seven-spins.

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