Shortening of primary operators in N-extended SCFT$_4$ and harmonic-superspace analyticity

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Abstract

We present the analysis of all possible shortenings which occur for composite gauge invariant conformal primary superfields in $SU(2,2/N)$ invariant gauge theories.

These primaries have top-spin range $\frac{N}{2} \leq J_{\max} < N$ with $J_{\max} = J_1 + J_2$, (J_1, J_2) being the $SL(2,\mathbb{C})$ quantum numbers of the highest spin component of the superfield.

In Harmonic superspace, analytic and chiral superfields give $J_{max} = \frac{N}{2}$ series while intermediate shortenings correspond to fusion of chiral with analytic in $N = 2$, or analytic with different analytic structures in $N = 3, 4$.

In the AdS/CFT language shortenings of UIR's correspond to all possible BPS conditions on bulk states.

An application of this analysis to multitrace operators, corresponding to multiparticle supergravity states, is spelled out.

1   Introduction

The recent interplay between supergravity in AdS_5 and superconformal $SU(\mathcal{N})$ Yang–Mills theories in the large $\mathcal{N}$ limit [44, 25, 54] has lead to a deeper investigation of $SU(2,2/N)$ superconformal algebras and their UIR's both on bulk states and on superfield boundary operators.

A complete identification of highest weight UIR's was given in ref. [15] for $N = 1$ and further extended to any N by Dobrev and Petkova [10] and also in [5, 45].

Since Yang–Mills theories are built only with a finite number of supersingleton fields, having $J_{max} \leq 1$ (these are the basic multiplets of the 4 dimensional superconformal theory) only a subclass of all possible UIR's are realized in QFT, nevertheless the variety of such representations is still rather rich and many different shortenings may occur.

Short multiplets have an important aspect in the AdS/CFT correspondence because they have "protected" conformal dimensions and therefore allow a reliable comparison between quantities computed in the bulk versus quantities derived in the CFT_4 [52, 12].

A particular example of such a phenomenon are the K–K masses of bulk states which belong to short $SU(2, 2/N)$ UIR's [26]. For such states in the $N = 4$ case, corresponding to IIB supergravity on $AdS_5 \times S_5$ [29], it is possible to give, at least for large $\mathcal{N}$, an "exact" operator realization in terms of 4d shortened superconformal fields [11, 2].

Another example of such a correspondence has been worked out in the literature [24, 6] by comparing IIB supergravity on $AdS_5 \times T_{11}$ [48] and a specific $SU(2, 2/1)$ invariant $SU(\mathcal{N}) \times SU(\mathcal{N})$ Yang–Mills theory constructed by Klebanov and Witten [42].

Already in this simple $N = 1$ example it was realized that $N = 1$ chiral superfields [13] are only a particular case of short representations. Indeed it was shown [6] that semishort multiplets occur in the K–K spectrum of IIB supergravity on $AdS_5 \times T_{11}$, with the very subtle implication that some square root formulae for the conformal dimensions, giving in general irrational numbers, become perfect squares for particular relations of the quantum numbers of the bulk states, precisely corresponding to semi-shortening conditions, which imply rational conformal dimensions.

For $N > 1$ $SCFT_4$'s the shortening and semi-shortening is even richer because maximal shortening (which means half of the total number of θ's) can occur either with chiral superfields or with "Grassmann (G–)analytic" superfields [18]-[32].

This may happen because a new class of UIR's occur for $N \geq 2$ which have no $N = 1$ analogue (class C) in the classification given below [10]).

The $N = 2$ hypermultiplets and the $N = 3, 4$ Yang–Mills field strength multiplets (supersingletons) belong precisely to this class of UIR's together with an infinite tower of recurrences. At the same, in superspace they are described by the new type of short (G–analytic) superfields (see [18]-[19] for $N = 2$ and [31, 32] for $N = 3, 4$).

A crucial ingredient to understand the occurrence of different shortenings for composite superconformal primaries in $N = 2, 3, 4$ theories is the use of harmonic superspaces [19, 20, 39, 31] with harmonic variables on $\frac{SU(N)}{U(1)^{N-1}}$, i.e., the coset given by non-Abelian R–symmetry modded by its Cartan subalgebra (the maximal torus).

Analytic superfields (in harmonic superspace) correspond to a new class of $N \geq 2$ UIR's which have no $N = 1$ analogue. Moreover for $N > 2$, since the above coset has many complex variables, different types of analyticity may occur and this allows for an even richer structure of shortenings when composites of superfields with different G–analytic structure are considered.

The present paper is organized as follows:

In Section 2 we review the unitarity bounds of highest weight UIR's of $SU(2,2/N)$ $(N \geq 2)$ and of $PSU(2,2/4)$ for $N = 4$.

In Section 3 we consider extended superspaces with harmonic variables in $\frac{SU(N)}{U(1)^{N-1}}$ for $N = 2,3,4$ in subsections 1, 2, 3 respectively. G–analytic properties of "supersingleton" representations (the would-be massless fields on the boundary) are explained together with different analytic structures occurring for $N = 2,3,4$.

Note that although $N = 3$ Yang–Mills is believed to be the same of $N = 4$, certainly this is not the case for the bulk theory since $N = 6$ supergravity on AdS_5 is not the same as $N = 8$ [30].

This can be understood from the fact that there is a ring of operators which reproduces the $N = 6$ supergravity states [12, 38].

Another fact is that $N = 3$ harmonic superspace provides an off-shell formulation of maximally supersymmetric Yang–Mills theory which is not available in its own $N = 4$ superspace [20, 40].

The three subsections are written in an independent way, so that the reader not interested can skip any of them.

In Section 4 we consider an application to multitrace operators in $N = 4$ Yang–Mills theory by showing that all such operators have some superconformal irreducible components which are short with different types of shortenings.

In the AdS_5/CFT_4 correspondence they should correspond to multiparticle BPS supergravity states preserving respectively $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$ supersymmetries. The first two types of states occur in the double-trace operators while the third type starts to occur for triple-trace operators.

In Section 5 a summary of our results is presented. Some technical material related to properties of harmonic superspace is collected in an appendix.

2   Highest weight UIR's of $SU(2,2/N)$: a review

The $SU(2,2/N)$ superalgebra has a 5-grading decomposition [3, 27, 28]:

$$\mathcal{L}_N = \mathcal{L}^1 + \mathcal{L}^{\frac{1}{2}} + \mathcal{L}^0 + \mathcal{L}^{-\frac{1}{2}} + \mathcal{L}^{-1} \qquad (2.1)$$

with respect to its maximal compact subalgebra $\mathcal{L}^0 = SU(2) \times SU(2) \times U(1) \times U(N)$.

A highest weight state is defined as

$$\mathcal{L}^{-\frac{1}{2}}|\Omega> = \mathcal{L}^{-1}|\Omega> = 0 \; . \qquad (2.2)$$

A highest weight UIR representation is specified by a UIR rep. of Ω with respect to $\mathcal{L}^0$. The eigenvalues of the two $U(1)$ respectively denote E_0 (the AdS energy) and the $U(1)$ R-charge.

The "compact basis" is suitable to discuss bulk states, i.e., UIR's on AdS_5. In the AdS/CFT correspondence the CFT operators are naturally described in the "non-compact basis"[28], in which the highest weight state is mapped into a space–time superfield whose lowest component $\phi(x,\theta)|_{x=\theta=0}$ is a irreducible representation of $SL(2,\mathbb{C}) \times O(1,1) \times U(N)$.

In this correspondence the (J_1, J_2) quantum numbers denote a $SL(2,\mathbb{C})$ irreducible representation and $E_0 \rightarrow \ell$ denotes the dilatational weight (conformal dimension).

A space–time superfield, whose lowest component corresponds to a highest weight state, is called a "primary" (or quasi-primary) conformal superfield. Needless to say that $SU(2,2/N)$ will have "supercasimir" operators which are at least $3 + N = \text{rank}(SU(2,2) \times U(N))$ ($3 + N - 1$ for $N = 4$ if we consider the superalgebra $PSU(2,2/4)$ [5] as we will do in the present paper).

In what follows it will be convenient to define "Poincaré" super-casimirs for the $SU(N)$ part of the superconformal algebra in order to study the irreducible content of the superfields in Harmonic superspace (HSS). The other quantum number properties will be straightforward.

Since a UIR of $SU(2,2/N)$ is denoted by its highest weight state, we will mainly denote such rep. [15] by $\mathcal{D}(\ell, J_1, J_2; r; a_1, \ldots, a_{N-1})$, where $a_1, \ldots, a_{N-1}$ are the Dynkin labels of $SU(N)$. For a given N we call $a_1 = a$, $a_2 = b$, $a_3 = c, \ldots$. Here ℓ is the conformal dimension of $\phi(x = 0, \theta = 0)$ and (J_1, J_2, r) its $SL(2, \mathbb{C})$ and $U(1)$ R-symmetry quantum numbers.

In the next sections we will only consider the cases with $J_1 = J_2 = 0$ and supersingletons with top spin $J \leq 1$.

The quantum numbers of the highest weight state are subjected to some unitarity bounds [10], whose thresholds correspond to the several possible shortenings of UIR's of $SU(2,2/N)$.

When the maximal possible number of bounds are fulfilled then the UIR becomes extrashort, in the sense that it gives the least possible number of states for a given set of unitarity bounds.

Examples of such extrashort UIR's are the "supersingletons" and the bulk "massless" reps., which in the CFT language correspond to boundary massless fields and to conserved current operators respectively [11, 14].

Supersingletons [16] are called "ultrashort" because their degrees of freedom are not enough to correspond to particle states on AdS bulk.

On these UIR's space-time derivative constraints are imposed on the conformal primary operators. For all other shortenings no space-time derivative constraints are imposed but rather a relation between different θ components of the conformal superfield at hand.

The highest weight UIR's of the superconformal algebras $SU(2,2/N)$ fall in three categories, depending on the quantum numbers of the highest weight state

$$\mathcal{D}(\ell, J_1, J_2; r, m_1, \ldots, m_{N-1}) \tag{2.3}$$

where ℓ and (J_1, J_2) label the dimension and the $SL(2, \mathbb{C})$ spin respectively, r is the $U(1)$ R–symmetry and $(m_1, \dots, m_{N-1})$ the Young tableaux (YT) labels of $SU(N)$ ($m = \sum_{k=1}^{N-1} m_k$).

Let us define the quantities:

$$
\begin{aligned}
X(J, r, 2\frac{m}{N}) &= 2 + 2J - r + \frac{2m}{N} \\
Y(r, 2\frac{m}{N}) &= -r + \frac{2m}{N}
\end{aligned} \tag{2.4}
$$

Then we have [10] $(J_1 = J_L, J_2 = J_R)$:

- A) $\ell \geq X(J_2, r, 2\frac{m}{N}) \geq X(J_1, -r, 2m_1 - 2\frac{m}{N})$,
  (or $J_1 \to J_2$, $r \to -r$, $2\frac{m}{N} \to 2m_1 - 2\frac{m}{N}$), for $J_1 J_2 \geq 0$.

- B) $\ell = Y(r, 2\frac{m}{N}) \geq X(J_1, -r, 2m_1 - 2\frac{m}{N})$,
  (or $J_1 \to J_2$, $r \to -r$, $2\frac{m}{N} \to 2m_1 - 2\frac{m}{N}$), for $J_1 J_2 = 0$, $J_1 = J$.

- C) $\ell = m_1$, $r = 2\frac{m}{N} - m_1$, $J_1 = J_2 = 0$

"Massless" representations in the AdS_5 bulk correspond to the threshold in A) when $\ell = 2 + J_1 + J_2$, $r = -J_1 + J_2$, in B) when $\ell = -r = 2 + J$ and in C) when $\ell = m_1 = 2$.

In these cases the CFT superfield is such that "current" components of the form $J_{\alpha_1 \cdots \alpha_{2s_1}, \dot{\alpha}_1 \cdots \dot{\alpha}_{2s_2}}$, with $\ell_J = 2 + s_1 + s_2$, are conserved:

$$
\partial^{\alpha_1 \dot{\alpha}_1} J_{\alpha_1 \cdots \alpha_{2s_1}, \dot{\alpha}_1 \cdots \dot{\alpha}_{2s_2}} = 0 \tag{2.5}
$$

We will call the superfield in question "current superfield". The "super-singletons" (massless conformal fields) occur in B) for $\ell = -r = 1 + J$ and in C) for $\ell = m_1 = 1$. These representations are "ultrashort" and the field components $O_{\alpha_1 \cdots \alpha_{2s}}$ obey the equatons of motion:

$$
\partial^{\alpha_1 \dot{\alpha}_1} O_{\alpha_1 \cdots \alpha_{2s}} = 0 \qquad (\Box\, O = 0 \text{ for } s = 0). \tag{2.6}
$$

It is a general fact of AdS reps. that "massless bulk" UIR's are contained in the product of two singletons [16].

In the SCFT language this means that a "current superfield" is bilinear in the supersingleton superfields as one can easily check.

Other shortenings, which do not involve space-time constraints on component fields, will be called "short multiplets" or "short superfields" where the shortening has a ring structure, namely it is preserved by multiplication of superfields.

As we will see in the next section, this corresponds to the concept of "chirality" and Grassmann analyticity in Harmonic superspace.

Shortenings of this type occur in B) and C).

Finally, there are other types of shortenings which are not of this type; we will call them "semishort". They can appear in A) B) C) and typically correspond to superfields which satisfy second or higher order constraints in covariant derivatives (second order constraints define the so-called linear superfields). In all the above cases no space-time constraints on component fields are implied.

Here we will consider the cases $N = 2, 3, 4$; the $N = 1$ case has been treated elsewhere [15, 17, 46].

We shall use Dynkin labels (DL) $[a_1, \ldots, a_{n-1}]$ for $SU(n)$, which are related to the YT labels $(m_1, \ldots, m_{n-1})$ as follows:

$$m_i = \sum_{k=1}^{n-i} a_k , \quad i = 1, \ldots, n - 1 . \tag{2.7}$$

A crucial ingredient in our analysis will be the use of harmonic superspace (x, θ, u), with "harmonic" variables u parametrizing the coset $SU(N)/U(1)^{N-1}$.

We will separately consider the cases $N = 2, 3, 4$ and for $N = 4$ we will restrict the analysis to the $PSU(2, 2/4)$ algebra $(r = 0)$ [10, 5], since it is the latter which is appropriate to $N = 4$ super Yang–Mills theory.

3   Extended harmonic superspaces and short superfields

We are interested in realizing the highest weight UIR's of $SU(2, 2/N)$ on superfields and harmonic superfields.

We shall consider an N–extended $D = 4$ superspace (without central charges) with Grassmann coordinates manifestly covariant with respect to the $SU(N)$ group

$$z = (x^{\alpha\dot\beta},\ \theta_i^\alpha,\ \overline\theta^{\dot\alpha i}) \tag{3.1}$$

using spinor indices of $SL(2, C)$ and indices in the fundamental representations of $SU(N)$ $i, k, \ldots = 1, \ldots, N$. (Note that the alternative convention $\theta^{\alpha i}$, $\overline\theta_i^{\dot\alpha}$ is sometimes used in the literature.)

The covariant opeartors in superspace are the spinor derivatives (see (A.18)). Using them one can define constrained superfields describing various irreducible representations of extended supersymmetries. The construction of supercurrents and superactions from the constrained superfields have been discussed in Refs. [52, 36].

For $N > 1$ the standard superspace (3.1) can be enlarged to a "harmonic superspace" (HSS) [19] by considering an extra set of "harmonic coordinates" u which provide an $SU(N)$-covariant parametrization of the coset space

$$\mathcal{M} = \frac{SU(N)}{S\left(U(n_1) \times \ldots \times U(n_P)\right)}\,, \qquad \left(\sum_{k=1}^P n_k = N\right). \tag{3.2}$$

Note that such spaces are known in the mathematical literature under the name of "flag manifolds" [50, 32, 43]. An exhaustive list of these space and the corresponding HSS's and of their properties for $N = 2, 3, 4$ is given in Ref. [39]. The choice of the subgroup depends on the practical use made of the harmonic variables. In our context it is crucial to use the highest-dimensional manifold of type (3.2) which occurs by dividing $SU(N)$ by its Maximal Torus:

$$\mathcal{M}_C = \frac{SU(N)}{(U(1))^{N-1}}\,. \tag{3.3}$$

This is a manifold of complex dimension $N(N - 1)/2$. The advantage of this choice is that the residual symmetry $(U(1))^{N-1}$ is the smallest one possible. This gives us maximal flexibility in defining subspaces of the full N–extended superspace in an $SU(N)$ covariant way. Such subspaces contain only a subset of the $4N$ Grassmann variables and are therefore called Grassmann (G–)analytic. They can be viewed as an alternative to the familiar chiral subspaces [13].

The $N = 2$ and $N = 3$ HSS's based on the cosets $SU(2)/U(1)$ and $SU(3)/U(1) \times U(1)$, correspondingly, have been introduced for the off-shell description of all the $N = 2$ supersymmetric theories and of $N = 3$ supersymmetric Yang-Mills theory (SYM) [19]-[40]. The realization of the superconformal group in these HSS's has been studied in Refs. [23, 21]. An $N = 4$ HSS involving the manifold $SU(4)/S(U(2) \times U(2))$ has been used in Ref. [31] to give an interpretation of the on-shell constraints of $N = 4$ SYM as combined G– and harmonic (H–)analyticities. There it was shown that the on-shell superfield strength of $N = 4$ SYM is an analytic (i.e., short) superfield (and, similarly, for $N = 3$ SYM). In Ref. [34] this observation was generalized to composite operators made out of $N = 4$ SYM field strenghts and it was suggested that this property might significantly restrict the correlation functions of analytic composite operators. This idea has subsequently been applied to the study of $N = 2$ and $N = 4$ CFT's in Refs. [35]. The notion of harmonic superspace and of G–analyticity was generalized to an arbitrary N in [31, 32]. Note that H–analyticity is also important in HSS's of lower space-time dimension [33, 56].

In what follows we shall consider the irreducible superfield representation of $SU(2, 2/N)$ using the G– and H–analyticities related to the choice (3.3).

3.1   The $N = 2$ case

$N = 2$ HSS has been introduced for the off-shell description of the hypermultiplet and gauge and supergravity multiplets [19]. Here we start by describing the shortening effect of G– and H–analyticities on the on-shell matter and gauge $N = 2$ multiplets satisfying their free equations of motion. We review the basic facts about the $SU(2)/U(1)$ HSS in the Appendix.

The hypermultiplet (or $N = 2$ matter multiplet), which is the supersingleton in the AdS literature, can be described by an ordinary superfield which is an $SU(2)$ doublet $q^i(z)$ and satisfies the on-shell constraints [53]:

$$D_\alpha^{(i} q^{j)}(z) = \overline{D}_{\dot\alpha}^{(i} q^{j)}(z) = 0 \ . \tag{3.4}$$

Now, let us project the $SU(2)$ doublets D^i_α, q^i and $\overline{D}_{i\dot\alpha}$ with the harmonics u^1_i and u^i_2

$$D^1_\alpha = D^i_\alpha u^1_i \,, \quad q^1 = q^i u^1_i \,, \quad \overline{D}_{2\dot\alpha} = \overline{D}_{i\dot\alpha} u^i_2 \,. \tag{3.5}$$

This allows us to equivalently rewrite the on-shell constraints (3.4) in the form of G–analyticity conditions in HSS:

$$D^1_\alpha q^1 = \overline{D}_{2\dot\alpha} q^1 = 0 \,. \tag{3.6}$$

The crucial point now is that by letting the superfield q^1 have a nontrivial harmonic dependence one can solve the constraints eqs. (3.6) in terms of a G–analytic superfield $q^1(x_A, \theta_2, \overline{\theta}^1, u) \equiv q^+$ in the appropriate analytic basis (A.17). This superfield describes the hypermultiplet off shell. It is an infinite-dimensional representation of supersymmetry because of the infinite harmonic expansion on the coset $SU(2)/U(1) \sim S^2$. In order to put it back on shell we need to restrict the arbitrary harmonic dependence down to the initial linear one (3.5). This is achieved with the help of the harmonic derivatives defined in the Appendix. We remark that the harmonic derivative D^1_2 commutes with the spinor ones from eq. (3.6),

$$[D^1_2, D^1_\alpha] = [D^1_2, \overline{D}_{2\dot\alpha}] = 0 \,, \tag{3.7}$$

i.e., preserves G–analyticity. Thus, the free equation of motion of the hypermultiplet takes the form of a harmonic (H–)analyticity condition:

$$D^1_2 q^1 = 0 \,, \tag{3.8}$$

where one uses the harmonic derivative in the analytic basis (see (A.20)).

This harmonic equation implies a number of constraints on the components of the superfield q^1. Take, for instance, the leading component $\phi^1(x, u) = q^1|_{\theta=0}$. Off shell it has an infinite harmonic expansion going over the irreducible products of the harmonics $u^{1,2}_i$:

$$\phi^1(x, u) = \sum_{n=0}^{\infty} \phi^{(i_1 \ldots i_{n+1} j_1 \ldots j_n)}(x) u^1_{i_1} \ldots u^1_{i_{n+1}} u^2_{j_1} \ldots u^2_{j_n} \,. \tag{3.9}$$

Now, the harmonic derivative converts any u^2 into u^1, so eq. (3.8) implies the vanishing of all the terms in (3.9) but the first one,

$$D^1_2 \phi^1(x, u) = 0 \quad \Rightarrow \quad \phi^1(x, u) = \phi^i(x) u^1_i \,. \tag{3.10}$$

The same argument can be applied to the higher-order components of the superfield. Thus, the spinor component $\psi_\alpha(x, u) = D_\alpha^2 q^1|_{\theta=0}$ satisfies the constraint

$$D_2^1 \psi_\alpha(x, u) = D_2^1 D_\alpha^2 q^1|_{\theta=0} = D_\alpha^1 q^1|_{\theta=0} = 0 \qquad (3.11)$$

(see (3.6)). Since this component is chargeless, the harmonic condition (3.11) implies that it is a singlet, $\psi_\alpha(x, u) = \psi_\alpha(x)$. In a similar way one can find the complete expansion of the on-shell superfield:

$$q^1 = \varphi^i(x) u_i^1 + \theta_2^\alpha \psi_\alpha(x) + \overline{\theta}^{1\dot\alpha} \overline{\chi}_{\dot\alpha}(x) + i\theta_2^\alpha \overline{\theta}^{1\dot\beta} \partial_{\alpha\dot\beta} \varphi^i(x) u_i^2 \qquad (3.12)$$

where all the components satisfy their free massless equations of motion. We clearly see that this superfield is "short" in the sense that it only depends on half of the Grassmann variables of the full $N = 2$ superspace. It is even "ultrashort" in the sense that the top spin in it is 1/2 instead of the maximal spin 1 allowed in a G–analytic scalar superfield.

It is useful to note the relations

$$(D^2)^2 q^1 = (\overline{D}_1)^2 q^1 = 0 \qquad (3.13)$$

which can again be derived from the basic constraints. They are the covariant form of the statement that the superfield is linear in θ_2 and $\overline{\theta}^1$, as can be seen from the expansion (3.12).

When rewritten in the central basis coordinates $x, \theta_i, \overline{\theta}^i$, the on-shell hypermultiplet superfield recovers its original form

$$q^1 = u_i^1 q^i(z) \qquad (3.14)$$

of an $SU(2)$ doublet. In fact, this observation can be given an invariant meaning as follows. We remark that the harmonic derivatives D_J^I form the algebra of $SU(2)$

$$[D_J^I, D_L^K] = \delta_J^K D_L^I - \delta_L^I D_J^K \qquad (3.15)$$

realized on the superfield q^1. As explained in the Appendix, the derivative D_2^1 is the positive root ("creation operator") of this algebra. Then the condition (3.8) simply defines the highest weight of an $SU(2)$ representation. The quantum number associated to this representation

("superisospin") coincides with this of the first component. To see this we write down the Casimir of this $SU(2)$,

$$C_2 = D_J^I D_I^J = \frac{1}{2}(D^0)^2 + D^0 + 2D_1^2 D_2^1 \qquad (3.16)$$

where $D^0 = D_1^1 - D_2^2$ is the $U(1)$ charge operator. Then, applying this Casimir to the on-shell superfield q^1 satisfying the H-analyticity constraint (3.8) and using the fact that it carries a definite $U(1)$ charge, $D^0 q^1 = q^1$, we obtain

$$C_2 q^1 = \frac{3}{2} q^1 . \qquad (3.17)$$

We see that q^1 is an eigenfunction of the Casimir, realizing the doublet (isospin 1/2) representation, just like its first component (see (3.10)). The important point here is that the harmonic derivatives D_J^I commute with the supersymmetry generators, therefore C_2 is the superisospin Casimir of the entire Poincaré supersymmetry algebra. Note that this algebra has another Casimir, that of superspin. We can apply similar arguments to compute the value of this Casimir on the superfield q^1. Indeed, the G-analyticity conditions (3.6) are equivalent to demanding that the positive odd roots of the Poincaré supersymmetry algebra annihilate q^1. This ensures that the superspin also takes a definite eigenvalue, so the on-shell superfield q^1 realizes an irrep of Poincaré supersymmetry (see in this context Ref. [47] for a general discussion of irreducibility conditions on superfields).

The above analysis can be repeated for other superfields satisfying both the G– and H–analyticity conditions but carrying different $U(1)$ charges. Take, for instance, the off-shell linear multiplet. In HSS it is described by a G–analytic superfield L^{11} of $U(1)$ charge $+2$ [22]. Unlike the hypermultiplet q^1, this time the H–analyticity condition

$$D_2^1 L^{11} = 0 \qquad (3.18)$$

does not put the superfield on shell. The only restriction on the components involving space-time derivatives is that the vector in L^{11} must be divergenceless. The $SU(2)$ Casimir now takes the eigenvalue 4 which corresponds to isospin 1 (triplet irrep). To put this superfield on shell, an additional constraint is required,

$$(D^2)^2 L^{11} = 0 . \qquad (3.19)$$

This H–analytic superfields of charge $+2$ is not ultrashort. H–analyticity for superfields of charge $q \geq +3$ still makes them irreducible but does not yield any constraints on the remaining components.

Now we turn to the other basic multiplet of $N = 2$ supersymmetry, that of SYM. The free on-shell ultrashort Maxwell multiplet is described by a chiral (harmonic independent) superfield satisfying an additional second-order constraint

$$\overline{D}_{\dot\alpha k} W = 0 , \quad D^{\alpha(i} D^{j)}_\alpha W = 0 . \tag{3.20}$$

In the chiral basis, the on-shell components of this superfield are

$$W = \phi(x) + \theta^\alpha_i \lambda^i_\alpha(x) + \theta^{(\alpha i} \theta^{\beta)}_i F_{\alpha\beta} . \tag{3.21}$$

This is another example of an ultrashort multiplet (its expansion goes only up to θ^2, as compared to θ^4 for a generic chiral $N = 2$ superfield).

We do not consider here the non-Abelian generalization of the HSS description of the hypermultiplet and gauge multiplet [19, 55]. It should be stressed that the linear H–analyticity condition (3.8) is valid for gauge-invariant superfields only, otherwise it should contain a harmonic connection.

We can now use the two objects above, the on-shell hypermultiplet q^1 and the SYM field strength W as building blocks which will allow us to construct all the short representations of $SU(2, 2/2)$. By adapting the series A), B), C) from Section 2 for $N = 2$ we have [14]:

A) $\ell \geq 2 + 2J_2 - r + 2I \geq 2 + 2J_1 + r + 2I, \quad J_1 J_2 \geq 0$;

B) $\ell = -r + 2I \geq 2 + 2J + r + 2I, \quad J_1 = J, J_2 = 0$;

C) $\ell = 2I, \quad J_1 = J_2 = r = 0 .$ \hfill (3.22)

A general long multiplet, belonging to the A) series, contains 4 θ's, 4 $\overline{\theta}$'s and $J_{max} = 2$ (in the case of $J_1 = J_2 = 0$ for the highest weight state, i.e., for the $\theta = \overline{\theta} = 0$ superfield component). In terms of our building blocks this series corresponds to chiral-antichiral multiplication of the type $W\overline{W}$, or to analytic-antianalytic multiplication of hypermultiplets of the type $q^1 q^2$ (where $q^2 = D^2_1 q^1$ is a superfield satisfying "antianalyticity" constraints), or to products of both.

Series B) for $J = 0$, $I = 0$ corresponds to chiral multiplets which can be obtained by the following operator series:

$$\text{Tr}[W^p] \tag{3.23}$$

$(p = 1, \ell = -r = 1$ is the on-shell SYM multiplet itself defined in (3.20)). These superfields depend on the 4 left-handed θ_i^α and one immediately sees that the top spin in their expansion belongs to the Lorentz representation $(1, 0)$. The above series is chiral (short) and for $N = 2$ may be called "tensor multiplet" tower [30], since the maximum spin is $(1, 0)$ (with $I = 0$ and $\ell_{(1,0)} = 1 + p$).

Series C) corresponds to the analytic multiplication of hypermultiplets:

$$\mathrm{Inv}(q^1)^{2I} \tag{3.24}$$

where the symbol Inv means a gauge invariant product. The case $I = 1/2$ corresponds to the on-shell hypermultiplet itself. The lowest component of these superfields is in the isospin I $SU(2)$ representation. The superfields depend on the 2 left-handed θ_2^α and the 2 right-handed $\overline{\theta}^{1\dot\alpha}$, so the top spin (for $I > \frac{1}{2}$) is a $(\frac{1}{2}, \frac{1}{2})$ vector in the isospin $I - 1$ $SU(2)$ representation with dimension $\ell_{(\frac{1}{2},\frac{1}{2})} = 2I + 1$. For $I = 1$ the vector is "conserved" and gives a "current" superfield (this is the linear multiplet (3.18)).

There is another intermediate shortening in the B) series $(J = 0)$ obtained by multiplying chiral with G–analytic superfields:

$$\mathrm{Inv}[W^p(q^1)^{2I}] \,. \tag{3.25}$$

This superfield is G–analytic in a weaker sense than either W or q^1, satisfying only the constraint (dropping the Inv symbol)

$$\overline{D}_{2\dot\alpha}[W^p(q^1)^{2I}] = 0 \,, \tag{3.26}$$

and so it depends on the 4 $\theta_{1,2}^\alpha$ and on the 2 $\overline{\theta}^{1\dot\alpha}$. Thus, the top spin in it is $\frac{3}{2} = (1, \frac{1}{2})$.

There are other even shorter multiplets when the component fields satisfy "space–time constraints" i.e., conservation laws (transversality) or equations of motion. This happens when the dimension takes a particular value.

In A) $W\overline{W}$ has $\ell = 2$ $(J_1 = J_2 = 0)$, $r = 0$, $I = 0$ and corresponds to the conserved stress-tensor multiplet. There is a similar object in the analytic-antianalytic multiplication of q^1 and $q^2 = D_1^2 q^1$.

The superfield $W^p q^1$ satisfies the additional linearity constraint (see (3.13) and (3.20))

$$(\overline{D}_1)^2 [W^p q^1] = 0 \, , \tag{3.27}$$

so it depends on the 4 θ's and only linearly on $\overline{\theta}_1$. In B) $\ell = 1$ and in C) $I = \frac{1}{2}$ correspond to the basic "super-singleton" UIR of $SU(2, 2/2)$. In the AdS/CFT language the $\ell = 2, r = J_1 = J_2 = I = 0$ (in A)) and $\ell = 2, r = J_1 = J_2 = 0, I = 1$ (in C)) correspond to massless graviton and gauge bosons ("current superfields"). Semishort multiplets, obeying to "A) treshold", also exist. They are $\overline{W}W^n, n > 1, (\overline{D}^{(i}\overline{D}^{j)}(\overline{W}W^n) = 0)$ with $\ell = 1 + n, r = 1 - n$ i.e., $\ell = 2 - r$.

3.2   The $N = 3$ case

3.2.1   The $N = 3$ super Yang–Mills multiplet

The $N = 3$ Yang–Mills multiplet is described by the field strength superfield [36, 20] $W_{ij}(x, \theta) \equiv \varepsilon_{ijk} W^k$ defined by anticommuting the gauge-covariant spinor derivatives:

$$\{\overline{\nabla}_{i\dot\alpha}, \overline{\nabla}_{j\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta} W_{ij} \tag{3.28}$$

or by the conjugate superfield $\overline{W}^{ij}$. It the Abelian case this superfield satisfies the following on-shell constraints:

$$D^i_\alpha W_{jl} = \frac{1}{2}(\delta^i_j D^k_\alpha W_{kl} - \delta^i_l D^k_\alpha W_{kj}), \tag{3.29}$$

$$\overline{D}_{i\dot\alpha} W_{jk} + \overline{D}_{j\dot\alpha} W_{ik} = 0 \tag{3.30}$$

and their complex conjugates.

The $SU(3)/U(1) \times U(1)$ HSS has been introduced for the off-shell description of the $N = 3$ Yang-Mills theory [20]. Here we shall use this superspace for the classification of the short on-shell $N = 3$ multiplets. Some basic facts about $N = 3$ HSS are given in the Appendix.

One can define three different harmonic projections of the Abelian

on-shell superfield W_{ij}:

$$W_{23} \equiv W^1 = u_2^i u_3^j W_{ij} \ , \tag{3.31}$$

$$W_{13} \equiv -W^2 = u_1^i u_3^j W_{ij} \ , \tag{3.32}$$

$$W_{12} = W^3 = u_1^i u_2^j W_{ij} \ . \tag{3.33}$$

By projecting the on-shell constraints (3.29), (3.30) with the appropriate harmonics one finds sets of G–analyticity constraints on each of these superfields. They lie in three different analytic superspaces with six odd coordinates (see the Appendix). The existence of such analytic superspace involving unequal numbers of left- and right-handed odd variables was first pointed out in [21] and then generalized to the so-called (N, p, q) superspaces in [31].

Consider, for example, the superfield W^1 satisfying the following conditions of G–analyticity [31]

$$\overline{D}_{2\dot\alpha} W^1 = \overline{D}_{3\dot\alpha} W^1 = D_\alpha^1 W^1 = 0 \tag{3.34}$$

meaning that

$$W^1 = W^1(x_A, \theta_2, \theta_3, \overline{\theta}^1, u) \tag{3.35}$$

in the appropriate analytic basis (A.17).

The G–analytic superfield W^1 is a harmonic superfield with an infinite expansion on the harmonic coset. In order to get back the original constrained harmonic-independent superfield $W_{ij}(x, \theta)$ we need to impose conditions of H–analyticity. To this end we should use only the harmonic derivatives corresponding to the positive roots of $SU(3)$ (see the Appendix). These are:

$$D_2^1 W^1 = D_3^2 W^1 = D_3^1 W^1 = 0 \ . \tag{3.36}$$

As expected, they form a closed algebra (CR structure) with the spinor derivatives in (3.34), i.e., preserve G–analyticity. Note that only the first of eqs. (3.36) is the true equation of motion. The second one is purely kinematical and the third one is a corollary of the first two, since $D_3^1 = [D_2^1, D_3^2]$.

The H–analyticity conditions (3.36) have the meaning of $SU(3)$ irreducibility conditions. Indeed, the derivatives D_J^I form the algebra of $SU(3)$

$$[D_J^I, \ D_L^K] = \delta_J^K D_L^I - \delta_L^I D_J^K \tag{3.37}$$

realized on the superfield W^1. Then (3.36) just defines the highest weight of an irrep. To find out which one, we can write down the Casimirs of this $SU(3)$,

$$C_2 = D_J^I D_I^J \,, \qquad C_3 = D_J^I D_K^J D_I^K \qquad (3.38)$$

and rearrange the derivatives so that all the analytic ones from eq. (3.36) are on the right. Then, applying these Casimirs to the on-shell superfield W^1 and using the fact that it carries a definite $U(1) \times U(1)$ charge,

$$D_1^1 W^1 = W^1 \,, \quad D_2^2 W^1 = D_3^3 W^1 = 0 \,, \qquad (3.39)$$

we find that W^1 is an eigenfunction of the Casimirs. Since the harmonic derivatives are supersymmetric invariant, we can switch back to the basis in superspace where the θ's are not projected with harmonics. There $W^1 = W^i u_i^1 = u_2^i u_3^j W_{ij}$ and we come back to the original form (3.31). Thus, the super-$SU(3)$ quantum numbers of the superfield W^1 coincide with those of its first component.

The G–analytic superfield is also an eigenfunction of the superspin Casimir. The reason is that it is annihilated by half of the odd generators (spinor derivatives), so it is a highest weight of the entire $N = 3$ Poincaré supersymmetry algebra. Moreover, the close examination of the components below shows that they are on shell, satisfying massless equations of motion. Thus, W^1 realizes an irrep of conformal supersymmetry as well.

It is easy to prove that W^1 also obeys linearity conditions with respect to each θ:

$$(D^2)^2 W^1 = (D^2 D^3) W^1 = (D^3)^2 W^1 = (\overline{D}_1)^2 W^1 = 0 \,. \qquad (3.40)$$

This is done by using the harmonic derivatives, e.g.,

$$D_2^1 (D^2)^2 W^1 = 2(D^1 D^2) W^1 = 0 \ \Rightarrow \ (D^2)^2 W^1 = 0 \,. \qquad (3.41)$$

Further, by examining the components of the superfield W^1 one finds that the top chargeless component lies very low in the θ expansion:

$$F_{\alpha\beta}^+ = D_{(\alpha}^2 D_{\beta)}^3 W^1 |_0 \,. \qquad (3.42)$$

Acting with harmonic derivatives on the higher-order spinor derivatives (components) of W^1 one can easily show that all of them are expressed in terms of space-time derivatives of the preceding components. In this way, one finally obtains the components of the ultrashort on-shell superfield W^1:

$$W^1 = \phi^1 + \overline{\theta}^{1\dot{\alpha}}\overline{\lambda}_{\dot{\alpha}} + \theta_2^\alpha \lambda_{3\alpha} - \theta_3^\alpha \lambda_{2\alpha} - i\theta_2^\alpha \overline{\theta}^{1\dot{\beta}}\partial_{\alpha\dot{\beta}}\phi^2$$

$$+\theta_2^\alpha \theta_3^\beta F_{\alpha\beta}^+ - i\overline{\theta}^{1\dot{\alpha}}\theta_2^\alpha \theta_3^\beta \partial_{(\alpha\dot{\alpha}}\lambda_{1\beta)} \ . \tag{3.43}$$

where the physical fields satisfy massless field equations.

One can treat the projection $W^2 = -W_{13}$ in a similar way. The spinor and harmonic derivatives annihilating W^2 are

$$D_\alpha^2, \ \overline{D}_{1\dot{\alpha}}, \ \overline{D}_{3\dot{\alpha}}, \ D_3^1, \ D_1^2, \ D_3^2 \ . \tag{3.44}$$

The harmonic conditions make the leading component $[W^2]|_0$ an irrep of $SU(3)$, and thus give definite super-$SU(3)$ quantum numbers to the whole superfield. The corresponding linearity conditions are

$$(D^1)^2 W^2 = (D^1 D^3) W^2 = (D^3)^2 W^2 = (\overline{D}_2)^2 W^2 = 0 \ . \tag{3.45}$$

The Abelian superfield W^2 lives in a rotated version of the G–analytic superspace (A.17):

$$W^2(x_A', \theta_1, \ \theta_3, \ \overline{\theta}^2, \ u) \tag{3.46}$$

$$x_A' = x + i\theta_1 \overline{\theta}^1 - i\theta_2 \overline{\theta}^2 + i\theta_3 \overline{\theta}^3 \ . \tag{3.47}$$

The components of W^2 are obtained from those of W^1 (3.43) by exchanging 1 with 2. It is evident that $W^2 = D_1^2 W^1$, so both superfields describe the same on-shell vector multiplet.

Finally, consider the harmonic projection of the $N = 3$ superfield $\overline{W}^{ij}$

$$\overline{W}^{12} = u_i^1 u_j^2 \overline{W}^{ij} \ . \tag{3.48}$$

The on-shell constraints on $\overline{W}^{ij}$ are equivalent to the following G–analyticity conditions:

$$D_\alpha^1 \overline{W}^{12} = D_\alpha^2 \overline{W}^{12} = \overline{D}_{3\dot{\alpha}}\overline{W}^{12} = 0 \ . \tag{3.49}$$

This Abelian superfield lives in yet another version of the G–analytic superspace (A.17),

$$\overline{W}^{12}(x''_A, \theta_3, \overline{\theta}^1, \overline{\theta}^2, u) \ . \tag{3.50}$$

In addition, it satisfies the harmonic constraints

$$D_3^2 \overline{W}^{12} = D_2^1 \overline{W}^{12} = D_3^1 \overline{W}^{12} = 0 \ . \tag{3.51}$$

From these constraints follow the linearity conditions

$$(D^3)^2 \overline{W}^{12} = (\overline{D}_1)^2 \overline{W}^{12} = (\overline{D}_2)^2 \overline{W}^{12} = 0 \ . \tag{3.52}$$

This superfield has the following components:

$$
\begin{aligned}
\overline{W}^{12} = {} & \overline{\phi}^{12} - \theta_3^\alpha \lambda_\alpha - \overline{\theta}^{1\dot\alpha} \overline{\lambda}_{\dot\alpha}^2 + \overline{\theta}^{2\dot\alpha} \overline{\lambda}_{\dot\alpha}^1 + i\theta_3^\alpha \overline{\theta}^{1\dot\beta} \partial_{\alpha\dot\beta} \overline{\phi}^{23} \\
& - i\theta_3^\alpha \overline{\theta}^{2\dot\beta} \partial_{\alpha\dot\beta} \overline{\phi}^{13} - \overline{\theta}^{1\dot\alpha} \overline{\theta}^{2\dot\beta} F_{\dot\alpha\dot\beta}^- + i\overline{\theta}_{\dot\alpha}^1 \overline{\theta}_{\dot\beta}^2 \theta_3^\alpha \partial_\alpha^{(\dot\alpha} \overline{\lambda}^{3\dot\beta)} \ . \tag{3.53}
\end{aligned}
$$

Once again, this is another equivalent description of the same $N = 3$ on-shell SYM multiplet.

3.2.2   Series of short $N = 3$ multiplets

The A), B), C) UIR's of the $SU(2, 2/3)$ algebra are given by adapting the quantities X, Y to the $N = 3$ case with $m_1 = a + b$, $m_2 = a$ where $[a, b]$ are the $SU(3)$ Dynkin labels.

It then follows that $(m = m_1 + m_2 = 2a + b)$:

-

$$\text{A)} \ \ \ell \geq 2 + 2J_2 - r + \frac{4}{3}a + \frac{2}{3}b \geq 2 + 2J_1 + r + \frac{2}{3}a + \frac{4}{3}b \tag{3.54}$$

which implies:

$$
\begin{aligned}
-r &\geq J_1 - J_2 - \frac{1}{3}(a - b) \\
\ell &\geq 2 + J_1 + J_2 + a + b
\end{aligned}
\tag{3.55}
$$

(or $r \to -r$, $J_1 \to J_2$, $a \to b$)

●

B) $\quad J_1 = J, J_2 = 0$

$$\ell = -r + \frac{4}{3}a + \frac{2}{3}b \geq 2 + 2J + r + \frac{2}{3}a + \frac{4}{3}b \quad (3.56)$$

which implies:

$$\begin{aligned} -r &\geq 1 + J - \frac{1}{3}(a-b) \\ \ell &\geq 1 + J + a + b \end{aligned} \quad (3.57)$$

(or $J_1 \to J_2,\ r \to -r.\ a \to b$)

●

C) $\quad J_1 = J_2 = 0, \quad \ell = a + b, \quad r = \frac{1}{3}(a-b) \quad (3.58)$

The Yang–Mills (supersingleton) multiplet corresponds to series C) for $a = 0$, $b = 1$, $r = -\frac{1}{3}$.

Now, let us realize these abstract short representations in terms of the SYM superfields W. The series C) for $a = 0$ corresponds to the tower with maximal shortening:

$$\mathrm{Tr}\,(W^1)^b = C[0, b] \,. \quad (3.59)$$

This is a superfield depending on 4 θ's and 2 $\overline{\theta}$'s, consequently the maximum spin ($b > 1$) is $J = \frac{3}{2}$ in the Lorentz representation $(1, \frac{1}{2})$. The first component is a scalar with dimension $\ell = b$ and r-charge $r = -b/3$ in the $[0, b]$ UIR of $SU(3)$. The case $b = 1$ is the ultrashort Yang–Mills singleton W^1.

The short H–analytic superfields for $b = 2, 3$ have the the following chargeless components:

$$(D^2)^2(D^3)^2 C[0, 2] \,, \quad D^2_{(\alpha} D^3_{\beta)} (\overline{D}_1)^2 C[0, 3] \,. \quad (3.60)$$

All the higher components are space-time derivatives of the lower ones. In the case $b \geq 4$ H–analyticity does not lead to any extrashortening, e.g., the superfield $C[0, 4]$ contains an independent top component

$$(D^2)^2(D^3)^2(\overline{D}_1)^2 C[0, 4] \,. \quad (3.61)$$

The complete C) series of short multiplets can be obtained by taking the products

$$(W^1)^b(\overline{W}^{12})^a = C[a, b] \tag{3.62}$$

(we omit the traces). The first components of these superfields contain *analytic* harmonics with $a + b$ indices) 1 and a indices 2 corresponding to the UIR $[a, b]$ of $SU(3)$. We obtain the generic short operator of the C) series with $\ell = a + b$, $r = \frac{1}{3}(a - b)$ and with $J_{\max} = 2 = (1, 1)$, since this operator contains 4 θ, 4 $\overline{\theta}$.

Now, let us consider $C[a, b]$ as an abstract G–analytic superfield

$$C[a, b] = C[a, b](\theta_2, \theta_3, \overline{\theta}^1, \overline{\theta}^2)$$

with the given $SU(3)$ quantum numbers. The H–analyticity conditions for these representations are the same as those for the building block W^1 (see (3.36)),

$$D_2^1 C[a, b] = D_3^2 C[a, b] = D_3^1 C[a, b] = 0 . \tag{3.63}$$

This is equivalent to imposing an $SU(3)$ irreducibility condition. Using both G– and H–analyticity one can derive various constraints on the components. For instance, we find the following highest chargeless components in the simplest cases:

$$D_{(\alpha}^2 D_{\beta)}^3 \overline{D}_{1(\dot\alpha} \overline{D}_{2\dot\beta)} C[1, 1] , \tag{3.64}$$

$$(D^2)^2 (D^3)^2 \overline{D}_{1(\dot\alpha} \overline{D}_{2\dot\beta)} C[1, 2] , \tag{3.65}$$

$$(D^2)^2 (D^3)^2 (\overline{D}_1)^2 \overline{D}_{2\dot\alpha} C[1, 3] . \tag{3.66}$$

From these levels of the expansion on the corresponding superfields should become short. Note, however, that there is an additional linearity constraint in the case $a = 1$,

$$(\overline{D}_2)^2 C[1, b] = 0 , \tag{3.67}$$

which follows from the properties of $\overline{W}^{12}$ in the product (3.62) but cannot be obtained from H–analyticity alone.

The multiplet which is dual to the graviton multiplet of $N = 6$ supergravity in AdS_5 is given in (3.64) ($C[1, 1]$). Indeed the top component is the spin 2 $(1, 1)$ graviton multiplet with $\ell = 4$. This is a "current superfield".

Note that (3.62) for $b = a$ is invariant under an additional r-phase $W_i \to e^{i\alpha} W_i$ which commutes with the $SU(2, 2/3)$ algebra [12, 37].

By selecting the r invariant singlets $C[a, a]$ we obtain a tower of spin 2 short multiplets which are $\frac{1}{3}$ BPS states of $N = 6$ supergravity [12, 36]. Other cases in this class of analytic representations have no semishortening.

The next series of representations is given by the products

$$(W^1)^{a+b}(W^2)^a = B[a, b] . \tag{3.68}$$

This superfield satisfies only one G–analyticity condition,

$$\overline{D}_{3\dot\alpha} B[a, b] = 0 \;\Rightarrow\; B[a, b] = B[a, b](\theta_1, \theta_2, \theta_3, \overline{\theta}^1, \overline{\theta}^2) . \tag{3.69}$$

This implies that the top spin in it is $J_{\max} = \frac{5}{2} = (\frac{3}{2}, 1)$. Further, the same H–analyticity constraints as in (3.36),

$$D_2^1 B[a, b] = D_3^2 B[a, b] = D_3^1 B[a, b] = 0 \tag{3.70}$$

imply that the first component belongs to the UIR $[a, b]$. The dimension and r-charge of this superfield are

$$\ell = 2a + b , \qquad r = -\frac{1}{3}(2a + b) . \tag{3.71}$$

One can prove the following constraints:

$$\overline{D}_{1(\dot\alpha} \overline{D}_{2\dot\beta)} B[a, b] = 0 , \tag{3.72}$$

$$(\overline{D}_1)^2 \overline{D}_{2\dot\alpha} B[a, b] = (\overline{D}_2)^2 \overline{D}_{1\dot\alpha} B[a, b] = 0 . \tag{3.73}$$

Thus, this representation can contain all 6 θ's and bilinear scalar combination $(\overline{\theta}^1 \overline{\theta}^2)$.

The superfields $B[1, b]$ satisfy the following additional conditions:

$$(D^1)^2 B[1, b] = (\overline{D}_2)^2 B[1, b] = 0 \tag{3.74}$$

which follow from the properties of W^2. The superfield $B[1, 0] = W^1 W^2$ is even further constrained:

$$(D^2)^2 B[1, 0] = (\overline{D}_1)^2 B[1, 0] = 0 . \tag{3.75}$$

We find the following highest chargeless components:

$$D^1_{(\alpha}D^3_{\beta)}\overline{D}_{1\dot\alpha}B[1,0] \ , \tag{3.76}$$

$$D^1_{\alpha}D^2_{\beta}(D^3)^2\overline{D}_{1\dot\alpha}B[1,1] \ , \tag{3.77}$$

$$(D^1D^2)(D^3)^2(\overline{D}_1)^2B[2,1] \ . \tag{3.78}$$

This means that these superfields are semishort.

These representations belong to the B) series $(J = 0)$.

The series A) corresponds typically to superfields with 6 θ, 6 $\overline{\theta}$ and for $J_1 = J_2 = 0$ contains multiplets with $J_{max} = 3 = (\frac{3}{2}, \frac{3}{2})$.

The last series (still corresponding to the B) shortening) can be constructed by taking the products

$$(W^1)^{p+q+n}(W^2)^{q+n}(\overline{W}^{12})^n = B'[q+2n,p] \ , \quad n \geq 1 \ . \tag{3.79}$$

It lives in the same superspace with 10 spinor coordinates as $B[a,b]$ and has its first component in the $SU(3)$ UIR $[q+2n,p]$. It is clear that one can find members of both series having their first components in the same UIR, $B'[q+2n,p]$ and $B[q+2n,p]$. However, the two short multiplets are not equivalent. The dimension and r-charge of the $B'[q+2n,p]$ series are $\ell = p + 2q + 3n$, $r = -\frac{1}{3}(p+2q+n)$ whereas for the $B[q+2n,p]$ series they are $\ell = p + 2q + 4n$, $r = -\frac{1}{3}(p+2q+4n)$.

In conclusion we can say that the short analytic $N = 3$ representations are defined by the choice of the lowest harmonic representation, the Grassmann dimension and the quantum numbers ℓ and r.

3.3   The $N = 4$ case

3.3.1   The $N = 4$ SYM multiplet

The $N = 4$ Yang–Mills multiplet is described by the field strength superfield $W^{ij}(x,\theta)$ satisfying the reality condition [52, 36]

$$W^{ij} = \frac{1}{2}\epsilon^{ijkl}W_{kl} \ , \qquad W_{kl} = \overline{W^{kl}}$$

and the following on-shell constraints:

$$\overline{D}_{i\dot{\alpha}}W^{jk} = \frac{1}{3}(\delta_i^j \overline{D}_{l\dot{\alpha}}W^{lk} - \delta_i^k \overline{D}_{l\dot{\alpha}}W^{lj}) \, , \qquad (3.80)$$

$$D_\alpha^i W^{jk} + D_\alpha^j W^{ik} = 0 \, . \qquad (3.81)$$

Note that both forms of W contain $F_{\alpha\beta}$ and $F_{\dot{\alpha}\dot{\beta}}$, so we do not use $\overline{W}$ for $N = 4$ superfields.

We shall rewrite these constraints in $N = 4$ HSS. To this end we have to chose one of the harmonic coset spaces for the group $SU(4)$ listed in Ref. [39]. It should be pointed out that a harmonic interpretation of the $N = 4$ SYM constraints has for the first time been proposed in Refs. [31]. It makes use of the harmonic coset $SU(4)/S(U(2) \times U(2))$. This is sufficient to show that the on-shell W is a G–analytic superfield depending only on half of the odd variables. However, the residual symmetry $S(U(2) \times U(2))$ in the approach of Ref. [31] turns out too restrictive for the analysis of all representations. In order to have maximal flexibility we shall use harmonics on the coset $SU(4)/U(1)^3$ (see the Appendix for details of the definition and the basic properties).

With the help of these harmonics we can introduce three independent projections of the on-shell field strength:

$$W^{12} \equiv u_i^1 u_j^2 W^{ij} = -W_{34} \qquad (3.82)$$

$$W^{13} \equiv u_i^1 u_j^3 W^{ij} = W_{42} \qquad (3.83)$$

$$W^{23} \equiv u_i^2 u_j^3 W^{ij} = W_{14} \, . \qquad (3.84)$$

It is easy to see that the constraints on W^{ij} imply that these three superfields belong to three different G–analytic subspaces of HSS. For example, the projection W^{12} satisfies the G–analyticity constraints corresponding to the following spinor derivatives:

$$D_\alpha^1 W^{12} = D_\alpha^2 W^{12} = \overline{D}_{3\dot{\alpha}}W^{12} = \overline{D}_{4\dot{\alpha}}W^{12} = 0 \, . \qquad (3.85)$$

In the appropriate basis in superspace (A.17) the analytic W^{12} has the form

$$W^{12} = W^{12}(x_A, \theta_3, \theta_4, \overline{\theta}^1, \overline{\theta}^2, u) \, . \qquad (3.86)$$

We see that W^{12} depends on only 8 out of the 16 θ's of the full $N = 4$ superspace. It is then obvious that its θ expansion can in principle go

up to spin 2:

$$W^{12} = \ldots + \theta_3^\alpha \theta_4^\beta \bar{\theta}_{\dot{\alpha}}^1 \bar{\theta}_{\dot{\beta}}^2 A_{\alpha\beta}^{\dot{\alpha}\dot{\beta}} + \ldots . \qquad (3.87)$$

This is an example of a short multiplet (a generic $N = 4$ superfield expansion goes up to spin 4). In fact, W^{12} is even shorter, as we shall see in the next subsection.

In order to achieve equivalence with the original constraints (3.80), (3.81) we have to eliminate the non-trivial harmonic dependence of W. This is done by imposing conditions of H–analyticity, in addition to G–analyticity. As in the cases $N = 2, 3$, we choose the set of six harmonic derivatives corresponding to the positive roots of $SU(4)$:

$$D_J^I W^{12} = 0 , \qquad I, J = 1, 2, 3, 4, \quad I < J . \qquad (3.88)$$

They define the highest weight of an $SU(4)$ irrep. In fact, among them only three are independent, $(D_2^1 , D_3^2 , D_4^3)W^{12} = 0$, but it is often convenient to use all the six. The implications of the condition $D_2^1 W^{12} = 0$ on the leading component in the θ expansion of $W^{12} = \phi^{12}(x, u) + \ldots$ are easy to see:

$$D_2^1(\phi^{ij}(x)u_i^1 u_j^2) = \phi^{ij}(x)u_i^1 u_j^1 = 0 \ \Rightarrow \phi^{ij} = -\phi^{ji} \qquad (3.89)$$

since the harmonic variables commute. In other words, the component ϕ is in the $SU(4)$ UIR $[0, 1, 0]$. The remaining harmonic conditions eliminate any dependence on the other harmonics in $\phi^{12}(x, u)$. The same argument shows that the remaining components of the superfield either belongs to UIR's of $SU(4)$(if they are not expressed in terms of the lower components or just vanish), so that in the end the entire superfield recovers its original trivial harmonic dependence shown in eq. (3.84).

The harmonic conditions (3.88) ensure that the superfield W^{12} forms a representation of supersymmetry with fixed $SU(4)$ super-quantum numbers. Indeed, the harmonic derivatives D_J^I form the algebra of $SU(4)$,

$$[D_J^I, D_L^K] = \delta_J^K D_L^I - \delta_L^I D_J^K \qquad (3.90)$$

realized on W^{12}. At the same time, these derivatives are super-covariant, i.e., commute with the supersymmetry generators. Therefore the $SU(4)$ Casimir operators

$$C_n = D_{I_2}^{I_1} D_{I_3}^{I_2} \ldots D_{I_1}^{I_n} , \qquad n = 2, 3, 4 \qquad (3.91)$$

are automatically super Casimirs. Now, the $SU(4)$ algebra (3.90) allows us to rewrite (3.91) in such a way that all the D_J^I with $I < J$ appear on the right, after which we can make use of the conditions (3.88). Thus, the Casimirs are reduced to polynomials of the charge operators D_I^I and take eigenvalues on the superfield W^{12} determined by its charges. The conclusion is that the supermultiplet described by W^{12} has definite $SU(4)$ quantum numbers which coincide with those of its first component.

In exactly the same way one can show that the other two projections of the field strength live in the two alternative G–analytic subspaces involving only 8 θ's each:

$$W^{13}(x_A', \theta_2, \theta_4, \overline{\theta}^1, \overline{\theta}^3, u) \,, \tag{3.92}$$
$$x_A' = x - i(\theta_1\overline{\theta}^1 + \theta_3\overline{\theta}^3 - \theta_2\overline{\theta}^2 - \theta_4\overline{\theta}^4) \,;$$
$$W^{23}(x_A'', \theta_1, \theta_4, \overline{\theta}^2, \overline{\theta}^3, u) \,, \tag{3.93}$$
$$x_A'' = x - i(\theta_1\overline{\theta}^1 + \theta_3\overline{\theta}^3 - \theta_2\overline{\theta}^2 - \theta_4\overline{\theta}^4) \,,$$

where the corresponding G–analytic bases in superspace have been used.

In addition, these G–analytic superfields satisfy H–analyticity conditions which can be obtained from eq. (3.88) by permuting the indices. As before, they make the superfield an irrep of $SU(4)$. It should be stressed that these analyticity conditions are flat (linear) for all W's in the Abelian theory or when applied to gauge-invariant composite operators of the type $\mathrm{Tr}W^n$ in the non-Abelian theory.

3.3.2   Series of short multiplets

The UIR's of the $PSU(2,2/4)$ superalgebra fall in three classes [10, 14]:

-

A) $\ell \geq 2 + J_1 + J_2 + a + b + c \,,$
$$J_2 - J_1 \geq \frac{1}{2}(c - a) \quad (\text{or } J_1 \leftrightarrow J_2, a \leftrightarrow c) \quad (3.94)$$

Massless bulk multiplets correspond to maximal shortenings with $J_2 = J_1, a = b = c = 0$

•

$$\text{B)} \quad \ell = \frac{1}{2}(c + 2b + 3a) \geq 2 + 2J + \frac{1}{2}(3c + 2b + a) \, ,$$

$$J_1 = J \, , J_2 = 0 \quad (3.95)$$

(or $J_1 \leftrightarrow J_2$, $a \leftrightarrow c$) with $\ell \geq 1 + J + a + b + c$, $1 + J \leq \frac{1}{2}(a - c)$

•

$$\text{C)} \quad \ell = 2a + b \, , \quad a = c \, , \quad J_1 = J_2 = 0 \quad (3.96)$$

Series C) contains the Yang–Mills multiplet for $a = 0$, $b = 1$ and the K–K tower of short multiplets for $b = p > 1$ and $a = c = 0$.

The discussion of the properties of the G–analytic superfield W^{12} above applies to any of its powers

$$(W^{12})^p = C[0, p, 0] \quad (3.97)$$

(or to $\text{Tr}(W^{12})^p$ in the non-Abelian case). The notation indicates the Dynkin $SU(4)$ labels $[0, p, 0]$ of the first component of the superfield.

These superfields satisfy the set of G–analyticity conditions

$$(D_\alpha^1, \ D_\alpha^2, \ \overline{D}_{3\dot\alpha}, \ \overline{D}_{4\dot\alpha}) C[0, p, 0] = 0 \, , \quad (3.98)$$

and are therefore short multiplets (maximal spin $2 = (1, 1)$). As before, the harmonic conditions

$$D_J^I C[0, p, 0] = 0 \, , \qquad I, J = 1, 2, 3, 4, \ \ I < J \quad (3.99)$$

ensure irreducibility under $SU(4)$. Indeed, consider the leading component

$$[(W^{12})^p]|_0 = \phi^{(i_1 \ldots i_p)(j_1 \ldots j_p)} u_{i_1}^1 \ldots u_{i_p}^1 u_{j_1}^2 \ldots u_{j_p}^2 \quad \xrightarrow{D_2^1}$$

$$\phi^{(i_1 \ldots i_p j_1)(j_2 \ldots j_p)} u_{i_1}^1 \ldots u_{i_p}^1 u_{j_1}^1 u_{j_2}^2 \ldots u_{j_p}^2 = 0 \, . (3.100)$$

This condition eliminates the symmetrization between the indices of the first and second set and thus renders the field irreducible, belonging to the $SU(4)$ UIR $[0, p, 0]$. The alternative proof of irreducibility makes use of the positive root or the Casimir argument above.

The series above corresponds to K–K towers of IIB supergravity on $AdS_5 \times S_5$ [41] and it was obtained using the oscillator method by Gunaydin and Marcus [26]. Its relation with analytic superfields with harmonic variables of $\frac{SU(4)}{SU(2) \times SU(2) \times U(1)}$ [34] was discussed in [2].

Another way of obtaining new short representations is to multiply two W's with different G–analyticities, e.g.,

$$[W^{12}(\theta_3, \theta_4, \bar{\theta}^1, \bar{\theta}^2)]^{p+q}[W^{13}(\theta_2, \theta_4, \bar{\theta}^1, \bar{\theta}^3)]^q = C[q, p, q] \qquad (3.101)$$

(we postpone the discussion of the role of the traces to the next section, and from now on we omit the traces). The lowest component of the corresponding irreducible superfield belongs to the UIR $[q, p, q]$ with $p + 2q$ indices 1, $p + q$ indices 2 and q indices 3 in the corresponding rows of the YT. (Note that interchanging W^{12} and W^{13} would give an equivalent series). It is clear that such superfields satisfy only a subset of the G–analyticity conditions above:

$$D_\alpha^1 C[q, p, q] = \overline{D}_4^{\dot{\alpha}} C[q, p, q] = 0 \qquad (3.102)$$

and thus depend on 12 out of the total of 16 θ's. As a consequence, the value of the spin in their expansions cannot exceed $3 = (\frac{3}{2}, \frac{3}{2})$. Next, we have to impose the set of H–analyticity constraints

$$D_J^I C[q, p, q] = 0 \, , \qquad I, J = 1, 2, 3, 4, \; I < J \qquad (3.103)$$

which are clearly compatible with the G–analyticity of $C[q, p, q]$. Note that they coincide with those for the preceding series (3.99) which is needed for consistency if we set $q = 0$. As before, the effect of these conditions is to single out the $SU(4)$ UIR $[q, p, q]$. Indeed, the leading component

$$\phi^{(i_1 \ldots i_{p+2q})(j_1 \ldots j_{p+q})(k_1 \ldots k_q)} u_{i_1}^1 \ldots u_{i_{p+2q}}^1 u_{j_1}^2 \ldots u_{j_{p+q}}^2 u_{k_1}^3 \ldots u_{k_q}^3 \qquad (3.104)$$

becomes irreducible after imposing the constraints involving D_2^1, D_3^1, D_3^2 (they remove all possible symmetrizations among the different sets of indices). The above series corresponds to the shortening C) in (3.96).

The third possibility corresponds to the shortening B) in (3.95). It involves all the three G–analytic W's:

$$C[q + 2n, p, q] = [(W^{12})^{p+q+n}(W^{13})^{q+n}(W^{23})^n] \, . \qquad (3.105)$$

This time there is only one G–analyticity condition left,

$$\overline{D}_{4\dot\alpha} C[q + 2n, p, q] = 0 .$$ (3.106)

Consequently, the superfield depends on 14 θ's (8 left-handed and 6 right-handed) and the spins in its expansion can go up to $7/2 = (2, \frac{3}{2})$. The H–analyticity constraints are

$$D_J^I C[q + 2n, p, q] = 0 , \qquad I, J = 1, 2, 3, 4, \quad I < J .$$ (3.107)

Once again, they look the same as those for the $C[0, p, 0]$ series (3.99) and for the $C[q, p, q]$ series (3.103) (needed for consistency). The irrep corresponding to the leading component now is $[q + 2n, p, q]$.

Concluding this subsection we note that there exist other G–analytic subspaces involving 10 out of the 16 θ's, for example, $\theta_{1,2}$, $\overline{\theta}^{1,2,3}$. However, superfields living in such subspaces cannot be obtained by multiplying W's.

3.3.3   Extra shortening of N=4 superfields

As in the cases $N = 2, 3$ before, the $N = 4$ representations can in some case be semishort. The simplest example is the superfield W^{12} itself, which is ultrashort. Due to the G– and H–analyticity constraints (3.85), (3.88) it describes the on-shell $N = 4$ ultrashort SYM multiplet containing six scalars $\phi^{ij} = -\phi^{ji}$, four spinors $\lambda^{\alpha i}$ (and their conjugates $\overline{\lambda}_i^{\dot\alpha}$) and the field strength $F_{\alpha\beta}^+$, $F_{\dot\alpha\dot\beta}^-$. Thus, its θ expansion is effectively shorter than that of a generic G–analytic superfield of the type (3.86).

$$W^{12} = \phi^{12} + \theta_3^\alpha \lambda_{4\alpha} - \theta_4^\alpha \lambda_{3\alpha} + \overline{\theta}^{1\dot\alpha}\overline{\lambda}_{\dot\alpha}^2 - \overline{\theta}^{2\dot\alpha}\overline{\lambda}_{\dot\alpha}^1 + \theta_3^\alpha\theta_4^\beta F_{\alpha\beta}^+ + \overline{\theta}^{1\dot\alpha}\overline{\theta}^{2\dot\beta} F_{\dot\alpha\dot\beta}^-$$

$$+ i\theta_3^\alpha\overline{\theta}^{1\dot\beta}\partial_{\alpha\dot\beta}\phi^{23} + i\theta_4^\alpha\overline{\theta}^{1\dot\beta}\partial_{\alpha\dot\beta}\phi^{24} - i\theta_3^\alpha\overline{\theta}^{2\dot\beta}\partial_{\alpha\dot\beta}\phi^{13} - i\theta_4^\alpha\overline{\theta}^{2\dot\beta}\partial_{\alpha\dot\beta}\phi^{14}$$

$$+ i\overline{\theta}^{1\dot\beta}\theta_3^{(\alpha}\theta_4^{\beta)}\partial_{\alpha\dot\beta}\lambda_{1\beta} + i\overline{\theta}^{2\dot\beta}\theta_3^{(\alpha}\theta_4^{\beta)}\partial_{\alpha\dot\beta}\lambda_{2\beta} + i\theta_3^\alpha\overline{\theta}^{1(\dot\alpha}\overline{\theta}^{2\dot\beta)}\partial_{\alpha\dot\beta}\overline{\lambda}_{\dot\beta}^3$$

$$+ i\theta_4^\alpha\overline{\theta}^{1(\dot\alpha}\overline{\theta}^{2\dot\beta)}\partial_{\alpha\dot\beta}\overline{\lambda}_{\dot\beta}^4 + \theta_3^{(\alpha}\theta_4^{\beta)}\overline{\theta}^{1(\dot\alpha}\overline{\theta}^{2\dot\beta)}\partial_{\alpha\dot\alpha}\partial_{\beta\dot\beta}\phi^{34} .$$ (3.108)

To see this take, for instance, the component at $\theta_3\overline{\theta}^1$. It can be defined using the spinor derivatives of W^{12} at $\theta = 0$, $A_{\alpha\dot\beta}^{23} = D_\alpha^3 \overline{D}_{1\dot\beta} W^{12}|_0$.

Now, W^{12} is subject to the H–analyticity conditions (3.88), in particular, $D_3^1 W^{12} = 0$. Applying this to the component and using the G–analyticity condition $D_\alpha^1 W^{12} = 0$, we find

$$D_3^1 A_{\alpha\dot\beta}^{23} = D_\alpha^1 \overline{D}_{1\dot\beta} W^{12}|_0 = i\partial_{\alpha\dot\beta} W^{12}|_0 = i\partial_{\alpha\dot\beta}\phi^{12} . \tag{3.109}$$

The resulting harmonic equation has the obvious solution $A_{\alpha\dot\beta}^{23} = -i\partial_{\alpha\dot\beta}\phi^{23}$.

Inspecting the superfield expansion (3.108) one sees that each θ (or $\bar\theta$) appears only linearly. This means that the superfield W^{12} satisfies Grassmann linearity conditions of the type, e.g.,

$$(D^3)^2 W^{12} = 0 . \tag{3.110}$$

Once again, this constraint can be easily derived by using the basic G– and H–analyticity properties of W^{12}. Indeed, denote $A^{1233} = (D^3)^2 W^{12}$ and hit it with the harmonic derivatives D_3^1, D_3^2:

$$D_3^1 A^{1233} = D_3^2 A^{1233} = 0 . \tag{3.111}$$

These two constraints ensure the $SU(4)$ reducibility of (the leading component of) A^{1233} by eliminating all the symmetrizations between indices projected with different harmonics. Then we can rewrite it as $A^{1233} = A_4^3$ and by hitting it with D_3^1 we find

$$D_3^1 A_4^3 = A_4^1 = 0 \;\Rightarrow\; A^{1233} = 0 . \tag{3.112}$$

In the same way one can readily prove the following relations

$$(D^3 D^4) W^{12} = (\overline{D}_1 \overline{D}_2) W^{12} = 0 , \tag{3.113}$$

$$(D^3)^2 W^{12} = (D^4)^2 W^{12} = (\overline{D}_1)^2 W^{12} = (\overline{D}_2)^2 W^{12} = 0 . \tag{3.114}$$

Another way to find out that the superfield W^{12} is ultrashort is to notice that the components

$$F_{\alpha\beta}^+ = D_{(\alpha}^3 D_{\beta)}^4 W^{12} , \qquad F_{\dot\alpha\dot\beta}^- = \overline{D}_{1(\dot\alpha}\overline{D}_{2\dot\beta)} W^{12} \tag{3.115}$$

are the highest (in this case the only) chargeless ones in the expansion. Adding one more spinor derivative (i.e.,, moving a step up in the

expansion) produces a charged component which is eliminated by the harmonic conditions. For example, take $\psi^3 = D^3 \overline{D}_1 \overline{D}_2 W^{12}|_0$ and hit it with D_3^1:

$$D_3^1 \psi^3_{\alpha\dot\alpha\dot\beta} = D_3^1 D_\alpha^3 \overline{D}_{1\dot\alpha} \overline{D}_{2\dot\beta} W^{12}|_0 = -i\partial_{\alpha\dot\alpha} \overline{D}_{2\dot\beta} W^{12}|_0 = -i\partial_{\alpha\dot\alpha} \overline{\chi}^1_{\dot\beta}$$

$$\Rightarrow \quad \psi^i_{\alpha\dot\alpha\dot\beta} = -i\partial_{\alpha\dot\alpha} \overline{\chi}^i_{\dot\beta} . \tag{3.116}$$

Thus, we can say that the expansion of the superfield W^{12} ends at the level of 2 θ's (in the sense that all the higher components are expressed in terms of derivatives of the lower ones). We call such superfields ultrashort.

The linearity property of W^{12} is of course lost when we start multiplying them. Nevertheless, $(W^{12})^2$ and $(W^{12})^3$ are still shorter than a generic superfield of the same G–analyticity type. According to the general discussion in section 2, $p = 2$ gives a "current" superfield while $p = 3$ gives a semishort multiplet. Indeed, let us examine the top component of $C[0, p, 0] \sim (W^{12})^p$:

$$\Phi^{(0,p-4,0)} \sim (D^3 D^4 \overline{D}_1 \overline{D}_2)^2 (W^{12})^p|_{\theta=0} . \tag{3.117}$$

For $p \geq 4$ this is a field containing an $SU(4)$ irrep which survives all the harmonic conditions (for $p = 4$ it becomes a singlet), so the superfield is not ultrashort. For $p = 3$ we find singlets at the level of 6 θ's:

$$(D^3)^2 (D^4)^2 \overline{D}_{1(\dot\alpha} \overline{D}_{2\dot\beta)} C[0, 3, 0] , \qquad D_{(\alpha}^3 D_{\beta)}^4 (\overline{D}_1)^2 (\overline{D}_3)^2 C[0, 3, 0] . \tag{3.118}$$

They are not affected by the harmonic conditions and indeed, by taking the expansion (3.108) of a single W^{12} to the third power we do find such components. Any higher component will carry a charge and will be killed by the harmonic conditions (for instance, $\psi^4 = D^3 (D^4 \overline{D}_1 \overline{D}_2)^2 (W^{12})^3|_{\theta=0}$ is annihilated by D_4^1). Thus, the expansion of $C[0, 3, 0]$ ends at 6 θ's.

Similarly, for $C[0, 2, 0] = (W^{12})^2$ we find the singlets

$$(D^3)^2 (D^4)^2 C[0, 2, 0] ,$$
$$(\overline{D}_1)^2 (\overline{D}_3)^2 C[0, 2, 0] ,$$
$$D_{(\alpha}^3 D_{\beta)}^4 \overline{D}_{1(\dot\alpha} \overline{D}_{2\dot\beta)} C[0, 2, 0] ,$$

which are indeed present in the square of the expansion (3.108). Thus, $C[0, 2, 0]$ is another extrashort superfield ending at 4 θ's, it is in fact a "current superfield".

In the cases of the series $C[q, p, q]$ with $q > 1$ and $C[q + 2n, p, q]$ with $n > 1$ the same analysis shows that the top component is always present. In the case $(W^{12})^{p+1}W^{13}$ the superfield is linear in θ_2 because this is the property of W^{13} and W^{12} does not depend on θ_2. Similarly, the product $(W^{12})^{p+q+1}(W^{13})^{q+1}W^{23}$ is linear in θ_1. The above cases correspond to semishortening. These superfields are even shorter for certain values of p and q, but this requires an additional analysis.

4   Multitrace operators and multiparticle states

The analysis of different classes of $N = 4$ conformal supermultiplets obeying different types of shortening conditions has an interesting application to some "states" which are not K–K states but have rather the interpretation of "multiparticle states" [1, 7] in the AdS/CFT correspondence.

In $N = 4$ Yang–Mills theory these states correspond to the decomposition of the product of some "short" (single trace) K–K multiplets into irreducible superconformal blocks. Such blocks necessarily contain multitrace (rather than single trace) [8, 9, 7] Yang–Mills gauge-invariant operators which in general are not in the same representations of the "short" K–K multiplets.

In this section we will make the analysis for the most general double-trace and triple-trace gauge invariant operators of $N = 4$ Yang–Mills. The extension to higher multitraces is in principle straightforward.

Let us denote by $\phi^{12}, \phi^{13}, \phi^{23}$ the lowest θ components of the three superfields W^{12}, W^{13}, W^{23}. Consider now the gauge invariant opera-

tors:

$$
\begin{aligned}
s) & \quad \mathrm{Tr}\,[(\phi^{12})^p] \\
d) & \quad \mathrm{Tr}\,[(\phi^{12})^{p+q}]\mathrm{Tr}\,[(\phi^{13})^q] \\
t) & \quad \mathrm{Tr}\,[(\phi^{12})^{p+q+n}]\mathrm{Tr}\,[(\phi^{13})^{q+n}]\mathrm{Tr}\,[(\phi^{23})^n]
\end{aligned}
\tag{4.1}
$$

Sequence $s)$ is the usual K–K tower of IIB supergravity on $AdS_5 \times S_5$. It gives all multiplets with $J_{max} = 2$ (more precisely a $(1,1)$ tensor in the $[0, p-2, 0]$ UIR of $SU(4)$) and the first component is

$$
\mathrm{Tr}(\phi^{\ell_1} \cdots \phi^{\ell_p}) \quad \text{symmetric traceless} \tag{4.2}
$$

where ϕ^ℓ is a scalar in the $[0,1,0]$ of $SU(4)$ defined as:

$$
\phi^\ell = W^\ell|_{\theta=0} \equiv (\gamma^\ell)_{ij} W^{ij}. \tag{4.3}
$$

Sequences $d)$ are the double-trace operators. The $\theta = 0$ term is contained in the product

$$
\mathrm{Tr}(\phi^{\ell_1} \cdots \phi^{\ell_{p+q}})\mathrm{Tr}(\phi^{m_1} \cdots \phi^{m_q}) \tag{4.4}
$$

where each single trace is symmetric traceless.

As an illustrative example consider for instance the product of two lowest components of the "current" multiplet:

$$
\mathrm{Tr}\left[\phi^{(\ell_1}\phi^{\ell_2)} - \frac{1}{6}\epsilon^{\ell_1\ell_2}\phi^{(m}\phi_{m)}\right] \mathrm{Tr}\left[\phi^{(\ell_3}\phi^{\ell_4)} - \frac{1}{6}\epsilon^{\ell_3\ell_4}\phi^{(n}\phi_{n)}\right] \tag{4.5}
$$

It contains the irreducible $SU(4)$ components:

$$
\begin{array}{cccccccc}
20 \times 20 & = & 105 & + & 84 & + & 20 & + & 1 \\
 & & [0,4,0] & & [2,0,2] & & [0,2,0] & & [0,0,0]
\end{array}
\tag{4.6}
$$

The (105) and (84) correspond to two short multiplets with top spin $(1,1)$ and $(\frac{3}{2}, \frac{3}{2})$ respectively, while the last two are long multiplets with top spin $(2,2)$. The latter acquire anomalous dimensions in perturbation theory as shown in refs. [7]-[4].

The first two UIRs are contained in $\mathrm{Tr}\,(\phi^{12})^2\,\mathrm{Tr}\,(\phi^{12})^2$ and $\mathrm{Tr}\,(\phi^{12})^2\,\mathrm{Tr}\,(\phi^{13})^2$ respectively, while the last two correspond to UIRs superfields with 8θ, $8\bar{\theta}$.

The multiplets in (4.4)decompose in long ($J_{max} = 4$) and short multiplets ($J_{max} = 2, 3$). The virtue of the multiplication in d) is that precisely it singles out all shortening occurring in (4.4).

Note that the operator

$$\mathrm{Tr}\,(\phi^{12})^{\ell}\mathrm{Tr}\,(\phi^{12})^{m} \qquad (4.7)$$

gives the same UIR as $\mathrm{Tr}(\phi^{12})^{\ell+m}$ i.e., $\mathcal{D}(\ell+m, 0, 0; 0, \ell+m, 0)$. This means that the $[0, p, 0]$ UIR obtained in any multitrace operator is a short multiplet. The same type of argument will apply to the other shortenings.

Analogously, an operator of the type

$$\mathrm{Tr}\,\left([\phi^{12}]^{\ell}[\phi^{13}]^{m}\right) \qquad (4.8)$$

would correspond to a single trace first component ($\theta = 0$) operator:

$$\mathrm{Tr}\,\phi^{a_1}\cdots\phi^{a_\ell}\phi^{b_1}\cdots\phi^{b_m} \qquad (4.9)$$

where some antisymmetrization of two a, b indices occurs so it would not be a superconformal primary operator in super Yang–Mills theory [54] (although it would be conformal primary in an Abelian theory where traces are removed, since in that case it would coincide with $(\phi^{12})^{\ell}(\phi^{13})^{m}$). Also note that in a rank 1 Abelian theory only s) would survive because in that case all antisymmetrizations of ϕ^{ℓ} would vanish. These considerations also imply that no single power of any ϕ^{12} (or ϕ^{13}, ϕ^{23}) should occur in the product (since $\mathrm{Tr}\,\phi = 0$) then implying that linear semishort operators (i.e., operators satisfying a $D^2 = 0$ constraint) do not occur in $SU(\mathcal{N})$ Yang–Mills theory.

From the general class of shortening we see that d) contains the irreducible pieces which correspond to the shortening of:

$$\mathrm{Tr}[(\phi^{12})^{p+q+k}] \times \mathrm{Tr}[(\phi^{13})^{q-k}], \quad 0 \le k \le q \qquad (4.10)$$

i.e., to highest weight states of the type $\mathcal{D}(2q+p, 0, 0; q-k, p+2k, q-k)$, the new one being the $k = 0$ one, with Dynkin label $[q, p, q]$ ($k = q-1$ is missing because $\mathrm{Tr}\,\phi^{13} = 0$). This is precisely the UIR singled out by the H-analyticity constraints (3.103)

All these states have quantized dimensions and lie in multiplets with $J_{max} = 3$, unless $k = q$, for which $J_{max} = 2$.
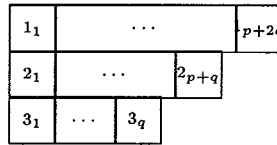
Figure 1: $[q, p, q]$ representation.

Let us apply this to the cases of double-traces with $d \leq 6$. The rational to restrict to s), d), t) families is the following:

- $\mathrm{Tr}[(\phi^{12})^2] \times \mathrm{Tr}[(\phi^{13})^2]$.

  In this case $p = 0$, $q = 2$, so the two short reps. are the $[0, 4, 0]$ ($J_{max} = 2$) and $[2, 0, 2]$ ($J_{max} = 3$). It has been confirmed by direct calculation that indeed two such objects are not renormalized in perturbative theory (at one loop).

- $\mathrm{Tr}[(\phi^{12})^3] \times \mathrm{Tr}[(\phi^{13})^2]$.

  In this case $p = 1$, $q = 2$. The short multiplets are in the $[0, 5, 0]$ and $[2, 1, 2]$.

- $\mathrm{Tr}[(\phi^{12})^3] \times \mathrm{Tr}[(\phi^{13})^3]$.

  In this case $p = 0$, $q = 3$. The short multiplets are in the $[0, 6, 0]$, $[3, 0, 3]$ and $[2, 2, 2]$.

- $\mathrm{Tr}[(\phi^{12})^4] \times \mathrm{Tr}[(\phi^{13})^2]$.

  In this case $p = 2$, $q = 2$. The short multiplets are in the $[0, 6, 0]$ and $[2, 2, 2]$.

It is thus obvious that the number of short multiplets is precisely q of which $q - 1$ have $J_{max} = 3$ and one has $J_{max} = 2$.

We now consider triple-trace operators where a new type of shortening ($J_{max} = \frac{7}{2}$) occurs.

The generic triple-trace operator is:

$$\mathrm{Tr}[(\phi^{12})^{p+q+n}]\mathrm{Tr}[(\phi^{13})^{q+n}]\mathrm{Tr}[(\phi^{23})^n] \tag{4.11}$$

with dimension $d = p + 2q + 3n$.

The above expressions single out the short multiplets contained in the following triple-trace operator composites:

$$\text{Tr}\,(\phi^{\ell_1}\cdots\phi^{\ell_{p+q+n}})\text{Tr}\,(\phi^{m_1}\cdots\phi^{m_{q+n}})\text{Tr}\,(\phi^{s_1}\cdots\phi^{s_n}) \qquad (4.12)$$

The new phenomenon here is that three types of short multiplets with $J_{max} = 2, 3$ and $\frac{7}{2}$ occur. The new short multiplet is a "chiral" superfield whose first component is in the $[q+2n, p, q]$ of $SU(4)$ and with a $J_{max} = \frac{7}{2}$ state in the $(2, \frac{3}{2})$ rep. of $SL(2, \mathbb{C})$. This is the UIR singled out by the constraints (3.107) From the analysis of the product of:

$$\text{Tr}\,[(W^{12})^{p+q+n}]\text{Tr}\,[(W^{13})^{q+n}]Tr\,[(W^{23})^n] \qquad (4.13)$$

it follows that the above triple-trace operators contain all the shortenings which occur in:

$$\text{Tr}\,[(\phi^{12})^{p+q+n+x}]\text{Tr}\,[(\phi^{13})^{q+n+k-x}]\text{Tr}\,[(\phi^{23})^{n-k}],$$
$$0 \le k \le n,\, 0 \le x \le q+2k \quad (4.14)$$

if $k \le p$, or

$$\text{Tr}\,[(\phi^{12})^{q+k+n+x}]\text{Tr}\,[(\phi^{13})^{p+q+n-x}]\text{Tr}\,[(\phi^{23})^{n-k}],$$
$$0 \le k \le n,\, 0 \le x \le p+q+k \quad (4.15)$$

for $k \ge p$.

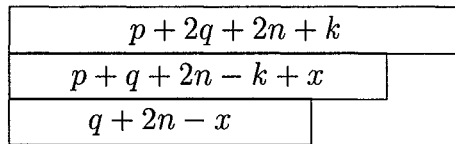| $p + 2q + 2n + k$		
$p + q + 2n - k + x$		
$q + 2n - x$		

Figure 2: $[q + 2n - x, p - k + 2x, q + 2k - x]$ representation, for $k \le p$; $0 \le k \le n$, $0 \le x \le q + 2k$.

The first triple trace operator is $(d = 6)$:

$$\text{Tr}\,[(\phi^{12})^2]\text{Tr}\,[(\phi^{13})^2]\text{Tr}\,[(\phi^{23})^2], \quad (p = q = 0; n = 2) \qquad (4.16)$$

For $k = 0$, $x = 0$ it gives the $[0, 0, 4] + [4, 0, 0]$ ($J_{max} = \frac{7}{2}$). For $k = 2$, it gives all the shortenings already occurred in $\text{Tr}\,[(\phi^{12})^{4+x}]\text{Tr}\,[(\phi^{13})^{2-x}]$. These are the $[0, 6, 0]$ and $[2, 2, 2]$ ($J_{max} = 2$ and 3 respectively).

| $p + 2q + 2n + k$ |
| $q + 2n + x$ |
| $p + q + 2n - k - x$ |

Figure 3: $[p+q+2n-k-x, -p+k+2x, p+q+k-x]$ representation, for $k \geq p$; $0 \leq k \leq n$, $0 \leq x \leq p+q+k$.

This completes the analysis of the shortenings of double and triple trace operators. Of course, due to their ring structure, higher multiple trace operators can be obtained by further multiplying structures as in s) d) t) then obtaining the same type of shortening as in the previous composite operators.

The bulk interpretation of these composite operators is that there are some multiparticle BPS states in the supergravity side.

The "non-renormalization" of the $[0, 4, 0]$ and $[2, 0, 2]$ short multiplets contained in the two graviton-multiplets particle state was shown in $N = 4$ Yang–Mills perturbation theory in ref. [4, 51]. The latter reference extended the analysis for the $(0, p, 0)$ block to all multitrace components. Its relation with shortening was established in ref. [2] and [14].

From the shortening conditions we see that while the usual K–K states are $\frac{1}{2}$ BPS (since the superfield does not depend on 4 θ_L, 4 $\overline{\theta}_R$) the new short classes correspond to $\frac{1}{4}$ BPS (2 θ_L, 2 $\overline{\theta}_R$) and $\frac{1}{8}$ BPS (2 $\overline{\theta}_R$). The lowest dimensional $\frac{1}{4}$ BPS operators occur for highest weight $\mathcal{D}(4, 0, 0; 2, 0, 2)$ (double-trace) while the lowest dimensional $\frac{1}{8}$ BPS states occur for highest weights $\mathcal{D}(6, 0, 0; 4, 0, 0)$ + c.c..

5   Conclusion

In the present paper we have analyzed all possible shortenings which are obtained by composite operators made out of the field strength gauge multiplets and hypermultiplets (for $N = 2$).

These shortenings are characterized by subspaces of HSS's which do not depend on a certain number of (fermionic) θ variables.

If a certain subspace (of the full superspace with $4N$ θ variables) does not depend on n fermionic coordinates, then a superfield on such a space is generally called $\frac{n}{4N}$ BPS in analogy with a particle state interpretation.

Moreover, if $n = n_L + n_R$ then the highest spin of such a superfield is

$$(J_1, J_2) = \left(\frac{2N - n_L}{4}, \frac{2N - n_R}{4} \right) . \tag{5.1}$$

We can summarize the set of subspaces, for each N, by the pair (n_L, n_R).

All possible shortenings found from the analysis of section 3 are summarized in the following table.

N	(n_L, n_R)	(J_1, J_2)	BPS
	$(2,2)$	$(\frac{1}{2}, \frac{1}{2})$	$\frac{1}{2}$
$N = 2$	$(0,4)$	$(1,0)$	$\frac{1}{2}$
	$(0,2)$	$(1, \frac{1}{2})$	$\frac{1}{4}$
	$(2,4)$	$(1, \frac{1}{2})$	$\frac{1}{2}$
$N = 3$	$(2,2)$	$(1,1)$	$\frac{1}{3}$
	$(0,2)$	$(\frac{3}{2}, 1)$	$\frac{1}{6}$
	$(4,4)$	$(1,1)$	$\frac{1}{2}$
$N = 4$	$(2,2)$	$(\frac{3}{2}, \frac{3}{2})$	$\frac{1}{4}$
	$(0,2)$	$(2, \frac{3}{2})$	$\frac{1}{8}$

All these representations refer to UIR's with highest weight states with $J_1 = J_2 = 0$. In this case the generic "long multiplets" (massive non-BPS states) have $J_1 = J_2 = 1, \frac{3}{2}, 2$ for $N = 2, 3, 4$ respectively.

Short multiplets have "protected dimensions" in conformal field theories. This is not the case for long multiplets whose dimension is then renormalized.

We found, as an application of these results, that some multiparticle

state channels, occurring in $AdS_5 \times S_5$ compactifications of IIB string theory, correspond to such short representations.

The AdS/CFT correspondence of supergravity with large $\mathcal{N}$ gauge theories then predicts that supergravity correlators in these channels would exhibit "canonical dimensions", then implying a new kind of "non-renormalization theorems" for $N = 4$ super Yang–Mills theory.

Acknowledgements

A   Appendix

A.1   Harmonic variables

We introduce harmonic variables on the coset $SU(N)/U(1)^{N-1}$ in the form of $SU(N)$ matrices u_i^I or their complex conjugates u_I^i. Here i is an index of the fundamental representation of $SU(N)$ whereas $I = 1, 2, \ldots, N$ is an index denoting the set of $N - 1$ $U(1)$ charges. The choice of the charges for $N = 2, 3, 4$ is as follows:

$$N = 2 : \ u_i^1 = u_i^{(1)} \ , \ u_i^2 = u_i^{(-1)} \ ;$$
$$N = 3 : \ u_i^1 = u_i^{(1,0)} \ , \ u_i^2 = u_i^{(-1,1)} \ , \ u_i^3 = u_i^{(0,-1)} \ ; \qquad (A.1)$$
$$N = 4 : \ u_i^1 = u_i^{(1,0,1)} \ , \ u_i^2 = u_i^{(-1,0,1)} \ , \ u_i^3 = u_i^{(0,1,-1)} \ , \ u_i^4 = u_i^{(0,-1,-1)} \ ;$$

the conjugates have the opposite charges, e.g., in $N = 2$ $u_1^i = u^{i(-1)}$, $u_2^i = u^{i(1)}$. The fact that the u's form an $SU(N)$ matrix implies the

following constraints:

$$u_i^I u_J^i = \delta_J^I \, , \tag{A.2}$$

$$u \in SU(N) : \qquad u_i^I u_I^k = \delta_i^k \, , \tag{A.3}$$

$$\varepsilon^{i_1 \cdots i_N} u_{i_1}^1 \ldots u_{i_N}^N = 1 \, . \tag{A.4}$$

Exceptionally, in the case $N = 2$ one can raise and lower the indices of the harmonics with the help of the Levi-Chivita symbol

$$\varepsilon_{ik}\varepsilon^{kl} = \delta_i^l \, , \quad \varepsilon^{12} = -\varepsilon_{12} = 1 \, .$$

This property allows us to identify the two sets of harmonics u_i^I and u_I^i:

$$u_i^1 \equiv \varepsilon_{ij} u_2^j \equiv u_{2i} \equiv u_i^+ \, , \qquad u_i^2 \equiv -\varepsilon_{ij} u_1^j \equiv -u_{1i} \equiv u_i^- \, .$$

Note that in ref. [19], where the $N = 2$ harmonic variables have been introduced for the first time, the notation $u_i^\pm$ was used. Here we prefer to have an uniform notation valid for any N.

The harmonic functions are supposed to transform homogeneously under $U(1)^{N-1}$, i.e., they carry definite $U(1)$ charges. This means that the dependence on the matrix variables u is considered modulo $U(1)^{N-1}$ transformations, which provides an $SU(N)$ covariant way to parametrize the coset $SU(N)/U(1)^{N-1}$. These functions are given by their infinite harmonic expansions on the coset. For instance, the function $f^1(u)$ will have the following expansion in $N = 2$:

$$f^1(u) = f^i u_i^1 + f^{(ijk)} u_i^1 u_j^1 u_k^2 + \cdots \tag{A.5}$$

going over the totally symmetrized multispinors (irreps) of $SU(2)$. In $N = 3$ this expansion is considerably richer,

$$f^1(u) = f^i u_i^1 + f_k^{(ij)} u_i^1 u_j^2 u_2^k + g_k^{(ij)} u_i^1 u_j^3 u_3^k + h_{(ij)} u_2^i u_3^j + \cdots \tag{A.6}$$

and goes over all possible irreps of $SU(3)$ such that after projection with u's the total charge will be that of f^1.

The harmonic coset $SU(N)/U(1)^{N-1}$ has $N(N - 1)/2$ complex dimensions. Correspondingly, there are as many covariant derivatives on it. In our $SU(N)$ covariant description of the coset these derivatives are made out of the operators

$$\partial_J^I = u_i^I \frac{\partial}{\partial u_i^J} - u_J^i \frac{\partial}{\partial u_I^i} \tag{A.7}$$

which respect the defining constraints (A.2), (A.3). The third constraint (A.4) implies that the charge-like operators ∂_I^I are not independent,

$$\sum_{I=1}^{N} \partial_I^I = 0 \ . \tag{A.8}$$

These derivatives act on the harmonics as follows:

$$\partial_J^I u_i^K = \delta_J^K u_i^I \ , \qquad \partial_J^I u_K^i = -\delta_K^I u_J^i \ . \tag{A.9}$$

So, the $N - 1$ $U(1)$ charges are counted by the derivatives

$$\begin{aligned}
N = 2: \quad & H = \partial_1^1 - \partial_2^2 \ ; \\
N = 3: \quad & H = \partial_1^1 - \partial_2^2 \ , \quad H' = \partial_2^2 - \partial_3^3 \ ; \\
N = 4: \quad & H = \partial_1^1 - \partial_2^2 \ , \quad H' = \partial_3^3 - \partial_4^4 \ , \quad H'' = \partial_1^1 + \partial_2^2 - \partial_3^3 - \partial_4^4 \ .
\end{aligned} \tag{A.10}$$

The remaining $N(N-1)/2$ derivatives ∂_J^I, $I < J$ (or their conjugates ∂_J^I, $I > J$) are the true harmonic derivatives on the coset $SU(N)/U(1)^{N-1}$.

It is important to realize that the set of N^2-1 derivatives ∂_J^I (taking into account the linear dependence (A.8)) form the algebra of $SU(N)$:

$$[\partial_J^I, \partial_L^K] = \delta_J^K \partial_L^I - \delta_L^I \partial_J^K \ . \tag{A.11}$$

The Cartan decomposition of this algebra $L^+ + L^0 + L^-$ is given by the sets

$$L^+ = \{\partial_J^I, \ I < J\}, \quad L^0 = \left\{\partial_I^I, \ \sum_{I=1}^{N} \partial_I^I = 0\right\}, \quad L^- = \{\partial_J^I, \ I > J\}. \tag{A.12}$$

It becomes clear that imposing the harmonic conditions

$$\partial_J^I f^{K_1 \ldots K_q}(u) = 0 \ , \quad I < J \tag{A.13}$$

on a harmonic function with a given set of charges $K_1 \ldots K_q$ defines a highest weight of an $SU(N)$ irrep. In other words, the harmonic expansion of such a function contains only one irrep which is determined by the combination of charges $K_1 \ldots K_q$. For instance, in $N = 2$ we have (see (A.5))

$$\partial_2^1 f^1(u) = 0 \ \Rightarrow \ f^1(u) = f^i u_i^1 \ , \tag{A.14}$$

and, similarly, in $N = 3$ (see (A.6))

$$\partial_2^1 f^1(u) = \partial_3^1 f^1(u) = \partial_3^2 f^1(u) = 0 \ \Rightarrow \ f^1(u) = f^i u_i^1 \ . \qquad \text{(A.15)}$$

Note that not all of the conditions (A.15) are independent since $\partial_3^1 = [\partial_2^1, \partial_3^2]$. Written down in a complex parametrization of the coset, conditions (A.13) take the form of harmonic (H-)analyticity conditions on the function $f(u)$. It is important to realize that for certain combinations of charges the condition (A.13) may not have a non-trivial solution. For example, the function $f^2(u)$ cannot be H-analytic since $\partial_2^1 f^2(u) = f^i u_i^1 = 0 \ \Rightarrow \ f^i = 0$.

A.2   Grassmann analyticity

The introduction of harmonic coordinates allows one to define various subspaces of the full N-extended superspace involving only a subset of the Grassmann coordinates without breaking $SU(N)$. Indeed, we can rewrite the supersymmetry transformations in terms of the harmonic-projected Grassmann variables as follows:

$$\begin{aligned}
\delta x^{\alpha\dot\beta} &= i(\theta_I^\alpha u_i^I \bar\epsilon^{\dot\beta i} - \epsilon_i^\alpha u_I^i \bar\theta^{\dot\beta I}) \ , \\
\delta \theta_I^\alpha &= \epsilon_i^\alpha u_I^i \ , \\
\delta \bar\theta^{\dot\beta I} &= u_i^I \bar\epsilon^{\dot\beta i}
\end{aligned} \qquad \text{(A.16)}$$

where $\theta_I^\alpha = \theta_i^\alpha u_I^i$, $\bar\theta^{\dot\beta I} = u_i^I \bar\theta^{\dot\beta i}$. Now, we can shift $x^{\alpha\dot\beta}$ in a variety of ways such that the transformation of the new variable does not involve some of the projections of θ or $\bar\theta$. Thus we obtain subspaces of the full superspace closed under supersymmetry. Such superspaces are called Grassmann (G-)analytic. Here are some examples:

$$N = 2: \quad x_A^{\alpha\dot\beta} = x^{\alpha\dot\beta} + i(\theta_2^\alpha \bar\theta^{2\dot\beta} - \theta_1^\alpha \bar\theta^{1\dot\beta}), \ \theta_2^\alpha, \ \bar\theta^{1\dot\alpha}; \qquad \text{(A.17)}$$

$$N = 3: \quad x_A^{\alpha\dot\beta} = x^{\alpha\dot\beta} + i(\theta_2^\alpha \bar\theta^{2\dot\beta} + \theta_3^\alpha \bar\theta^{3\dot\beta} - \theta_1^\alpha \bar\theta^{1\dot\beta}), \ \theta_2^\alpha, \ \theta_3^\alpha, \ \bar\theta^{1\dot\alpha};$$

$$N = 4: \quad x_A^{\alpha\dot\beta} = x^{\alpha\dot\beta} + i(\theta_3^\alpha \bar\theta^{3\dot\alpha} + \theta_4^\alpha \bar\theta^{4\dot\alpha} - \theta_1^\alpha \bar\theta^{1\dot\beta} - \theta_2^\alpha \bar\theta^{2\dot\beta}),$$
$$\theta_3^\alpha, \ \theta_4^\alpha, \ \bar\theta^{1\dot\alpha}, \ \bar\theta^{2\dot\alpha} \ .$$

In these examples the G-analytic superspace has the minimal odd dimension possible, i.e., half of the total number $4N$. In this sense the

G-analytic superspaces are analogs of chiral superspace, which also involves the left- or right-handed half of the odd variables. However, an important difference is that in the cases $N > 2$ one can also have G-analytic superspaces with intermediate odd dimensions, i.e., 8 and 10 in $N = 3$ and 10, 12 and 14 in $N = 4$. The reason is that the harmonics on the coset $SU(N)/U(1)^{N-1}$ allow one to break the spinor variables up into N independent projections, whereas the chiral projection always picks a spinor in the fundamental representation of $SU(N)$.

An equivalent definition of G-analyticity is to consider superfields satisfying constraints involving the spinor derivatives D_α^i and $\overline{D}_{i\dot\alpha}$. These derivatives commute with supersymmetry and satisfy the following algebra

$$\{D_\alpha^k, D_\beta^l\} = 0 \ ,$$
$$\{\overline{D}_{k\dot\alpha}, \overline{D}_{l\dot\beta}\} = 0 \ , \qquad\qquad (A.18)$$
$$\{D_\alpha^k, \overline{D}_{l\dot\beta}\} = i\delta_l^k \partial_{\alpha\dot\beta} \ .$$

which resembles the supersymmetry algebra. Now, projecting them with harmonics, we can impose a number of G-analyticity conditions on the superfields $\Phi(x, \theta, \overline{\theta})$. For example, the conditions corresponding to the subspaces (A.17) are

$$N = 2: \qquad D_\alpha^1 \Phi = \overline{D}_{2\dot\alpha} \Phi = 0 \ ;$$
$$N = 3: \qquad D_\alpha^1 \Phi = \overline{D}_{2,3\dot\alpha} \Phi = 0 \ ; \qquad (A.19)$$
$$N = 4: \qquad D_\alpha^{1,2} \Phi = \overline{D}_{3,4\dot\alpha} \Phi = 0 \ .$$

It is clear that this can be done with any subset of D's and $\overline{D}$'s as long as they anticommute.

The role of the shifts of x in (A.17) is to define a G-analytic basis in which the derivatives in (A.19) become torsion-free, e.g.,

$$D_\alpha^1 = \partial_\alpha^1 \ , \qquad \overline{D}_{2\dot\alpha} = \overline{\partial}_{2\dot\alpha} \ , \quad \text{etc.}$$

Of course, the spinor derivatives which do not belong to the analytic set still involve space-time derivatives in this basis. More important, the harmonic derivatives acquire torsion terms in the G-analytic basis. Thus, in the bases (A.17) one has

$$D_3^1 = \partial_3^1 - i\theta_3^\alpha \overline{\theta}^{1\dot\beta} \partial_{\alpha\dot\beta} - \theta_3^\alpha \partial_\alpha^1 + \overline{\theta}^{1\dot\alpha} \overline{\partial}_{3\dot\alpha} \ , \quad \text{etc.} \qquad (A.20)$$

This implies that the condition of H-analyticity on harmonic superfields $\Phi(x_A, \theta, \overline{\theta}, u)$ involves space-time derivatives of the components.

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