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Should start-up companies be cautious? Inventory policies which maximise survival probabilities

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Abstract

New start-up companies, which are considered to be a vital ingredient in a successful economy, have a different objective than established companies. They want to maximise their chance of long-term survival. We examine the implications for their operating decisions of this different criterion by considering an abstraction of the inventory problem faced by a start-up manufacturing company. The problem is modelled under two criteria as a Markov decision process and the characteristics of the optimal policies under the two criteria are compared. It is shown that although the start-up company should be more conservative in its component purchasing strategy than if it were a well established company it should not be too conservative. Nor is its strategy monotone in the amount of capital it has available. The models are extended to allow for interest on investment and inflation.

1   Introduction

In this age of innovation and entrepreneurship it is important to identify strategies that ensure the success of newly formed companies. There has been considerable debate about what these strategies are and how they differ from those that well-established companies should employ [7]. This paper seeks to address these issues by looking at a specific operational decision which has to be considered by most manufacturing start-up companies, namely their inventory strategy. The model considered is a simplified version of the problem facing a real

start-up manufacturing company.

Modelling the inventory control problem has been one of the most successful and widely used applications of operational research techniques in commerce and industry. The texts [1, 8, 10] and the survey [4] identify the types of inventory models that have been used and their wide range of application. In almost all cases the objective has been to minimise the cost or maximise the profit to the firm, possibly with some constraint on the level of demand to be satisfied. There are a variety of such models, some with average cost or profit over an infinite time horizon, some with discounted infinite horizon criteria and others with finite horizon criteria, but the objective is always to optimise the cost or profit to the organisation.

The thesis behind this paper is that for newly created firms it is their probability of survival rather than their profits that is their main objective. Thus we consider a very simple inventory problem and consider what are the optimal strategies for maximising the survival of the organisation and what are the optimal strategies for maximising the average profit per time period. The former are taken to be the strategies that newly formed companies are likely to be interested in, while we assume the latter are the strategies appropriate for well established firms with sufficient assets backing them to allow longer term goals. We are interested in the differences in these strategies and whether new firms should be more cautious in their outlook or should be willing to take more risks than the longer established companies. Clearly there are other issues of importance to start-up companies such as pricing and competition but we wish to concentrate on the affect on their survival of one of the standard operating decisions.

There has been little work in this area, but recently there has been increasing interest in connecting financial decisions and operating decisions for manufacturing firms. Li et al [6] examine the relationship between production decisions and dividend policies, while Birge and Zhang [3] seek to use option theory to introduce risk into inventory problems. However all these papers assume there is infinite borrowing power. The paper by Buzacott and Zhang [5] is one of the few that looks at the interface of finance and production for small firms with limited borrowing. They use a mathematical programming model to maximise profit over

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a finite horizon looking at inventory and borrowing decisions, but they assume that the demand for the product is known.

In section two we present three simple models of the real inventory problem facing a new manufacturing company with unknown random demand — for details of other problems that faced such a company see [2]. We formulate the inventory problem as dynamic programming models under the criteria of maximising the survival probability and of maximising the average profit. In sections 3 and 4 we derive properties of the optimal strategies under these criteria. In section 5 we compare the forms of these optimal strategies, while in section 6 we extend the models to allow for interest rates and inflation, which take the place of holding costs in these models. Finally we use the comparison results of section 5 to draw conclusions about the type of strategy that it is sensible for start-up firms to adopt.

2   Inventory models

The manufacturer makes one type of unit only from components that it has to purchase from other manufacturers. The lead time for ordering components is fixed and is taken as 1 time period. The demand for the unit is random with independent identical distributions each time period. If the demand for the unit in a period exceeds the number of components available at the start of the period then the excess demand is lost. Figure 1 shows a time line for the events in one period.
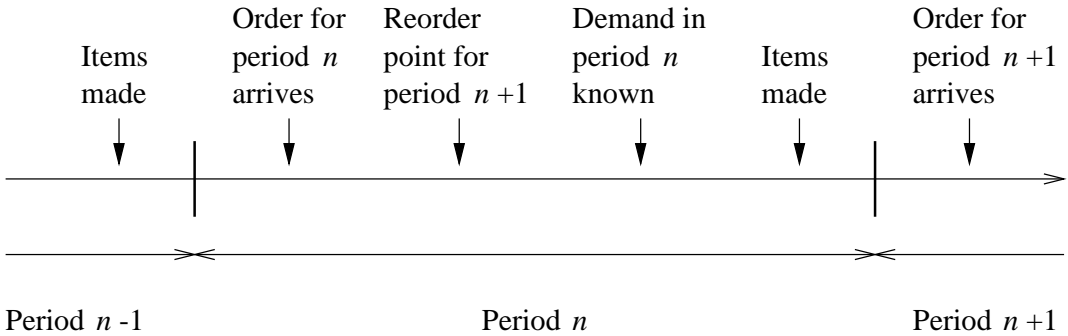

Figure 1: A time line for the events in one period

Let: S be the selling price of each unit;

C be the unit cost of buying the components required for one unit — to simplify the exposition we write as if each unit produced requires a single component from this point on;

H be the overhead costs per period (e.g. cost of staff and premises) which are incurred irrespective of the activity of the firm;

$p(d)$ be the probability that the demand for units in a period is d;

$M = \max\{d \mid p(d) > 0\}$ be the maximum possible demand that can be satisfied in a period (this can be interpreted as the maximum production capacity in a period);

$\bar{d} = \sum_{d=0}^{M} dp(d)$ be the average demand per period.

We will assume throughout the paper that $(S - C)\bar{d} > H$, so that in the long run the firm can be profitable. Notice that there is no holding cost in this inventory formulation. This is because the main component of holding cost is the cost of capital and the loss of capital from holding extra inventory is reflected in the lower capital available to the firm, which is our state variable. What we are really assuming here is that the interest rate that one gets from capital invested in the bank is the same as the inflation rate on the cost of components, the selling price of the unit and the overhead cost. In section six we will investigate what happens if this assumption is relaxed.

The decision the firm has to make is how many components to order each period. These components will be delivered at the beginning of the next period. Thus there is a constant lead time of one time period on the delivery of components. We assume units cannot be manufactured if components are not available. These assumptions really mean that the manufacturing time and any variation in lead time is small compared with the lead time itself. Ordering too many components ties up the firm's capital in stock which is not required; ordering too few components leads to unsatisfied demand and hence loss of profit.

Consider first that the manufacturer is a newly set-up firm. In this case the firm's objective is assumed to be to maximise its chance of survival. A firm will not survive if it uses up all its capital. Hence at any point the state of the firm is described by two variables — i, the number of components in stock, and x, the capital of the firm. One could allow for overdrafts by defining capital to be the amount above the overdraft limit. Define $q(n, i, x)$ to be the maximum probability that the firm will survive n more periods given it has i components in stock and x units of capital. We assume that storage constraints put some upper limit on i and hence we have a finite action space since we cannot sensibly order more than this amount. $q(n, i, x)$ is the optimal function for a finite horizon dynamic programming problem with a countable state space (x can be assumed to have discrete levels) and a finite action space — the amount k to order. Thus it has an optimal non-stationary policy (see [9, p90]). Moreover the survival probability $q(n, i, x)$ satisfies the following dynamic programming optimality equation.

$$q(n, i, x) = \max_k \left\{ \sum_{d=0}^{M} p(d) q(n-1, i+k-\min(i,d), x+S\min(i,d)-kC-H) \right\} \quad (1)$$

with boundary conditions $q(n, i, x) = 0 \ \forall x < 0$ and $q(0, i, x) = 1 \ \forall i, x \geq 0$. These boundary conditions could be modified to allow for the use of inventory as collateral for borrowing.

Define $k(n, i, x) = \text{argmax} \left\{ \sum_{d=0}^{M} p(d) q(n-1, i+k-\min(i,d), x+S\min(i,d)-kC-H) \right\}$.

If we assume the limit of $q(n, i, x)$ as $n \to \infty$ exists (this will follow from Lemma 3 (i)), the limit $q(i, x) = \lim_{n \to \infty} q(n, i, x)$ is the probability that the firm will survive forever given that it has i components in stock and x units of capital. This is a positive bounded dynamic programming model as defined in [9, Chapter 7] and taking the limit as $n \to \infty$ in (1) shows that $q(i, x)$ satisfies the optimality equation

$$q(i, x) = \max_k \left\{ \sum_{d=0}^{M} p(d) q(i+k-\min(i,d), x+S\min(i,d)-kC-H) \right\} \quad (2)$$

with boundary conditions $q(i, x) = 0 \ \forall x < 0$. Define $k(i, x)$ to be the optimal action in the state (i, x) in the infinite horizon case and so be the value of k that maximises the right hand side of (2).

If the manufacturing firm is well established, then we assume its objective is to maximise the average reward per period. In this case there is no constraint on the amount of capital the firm has since it is assumed that it has enough to finance any purchase it wants. Thus the state of the firm at the start of any period is described completely by the number of components in stock. Hence this is a countable state, finite action, unichained Markov decision process and so the standard results for average reward Markov decision processes hold (see [9]). Let g be the average reward per period under the inventory policy that optimises the average reward per period and let $v(i)$ be the bias term of starting with i in stock. Then the optimality equation of the dynamic programming model of the situation with this criterion is

$$g + v(i) = \max_k \left\{ \sum_{d=0}^{M} p(d) \left(S \min(i,d) - kC - H + v(i + k - \min(i,d)) \right) \right\}. \qquad (3)$$

Define $k(i) = \text{argmax} \left\{ \sum_{d=0}^{M} p(d) \left(S \min(i,d) - kC - H + v(i + k - \min(i,d)) \right) \right\}.$

In the next section we analyse the survival models of (1) and (2), and in section 5 we compare their optimal policies with the optimal policy for the average reward model of (3). This will give the opportunity to examine whether it is sensible to be more or less conservative in strategy when one is a start-up company than when one is well established.

3 Properties of the survival models for a start-up company

There are some obvious properties which one expects of the survival probability $q(n,i,x)$ as n, i and x vary and these are confirmed in Lemma 1.

Lemma 1

i) $q(n,i,x) \geq q(n+1,i,x)$ and hence $q(n,i,x)$ is non-increasing in n.

ii) $q(n,i,x)$ is non-decreasing in x.

iii) $q(n,i,x)$ is non-decreasing in i.

Proof

The proof of all three parts is by induction on n. Since $q(n, i, x) = 0$ when $x < 0$ for all n, $q(0, i, x) = 1$ when $x \geq 0$ and $q(1, i, x) \leq 1$ when $x \geq 0$, all three hypotheses hold in the case $n = 0$.

Assume all three hypotheses hold for n, and use $\max_i \{a_i\} - \max_i \{b_i\} \leq \max_i \{a_i - b_i\}$ to show

(i) $q(n+1, i, x) - q(n, i, x) \leq$
$$\max_k \left\{ \sum_{d=0}^{M} p(d) \Big(q(n, i + k - \min(i, d), x + S\min(i, d) - kC - H) \right.$$
$$\left. - q(n - 1, i + k - \min(i, d), x + S\min(i, d) - kC - H) \Big) \right\} \leq 0.$$

Hence hypothesis (i) holds for $n + 1$.

(ii) $q(n+1, i, x) - q(n+1, i, x + a) \leq$
$$\max_k \left\{ \sum_{d=0}^{M} p(d) \Big(q(n, i + k - \min(i, d), x + S\min(i, d) - kC - H) \right.$$
$$\left. - q(n, i + k - \min(i, d), x + a + S\min(i, d) - kC - H) \Big) \right\} \leq 0 \text{ (where } a > 0\text{)}.$$

Hence hypothesis (ii) holds for $n + 1$.

(iii) $q(n+1, i, x) - q(n+1, i+1, x) \leq \max_k \left\{ \sum_{d=0}^{i} p(d) \Big(q(n, i + k - d, x + Sd - kC - H) \right.$
$$- q(n, i + 1 + k - d, x + Sd - kC - H) \Big)$$
$$\left. + \sum_{d=i+1}^{M} p(d) \Big(q(n, k, x + Si - kC - H) - q(n, k, x + Si + S - kC - H) \Big) \right\} \leq 0.$$

where the second inequality holds because of (ii) as well as the induction hypothesis of (iii). Hence hypothesis (iii) holds for $n + 1$.

One point of interest is how $q(n, i, x)$ depends on x and in particular for what values of x is there no chance of survival and for what values of x is survival certain. If one assumes $p(0) > 0$ then the worst case is a continual zero demand for the product. Even in this case a

firm can survive if its initial capital is enough to meet the overheads in each period. Hence $q(n, i, x) = 1$ if $x \geq Hn$.

At the other extreme the best case is a perpetual demand of M for the product. Each period the firm only makes a profit and hence improves its financial position if the amount ordered (and hence sold if demand is M) is at least $k^* = \lfloor H/(S - C) \rfloor + 1$ (so that $(S - C)k^* > H$) where $\lfloor x \rfloor$ denotes the integer part of x. Hence one can never survive in the long run if the first order has to be for less than k^*, so that $q(n, i, x) = 0$ for some n if $x + S\min(i, M) - H < Ck^*$ or, equivalently, if $x < Ck^* + H - S\min(i, M)$.

If $x \geq Ck^* + H - S\min(i, M)$ then one can order k^* at the first period and survive if the demand in the first period is at least $\min(i, M)$. This means that if the demand in the second period is at least k^*, one will make a profit and so be able to order k^* for the next period. Repeating the argument shows that $q(n, i, x) > Q(i)Q(k^*)^{n-1}$ where $Q(r)$ is the probability that the demand in a period is at least r.

Before proving results about the infinite horizon survival probabilities $q(i, x)$, we need to prove some more properties of the probability function $q(n, i, x)$. These say that capital is preferable to inventory. In all cases having S more in capital is better than having 1 more in inventory, while if the inventory is high enough, having C more in capital is better than having 1 more in inventory.

Lemma 2

i) For any $i \geq M$, $q(n, i, x) \geq q(n, i + j, x - jC) \; \forall j, x > 0$.

ii) For all i, $q(n, i + j, x) \leq q(n, i, x + jS) \; \forall j, x > 0$.

Proof

i) Suppose $k(n, i + j, x - jC) = \delta$ and consider the action that orders $j + \delta$ items in state (n, i, x).

$$q(n, i, x) \geq \sum_{d=0}^{M} p(d)q\left(n - 1, i + j + \delta - d, x + Sd - (j + \delta)C - H\right)$$

$$= \sum_{d=0}^{M} p(d) q\left(n-1, i+j+\delta-d, x-jC+Sd-\delta C-H\right)$$

$$= q\left(n, i+j, x-jC\right)$$

where as $i \geq M, d = \min(i, d)$.

ii) It is sufficient to prove (ii) for $j = 1$ and the proof will be by induction on n. The result is trivially true for $n = 0$. Assume true for $n - 1$. Let $k(n, i+1, x) = \delta$ and consider the action that orders δ in state $(n, i, x + S)$.

$$\begin{aligned}
q(n, i, x+S) \quad \geq \quad & \sum_{d=0}^{i} p(d) q\left(n-1, i+\delta-d, x+S+Sd-\delta C-H\right) \\
& + \sum_{d=i+1}^{M} p(d) q\left(n-1, \delta, x+S+Si-\delta C-H\right) \\
\geq \quad & \sum_{d=0}^{i} p(d) q\left(n-1, i+1+\delta-d, x+Sd-\delta C-H\right) \\
& + \sum_{d=i+1}^{M} p(d) q\left(n-1, \delta, x+S(i+1)-\delta C-H\right) \\
= \quad & q\left(n, i+1, x\right)
\end{aligned}$$

where the second inequality follows from the induction hypothesis. This proves the induction hypothesis holds for n and the result follows.

We are now in a position to consider the probability the firm will survive over an infinite horizon $q(i, x)$. Firstly we describe the properties of the function $q(i, x)$.

Lemma 3

i) $q(i, x) = \lim_{n \to \infty} q(n, i, x)$ exists.

ii) For any $i \geq M$, $q(i, x) \geq q(j + i, x - jC)$ $\forall j, x > 0$.

iii) For all i, $q(i + j, x) \leq q(i, x + jS)$ $\forall j, x > 0$.

iv) $q(i, x)$ is non-decreasing in x.

v) $q(i, x)$ is non-decreasing in i.

Proof

$q(n, i, x)$ is bounded above by 1 and below by 0, and from Lemma 1 (i) is monotonic non-increasing in n. As bounded monotonic sequences converge, (i) follows. (ii) and (iii) follow immediately by taking the limit in the results of Lemma 2, and (iv) and (v) follow by taking the limit in the results of Lemma 1 parts (ii) and (iii).

However we still have to prove that these results are non-trivial, namely that there are levels of capital x where survival is a real possibility, i.e. $q(i, x) \neq 0$.

Theorem 1

For all inventory levels i, $q(i, x) > 0$ for some finite x.

Proof

Assume we have enough capital to begin a policy of ordering up to $2M$, so that the initial wealth is at least $2MC + H$. Consider applying the policy of ordering up to $2M$ starting in state $(2M, x)$ when one is trying to survive n periods.

$$
\begin{aligned}
q(n, 2M, x) &\geq \sum_{d=0}^{M} p(d) q\left(n-1, 2M-d, x+Sd-H\right) \\
&\geq \sum_{d=0}^{M} p(d) q\left(n-1, 2M, x+(S-C)d-H\right) \quad (4)
\end{aligned}
$$

where the second inequality follows from Lemma 2 (i).

Let $f(x)$ be a solution of the finite difference equation

$$
f(x) = \sum_{d=0}^{M} p(d) f\left(x+(S-C)d-H\right) \quad (5)
$$

where $f(x) = 0 \; \forall x \leq 0$ and $f(x) \leq 1 \; \forall x$. Trivially $q(0, 2M, x) \geq f(x)$ and assuming $q(n-1, 2M, x) \geq f(x)$ then equation (4) implies that

$$
\begin{aligned}
q(n, 2M, x) &\geq \sum_{d=0}^{M} p(d) q\left(n-1, 2M, x+(S-C)d-H\right) \\
&\geq \sum_{d=0}^{M} p(d) f\left(x+(S-C)d-H\right) = f(x).
\end{aligned}
$$

Hence $q(n, 2M, x) \geq f(x)$ for all n and in the limit $q(2M, x) \geq f(x)$.

Assume x is greater than $2MS$. By Lemma 3 (iii), $q(i, x) \geq q(2M, x - (2M - i)S)$ and, by Lemma 3 (iv), $q(2M, x - (2M - i)S) \geq q(2M, x - 2MS) \geq f(x - 2MS)$. So if we can prove that $f(x)$ is positive for $x > 0$ the result follows.

Let g be the greatest common factor of H and $S - C$ so $H = hg$ and $(S - C) = ag$ where a and h are integer. Let m be the integer part of h divided by a and let $r = h - ma$. The solution of the difference equation (5) satisfies $f(x) = \sum_i A_i z_i^x$ where A_i are constants and z_i are the roots of the equation

$$z^x = \sum_{d=0}^{M} p(d) z^{x+(S-C)d-H} \text{ or } \sum_{d=0}^{M} p(d) z^{agd} - z^{hg} = 0.$$

This factors into

$$(z^g - 1) \left(\sum_{k=m+2}^{M} \left(\sum_{s=k}^{M} p(s) \right) \left(\sum_{j=1}^{a} z^{g(ka-j)} \right) + \left(\sum_{s=m+1}^{M} p(s) \right) \left(\sum_{j=1}^{a-r} z^{g((m+1)a-j)} \right) \right.$$
$$\left. - \left(\sum_{s=1}^{m} p(s) \right) \left(\sum_{j=a-r+1}^{a} z^{g((m+1)a-j)} \right) - \sum_{k=1}^{m} \left(\sum_{s=0}^{k-1} p(s) \right) \left(\sum_{j=1}^{a} z^{g(ka-j)} \right) \right) = 0.$$

The second factor in this expression is $-p(0)$ at $z = 0$ and at $z = 1$ it is

$$\sum_{k=0}^{M} \left(a(k - m) - r \right) p(k) = g^{-1} \sum_{k=0}^{M} \left(k(S - C) - H \right) p(k) = \left((S - C)\bar{d} - H \right)/g > 0.$$

Hence there must be at least one root of the equation between 0 and 1, as well as the root at 1. Let this root be z_1. So a solution of equation (5) is $f(x) = A + Bz_1^x$. To satisfy $f(0) = 0$ we require $A = -B$, so $f(x) = 1 - z_1^x$ is a possible solution. From above, for $x > 2MS$, $q(i, x) \geq f(x - 2MS) = 1 - z_1^{x-2MS} > 0$ and the result follows.

4 Properties of the average reward model for a mature company

For a mature company, there should be little concern about survival in the short term. Survival in the long term depends on average profitability and so such companies should have as their objective the maximisation of the average profit per period. As suggested in section two this leads to the optimality equation (3) where g is the maximum average reward and $v(i)$ is the bias (extra reward) of starting with i components in stock.

Lemma 4

i) The optimal average reward and bias terms which satisfy the average reward model of (3)

are

$$
\begin{aligned}
g &= (S - C)\,\bar{d} - H \\
v(i) &= \begin{cases} Ci + (S - C)\,\bar{d} \text{ if } i > M \\ Si - (S - C)\sum_{d=0}^{i}(i - d)\,p(d) \text{ if } i \leq M \end{cases}
\end{aligned} \tag{6}
$$

ii) An optimal policy for the average reward model of (3) is

$$
k(i) = \begin{cases} 2M - i \text{ if } i > M \\ M \text{ if } i \leq M \end{cases} \tag{7}
$$

Proof

The proof uses the policy iteration algorithm for dynamic programming models (see [9]). First evaluate the policy described by (7). When this policy is applied, the average reward and bias terms satisfy the following equation.

$$
g + v(i) = \begin{cases} \sum_{d=0}^{M} p(d)\Big(Sd - (2M - i)C - H + v(2M - d)\Big) \text{ for } i > M \\ \sum_{d=0}^{M} p(d)\Big(S\min(i, d) - MC - H + v(i + M - \min(i, d))\Big) \text{ for } i \leq M \end{cases}
$$

It is easy to verify by substitution that the values of g and $v(i)$ from (6) satisfy this equation.

Now apply a policy improvement step to verify that the policy is optimal. For $i > M$ the policy improvement step looks for the action k which maximises

$$
\sum_{d=0}^{M} p(d)\left(Sd - kC - H + v\left(i + k - d\right)\right).
$$

Since

$$
v(i + 1) - v(i) = S\sum_{d=i+1}^{M} p(d) + C\sum_{d=0}^{i} p(d) > C \text{ if } i < M
$$

and $v(i + 1) - v(i) = C$ if $i \geq M$, this expression is maximised when k is chosen so that $i + k - d \geq M$ for all possible values of demand, d. Hence any $k \geq 2M - i$ is optimal.

For $i \leq M$ the policy improvement step looks for the action k that maximises

$$
\sum_{d=0}^{i} p(d)\left(Sd - kC - H + v(i + k - d)\right) + \sum_{d=i+1}^{M} p(d)\left(Si - kC - H + v(k)\right)
$$

Since $v(i+1) - v(i) > C$ if $i < M$ and $v(i+1) - v(i) = C$ if $i \geq M$, this expression is maximised when k is chosen so that $k \geq M$. Hence the policy given by (7) is optimal.

5 Comparison of optimal policies for maximising survival and average reward

In this section the models described in the previous two sections are compared to investigate if a company should be more cautious and risk averse in its initial 'survival' phase than in its mature 'profit maximising' phase. Caution in this case is shown in the ordering policy. If one only orders a few items the depletion in reserves is not that great, but one gives up the chance of high profits if the demand in the next period turns out to be high. If one orders a large number of items then this depletes the reserves much more, but gives the chance of a considerable profit if the demand is high.

There are three properties that one might expect of the optimal ordering policy in the survival phase that then have interpretations in terms of the caution appropriate to such circumstances.

- Are $k(i,x)$ and $k(n,i,x) \leq k(i)$? If so, the optimal survival policy is always more cautious than the optimal profit maximising policy.

- Are $k(i,x)$ and $k(n,i,x) \geq m > 0$? If so then there is a limit to how cautious one should be.

- Are $k(i,x)$ and $k(n,i,x)$ non-decreasing in x? If so, one becomes less cautious the more capital reserves one has available.

As we shall show, two of these assertions are true but the third is false. The first two assertions that the optimal survival policy should be more cautious than the optimal profit maximising policy but there is a limit to how cautious it should be, are established by the following two theorems.

Theorem 2

i) $k(n, i, x) \leq k(i)$ for all n, i and x.

ii) $k(i, x) \leq k(i)$ for all i, and x.

Proof

(i) Define $Q(k, n, i, x) = \left\{ \sum_{d=0}^{M} p(d)q \left(n - 1, i + k - \min(i, d), x + S\min(i, d) - kC - H \right) \right\}$

From Lemma 3 we have to show that if $i \leq M$, $k(n, i, x) \leq M$ and if $i > M$, then $k(n, i, x) \leq 2M - i$.

First for $i \leq M$, consider the order quantity $k = M + \delta$ where $\delta \geq 0$.

$$
\begin{aligned}
Q(M + \delta, n, i, x) &= \sum_{d=0}^{i} p(d)q \left(n - 1, i + (M + \delta) - d, x + Sd - (M + \delta)C - H \right) \\
&\quad + \sum_{d=i+1}^{M} p(d)q \left(n - 1, M + \delta, x + Si - (M + \delta)C - H \right) \\
&\leq \sum_{d=0}^{i} p(d)q \left(n - 1, i + M - d, x + Sd - MC - H \right) \\
&\quad + \sum_{d=i+1}^{M} p(d)q \left(n - 1, M, x + Si - MC - H \right) \\
&= Q(M, n, i, x)
\end{aligned}
$$

where the inequality follows from Lemma 2 (i). Hence the result holds in this case.

For $i > M$, consider the order quantity $k = 2M - i + \delta$ where $\delta \geq 0$.

$$
\begin{aligned}
Q(2M - i + \delta, n, i, x) &= \sum_{d=0}^{M} p(d)q \left(n - 1, 2M + \delta - d, x + Sd - (2M - i + \delta)C - H \right) \\
&\leq \sum_{d=0}^{M} p(d)q \left(n - 1, 2M - d, x + Sd - (2M - i)C - H \right) \\
&= Q(2M - i, n, i, x)
\end{aligned}
$$

where again the inequality follows from Lemma 2 (i). Hence $k(n, i, x) \leq k(i)$.

(ii) The proof that $k(i, x) \leq k(i)$ follows in exactly the same way using Lemma 3 rather than Lemma 2.

The idea that it does not pay a start-up firm to be too conservative in its ordering policy is made precise in the following way. If, when the firm has no components in stock, the optimal policy is to order fewer components than the number the firm needs to sell each period in order to break even, then there is no chance that the firm will survive. A similar result about the level the firm should order up to holds when there are some components in stock. These results are made explicit in the following theorem.

Theorem 3

Define $\tilde{d}$ to be the largest integer less than $H/(S-C)$.

i) If $k(0,x) \leq \tilde{d}$ then $q(0,\tilde{x}) = 0, \forall \tilde{x} \leq x$.

ii) If $i \leq \tilde{d}$ and $k(i,x) \leq \tilde{d} - i$, then $q(\tilde{i},\tilde{x}) = 0, \forall \tilde{i} \leq i$, and $\tilde{x} \leq x$.

Proof

(i) Suppose $k(0,x) = \alpha \leq \tilde{d}$ is the largest order quantity that maximises the survival probability for state $(0,x)$. Let $\epsilon = H - (S-C)\tilde{d}$ and note that $\epsilon > 0$.

$$q(0,x) = q(\alpha, x - C\alpha - H) \leq q(0, x + (S-C)\alpha - H) \leq q(0, x + (S-C)\tilde{d} - H) = q(0, x - \epsilon)$$

where the first inequality follows from Lemma 3 (iii) and the second from Lemma 3 (iv). Assume $k(0, x - \epsilon) = \gamma > \tilde{d}$.

$$q(0,x) > q(\gamma, x - C\gamma - H) \geq q(\gamma, x - \epsilon - C\gamma - H) = q(0, x - \epsilon)$$

where the first inequality follows from the fact that γ is not optimal in state $(0,x)$ and the second from Lemma 3 (iv). This contradicts $q(0,x) \leq q(0, x - \epsilon)$, so $k(0, x - \epsilon) \leq \tilde{d}$.

Repeating the argument shows that $q(0, x - \epsilon) \leq q(0, x - 2\epsilon) \leq \ldots \leq q(0, y)$ where $y \leq 0$ and hence $q(0,x) = 0$. Lemma 3 (iv) then implies that $q(0, \tilde{x}) = 0$ for all $\tilde{x} \leq x$.

(ii) Let $k(i,x) = \delta \leq \tilde{d} - i$ be the largest order quantity that maximises the survival probability for state (i,x).

$$q(i,x) \;=\; \sum_{d=0}^{i} p(d)q\left(i - d + \delta, x + Sd - C\delta - H\right) + \sum_{d=i+1}^{M} p(d)q\left(\delta, x + Si - C\delta - H\right)$$

$$\leq \quad \sum_{d=0}^{i} p(d)q\left(0, x + Sd - C\delta - H + S\left(i - d + \delta\right)\right)$$

$$+ \sum_{d=i+1}^{M} p(d)q\left(0, x + Si - C\delta - H + S\delta\right) \text{ by Lemma 3 (iii)}$$

$$= \quad q\left(0, x + Si + \left(S - C\right)\delta - H\right)$$

$$\leq \quad q(0, x + Si + \left(S - C\right)\left(\tilde{d} - i\right) - H) \text{ by Lemma 3 (iv)}$$

$$= \quad q(0, x + Ci + \left(S - C\right)\tilde{d} - H)$$

$$\leq \quad q\left(0, x + Ci\right) \text{ by Lemma 3 (iv)}$$

Assume $k(0, x + Ci) = i + \delta + \gamma$ where $\gamma > 0$.

$$q(i, x) \quad > \quad \sum_{d=0}^{i} p(d)q\left(i - d + \delta + \gamma, x + Sd - C\delta - C\gamma - H\right)$$

$$+ \sum_{d=i+1}^{M} p(d)q\left(\delta + \gamma, x + Si - C\delta - C\gamma - H\right) \text{ by the definition of } \delta$$

$$\geq \quad \sum_{d=0}^{i} p(d)q\left(i + \delta + \gamma, x - C\delta - C\gamma - H\right)$$

$$+ \sum_{d=i+1}^{M} p(d)q\left(i + \delta + \gamma, x - C\delta - C\gamma - H\right) \text{ by Lemma 3 (iii)}$$

$$= \quad q\left(i + \delta + \gamma, x - C\delta - C\gamma - H\right)$$

$$= \quad q\left(0, x + Ci\right)$$

This contradicts $q(i, x) \leq q(0, x + Ci)$, so $k(0, x + Ci) \leq i + \delta \leq \tilde{d}$.

Hence by (i), $q(0, x + Ci) = 0$ and, as we have proved $q(i, x) \leq q(0, x + Ci)$, $q(i, x) = 0$. The monotonicity results of Lemma 3 (iii) and (iv) then imply that $q(\tilde{i}, \tilde{x}) = 0$ for all $\tilde{i} \leq i$ and $\tilde{x} \leq x$.

The third seemingly reasonable property, namely that $k(n, i, x)$ and $k(i, x)$ are non-decreasing in x, is not true. Consider the following examples.

Example 1

Let $H = 5, C = 2, S = 5, M = 2$ and suppose we want to maximise $q(2, i, x)$, the probability of surviving 2 periods.

$$
\begin{aligned}
q(2, 1, 4) &= \max_k \{p(0)q(1, 1 + k, -1 - 2k) + (1 - p(0))q(1, k, 4 - 2k)\} \\
&= (1 - p(0)) \max \{q(1, 0, 4), q(1, 1, 2), q(1, 2, 0)\} \\
q(2, 1, 5) &= \max_k \{p(0)q(1, 1 + k, -2k) + (1 - p(0))q(1, k, 5 - 2k)\} \\
&= \max \{p(0)q(1, 1, 0) + (1 - p(0))q(1, 0, 5), (1 - p(0))q(1, 1, 3), (1 - p(0))q(1, 2, 1)\}
\end{aligned}
$$

Due to the lead-time, the optimal action with one period to go is always to order 0 components, so

$$
\begin{aligned}
q(1, 0, 4) &= q(0, 0, -1) = 0; \\
q(1, 1, 2) &= p(0)q(0, 1, -3) + (1 - p(0))q(0, 0, 2) = 1 - p(0); \\
q(1, 2, 0) &= p(0)q(0, 2, -5) + p(1)q(0, 1, 0) + (1 - p(0) - p(1))q(0, 0, 5) = 1 - p(0); \\
q(1, 1, 0) &= p(0)q(0, 1, -5) + (1 - p(0))q(0, 0, 0) = 1 - p(0); \\
q(1, 0, 5) &= q(0, 0, 0) = 1; \\
q(1, 1, 3) &= p(0)q(0, 1, -2) + (1 - p(0))q(0, 0, 3) = 1 - p(0); \\
q(1, 2, 1) &= p(0)q(0, 2, -4) + p(1)q(0, 1, 1) + (1 - p(0) - p(1))q(0, 0, 6) = 1 - p(0).
\end{aligned}
$$

If $0 < p(0) < 1$, $q(2, 1, 4) = (1 - p(0))^2$, $k(2, 1, 4) = 1$ or 2, $q(2, 1, 5) = (1 - p(0))(1 + p(0))$ and $k(2, 1, 5) = 0$. Since $k(2, 1, 4) > k(2, 1, 5)$, the conjecture is disproved.

One might think this is due to the end effect of there only being a few periods to go. This is not true.

Example 2

Figure 2 shows the optimal policy $k(1000, 0, x)$ for an example with $H = 10, C = 3, S = 5$ and a Poisson demand process with mean 7.5 truncated at 20 (i.e. the probability that the demand is higher than 20 is added to the probability that the demand equals 20). Although
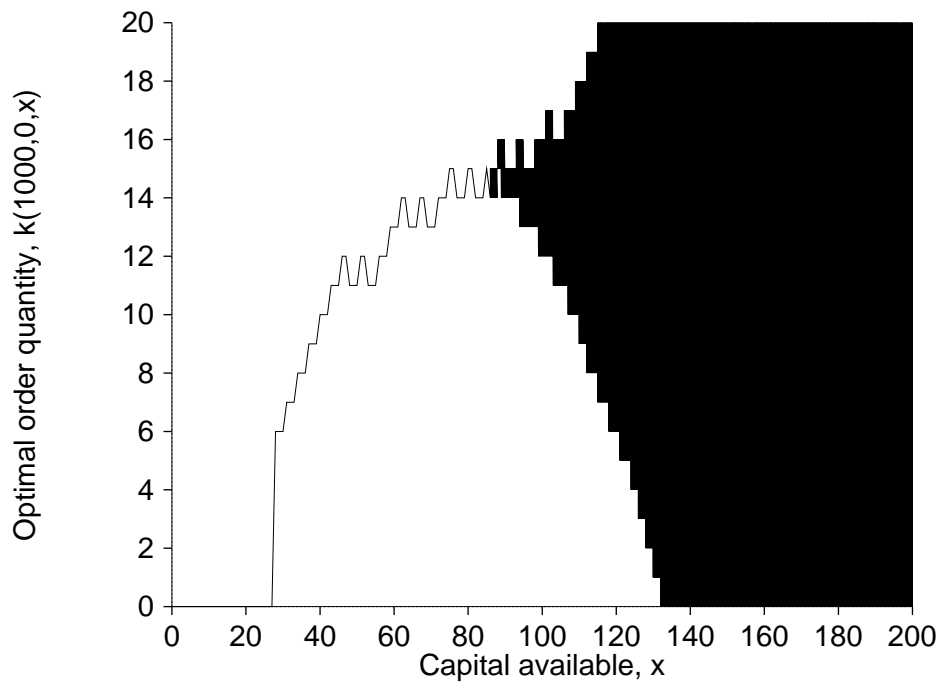
17

Figure 2: Order quantity against capital available when inventory level is zero for the optimal survival strategy
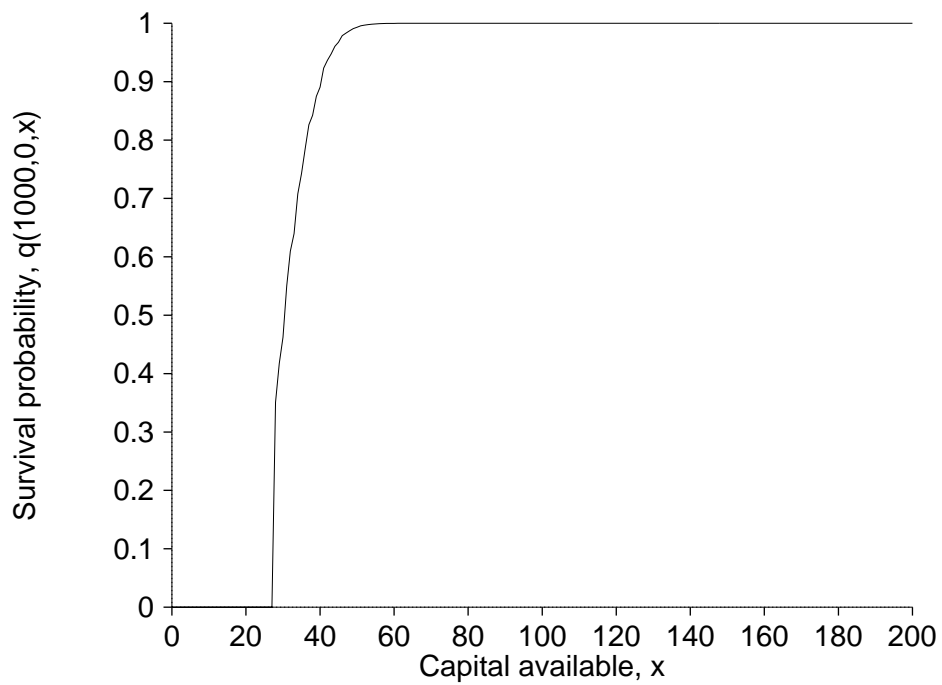

Figure 3: Survival probability against capital available when inventory level is zero for the optimal survival strategy

the problem appears to have an infinite state space, Lemma 4 and Theorem 2 mean that the value function need not be evaluated for $i > 40$. Since H, C and S are integer we need only solve for integer values of the capital available x. Also as we cannot lose more than an average of 10 per period there is no point in evaluating the 1,000 period problem with capital levels of more than 10,000. Where the optimal order quantity is not unique, the minimum and maximum optimal order quantities have been plotted and the area in between (which also corresponds to optimal order quantities) has been shaded. So when the capital available is less than 86, there is a unique order quantity and when the capital available is greater than 131, the next order has not discernible effect on the survival probability. The saw tooth effect between $x = 45$ and $x = 106$ shows that the optimal policy is not monotonic in the capital available. The first instance of the monotonicity property failing is $k(1000, 0, 47) = 12$ and $k(1000, 0, 48) = 11$. This can be partly explained by looking at the short term effect of the order decision. In state $(0, 47)$ it is necessary for survival to sell 2 items in the next period regardless of whether one orders 11 or 12 items in this period (state in next period is $(12, 4)$ and $(12, 1)$ respectively). In state $(0, 48)$ it is necessary for survival to sell 1 item in the next period if one orders 11 items in this period (state in next period is $(11, 5)$), but it is necessary to sell 2 if one orders 12 (state in next period is $(12, 2)$). In state $(0, 48)$ one is willing to sacrifice the additional expected revenue from the 12th item ordered for the higher chance of survival. Similar results hold for all other points at which the monotonicity property breaks down.

Figure 2 illustrates some of the other features of an optimal policy that have been discussed in this paper. The order quantity $k(0) = 20$ is optimal in the average reward model of this example when the inventory level is 0, see equation (7). Figure 2 confirms that $k(1000, 0, x) \leq k(0)$. Interestingly $k(1000, 0, x) = 20$ for $x \geq 115$, indicating that when the capital available reaches 115, the optimal policy in the survival model is no longer less cautious than the optimal policy in the average reward model. Figure 3 shows the survival probabilities $q(1000, 0, x)$ for this example. It shows that there is no chance of survival when $x < 28$, but the survival probability jumps to 0.35 when $x = 28$ and continues to increase
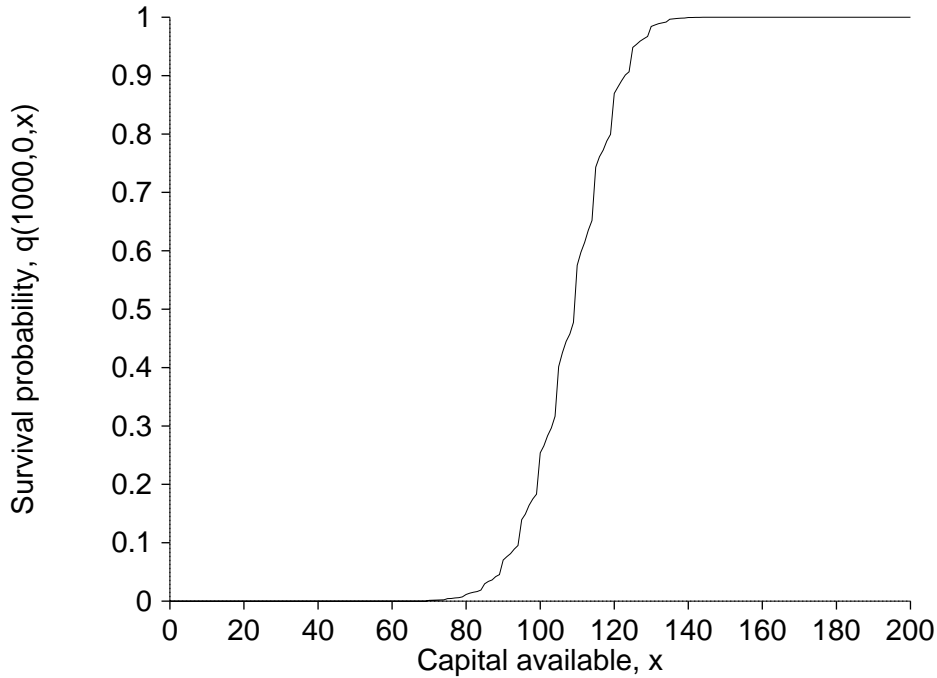
Figure 4: Survival probability against capital available when inventory level is zero for the profit maximising strategy

rapidly, so that when $x = 61$ it exceeds 0.9999. For this example, the largest integer less than $H/(S - C) = 5$ is $\tilde{d} = 4$. Figure 2 confirms that $k(1000, 0, x) > \tilde{d}$ when $q(1000, 0, x) > 0$.

One question of interest is "how sub-optimal in terms of long run reward is the policy that maximises the probability of survival". From Figure 2 it can be seen that the two policies agree once the capital available to the firm exceeds 115. Once the capital goes below 28 the order quantity of the optimal survival policy is 0 and so the loss of profit is $g = (5 - 3)7.5 - 10 = 5$. In the long run the amount of capital the firm has must reach one of these two regions, and though the former is not quite an absorbing state, the error in assuming it is will be very small. Hence the loss in average profit by using the optimal survival policy is approximately $0 \times q(0, x) + 5\,(1 - q(0, x))$.

The complimentary question is "what are the survival probabilities if one uses the policy that maximises the average reward per period". Applying the optimal policy of equation (7) in this example gives the survival probabilities in Figure 4. Comparing Figures 3 and 4 we see that under the profit maximising strategy one needs initial capital of 75 to have

any chance of surviving where as under the optimal survival strategy one only needs initial capital of 29. Further to have a 99% chance of survival, one needs initial capital of 133 under the profit maximising strategy, but only of 49 under the optimal survival strategy. These differences illustrate that it is more sensible for a start-up company with limited capital to be guided by a survival strategy than a profit maximising one. For capital of 150 or more, the survival probability when the profit maximising strategy is used is within 10^{-5} of 1. This suggests that the firm should consider switching to a profit maximising strategy by the time the capital available reaches 150.

6   Models with interest rate and inflation included

The models in the previous sections do not include any holding costs. Holding costs are inappropriate for the survival models since the main component of holding cost is the cost of capital and in these models we explicitly incorporate the capital available into the state of the system. What has been assumed though in these models is that the interest rate r per period that the capital can attract when invested in a risk free financial instrument is equal to the cost/price inflation rate f per period. If this assumption is relaxed the equations of section 2 change as follows. Define the survival probability $q(n, t, i, x)$ to be the probability that the firm will survive another n periods given that it is t periods since the firm was set-up and the firm has i components and x units of capital still available. This survival probability satisfies the following variant of the dynamic programming optimality equation (1).

$$q(n,t,i,x) = \max_{k} \left\{ \sum_{d=0}^{M} p(d)\, q\left(n-1, t+1, i+k-\min(i,d),\right.\right.$$

$$\left.\left. (1+r)x + (1+f)^{t+1}(S\min(i,d) - kC - H) \right) \right\} \quad (8)$$

with boundary conditions $q(n, t, i, x) = 0 \ \forall x < 0$ and $q(0, t, i, x) = 1 \ \forall i, x \geq 0$.

The value of capital x on the left hand side of this equation is the value of capital at one period earlier than the period at which the capital on the right hand side of the

equation is valued. This can be overcome by always quoting the amount of capital as its value discounted back to some standard date, say the date at which the firm was set up. We assume the appropriate discount factor is $1/(1+f)$ since, as f is the inflation rate, this will make the discounted costs constant over time. Define $q^*(n, t, i, x)$ to be the probability that the firm will survive another n periods given that it is t periods since the firm was set-up and the firm has available i components and capital corresponding to an amount x at the set-up date. If $\beta = (1+r)/(1+f)$ the optimality equation becomes

$$q^*(n, t, i, x) = \max_k \left\{ \sum_{d=0}^{M} p(d)\, q^*(n-1, t+1, i+k - \min(i, d), \right.$$

$$(1+r)(1+f)^t x/(1+f)^{t+1} + (1+f)^{t+1}(S\min(i,d) - kC - H)/(1+f)^{t+1} \Big) \bigg\}$$

$$= \max_k \left\{ \sum_{d=0}^{M} p(d) q^*(n-1, t+1, i+k - \min(i,d), \beta x + S\min(i,d) - kC - H) \right\} \quad (9)$$

with boundary conditions $q^*(n, t, i, x) = 0 \ \forall x < 0$ and $q^*(0, t, i, x) = 1 \ \forall i, x \geq 0$.

Notice the boundary conditions remain the same because a positive amount of capital discounts back to a positive amount of capital at the start date while zero capital discounts back to zero capital. Note that the solution of (9) is independent of t and so we may denote the solution $q^*(n, i, x)$.

As in section 2 taking the limit as $n \to \infty$ in (9) shows that $q^*(i, x)$ the probability of surviving forever satisfies the optimality equation

$$q^*(i, x) = \max_k \left\{ \sum_{d=0}^{M} p(d) q(i+k - \min(i,d), \beta x + S\min(i,d) - kC - H) \right\} \quad (10)$$

There are three different cases to look at for this equation, namely (1) $r = f$ so $\beta = 1$; (2) $r < f$ so $\beta < 1$; (3) $r > f$ so $\beta > 1$. Before examining these more in detail note that the solutions of (9) and (10) have similar properties to those described in Lemma 3.

Lemma 5

Let $q^*(n, i, x)$ and $q^*(i, x)$ be solutions of (9) and (10).

i) $q^*(i, x) = \lim_{n \to \infty} q^*(n, i, x)$ exists.

ii) $q^*(i, x)$ is non-decreasing in x.

iii) $q^*(i, x)$ is non-decreasing in i.

iv) $q^*(i, x)$ is non-decreasing in β.

Proof

The proofs of i), ii) and iii) follow exactly as in Lemma 3. The proof of iv) follows in the same way using the non-decreasing property of $q^*(i, x)$ in x.

Turning to the three cases:

Case 1 $r = f$ or $\beta = 1$

As this case corresponds to $\beta = 1$, the model is equivalent to the model considered in Sections 2 to 5.

Case 2 $r < f$ or $\beta < 1$

It is clear that in this case it is better to hold components than the money for them as the latter does not inflate as quickly as the former. Thus the order quantities are likely to be higher. It is also the case that there is no initial capital which means that survival is guaranteed without trading. In this case to maximise infinite horizon discounted profit one would order as much as possible, restricted only by physical capacity constraints. Therefore if the objective is to maximise survival probability with limited available capital, it must be the case that the optimal order quantity does not exceed the optimal order quantity in the profit maximising model.

As an example of this case consider H, C, S and demand process as in example 2, but $r = 5\%$ and $f = 10\%$, so that $\beta = 0.955$. Figure 5 shows that the optimal policy is unique. This is because one needs to trade as much as one dares in order to survive. When small amounts of capital ($x \leq 56$) are available, it is optimal to order as much as that capital will allow. However when larger amounts of capital are available, one sometimes chooses to hold a little back in order to improve the chance of short term survival. As in example 2,
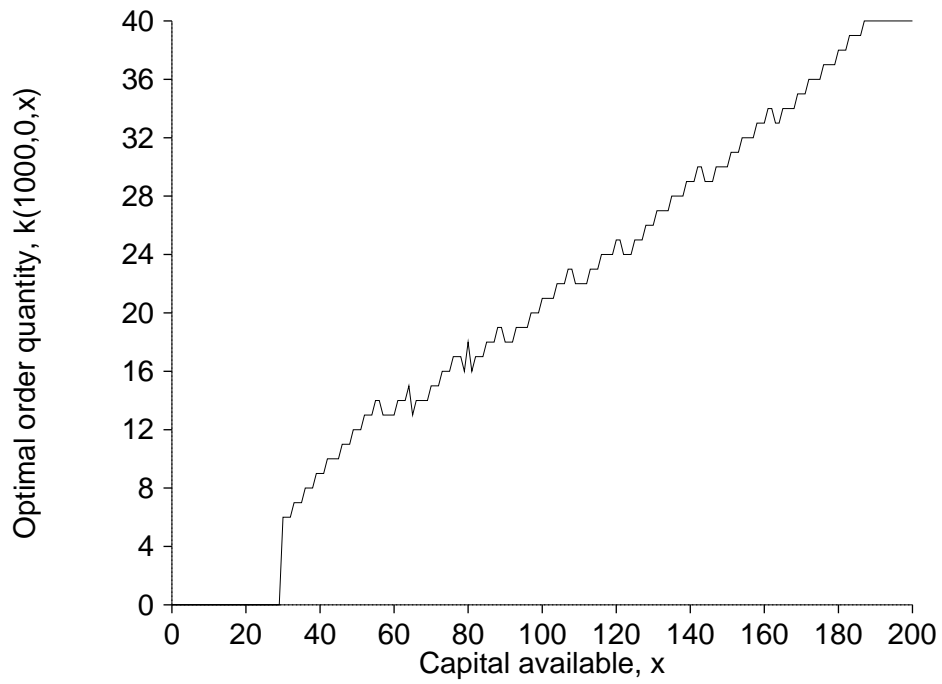
Figure 5: Order quantity against capital available when inventory level is zero for the optimal survival strategy ($\beta = 0.955$)
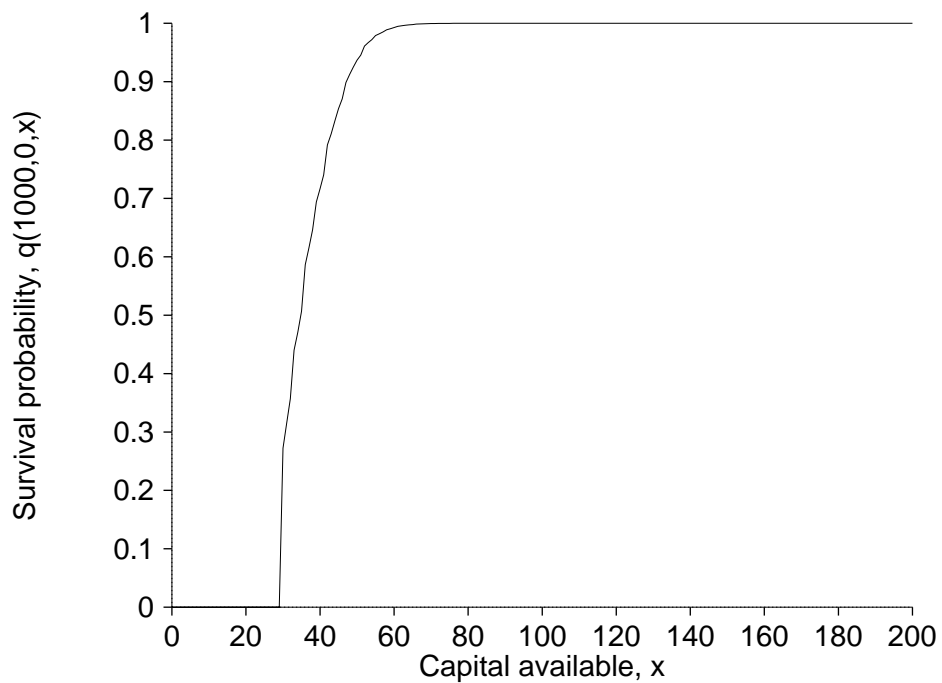

Figure 6: Survival probability against capital available when inventory level is zero for the optimal survival strategy ($\beta = 0.955$)

this explains the non-monotonicity of the order quantity. Figure 6 shows that the maximum survival probability for any initial capital is less in this case than in case 1, and hence verifies Lemma 5 (iv). However it is worth noting that the shape of the survival probability curve is very similar to that in case 1.

Case 3 $r > f$ or $\beta > 1$

Notice from Lemma 5 (iv) that for any initial capital the maximum survival probability in this case is no less than in the other cases. In fact the maximum survival probability does become 1 for some initial capital level unlike the other cases. This follows easily because one can survive with certainty without trading provided $x > H/(\beta - 1)$. Hence the optimal order quantity may be zero in this case which of course does not exceed the optimal order quantity in the profit maximising model. However at this point, survival is no longer the issue and even ordering large quantities is essentially risk free. Thus the objective should transfer to that of maximising profit.

To examine what happens if captial is below this value consider an example of this case with H, C, S and demand process as in example 2, but $r = 20\%$ and $f = 10\%$, so that $\beta = 1.091$. Figure 7 shows that, when the capital is small, increases in capital increase order quantity, but thereafter the survival strategy becomes more cautious until it reaches a level where survival is almost certain. Figure 8 confirms the result of Lemma 5 (iv) that the survival probability is never less than the survival probability for the other two cases.

In all three cases there is a minimum order quantity needed for there to be any chance of survival. Hence there is a limit to how cautious one should be, even when long term survival is the main objective.

7 Conclusion

The problem considered here is a fairly simple one but it illustrates that if small companies are more interested in surviving than maximising their average reward, they should employ more conservative strategies for ordering component parts. They should be willing to forego
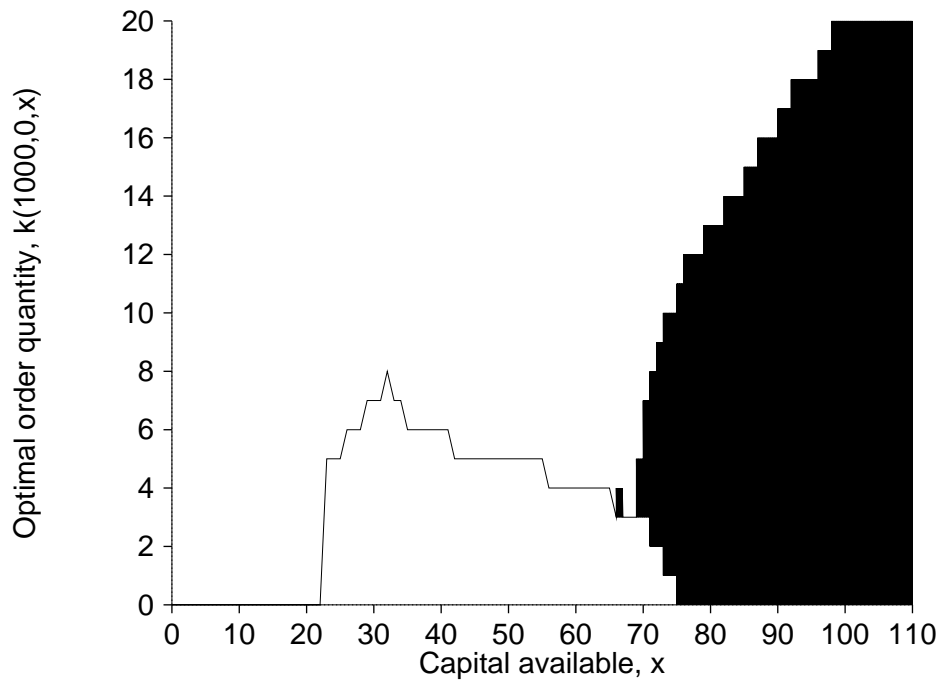
Figure 7: Order quantity against capital available when inventory level is zero for the optimal survival strategy ($\beta = 1.091$)
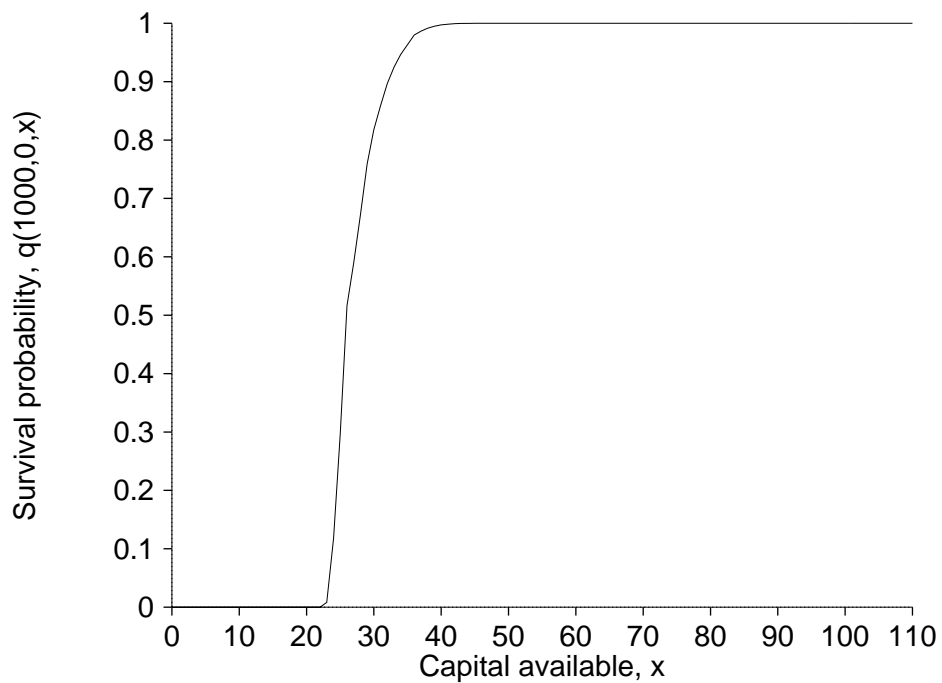

Figure 8: Survival probability against capital available when inventory level is zero for the optimal survival strategy ($\beta = 1.091$)

the profits that a high demand in the next few periods will bring to increase their chance of surviving a spell of lean order books. Similar analysis could be undertaken for the problem of what size of batches to make. What is surprising is that the probability of survival as a function of capital x is very close to being a step function, see Figure 3. It does seem that in the usual case where the minimum demand is below the sustainability level D^*, there is a critical amount of initial capitalisation so that below that amount there is essentially no chance of survival while above that amount the chances are very high.

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