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Sigmoid functions for the smooth approximation to the absolute value function

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ABSTRACT. We present smooth approximations to the absolute value function $|x|$ using sigmoid functions. In particular, $x\,\mathrm{erf}(x/\mu)$ is proved to be a better smooth approximation for $|x|$ than $x\tanh(x/\mu)$ and $\sqrt{x^2+\mu}$ with respect to accuracy. To accomplish our goal we also provide sharp hyperbolic bounds for the error function.

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1. Introduction

An S-shaped function which usually monotonically increases on $\mathbb{R}$ (the set of all real numbers) and has finite limits as $x \to \pm\infty$ is known as a sigmoid function. Rigorously, a sigmoid function is bounded and differentiable real function that is defined for all real input values and has a non-negative derivative at each point[4]. It has bell-shaped first derivative. A sigmoid function is constrained by two parallel and horizontal asymptotes. Some examples of sigmoid functions include half-logistic function, i.e. $(1-e^{-x})/(1+e^{-x}) = 2[1/(1+e^{-x}) - 1/2], \tanh(x), \tan^{-1} x$, Gudermannian function, i.e. $\mathrm{gd}(x)$, error function, i.e. $\mathrm{erf}(x)$,

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$x(1+x^2)^{-1/2}$ etc. Sigmoid functions have many applications including the one in artificial neural networks. The one-sided Hausdorff distance [10] between $sign(x)$ and half-logistic function with "polynomial variable transfer" is considered in [6, 7]. We describe below some of these sigmoid functions.

The Gudermannian function is defined as follows:

$$\mathrm{gd}(x) = \int_0^x \frac{1}{\cosh(t)}\,dt.$$

Alternatively,

$$\mathrm{gd}(x) = \sin^{-1}(\tanh(x)) = \tan^{-1}(\sinh(x)).$$

The error function or Gaussian error function is defined as follows:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt.$$

The Gudermannian and error functions are special functions and they have many applications in mathematics and applied sciences. All the above mentioned sigmoid functions are differentiable and their limits as $x \to \pm\infty$ are listed below:

$$\lim_{x\to-\infty} 2\left[\frac{1}{1+e^{-x}} - \frac{1}{2}\right] = -1, \lim_{x\to+\infty} 2\left[\frac{1}{1+e^{-x}} - \frac{1}{2}\right] = 1,$$

$$\lim_{x\to-\infty} \tanh(x) = -1, \lim_{x\to+\infty} \tanh(x) = 1,$$

$$\lim_{x\to-\infty} \tan^{-1}(x) = -\frac{\pi}{2}, \lim_{x\to+\infty} \tan^{-1}(x) = \frac{\pi}{2},$$

$$\lim_{x\to-\infty} \mathrm{gd}(x) = -\frac{\pi}{2}, \lim_{x\to+\infty} \mathrm{gd}(x) = \frac{\pi}{2},$$

$$\lim_{x\to-\infty} \mathrm{erf}(x) = -1, \lim_{x\to+\infty} \mathrm{erf}(x) = 1,$$

$$\lim_{x\to-\infty} \frac{x}{\sqrt{1+x^2}} = -1, \lim_{x\to+\infty} \frac{x}{\sqrt{1+x^2}} = 1.$$

Due to these properties, it is easy to see that the functions $2x\left[1/(1+e^{-x/\mu}) - 1/2\right]$, $x\tanh(x/\mu), (2/\pi)x\tan^{-1}(x/\mu), (2/\pi)x\,\mathrm{gd}(x/\mu), x\,\mathrm{erf}(x/\mu)$ and $x^2(x^2+\mu^2)^{-1/2}$ as $\mu \to 0$ can be used as smooth approximations for the absolute function $|x|$. In [8], $\sqrt{x^2+\mu}$ is proved to be a computationally efficient smooth approximation of $|x|$ since it involves less number of algebraic operations. In spite of being this, as far as accuracy is concerned some of the above-mentioned functions are better transcendental approximations to $|x|$. In [2], $x\tanh(x/\mu)$ was proposed by first author and it is recently proved [3] that this approximation is better than $\sqrt{x^2+\mu}$ in terms of accuracy by first author and B. K. Khairnar. One of the users of Mathematics Stack Exchange [9] suggested $x\,\mathrm{erf}(x/\mu)$ as a smooth approximation to $|x|$. However, that user did not give the logical proof or did not compare this approximation with existing ones. In fact, it is better than $\sqrt{x^2+\mu}$ or $\sqrt{x^2+\mu^2}$ in terms of accuracy; but it is not proved at [9]. To prove this fact is the main goal of this paper. We shall prove this thing by showing $x\,\mathrm{erf}(x/\mu)$ to be better than $x\tanh(x/\mu)$. We avoid logical proofs for other approximations presented above since they are not as good as $x\tanh(x/\mu)$ or $x\,\mathrm{erf}(x/\mu)$ for accuracy which can be seen in the figures given at the end of this article.

The rest of the paper is organized in the following manner. Section 2 presents the main results, with proofs. Two tight approximations are then compared numerically and graphically in Section 3. A conclusion is given in Section 4.

2. Main Results with Proofs

We need the following lemmas to prove our main result.

Lemma 2.1. *(l'Hôpital's Rule of Monotonicity [1]): Let $f, g : [c, d] \to \mathbb{R}$ be two continuous functions which are differentiable on (c, d) and $g' \neq 0$ in (c, d). If $f'(x)/g'(x)$ is increasing (or decreasing) on (c, d), then the functions $(f(x) - f(c))/(g(x) - g(c))$ and $(f(x) - f(d))/(g(x) - g(d))$ are also increasing (or decreasing) on (c, d). If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.*

Lemma 2.2. *For $x \in \mathbb{R}$, the following inequality holds:*

$$x^2 e^{-x^2} \leqslant \frac{1}{e}.$$

Proof: Suppose that

$$h(x) = x^2 e^{-x^2}.$$

By differentiation we get

$$h'(x) = 2x\, e^{-x^2}(1 - x^2).$$

This implies $x = 0$, ± 1 are the critical points for $h(x)$. Again differentiation gives

$$h''(x) = 2e^{-x^2}(1 - x^2) - 4x^2 e^{-x^2}(2 - x^2)$$

Hence,

$$h''(0) = 2,\ h''(-1) = -\frac{4}{e},\ h''(1) = -\frac{4}{e}.$$

By second derivative test, $h(x)$ has minima at $x = 0$ and maxima at $x = \pm 1$. Therefore 0 is the minimum value and $1/e$ is the maximum value of $h(x)$, ending the proof of Lemma 2.2.

Lemma 2.3. *For $x \in \mathbb{R} - \{0\}$, one has*

$$|\mathrm{erf}(x)| + \frac{\alpha}{|x|} > 1, \tag{2.1}$$

with $\alpha = 2/(e\sqrt{\pi}) \approx 0.4151075$.

Proof: We consider two cases depending on the sign of x as follows:

Case(1): For $x > 0$, let us consider the function

$$f(x) = \mathrm{erf}(x) + \frac{\alpha}{x} - 1$$

which on differentiation gives

$$f'(x) = \frac{2}{\sqrt{\pi}}e^{-x^2} - \frac{\alpha}{x^2} = \frac{2}{\sqrt{\pi}}\left[e^{-x^2} - \frac{1}{ex^2}\right].$$

By Lemma 2.2, $f'(x) \leqslant 0$ and hence $f(x)$ is decreasing on $(0, +\infty)$. So, for any $x > 0$, $f(x) > f(+\infty^-)$, i.e.

$$\operatorname{erf}(x) + \frac{\alpha}{x} > 1.$$

Case(2): For $x < 0$ let us consider the function $g(x) = \operatorname{erf}(x) + \alpha/x + 1$. As in Case(1), $g'(x) \leqslant 0$ and is decreasing in $(-\infty, 0)$. Hence, for any $x < 0$, $g(x) < g(-\infty^+)$. So we get

$$\operatorname{erf}(x) + \frac{\alpha}{x} < -1,$$

which completes the proof of Lemma 2.3.

Proposition 2.1. *Let $\mu > 0$ and $\alpha = 2/(e\sqrt{\pi}) \approx 0.4151075$. For $x \in \mathbb{R}$, the approximation $F(x) = x\operatorname{erf}(x/\mu)$ to $|x|$ satisfies*

$$F'(x) = \frac{2x}{\sqrt{\pi}\,\mu}\,e^{-\frac{x^2}{\mu^2}} + \frac{1}{x}F(x)$$

and

$$||x| - F(x)| < \alpha\mu. \tag{2.2}$$

Proof: We have

$$F'(x) = \frac{2x}{\sqrt{\pi}\,\mu}\,e^{-\frac{x^2}{\mu^2}} + \operatorname{erf}\left(\frac{x}{\mu}\right) = \frac{2x}{\sqrt{\pi}\,\mu}\,e^{-\frac{x^2}{\mu^2}} + \frac{1}{x}F(x).$$

For $x = 0$ the inequality (2.2) is obvious. For $x \neq 0$, it follows from Lemma 2.3 that

$$\begin{aligned} ||x| - F(x)| &= \left||x| - \left|x\operatorname{erf}\left(\frac{x}{\mu}\right)\right|\right| = |x|\,\left|1 - \left|\operatorname{erf}\left(\frac{x}{\mu}\right)\right|\right| \\ &= |x|\,\left[1 - \left|\operatorname{erf}\left(\frac{x}{\mu}\right)\right|\right] < |x|\,\alpha\,\left|\frac{\mu}{x}\right| = \alpha\mu. \end{aligned}$$

The proof of Proposition 2.1 is completed.

In the following Proposition, we give sharp bounds for error function $\operatorname{erf}(x)$ implying that the present approximation to $|x|$ is better than $x\tanh(x/\mu)$.

Proposition 2.2. *For $x > 0$, it is true that*

$$\tanh(x) < \operatorname{erf}(x) < \frac{2}{\sqrt{\pi}}\tanh(x). \tag{2.3}$$

Proof: Consider the function

$$G(x) = \frac{\operatorname{erf}(x)}{\tanh(x)} = \frac{G_1(x)}{G_2(x)},$$

where $G_1(x) = \operatorname{erf}(x)$ and $G_2(x) = \tanh(x)$ with $G_1(0) = G_2(0) = 0$. On differentiating we get

$$\frac{G_1'(x)}{G_2'(x)} = \frac{2}{\sqrt{\pi}} e^{-x^2} \cosh^2(x) = \frac{2}{\sqrt{\pi}} \lambda(x),$$

where $\lambda(x) = e^{-x^2} \cosh^2(x)$, derivative of which is given by

$$\lambda'(x) = 2e^{-x^2} \cosh(x) \left[\sinh(x) - x\cosh(x)\right].$$

Since $\sinh(x)/x < \cosh(x)$ (see, for instance, [5]), we have $\lambda'(x) < 0$ and hence $\lambda(x)$ is decreasing in $(0, +\infty)$. By Lemma 2.1, $G(x)$ is also decreasing in $(0, +\infty)$. So, for $x > 0$,

$$G(0^+) > G(x) > G(+\infty^-).$$

It is easy to evaluate $G(0^+) = 2/\sqrt{\pi}$ by l'Hospital's rule and $G(+\infty^-) = 1$. This ends the proof of Proposition 2.2.

3. Comparison between two approximations

By virtue of Proposition 2.2, for all $x \in \mathbb{R}$ and $\mu > 0$, we get the following chain of inequalities:

$$x \tanh\left(\frac{x}{\mu}\right) < x \operatorname{erf}\left(\frac{x}{\mu}\right) < |x| < \sqrt{x^2 + \mu}. \tag{3.1}$$

Again in [3], it is proved that $x\tanh(x/\mu)$ is better than $\sqrt{x^2+\mu}$ or $\sqrt{x^2+\mu^2}$ with respect to accuracy. Consequently, $x \operatorname{erf}(x/\mu)$ is better than $\sqrt{x^2+\mu}$ or $\sqrt{x^2+\mu^2}$ in the same regard. Numerical and graphical studies support the theory.

In Table 1, we compare numerically some of these approximations by investigating global L_2 error which is given by

$$e(f) = \int_{-\infty}^{+\infty} \left[|x| - f(x)\right]^2 dx,$$

where $f(x)$ denotes an approximation to $|x|$. With this criterion, a lower $e(f)$ value indicates a better approximation. Table 1 indicates that $x \operatorname{erf}(x/\mu)$ is the best of the considered approximations (for $\mu = 0.1$ and $\mu = 0.01$, but other values can be considered for μ, with the same conclusion).

TABLE 1. Global L_2 errors $e(f)$ for the functions $f(x)$.

	$\mu = 0.1$			
$f(x)$	$2x\left[\dfrac{1}{1+e^{-x/\mu}} - \dfrac{1}{2}\right]$	$\dfrac{2}{\pi}x\,\mathrm{gd}\left(\dfrac{x}{\mu}\right)$	$x\tanh\left(\dfrac{x}{\mu}\right)$	$x\,\mathrm{erf}\left(\dfrac{x}{\mu}\right)$
$e(f)$	≈ 0.00126521	≈ 0.000754617	≈ 0.000158151	≈ 0.000087349
	$\mu = 0.01$			
$f(x)$	$2x\left[\dfrac{1}{1+e^{-x/\mu}} - \dfrac{1}{2}\right]$	$\dfrac{2}{\pi}x\,\mathrm{gd}\left(\dfrac{x}{\mu}\right)$	$x\tanh\left(\dfrac{x}{\mu}\right)$	$x\,\mathrm{erf}\left(\dfrac{x}{\mu}\right)$
$e(f)$	$\approx 1.26521 \times 10^{-6}$	$\approx 7.54617 \times 10^{-7}$	$\approx 1.58151 \times 10^{-7}$	$\approx 8.7349 \times 10^{-8}$

By considering the setting of Table 1, Figures 1 and 2 also support our theoretical findings.

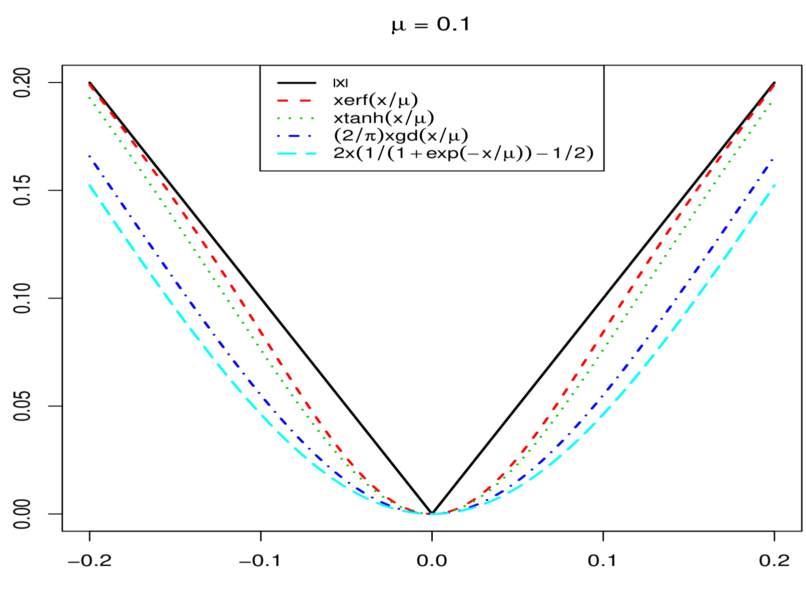

FIGURE 1. Graphs of the functions in Table 1 with $\mu = 0.1$ for $x \in (-0.2, 0.2)$.

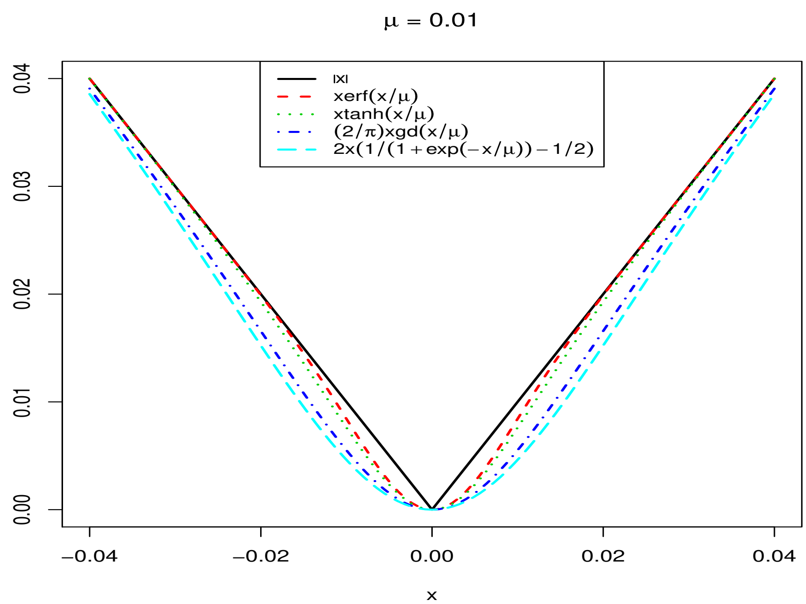

FIGURE 2. Graphs of the functions in Table 1 with $\mu = 0.01$ for $x \in (-0.04, 0.04)$.

4. Conclusion

Sigmoid functions can be used for smooth approximations of $|x|$. In particular, $x\,\mathrm{erf}(x/\mu)$ is proved to be a better smooth approximation for $|x|$ with respect to accuracy. Our main results can be used successfully in other areas of scientific knowledge.

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