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SIGN–PATTERNS WHICH REQUIRE A POSITIVE EIGENVALUE

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(This version contains the proof of Lemma 5.1)

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Abstract

We investigate matrices which have a positive eigenvalue by virtue of their sign–pattern and regardless of the magnitudes of the entries. When all the off–diagonal entries are nonzero, we show that an $n \times n$ sign–pattern, $n \neq 3, 4$, requires a positive eigenvalue if and only if it has at least one nonnegative diagonal entry and every cycle of length greater than one in its signed digraph is positive. When $n = 3, 4$, or when not all off–diagonal entries are nonzero, positivity of the cycles of length greater than one is no longer necessary. In the course of proving these results we observe certain necessary and certain sufficient

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conditions for a general sign–pattern to require a positive eigenvalue. We also identify and construct more classes of sign–patterns which require a positive eigenvalue.

1 INTRODUCTION

In [10], Schneider extends the Perron–Frobenius theory of nonnegative matrices by showing that a non–nilpotent complex matrix A has a positive eigenvalue if and only if the cone $w_k(A)$ generated by $A^k, A^{k+1}, \ldots$, is pointed, for some (and hence for all) positive integers k. Moreover, it is shown in [10] that the spectral radius of A is an eigenvalue, with the property that its index is at least as large as the index of any equimodular eigenvalue, if and only if the closure of $w_k(A)$ is a pointed cone. The classical result that the spectral radius is an eigenvalue of any (entrywise) nonnegative matrix follows readily. A combinatorial converse of the latter is shown in Eschenbach and Johnson [3], where the authors prove that the sign–pattern class of a matrix requires the spectral radius to be an eigenvalue if and only if all the cycles in its signed digraph are positive.

Motivated by the above facts we raise the following question: what are the sign–patterns which require every matrix with one of those sign–patterns to possess a positive eigenvalue?

We point out that the set of $n \times n$ sign–patterns which require a positive eigenvalue contains those which require the spectral radius be an eigenvalue but also contains sign–patterns with negative cycles. In particular, it will be evident from our results that diagonal entries can be negative and when $n = 3, 4$, or when the sign–pattern has some zero entries, negative cycles of length greater than 1 can be present. We will show that an $n \times n$, $n \neq 3, 4$, sign–pattern all of whose off–diagonal entries are nonzero requires a positive eigenvalue exactly when it is similar by a signature matrix to a pattern with positive off–diagonal entries, and has at least one nonnegative diagonal entry. A particular implication of this result is that for a sign–pattern with only nonzero off–diagonal entries, requiring a nonnegative eigenvalue is equivalent to requiring a positive eigenvalue. Thus, many of our results are developed for patterns that require a nonnegative eigenvalue. When $n \leq 4$ the analysis is done separately. It is worth remarking that the cases $n = 3, 4$ are inherently different.

The structure of our paper is as follows. In Section 2 we include most definitions and notation. In Section 3 we collect some useful observations regarding general sign–patterns that require a positive eigenvalue. Our main result in Section 5 (Theorem 5.2), is proved by induction on n based on the results for the cases $n \leq 4$ and on graph theoretic results found in Section 4. In Section 6 we present a method for constructing general sign–patterns that require a positive eigenvalue, and conclude with discussion about our question and the solution to a related qualitative problem.

2 DEFINITIONS AND NOTATION

We let $\Gamma = (V, E)$ denote a **digraph** with vertex set $V = \{1, 2, \ldots, n\}$ and directed arc set $E \subseteq \{(j, k) \mid j, k \in V\}$. A **path of length** $t \geq 1$ from j to k in Γ is a sequence of t arcs $(r_i, r_{i+1}) \in E$, for $i = 1, ..., t$, between distinct vertices $j = r_1, r_2, ..., r_t, r_{t+1} = k$.

A **cycle of length** $t \geq 2$ (or a t–**cycle**) is a path $r_1, \ldots, r_t$ together with the arc (r_t, r_1). A loop from a vertex to itself is considered to be a cycle of length 1.

The digraphs we consider are **signed**, meaning that every arc is assigned a weight $+$ or $-$. A signed path is symbolized e.g., by

$$i \xrightarrow{-} j \xrightarrow{+} k.$$

Naturally, we say that a path or a cycle is positive (negative) if the product of the signs of its arcs is positive (negative). We say Γ is **sign–symmetric** if all the cycles of length 2 are positive (i.e., for all distinct i, j, the arc $i \longrightarrow j$ has the same sign as the arc $j \longrightarrow i$). In such a case we use symbols such as

$$1 \xleftrightarrow{+} 2 \xleftrightarrow{+} 3 \xleftrightarrow{x} 4 \xleftrightarrow{+} 5,$$

where x is either $+$ or $-$.

A **complete signed digraph** is a signed digraph with all possible directed arcs between distinct vertices present. Notice that a complete signed digraph may or may not have loops.

For a matrix $A = (a_{ij}) \in M_n(\mathbb{R})$, the **signed digraph** of A, denoted by $D(A) = (V, E)$, has vertex set $V = \{1, 2, \ldots, n\}$, arc set $E = \{(i, j) \mid a_{ij} \neq 0\}$, and every arc (i, j) is weighted by the sign of a_{ij}.

An $n \times n$ array consisting of $+$, $-$, or 0 entries is called a **sign–pattern** and is denoted by $\mathcal{P}$. For an $n \times n$ matrix A, we say that $A \in \mathcal{P}$ if the signs

(+, −, or 0) of the entries in A agree with the corresponding entries in $\mathcal{P}$. The signed digraph of $\mathcal{P}$, $D(\mathcal{P})$, is $D(A)$ for some (and hence any) $A \in \mathcal{P}$. In what follows, when an entry of a sign–pattern is to be determined it will be denoted by ?, and when its value is immaterial to our argument it will be denoted by $\star$. The short–hand notation ≥ 0 (resp. ≤ 0) is used as an entry of a sign–pattern to represent either of the two possible sign–patterns obtained by replacing this entry with 0 or + (resp. −).

We say that $\mathcal{P}$ **requires** (resp. **allows**) a particular property if every (resp. some) matrix in $M_n(\mathbb{R})$ whose sign–pattern is $\mathcal{P}$ possesses this property.

Given $\alpha \subseteq \{1, 2, \ldots, n\}$, we let $A[\alpha]$ and $A(\alpha)$ denote the complementary principal submatrices of $A \in M_n(\mathbb{R})$, indexed by α and $\{1, 2, \ldots, n\} \setminus \alpha$, respectively. Similar notation is adopted for a sign–pattern $\mathcal{P}$. Also $\sigma(A)$ and $\rho(A)$ denote the spectrum and the spectral radius, respectively. The **spectral abscissa** of A is

$$\lambda_A = \max\{\text{Re } \lambda \mid \lambda \in \sigma(A)\}.$$

As is defined in [10], given a nonnegative integer k, the **intrinsic cone** of A, $w_k(A)$, consists of all nonnegative linear combinations of $A^k, A^{k+1}, \ldots$. By definition, $w_k(A)$ is **pointed** if $w_k(A) \bigcap (-w_k(A)) = \{0\}$.

A diagonal matrix S whose diagonal entries are ± 1 is called a **signature matrix**. Of course, $S^{-1} = S$, and SAS^{-1} is referred to as a **signature similarity** of A. It is well known that a signature similarity gives rise to a re–signing of the arcs of the signed digraph of a matrix (or of a sign–pattern) in a way such that the signs of all cycles are preserved. If A is irreducible and all the cycles of $D(A)$ are positive, then A is signature similar to a nonnegative matrix (see Engel and Schneider [2]).

3 GENERAL OBSERVATIONS

We begin with some general useful observations on sign–patterns that require a nonnegative eigenvalue.

Lemma 3.1 If $\mathcal{P}$ is an $n \times n$ sign-pattern that requires a nonnegative eigenvalue, then for all $\alpha \subseteq \{1, 2, \ldots, n\}$, either $\mathcal{P}[\alpha]$ or $\mathcal{P}(\alpha)$ requires a nonnegative eigenvalue. Moreover, if the diagonal entries of $\mathcal{P}$ are nonzero, then either $\mathcal{P}[\alpha]$ or $\mathcal{P}(\alpha)$ requires a positive eigenvalue.

Proof:
Suppose that for some $\alpha \subseteq \{1, 2, \ldots, n\}$, $\mathcal{P}[\alpha]$ and $\mathcal{P}(\alpha)$ allow matrices without a nonnegative eigenvalue. Then $A \in \mathcal{P}$ can be chosen such that $A[\alpha]$ and $A(\alpha)$ have no nonnegative eigenvalues and such that its nonzero entries off these principal submatrices are arbitrarily close to zero. By continuity of the spectrum, such an A can be chosen to have no nonnegative eigenvalues, a contradiction.

Suppose now that the diagonal entries of $\mathcal{P}$ are nonzero and that $A[\alpha]$ and $A(\alpha)$ have no positive eigenvalues, but possibly a zero eigenvalue. Subtracting a small enough positive multiple of the identity from $A[\alpha]$ and $A(\alpha)$ we have that they still belong to $\mathcal{P}[\alpha]$ and $\mathcal{P}(\alpha)$, respectively, but neither has a nonnegative eigenvalue, contradicting the first part of the proof. $\square$

Lemma 3.2 If $\mathcal{P}$ is an $n \times n$ sign-pattern that requires a nonnegative eigenvalue, then $D(\mathcal{P})$ has no negative cycle of length n.

Proof:
If $D(\mathcal{P})$ has a negative cycle of length n, choose $A \in \mathcal{P}$ such that the modulii of the entries corresponding to this cycle are equal to 1 and all other nonzero entries are of arbitrarily small modulus. By continuity, we then obtain a matrix whose eigenvalues are arbitrarily close to the n roots of -1 and hence none is nonnegative. $\square$

A generalized diagonal of an $n \times n$ array is a set of n entries of the array such that no two of them lie on the same row or column. It is well known that a generalized diagonal always has a unique partition into subsets such that the elements of each partition correspond to a cycle in the complete digraph with loops on n vertices. We refer to such a partition as the the cyclic decomposition of the generalized diagonal.

Proposition 3.3 No cyclic decomposition of a generalized diagonal of a sign–pattern that requires a nonnegative eigenvalue consists entirely of negative cycles. In particular, every sign-pattern $\mathcal{P}$ that requires a nonnegative eigenvalue has at least one nonnegative diagonal entry.

Proof:
Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_k$ is the cyclic decomposition of a generalized diagonal of a sign–pattern $\mathcal{P}$ that requires a nonnegative eigenvalue, and suppose

that there is a negative cycle on the vertices in α_ℓ, $\ell = 1, 2, \ldots, k$ in $D(\mathcal{P})$. Lemma 3.1 applied to $\mathcal{P}$ and its principal sub–patterns implies that $\mathcal{P}[\alpha_i]$ must require a nonnegative eigenvalue, for some $i \in \{1, 2, \ldots k\}$, contradicting Lemma 3.2. The last statement of the proposition follows from the fact that the main diagonal of $\mathcal{P}$ is a generalized diagonal. □

We continue with a sufficient condition for $\mathcal{P}$ to require a positive eigenvalue.

Proposition 3.4 Suppose that all cycles of length at least two in the signed digraph of an irreducible sign–pattern $\mathcal{P}$ are positive, and that $\mathcal{P}$ has at least one nonnegative diagonal entry. Then $\mathcal{P}$ requires that the spectral abscissa be a positive eigenvalue.

Proof:
As we have mentioned, if $\mathcal{P}$ is as prescribed above, then it is signature similar to a sign–pattern with nonnegative off–diagonal entries. Then, by the Perron–Frobenius theorem (see e.g., [1]), if $A \in \mathcal{P}$, its spectral abscissa, λ_A, is an eigenvalue of A. Moreover, it is well known that because A is irreducible, λ_A is strictly greater than every diagonal entry of A and hence is positive. □

4 PROPERTIES OF COMPLETE SIGNED DIGRAPHS

In [9], Popescu shows that for a complete signed undirected graph one of three alternatives holds. Either all the cycles are positive, or when all signs are reversed all the cycles are positive, or the graph contains a negative k–cycle and a positive k–cycle, for all $3 \leq k \leq n$. In this section, we present the proof of a similar property of complete signed digraphs, which is useful in our study of sign–patterns. We will show that if a complete signed digraph contains a negative cycle of length at least 2, then for every $2 \leq k \leq n-1$, either there is a negative cycle of length k or a negative cycle of length $k+1$, or both. We begin with the following result.

Proposition 4.1 Let Γ be a complete signed digraph with a vertex k such that all cycles of length at least 2 of Γ through k are positive. Then all the cycles of length at least 2 of Γ are positive.

Proof:
Let Γ be a digraph on n vertices as prescribed above. Without loss of generality we may assume that every cycle through n is positive. There is nothing to show if $n \leq 2$. Suppose $n \geq 3$. Since n is part of any cycle of length n, Γ does not contain an n–cycle that is negative. Suppose, without loss of generality, that $1 \longrightarrow 2 \longrightarrow \ldots \longrightarrow j \longrightarrow 1$ is a negative cycle with $2 \leq j \leq n-1$. By our initial assumption, $1 \longrightarrow 2 \longrightarrow n \longrightarrow 1$ is a positive cycle, so $2 \longrightarrow n \longrightarrow 1$ has the same sign as $1 \longrightarrow 2$, and since the 2–cycles through n are positive, $1 \longrightarrow n \longrightarrow 2$ has the same sign as $1 \longrightarrow 2$. But then $1 \longrightarrow n \longrightarrow 2 \longrightarrow 3 \longrightarrow \ldots \longrightarrow j \longrightarrow 1$ would be a negative cycle through n. Contradiction. $\square$

Proposition 4.2 Let Γ be a complete signed digraph on n vertices. If Γ has a negative cycle of length j for some $2 \leq j \leq n$, then for all $2 \leq k \leq n-1$, Γ either contains a negative cycle of length k or a negative cycle of length $k+1$.

Proof:
Without loss of generality, let $1 \longrightarrow 2 \longrightarrow \ldots \longrightarrow j \longrightarrow 1$ be a negative cycle of length j. By performing a signature similarity of the corresponding sign–pattern, we can without loss of generality assume that $i \xrightarrow{+} i+1$ for $i = 1, 2, \ldots, j-1$ and $j \xrightarrow{-} 1$.

Claim 1 If $3 \leq j \leq n-2$, then there is a negative cycle of length $j+1$ or of length $j+2$.

Proof of Claim 1: Suppose that there is no negative cycle of length $j+1$. Let x represent the sign of the arc $1 \longrightarrow j+1$. If also $j+1 \xrightarrow{x} 2$, then $1 \xrightarrow{x} j+1 \xrightarrow{x} 2 \xrightarrow{+} \ldots \xrightarrow{+} j \xrightarrow{-} 1$, is a negative cycle of length $j+1$, a contradiction. Hence the sign of $j+1 \longrightarrow 2$ must be $-x$. By the same reasoning if $2 \xrightarrow{y} j+2$, then $j+2 \xrightarrow{-y} 3$. But then the cycle

$$1 \xrightarrow{x} j+1 \xrightarrow{-x} 2 \xrightarrow{y} j+2 \xrightarrow{-y} 3 \xrightarrow{+} \ldots \xrightarrow{+} j \xrightarrow{-} 1$$

is a negative cycle of length $j+2$, establishing the claim.

Claim 2 If $n \geq 4$ and $j = 2$, then there is a negative cycle of length 3 or of length 4.

Proof of Claim 2: Suppose that $1 \xrightarrow{+} 2 \xrightarrow{-} 1$ is a negative 2–cycle and that there is no negative 3–cycle. Without loss of generality we can assume that $3 \xrightarrow{+} 4$. Let $1 \xrightarrow{x} 3$. Then

- $4 \xrightarrow{-x} 2$, otherwise $1 \xrightarrow{x} 3 \xrightarrow{+} 4 \longrightarrow 2 \xrightarrow{-} 1$ would be a negative 4–cycle and we are done.
- $3 \xrightarrow{-x} 2$, otherwise $1 \xrightarrow{x} 3 \longrightarrow 2 \xrightarrow{-} 1$ would be a negative 3–cycle.
- $2 \xrightarrow{-x} 3$, otherwise $2 \longrightarrow 3 \xrightarrow{+} 4 \xrightarrow{-x} 2$ would be a negative 3–cycle.
- $3 \xrightarrow{-x} 1$, otherwise $1 \xrightarrow{+} 2 \xrightarrow{-x} 3 \longrightarrow 1$ would be a negative 3–cycle.
- $1 \xrightarrow{x} 4$, otherwise $1 \longrightarrow 4 \xrightarrow{-x} 2 \xrightarrow{-} 1$ would be a negative 3–cycle.
- $4 \xrightarrow{-} 3$, otherwise $1 \xrightarrow{x} 4 \longrightarrow 3 \xrightarrow{-x} 1$ would be a negative 3–cycle.
- $2 \xrightarrow{x} 4$, otherwise $4 \xrightarrow{-} 3 \xrightarrow{-x} 2 \longrightarrow 4$ would be a negative 3–cycle.
- $4 \xrightarrow{x} 1$, otherwise $4 \longrightarrow 1 \xrightarrow{+} 2 \xrightarrow{x} 4$ would be a negative 3–cycle.

Then $1 \xrightarrow{x} 3 \xrightarrow{-x} 2 \xrightarrow{x} 4 \xrightarrow{x} 1$ is a negative 4–cycle.

Claim 3 If $4 \leq j \leq n$, then there is a negative cycle of length $j-1$ or of length $j-2$.

Proof of Claim 3: If there is negative 2–cycle, then by Claim 2 and by recursively applying Claim 1, there is a negative cycle of length $j-1$ or $j-2$. So let us assume that there are no negative 2–cycles and thus $i \xrightarrow{+} i+1 \xrightarrow{+} i$ for $i = 1, 2, \ldots, j-1$. Suppose also there is no negative cycle of length $j-1$. Then $1 \xrightarrow{-} 3$, otherwise $1 \longrightarrow 3 \longrightarrow$ $+$ $4 \xrightarrow{+} \ldots \xrightarrow{+} j \xrightarrow{-} 1$ is a negative cycle of length $j-1$. If $j = 4$, then $1 \xrightarrow{-} 3 \xrightarrow{+} 2 \xrightarrow{+} 1$ is a negative cycle of length 3. If $j \geq 5$, then $3 \xrightarrow{-} 5$, otherwise

$$1 \xrightarrow{+} 2 \xrightarrow{+} 3 \longrightarrow 5 \xrightarrow{+} \ldots \xrightarrow{+} j \xrightarrow{-} 1$$

would be a negative cycle of length $j-1$. But then

$$1 \xrightarrow{-} 3 \xrightarrow{-} 5 \xrightarrow{+} \ldots \xrightarrow{+} j \xrightarrow{-} 1$$

is a negative cycle of length $j-2$.

Now the proof of the theorem follows by recursively applying Claims 1, 2 and 3 to Γ. $\square$

Remark 4.3 The complete signed graph obtained by signing every arc negatively, has the property that all the cycles of even length are positive and all the cycles of odd length are negative. This example illustrates that there need not be a negative cycle of every length in a complete signed digraph with a negative cycle.

5 SIGN–PATTERNS WITH NONZERO OFF–DIAGONAL ENTRIES

For the purpose of induction, we begin with the results of the analysis of the cases $n \leq 4$. The cases $n=3$ and $n=4$ are inherently different from all other cases. Due to their technical nature, the proof of the following lemma is not included in this manuscript but is available from the authors on request.

Lemma 5.1 Let $\mathcal{P}$ be an $n \times n$ sign–pattern with only nonzero off–diagonal entries, where $2 \leq n \leq 4$. Then the following are equivalent:

(i) $\mathcal{P}$ requires a positive eigenvalue.

(ii) $\mathcal{P}$ requires a nonnegative eigenvalue.

(iii) $\mathcal{P}$ satisfies one of conditions (A), (B), or (C) below.

(A) At least one diagonal entry of $\mathcal{P}$ is nonnegative and all the k–cycles, with $k \geq 2$, in $D(\mathcal{P})$ are positive.

(B) $n=3$ and $\mathcal{P}$ is similar, by signatures and permutations, to

$$\begin{bmatrix} \geq 0 & - & + \\ + & \leq 0 & - \\ + & + & \geq 0 \end{bmatrix}.$$

(C) $n = 4$ and $\mathcal{P}$ is similar, by signatures and permutations, to

$$\begin{bmatrix} 0 & + & + & + \\ + & \geq 0 & + & - \\ - & + & 0 & - \\ - & - & + & 0 \end{bmatrix}.$$

Theorem 5.2 Let $\mathcal{P}$ be an $n \times n$ sign–pattern with only nonzero off–diagonal entries, where $n \geq 5$. Then $\mathcal{P}$ requires a nonnegative eigenvalue if and only if both of the following hold:

(i) At least one diagonal entry of $\mathcal{P}$ is nonnegative.

(ii) All the k–cycles, with $k \geq 2$, in $D(\mathcal{P})$ are positive.

Proof:
Sufficiency of (i) and (ii) follows from Proposition 3.4. We continue with necessity of (i) and (ii). Recall that by Proposition 3.3, $\mathcal{P}$ has a nonnegative diagonal entry.

Suppose $D(\mathcal{P})$ contains a negative cycle of length 2 or greater. Then by Proposition 4.2, $D(\mathcal{P})$ must contain a negative cycle of length 2 or of length 3. Let us now assume that any $m \times m$ sign–pattern, $5 \leq m < n$, which requires a nonnegative eigenvalue satisfies (i) and (ii) in the statement of the theorem. We will proceed by isolating a principal submatrix P_{11} indexed by the vertices of a negative cycle of length 2 or of length 3, and then use Lemma 5.1 and this assumption to determine the complementary principal submatrix P_{22}. We will show that each of the possible cases leads to a contradiction thus establishing that there cannot be a negative cycle of length 2 or greater.

Case I : Suppose $D(\mathcal{P})$ does not contain a negative cycle of length 2. Then it must contain a negative cycle of length 3. So without loss of generality we have that

$$\mathcal{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \text{ where } P_{11} = \begin{bmatrix} \star & + & - \\ + & \star & + \\ - & + & \star \end{bmatrix}.$$

By Lemma 3.1 and Lemma 3.2 we see that P_{22} must require a positive eigenvalue. Since $\mathcal{P}$ does not contain a negative cycle of length 2, Lemma 5.1 (if $5 \leq n \leq 7$) or our assumption applied to P_{22} (if $n \geq 8$) imply that there is only one subcase to consider, namely,

$$P_{22} = \left[\begin{array}{ccccc} \geq 0 & + & + & \dots & + \\ + & \star & + & \dots & + \\ \vdots & & \ddots & & \vdots \\ + & + & + & \dots & \star \end{array} \right].$$

Label $3 \xrightarrow{x} 4$. Then

- $n \xleftrightarrow{x} 1$, otherwise $1 \xrightarrow{+} 2 \xrightarrow{+} 3 \xrightarrow{x} 4 \xrightarrow{+} 5 \xrightarrow{+} \dots$ $\dots \xrightarrow{+} n \longrightarrow 1$ would be a negative n–cycle.
- $4 \xleftrightarrow{-x} 2$, otherwise $2 \xrightarrow{+} 3 \xrightarrow{-} 1 \xrightarrow{x} n \xrightarrow{+} n-1 \xrightarrow{+} \dots$ $\dots 5 \xrightarrow{+} 4 \longrightarrow 2$ would be a negative n–cycle.
- $n \xleftrightarrow{x} 3$, otherwise $3 \xrightarrow{-} 1 \xrightarrow{+} 2 \xrightarrow{-x} 4 \xrightarrow{+} \dots \xrightarrow{+} n \longrightarrow 3$ would be a negative n–cycle.
- $n \xleftrightarrow{-x} 2$, otherwise $2 \xrightarrow{+} 1 \xrightarrow{-} 3 \xrightarrow{x} 4 \xrightarrow{+} \dots \xrightarrow{+} n \longrightarrow 2$ would be a negative n–cycle.
- $n-1 \xleftrightarrow{-x} 2$ has been shown above if $n = 5$, while if $n \geq 6$, it still must hold otherwise $2 \xrightarrow{+} 1 \xrightarrow{-} 3 \xrightarrow{x} 4 \xrightarrow{+} \dots$ $\dots \xrightarrow{+} n-2 \xrightarrow{+} n \xrightarrow{+} n-1 \longrightarrow 2$ would be a negative n–cycle.

But then $3 \xrightarrow{x} 4 \xrightarrow{+} \dots \xrightarrow{+} n-1 \xrightarrow{-x} 2 \xrightarrow{-x} n \xrightarrow{x} 1 \xrightarrow{-} 3$ is a negative n–cycle. Contradiction.

Case II Suppose that $D(\mathcal{P})$ contains a negative 2–cycle. Then without loss of generality we have that

$$\mathcal{P} = \left[\begin{array}{cc} P_{11} & P_{12} \\ P_{21} & P_{22} \end{array} \right], \quad \text{where} \quad P_{11} = \left[\begin{array}{cc} \star & - \\ + & \star \end{array} \right].$$

By Lemma 3.1 and Lemma 3.2 we see that P_{22} must require a positive eigenvalue. Lemma 5.1 (if $5 \leq n \leq 6$) or our assumption applied to P_{22} (if $n \geq 7$) imply that there are three subcases to consider.

Case 1: $n = 5$ and
$$P_{22} = \begin{bmatrix} \geq 0 & - & + \\ + & \leq 0 & - \\ + & + & \geq 0 \end{bmatrix}.$$

Label $1 \xrightarrow{y} 3$, $2 \xrightarrow{x} 3$. By Lemma 3.2 $D(\mathcal{P})$ cannot contain a negative 5–cycle. Then

- $5 \xrightarrow{-x} 1$, otherwise $1 \xrightarrow{-} 2 \xrightarrow{x} 3 \xrightarrow{-} 4 \xrightarrow{-} 5 \longrightarrow 1$ would be a negative 5–cycle.
- $2 \xrightarrow{x} 4$, otherwise $1 \xrightarrow{-} 2 \longrightarrow 4 \xrightarrow{+} 3 \xrightarrow{+} 5 \xrightarrow{-x} 1$ would be a negative 5–cycle.
- $5 \xrightarrow{y} 2$, otherwise $2 \xrightarrow{+} 1 \xrightarrow{y} 3 \xrightarrow{-} 4 \xrightarrow{-} 5 \longrightarrow 2$ would be a negative 5–cycle.
- $1 \xrightarrow{y} 4$, otherwise $1 \longrightarrow 4 \xrightarrow{+} 3 \xrightarrow{+} 5 \xrightarrow{y} 2 \xrightarrow{+} 1$ would be a negative 5–cycle.
- $3 \xrightarrow{-y} 2$, otherwise $2 \xrightarrow{+} 1 \xrightarrow{y} 4 \xrightarrow{-} 5 \xrightarrow{+} 3 \longrightarrow 2$ would be a negative 5–cycle.

But then $1 \xrightarrow{y} 3 \xrightarrow{-y} 2 \xrightarrow{x} 4 \xrightarrow{-} 5 \xrightarrow{-x} 1$ is a negative 5–cycle. Contradiction.

Case 2: $n = 6$ and
$$P_{22} = \begin{bmatrix} 0 & + & + & + \\ + & \geq 0 & + & - \\ - & + & 0 & - \\ - & - & + & 0 \end{bmatrix}.$$

Label $2 \xrightarrow{x} 3$. By Lemma 3.2 $D(\mathcal{P})$ cannot contain a negative 6–cycle. Then

- $6 \xrightarrow{x} 1$, otherwise $1 \xrightarrow{-} 2 \xrightarrow{x} 3 \xrightarrow{+} 4 \xrightarrow{+} 5 \xrightarrow{-} 6 \longrightarrow 1$ would be a negative 6–cycle.
- $5 \xrightarrow{x} 1$, otherwise $1 \xrightarrow{-} 2 \xrightarrow{x} 3 \xrightarrow{+} 4 \xrightarrow{-} 6 \xrightarrow{+} 5 \longrightarrow 1$ would be a negative 6–cycle.
- $2 \xrightarrow{x} 4$, otherwise $1 \xrightarrow{-} 2 \longrightarrow 4 \xrightarrow{+} 5 \xrightarrow{-} 3 \xrightarrow{+} 6 \xrightarrow{x} 1$ would be a negative 6–cycle.

But then $1 \xrightarrow{-} 2 \xrightarrow{x} 4 \xrightarrow{+} 3 \xrightarrow{+} 6 \xrightarrow{+} 5 \xrightarrow{x} 1$ is a negative 6–cycle. Contradiction.

Case 3 : $n \geq 5$ and

$$P_{22} = \begin{bmatrix} \geq 0 & + & + & \dots & + \\ + & \star & + & \dots & + \\ \vdots & & \ddots & & \vdots \\ + & + & + & \dots & \star \end{bmatrix}.$$

Label $1 \xrightarrow{y} 3$, $2 \xrightarrow{x} 3$. By Lemma 3.2, $D(\mathcal{P})$ cannot contain a negative n–cycle. Then

- $n \xrightarrow{-x} 1$, otherwise $1 \xrightarrow{-} 2 \xrightarrow{x} 3 \xrightarrow{+} \dots \xrightarrow{+} n \longrightarrow 1$ would be a negative n–cycle.
- $2 \xrightarrow{x} 4$, otherwise $1 \xrightarrow{-} 2 \longrightarrow 4 \xrightarrow{+} 3 \xrightarrow{+} 5 \xrightarrow{+} \dots \xrightarrow{+} n \xrightarrow{-x} 1$ would be a negative n–cycle.
- $n \xrightarrow{y} 2$, otherwise $2 \xrightarrow{+} 1 \xrightarrow{y} 3 \xrightarrow{+} \dots \xrightarrow{+} n \longrightarrow 2$ would be a negative n–cycle.
- $1 \xrightarrow{y} 4$, otherwise $1 \longrightarrow 4 \xrightarrow{+} 3 \xrightarrow{+} 5 \xrightarrow{+} \dots \xrightarrow{+} n \xrightarrow{y} 2 \xrightarrow{+} 1$ would be a negative n–cycle.
- $3 \xrightarrow{y} 2$, otherwise $2 \xrightarrow{+} 1 \xrightarrow{y} 4 \xrightarrow{+} \dots \xrightarrow{+} n \xrightarrow{+} 3 \longrightarrow 2$ would be a negative n–cycle.

But then $1 \xrightarrow{y} 3 \xrightarrow{y} 2 \xrightarrow{x} 4 \xrightarrow{+} \dots \xrightarrow{+} n \xrightarrow{-x} 1$ is a negative n–cycle. Contradiction.

Now we will use the above cases to establish inductively that $D(\mathcal{P})$ is as claimed. If $n = 5$, then by Lemma 5.1 , Case I, Case II.1, and Case II.3, $D(\mathcal{P})$ cannot contain a negative k–cycle for $k \geq 2$. If $n = 6$, then by Lemma 5.1, Case I, Case II.2, and Case II.3, $D(\mathcal{P})$ cannot contain a negative k–cycle for $k \geq 2$. If $n = 7$, then by Lemma 5.1, the case $n = 5$ above, Case I, and Case II.3, $D(\mathcal{P})$ cannot contain a negative k–cycle for $k \geq 2$. Finally, if $n \geq 8$, our results for $n = 5, 6, 7$, Case I, Case II.3, and induction imply that $D(\mathcal{P})$ must be as claimed.

□

The following theorem summarizes the characterizations of sign–patterns with nonzero off–diagonal entries that require a positive eigenvalue.

Theorem 5.3 *Let $\mathcal{P}$ be an $n \times n$ sign–pattern, $n \geq 5$, with only nonzero off–diagonal entries, and let $\Gamma = D(\mathcal{P})$. Then the following are equivalent:*

(i) $\mathcal{P}$ requires a positive eigenvalue.

(ii) $\mathcal{P}$ requires a nonnegative eigenvalue.

(iii) $\mathcal{P}$ is signature similar to a matrix with positive off–diagonal entries and at least one nonnegative diagonal entry.

(iv) $\mathcal{P}$ requires that the spectral abscissa of every $A \in \mathcal{P}$ is a positive eigenvalue.

(v) Γ has a vertex i such that all cycles of length at least 2 via i are positive, and at least one diagonal entry of $\mathcal{P}$ is nonnegative.

(vi) All cycles of length at least 2 in Γ are positive, and at least one diagonal entry of $\mathcal{P}$ is nonnegative.

(vii) For all nonnegative integers k, and for all $A \in \mathcal{P}$, the intrinsic cone $w_k(A)$ is pointed.

Proof:
The equivalence of (i) and (ii) is trivial in one direction and in the other direction follows from Theorem 5.2, and Proposition 3.4. The equivalence of (ii), (iii), and (iv) is also a consequence of Theorem 5.2, Proposition 3.4, and the comments in Section 2. The implications (i) $\Longrightarrow$ (v) $\Longrightarrow$ (vi) $\Longrightarrow$ (i), are deduced from Theorem 5.2 and Proposition 4.1. The equivalence of (i) and (vii) follows from the results in [10]. □

We comment that the above theorem also holds for $n = 2$, and that for $n = 3,\ 4$, clauses (i), (ii), and (vii) are equivalent for sign–patterns with only nonzero off–diagonal entries.

6 DISCUSSION

In previous sections we have discussed some general necessary and some sufficient conditions for a sign–pattern to require a positive eigenvalue. We have also characterized such patterns with nonzero off–diagonal entries. In this section we identify some classes of sign–patterns that require a positive eigenvalue, present a method of construction, and also discuss a related qualitative problem.

One well studied class of sign–patterns are the **sign nonsingular** patterns, which are sign–patterns that require nonsingularity (see Klee, Ladner, and Manber [8] for the notion of an L–matrix and relevant references). Any sign nonsingular pattern $\mathcal{P}$ requires that for all $A \in \mathcal{P}$, $\det A$ has a fixed sign, either $+$ or $-$. Based on this fact we make the following observation:

Observation 6.1 Let $\mathcal{P}$ be an $n \times n$ sign nonsingular pattern, with n being an even (odd) integer, and such that $\det A < 0$ (> 0), for all $A \in \mathcal{P}$. Then $\mathcal{P}$ requires a positive eigenvalue.

Proof:
Let $f(\lambda)$ be the characteristic polynomial of $A \in \mathcal{P}$. Since the constant term of $f(\lambda)$ is $(-1)^n \det A$, we have that if n is even (odd) and $\det A < 0$ (> 0), then $f(0) < 0$. But since $f(\lambda)$ is monic, for large enough λ, $f(\lambda) > 0$, hence $f(\lambda)$ has a positive root. $\square$

Based on the above observation we can construct irreducible sign–patterns that require a positive eigenvalue and have a negative cycle of length greater than one. For example, consider

$$\mathcal{P} = \begin{bmatrix} + & + & 0 & 0 & 0 \\ 0 & + & + & 0 & 0 \\ 0 & 0 & + & + & 0 \\ - & 0 & 0 & + & + \\ 0 & 0 & 0 & - & + \end{bmatrix}.$$

It can be verified that $\mathcal{P}$ requires positive determinant and hence, by Observation 6.1, $\mathcal{P}$ requires a positive eigenvalue. Notice that $D(\mathcal{P})$ has a negative 4–cycle.

Suppose now that $\mathcal{P}$ is an irreducible $n \times n$ sign–pattern which requires a positive eigenvalue, and which has the additional property that for some $1 \leq i \leq n$, both row i and column i of $\mathcal{P}$ contain just one nonzero entry. Note that if this is the case, then necessarily the i–th diagonal entry of $\mathcal{P}$ is zero, since $\mathcal{P}$ is irreducible. We can construct an irreducible sign–pattern of order $n+1$, which also requires a positive eigenvalue as follows. Let e_i denotes the i–th standard basis vector in $\mathbb{R}^n$ and let

$$\mathcal{Q} = \begin{bmatrix} \mathcal{P} & \mathcal{P}e_i \\ e_i^t\mathcal{P} & 0 \end{bmatrix}.$$

Evidently $\mathcal{Q}$ is also irreducible. Suppose that $A \in \mathcal{Q}$. Since $\mathcal{P}\ e_i$ and $e_i^t\mathcal{P}$ each have a single nonzero entry, it follows that there exist $\alpha, \beta > 0$ such that

$$A = \begin{bmatrix} B & \alpha B e_i \\ \beta e_i^t B & 0 \end{bmatrix},$$

for some $B \in \mathcal{P}$. Thus

$$A = \begin{bmatrix} B \\ \beta e_i^t B \end{bmatrix} \ [\, I \quad \alpha e_i \,],$$

which has the same nonzero eigenvalues as

$$C = [\, I \quad \alpha e_i \,] \ \begin{bmatrix} B \\ \beta e_i^t B \end{bmatrix}.$$

(see e.g., [5]). But $C = B + \alpha\beta e_i e_i^t B \in \mathcal{P}$, and it now follows that A has a positive eigenvalue. Consequently, $\mathcal{Q}$ requires a positive eigenvalue.

We can use the above construction iteratively to produce sign–patterns requiring a positive eigenvalue, which are neither sign nonsingular nor signature similar to a pattern with nonnegative off–diagonal entries. For example, starting with

$$\mathcal{P} = \begin{bmatrix} 0 & - & + \\ + & 0 & 0 \\ 0 & + & 0 \end{bmatrix},$$

which requires a positive eigenvalue (by Observation 6.1), and using the construction repeatedly with $i = 3$, we find that the $n \times n$ pattern

$$\begin{bmatrix} 0 & - & + & + & \dots & + \\ + & 0 & 0 & 0 & \dots & 0 \\ 0 & + & 0 & 0 & \dots & 0 \\ 0 & + & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & + & 0 & 0 & \dots & 0 \end{bmatrix}$$

is irreducible, requires a positive eigenvalue, is not sign nonsingular, and is not signature similar to a pattern with nonnegative off–diagonal entries (since its graph has a negative 2–cycle). In fact, the above pattern also requires singularity if $n \geq 4$ since the third and fourth rows are linearly dependent.

The difficulty in answering the question we raised in general, we believe, lies mainly in the fact that the implications of assuming that a pattern requires a positive eigenvalue are indeed of a qualitative nature but also, as the results in [10] suggest, of a subtle quantitative nature (involving the powers of the matrix).

We conclude with an answer to the related question of identifying the sign–patterns that allow a positive eigenvalue.

Observation 6.2 An $n \times n$ sign–pattern $\mathcal{P}$ allows a positive eigenvalue if and only if $D(\mathcal{P})$ has a positive cycle.

Proof:
If $D(\mathcal{P})$ has a positive k–cycle, choose $A \in \mathcal{P}$ such that the magnitudes of the entries corresponding to this cycle are equal to 1, and all other nonzero entries are of arbitrarily small magnitude. By continuity, we then obtain a matrix with k eigenvalues arbitrarily close to the k–th roots of unity and hence one of them is a simple positive eigenvalue.

Conversely, suppose that no cycle in $D(\mathcal{P})$ is positive. Then the sign of a k–cycle in $D(-\mathcal{P})$ is $(-1)^{k+1}$. As a consequence, if $A \in \mathcal{P}$, all the principal minors of $-A$ are nonnegative (see the proof of Theorem 1.9 in Eschenbach and Johnson [4]). It follows that $-A$ cannot have a negative eigenvalue (see Kellogg [7] and Johnson, Olesky, Tsatsomeros, and van den Driessche [6]). $\square$

As it is remarked in [3], the sign–patterns $\mathcal{P}$ whose signed digraphs have a positive cycle are exactly those that allow the spectral radius be an eigenvalue. In other words, allowing a positive eigenvalue is equivalent to allowing the spectral radius be an eigenvalue.

PROOF OF LEMMA 5.1

Lemma 5.1 Let $\mathcal{P}$ be an $n \times n$ sign–pattern with only nonzero off–diagonal entries, where $2 \leq n \leq 4$. Then the following are equivalent:

(i) $\mathcal{P}$ requires a positive eigenvalue.

(ii) $\mathcal{P}$ requires a nonnegative eigenvalue.

(iii) $\mathcal{P}$ satisfies one of conditions (A), (B), or (C) below.

(A) At least one diagonal entry of $\mathcal{P}$ is nonnegative and all the k–cycles, with $k \geq 2$, in $D(\mathcal{P})$ are positive.

(B) $n = 3$ and $\mathcal{P}$ is similar, by signatures and permutations, to

$$\begin{bmatrix} \geq 0 & - & + \\ + & \leq 0 & - \\ + & + & \geq 0 \end{bmatrix}.$$

(C) $n = 4$ and $\mathcal{P}$ is similar, by signatures and permutations, to

$$\begin{bmatrix} 0 & + & + & + \\ + & \geq 0 & + & - \\ - & + & 0 & - \\ - & - & + & 0 \end{bmatrix}.$$

Proof:
(i) implies (ii) is trivial. We will show that (ii) implies (iii) and (iii) implies (i) by separating the cases $n = 2, 3, 4$.

<u>The case $n = 2$</u>

(ii) implies (iii): Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

If $\mathcal{P}$ requires a nonnegative eigenvalue, then $bc > 0$ and at least one of a and d must be nonnegative, by Proposition 3.3.

(iii) implies (i): Follows from Proposition 3.4.

The case $n = 3$

(ii) implies (iii): Assume that $\mathcal{P}$ requires a nonnegative eigenvalue. By Proposition 3.3, $\mathcal{P}$ has at least one nonnegative diagonal entry.

Suppose then that $\mathcal{P}$ has a negative k–cycle, where $k \geq 2$. It follows from Lemma 3.2 that $k = 2$. Without loss of generality we may assume that $1 \xrightarrow{-} 2 \xrightarrow{+} 1$ and that $1 \xrightarrow{+} 3$. Let $2 \xrightarrow{x} 3$. Then

- $3 \xrightarrow{-x} 1$, otherwise $1 \xrightarrow{-} 2 \xrightarrow{x} 3 \longrightarrow 1$ would be a negative 3–cycle.
- $3 \xrightarrow{+} 2$, otherwise $1 \xrightarrow{+} 3 \longrightarrow 2 \xrightarrow{+} 1$ would be a negative 3–cycle.

If $x = -$, then

$$\mathcal{P} = \begin{bmatrix} ? & - & + \\ + & ? & - \\ + & + & ? \end{bmatrix}.$$

The case with $x = +$ is permutationally similar to the case with $x = -$; hence we need only consider the former. From Proposition 3.3 we have that the (1,1) and (3,3) entries of $\mathcal{P}$ must be nonnegative. Thus

$$\mathcal{P} = \begin{bmatrix} \geq 0 & - & + \\ + & ? & - \\ + & + & \geq 0 \end{bmatrix}.$$

Suppose that the (2,2) entry of $\mathcal{P}$ is positive. Then the matrix

$$A = \begin{bmatrix} 0 & -100 & 1 \\ 1 & 3 & -.01 \\ 100 & 1 & 0 \end{bmatrix},$$

does not have a nonnegative eigenvalue, and any matrix obtained from A by substituting a sufficiently small positive number into either the (1,1) entry or the (3,3) entry, or both, does not have a nonnegative eigenvalue. Hence the (2,2) entry may only be nonpositive, establishing (iii).

(iii) implies (i): If $\mathcal{P}$ is a pattern which satisfies (A), then $\mathcal{P}$ requires a positive eigenvalue by Proposition 3.4.

If $\mathcal{P}$ is the pattern in (B), let

$$A = \begin{bmatrix} a & -b & c \\ d & -e & -f \\ g & h & j \end{bmatrix} \in \mathcal{P},$$

where $a, e,$ and j are nonnegative and $b, c, d, f, g,$ and h are positive. Let $f(\lambda)$ be the characteristic polynomial of A. Then, if $a \leq j$,

$$f(a) = -[bd(j-a) + bfg + cdh + cg(a+e)] < 0,$$

and hence since

$$\lim_{\lambda \longrightarrow \infty} f(\lambda) = +\infty,$$

A must have a positive eigenvalue. Similarly, the sign of $f(j)$ when $a \geq j$ ensures that A has a positive eigenvalue.

<u>The case $n = 4$</u>

(ii) implies (iii): By Proposition 3.3 $\mathcal{P}$ has at least one nonnegative diagonal entry.

Suppose $D(\mathcal{P})$ has a negative k–cycle where $k \geq 2$. Then by Proposition 4.2, $D(\mathcal{P})$ has either a negative 3–cycle or a negative 4–cycle. If $D(\mathcal{P})$ had a negative 4–cycle, then by Lemma 3.2 $\mathcal{P}$ would not require a nonnegative eigenvalue. Hence $D(\mathcal{P})$ contains a negative 3–cycle. Using signature and permutation similarities we can without loss of generality assume that

$$1 \xrightarrow{+} 2 \xrightarrow{+} 3 \xrightarrow{-} 1 \quad \text{and} \quad 1 \xrightarrow{+} 4.$$

If we label

$$4 \xrightarrow{x} 3, \quad 3 \xrightarrow{y} 4, \quad 3 \xrightarrow{z} 2,$$

it follows that

- $4 \xrightarrow{-} 2$, otherwise $1 \xrightarrow{+} 4 \longrightarrow 2 \xrightarrow{+} 3 \xrightarrow{-} 1$ would be a negative 4–cycle.
- $2 \xrightarrow{-x} 4$, otherwise $4 \xrightarrow{x} 3 \xrightarrow{-} 1 \xrightarrow{+} 2 \longrightarrow 4$ would be a negative 4–cycle.
- $4 \xrightarrow{y} 1$, otherwise $3 \xrightarrow{y} 4 \longrightarrow 1 \xrightarrow{+} 2 \xrightarrow{+} 3$ would be a negative 4–cycle.

- $2 \xrightarrow{xz} 1$, otherwise $3 \xrightarrow{z} 2 \longrightarrow 1 \xrightarrow{+} 4 \xrightarrow{x} 3$ would be a negative 4–cycle.
- $1 \xrightarrow{-xyz} 3$, otherwise $1 \longrightarrow 3 \xrightarrow{z} 2 \xrightarrow{-x} 4 \xrightarrow{y} 1$ would be a negative 4–cycle.

Applying Proposition 3.3 to the pairs of 2–cycles below, the following implications hold:

(1) $\quad 1 \xrightarrow{+} 2 \xrightarrow{xz} 1,\ \ 3 \xrightarrow{y} 4 \xrightarrow{x} 3 \quad \Longrightarrow \quad xy > 0$ or $xz > 0$ or both.

(2) $\quad 1 \xrightarrow{-xyz} 3 \xrightarrow{-} 1,\ \ 2 \xrightarrow{-x} 4 \xrightarrow{-} 2 \quad \Longrightarrow \quad xyz > 0$ or $x > 0$ or both.

(3) $\quad 2 \xrightarrow{+} 3 \xrightarrow{z} 2,\ \ 1 \xrightarrow{+} 4 \xrightarrow{y} 1 \quad \Longrightarrow \quad y > 0$ or $z > 0$ or both.

It follows from equations (1)–(3) that if $yz < 0$, then one of y, z is positive and either $x > 0$ or $x < 0$. If $yz > 0$, then they are both positive and necessarily $x > 0$. We now analyze these two cases:

Case I $yz < 0$: Using Proposition 3.3, we see that all diagonal entries of $\mathcal{P}$ must be ≥ 0 since there are negative cycles on all 3–tuples of the vertices of the digraph. The following sign–patterns arise:

$$\begin{bmatrix} \geq 0 & + & + & + \\ + & \geq 0 & + & - \\ - & + & \geq 0 & - \\ - & - & + & \geq 0 \end{bmatrix}, \quad \begin{bmatrix} \geq 0 & + & + & + \\ - & \geq 0 & + & - \\ - & - & \geq 0 & + \\ + & - & + & \geq 0 \end{bmatrix},$$

$$\begin{bmatrix} \geq 0 & + & - & + \\ + & \geq 0 & + & + \\ - & - & \geq 0 & + \\ + & - & - & \geq 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} \geq 0 & + & - & + \\ - & \geq 0 & + & + \\ - & + & \geq 0 & - \\ - & - & - & \geq 0 \end{bmatrix}.$$

These four patterns are mutually similar by signature and permutation matrices, so it suffices to consider the first pattern only. The following three matrices do not have nonnegative eigenvalues.

$$A = \begin{bmatrix} 100 & 100 & 10 & .01 \\ 0.1 & 0 & 100 & -1 \\ -10 & 10 & 0 & -100 \\ -.01 & -10 & .1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 5 & 5 & 10 \\ 5 & 0 & 1 & -.01 \\ -.1 & 10 & 0 & -1 \\ -.1 & -5 & .1 & 10 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 0.01 & 10 \\ 0.01 & 0 & 1 & -100 \\ -0.01 & 100 & 30 & -1 \\ -10 & -1 & 100 & 0 \end{bmatrix}.$$

Matrices A, B, C, and any matrices formed by adding sufficiently small positive numbers to any of the diagonal entries of A, B, or C, show that if at least one of the $(1,1)$, $(3,3)$, or $(4,4)$ entries of the first pattern is positive, then the pattern does not require a nonnegative eigenvalue. Hence $\mathcal{P}$ is as claimed.

Case II $yz > 0$: As in Case I, the diagonal entries of $\mathcal{P}$ must be ≥ 0, hence

$$\mathcal{P} = \begin{bmatrix} \geq 0 & + & - & + \\ + & \geq 0 & + & - \\ - & + & \geq 0 & + \\ + & - & + & \geq 0 \end{bmatrix}.$$

But $\mathcal{P}$ is signature similar to

$$\begin{bmatrix} \geq 0 & - & - & - \\ - & \geq 0 & - & - \\ - & - & \geq 0 & - \\ - & - & - & \geq 0 \end{bmatrix},$$

which does not require a nonnegative eigenvalue. This can be seen by considering the following matrix, which has no positive eigenvalues, and any matrix formed from it by adding sufficiently small positive numbers to any of its diagonal entries:

$$\begin{bmatrix} 0 & -100 & -1 & -0.1 \\ -1 & 0 & -0.1 & -1 \\ -1 & -0.1 & 0 & -10 \\ -10 & -1 & -10 & 0 \end{bmatrix}.$$

The proof of the (ii) implies (iii) is now complete.

(iii) implies (i): If (A) holds, then, by Proposition 3.4, $\mathcal{P}$ requires a positive eigenvalue.

We now have to consider the pattern descibed in (C). First suppose that the $(2,2)$ entry is zero. Let

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & -a_{24} \\ -a_{31} & a_{32} & 0 & -a_{34} \\ -a_{41} & -a_{42} & a_{43} & 0 \end{bmatrix}$$

be any such matrix in the pattern. Then det$A$ equals

$$\begin{aligned}
&-a_{21}a_{12}a_{34}a_{43} - a_{21}a_{32}a_{14}a_{43} - a_{21}a_{42}a_{13}a_{34} - a_{31}a_{12}a_{24}a_{43} \\
&-a_{31}a_{42}a_{13}a_{24} - a_{31}a_{42}a_{14}a_{23} - a_{41}a_{12}a_{23}a_{34} - a_{41}a_{32}a_{13}a_{24} \\
&-a_{41}a_{32}a_{14}a_{23}
\end{aligned}$$

which is negative. Hence, by considering the characteristic polynomial, A must have a positive eigenvalue.

Next suppose that the $(2,2)$ entry of A is positive, that is

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & -a_{24} \\ -a_{31} & a_{32} & 0 & -a_{34} \\ -a_{41} & -a_{42} & a_{43} & 0 \end{bmatrix}.$$

If det$A$ is negative, then A requires a positive eigenvalue. Assume det$A$ is positive. Since det$A$ equals

$$\begin{aligned}
&a_{41}a_{22}a_{13}a_{34} - a_{21}a_{12}a_{34}a_{43} - a_{21}a_{32}a_{14}a_{43} - a_{21}a_{42}a_{13}a_{34} \\
&-a_{31}a_{12}a_{24}a_{43} - a_{31}a_{22}a_{14}a_{43} - a_{31}a_{42}a_{13}a_{24} - a_{31}a_{42}a_{14}a_{23} \\
&-a_{41}a_{12}a_{23}a_{34} - a_{41}a_{32}a_{13}a_{24} - a_{41}a_{32}a_{14}a_{23},
\end{aligned}$$

the sole positive summand must dominate the remaining sum. Thus, if $B = (b_{ij}) = (\det A)A^{-1}$, we can conclude that:

$$
\begin{aligned}
b_{11} &= a_{42}a_{23}a_{34} + a_{22}a_{34}a_{43} - a_{32}a_{24}a_{43} > 0,\\
b_{12} &= -a_{12}a_{34}a_{43} - a_{32}a_{14}a_{43} - a_{42}a_{13}a_{34} < 0,\\
b_{13} &= a_{12}a_{24}a_{43} + a_{22}a_{14}a_{43} + a_{42}a_{13}a_{24} + a_{42}a_{14}a_{23} > 0,\\
b_{14} &= -a_{22}a_{13}a_{34} + a_{12}a_{23}a_{34} + a_{32}a_{13}a_{24} + a_{32}a_{14}a_{23} < 0,\\
b_{21} &= -a_{21}a_{34}a_{43} - a_{31}a_{24}a_{43} - a_{41}a_{23}a_{34} < 0,\\
b_{22} &= a_{41}a_{13}a_{34} - a_{31}a_{14}a_{43} > 0,\\
b_{23} &= -a_{21}a_{14}a_{43} - a_{41}a_{13}a_{24} - a_{41}a_{14}a_{23} < 0,\\
b_{24} &= a_{21}a_{13}a_{34} + a_{31}a_{13}a_{24} + a_{31}a_{14}a_{23} > 0,\\
b_{31} &= a_{41}a_{22}a_{34} - a_{21}a_{34}a_{42} - a_{31}a_{24}a_{42} - a_{41}a_{24}a_{32} > 0,\\
b_{32} &= -a_{31}a_{14}a_{42} - a_{41}a_{12}a_{34} - a_{41}a_{14}a_{32} < 0,\\
b_{33} &= a_{41}a_{12}a_{24} + a_{41}a_{14}a_{22} - a_{21}a_{14}a_{42} > 0,\\
b_{34} &= -a_{21}a_{12}a_{34} - a_{21}a_{14}a_{32} - a_{31}a_{12}a_{24} - a_{31}a_{14}a_{22} < 0,\\
b_{41} &= -a_{21}a_{32}a_{43} - a_{31}a_{22}a_{43} - a_{31}a_{23}a_{42} - a_{41}a_{23}a_{32} < 0,\\
b_{42} &= a_{31}a_{12}a_{43} + a_{31}a_{13}a_{42} + a_{41}a_{13}a_{32} > 0,\\
b_{43} &= -a_{41}a_{13}a_{22} + a_{21}a_{12}a_{43} + a_{21}a_{13}a_{42} + a_{41}a_{12}a_{23} < 0,\\
b_{44} &= a_{21}a_{13}a_{32} - a_{31}a_{12}a_{23} + a_{31}a_{13}a_{22} > 0.
\end{aligned}
$$

Thus the sign–pattern of A^{-1} is

$$
\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix},
$$

which by Proposition 3.4 requires a positive eigenvalue, implying that A requires a positive eigenvalue.

Finally, in the case that $\det A = 0$, let $f(\lambda)$ be the characteristic polynomial of A, which necessarily has zero as a root. The coefficient of λ in f is

$$
\begin{aligned}
&a_{24}a_{43}a_{32} - a_{22}a_{34}a_{43} + a_{12}a_{31}a_{23}\\
&-a_{13}a_{22}a_{31} + a_{14}a_{31}a_{43} - a_{13}a_{34}a_{41}\\
&+a_{14}a_{21}a_{42} - a_{14}a_{22}a_{41} - a_{23}a_{34}a_{42}\\
&-a_{12}a_{41}a_{24} - a_{13}a_{21}a_{32}.
\end{aligned}
$$

From the fact that $\det A = 0$, it follows that this expression is negative. Hence $f'(0) < 0$ so that $f(\lambda)$ is negative for sufficiently small positive values of λ. Thus A has a positive eigenvalue. This establishes (i). $\square$

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