SIGNAL RECONSTRUCTION FROM FOURIER TRANSFORM SIGN INFORMATION

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Signal Reconstruction from Fourier Transform Sign Information*

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Abstract

In this paper, we present new results on the reconstruction of signals from only the sign of the real part of the Fourier transform. Specifically, we develop new theoretical results which state conditions under which two-dimensional signals are uniquely specified to within a scale factor with this information and show that these conditions include a broad class of signals. Furthermore, we apply this result to the problem of reconstructing two-dimensional signals from their zero crossings. We also present two algorithms for reconstructing a signal from sign information in either the time or frequency domain.

1. Introduction

Signal reconstruction from partial Fourier domain information has been of interest to a number of different authors both for particular applications and for its inherent theoretical value [1]. Previous work in this area has involved developing conditions under which signals are uniquely specified with Fourier transform magnitude or phase [2,3,4] or signed-magnitude [5] information and developing practical algorithms for recovering signals from this information. In this paper, we consider the problem of reconstructing signals from only Fourier transform (or inverse Fourier transform) sign information. This sign information can be viewed as one bit of the Fourier transform phase, without any magnitude information. Alternatively, this information can be viewed as the zero crossings of the real part of the Fourier or inverse Fourier transform. This latter viewpoint suggests a number of practical applications pertaining

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to the reconstruction of signals from zero crossing information. One such application occurs when a signal is clipped or otherwise distorted in such a way as to preserve the zero crossing or level crossing information, and it is desired to recover the original signal. Another application occurs in the theory of vision where studies have stressed the importance of edge detection as a means of classifying and identifying images but have not succeeded in developing a strong theoretical basis for this work [6]. A third application occurs in some design problems where one might want to specify a filter response [7] or antenna pattern [8] in terms of zero crossing or null points (such as for interpolation) and derive the remainder of the response from these.

In our previous work, we have established the importance of the sign of the real part of the Fourier transform (also referred to as one bit of Fourier transform phase) both experimentally and theoretically [1]. On the experimental side, we have shown with images that if the correct one bit of phase is combined with a unity or average magnitude, the resulting image maintains many of the features of the original image, and in fact, is identical to the phase-only version of the even (symmetric) component of the image. Furthermore, if an image is synthesized from the correct magnitude and phase with the most significant bit of the phase randomized, the result is unintelligible. On the theoretical side, we have demonstrated that much stronger results can be obtained for unique specification of signals with signed Fourier transform magnitude than with Fourier transform magnitude alone.

We have also previously established conditions under which a one-dimensional signal is uniquely specified to within a scale factor with Fourier transform sign information [1], although these conditions are fairly restrictive. One-dimensional signals are uniquely specified with this information only if the real part of the Fourier transform contains a sufficient number of sign changes, or zero crossings. In the two-dimensional case, a "zero-crossing" is actually a contour in the (ω_1, ω_2) plane and thus consists of an infinite number of points. Thus, it is reasonable to expect that in the two-dimensional problem, sign information in the Fourier domain is more likely to uniquely specify a signal than in the corresponding one-dimensional problem.

In this paper, we present new results* on the unique specification of two-dimensional signals with Fourier transform sign information. Specifically, we develop conditions under which a two-dimensional signal is uniquely specified to within a scale factor with sign information alone. Furthermore, we show that these conditions include a broad class of signals. These results are presented in section 2. We also develop a number of extensions to these results and discuss the problem of unique specification with sign information available only at a finite set of discrete frequencies.

In section 3, we apply our results to the problem of recovering a two-dimensional signal from its zero crossings and more generally, from its threshold crossings. A considerable amount of research has been devoted to the problem of reconstructing one-dimensional signals from zero crossings [13,14] and much less to the corresponding two-dimensional problem. The problem of reconstruction from zero crossings is a dual to the problem of reconstruction from one bit of phase since it involves recovering a signal from sign information in the signal domain rather than in the Fourier domain and thus the results developed in section 2 are directly applicable. In section 4, we discuss two potential algorithms for recovering a signal from sign information in either domain.

*Some of the results presented here were first presented at the Int'l Conf. on Acoustics, Speech, and Signal Processing in March 1984 [9].

2. Unique Specification with Fourier Transform Sign Information

In this section we present new theoretical results on the unique specification of two-dimensional signals with Fourier transform sign information. After introducing some notation, we will present a number of results which apply when the sign information is available at all frequencies and then extend these results to situations where the sign information is available only at a discrete set of frequencies.

Notationally, we will use $x[n_1, n_2]$ to denote a two-dimensional discrete-time sequence, $X(z_1, z_2)$ to denote its z-transform, and $X(\omega_1, \omega_2)$ to denote its Fourier transform, i.e.:

$$X(z_1, z_2) = \sum_{n_1} \sum_{n_2} x[n_1, n_2] z_1^{-n_1} z_2^{-n_2} \tag{1}$$

$$X(\omega_1, \omega_2) = \sum_{n_1} \sum_{n_2} x[n_1, n_2] e^{-j\omega n_1} e^{-j\omega n_2}$$

The Fourier transform sign information, or one bit of phase, will be defined as:

$$S_x(\omega_1, \omega_2) = \begin{cases} 1 & \text{if } \mathrm{Re}\{X(\omega_1, \omega_2)\} \geq 0 \\ -1 & \text{otherwise} \end{cases} \tag{2}$$

We will also refer to the even (symmetric) and odd (antisymmetric) components of a signal, defined as:

$$x_e[n_1, n_2] = \frac{x[n_1, n_2] + x[-n_1, -n_2]}{2} \tag{3}$$

$$x_o[n_1, n_2] = \frac{x[n_1, n_2] - x[-n_1, -n_2]}{2}$$

Similarly, $X_e(z_1, z_2)$ will denote the z-transform of $x_e[n_1, n_2]$. We will refer to z-transforms as symmetric if they correspond to symmetric sequences, that is, $X(z_1, z_2)$ is symmetric if $X(z_1, z_2) = X(z_1^{-1}, z_2^{-1})$. A factor of $X(z_1, z_2)$ will be said to be a real symmetric factor if it is symmetric as defined above and if all of its coefficients are real. This does not imply that $X(z_1, z_2)$ will take on only real values but does imply that $X(z_1, z_2)$ will be real on the unit surface $|z_1| = |z_2| = 1$. Furthermore, the set of real symmetric factors of a z-transform includes all possible factors which satisfy the definition above and is not limited to irreducible factors.

2.1. Unique Specification with $S_x(\omega_1, \omega_2)$

In this section, we will develop results on the unique specification of signals with the Fourier transform sign information $S_x(\omega_1, \omega_2)$, when this information is available for all frequencies. These results can be stated in a number of different forms since it is possible to uniquely specify a signal with Fourier transform sign information under a number of different sets of constraints. After stating and proving our primary result, we develop a number of extensions for different types of sequences and different definitions of $S_x(\omega_1, \omega_2)$. We also show that the results developed here apply to a broad class of two-dimensional signals.

2.1.1. Primary Result

Since our uniqueness theorem relies primarily on a well-established result from algebraic geometry, we shall first state this result without proof. The detailed proof is available in references [10] and [11].

Theorem 1 [10,11]. If $X(z_1, z_2)$ and $Y(z_1, z_2)$ are two-dimensional polynomials of degrees r and s with no common factors of degree > 0, then there are at most rs solutions to the following equations:

$$X(z_1, z_2) = 0 \tag{4}$$

$$Y(z_1, z_2) = 0$$

In this theorem, the degree of a polynomial in two variables is defined in terms of the sum of the degrees in each variable (for each term), that is, the degree of a two-dimensional polynomial $p(x, y)$ is equivalent to the degree of the one-dimensional polynomial $p(x, x)$. For example, a two-dimensional sequence with support over $0 \leq n_1, n_2 \leq N$ would have a z-transform which is a polynomial of degree $2N$.

Essentially, Theorem 1 places an upper bound on the number of points where two two-dimensional polynomials can both be zero if they do not have a common factor. As we state and prove in Theorem 2, this result can be applied directly to the problem of unique specification of signals with $S_x(\omega_1, \omega_2)$. Consider, for example, two signals $x[n_1, n_2]$ and $y[n_1, n_2]$ for which $S_x(\omega_1, \omega_2) = S_y(\omega_1, \omega_2)$. First of all, we note that since the real part of the Fourier transform only contains information about the even component of the sequence, we must require that $x[n_1, n_2]$ be even or be defined only over a nonsymmetric half-plane* so it can be recovered from its even part. Also, we note that if $\text{Re}\{X(\omega_1, \omega_2)\} > 0$ for all (ω_1, ω_2), then we could not expect sign $(\text{Re}\{X(\omega_1, \omega_2)\})$ to be sufficient to reconstruct the original signal. Thus, we will also assume that $\text{Re}\{X(\omega_1, \omega_2)\}$ and $\text{Re}\{Y(\omega_1, \omega_2)\}$ are positive in some regions of the (ω_1, ω_2) plane and negative in other regions. The boundary between these regions is a contour where $\text{Re}\{X(\omega_1, \omega_2)\} = \text{Re}\{Y(\omega_1, \omega_2)\} = 0$. With some algebra, we can show that this implies two polynomials in (z_1, z_2) are zero over the same infinite set of points, and thus from Theorem 1, these polynomials must contain a common factor. If we also assume that these polynomials are nonfactorable, then they must be equal to within a scale factor, and $x[n_1, n_2]$ and $y[n_1, n_2]$ must be equal to within a scale factor.

*A nonsymmetric half-plane (NSHP) region of support is defined to mean that if (n_1, n_2) is in the region of support, then $(-n_1, -n_2)$ is not in the region of support unless $n_1 = n_2 = 0$.

Specifically, we state the following theorem:

Theorem 2. Let $x[n_1,n_2]$ and $y[n_1,n_2]$ be real two-dimensional sequences with support over a finite non-symmetric half-plane, with $S_x(\omega_1,\omega_2) \equiv S_y(\omega_1,\omega_2)$. If Re $\{X(\omega_1,\omega_2)\}$ takes on both positive and negative values and $X_e(z_1,z_2)$ and $Y_e(z_1,z_2)$ are nonfactorable, then $x[n_1,n_2] = cy[n_1,n_2]$ for some positive constant c.

Proof: From the theorem statement, we know that $S_x(\omega_1,\omega_2) = S_y(\omega_1,\omega_2)$; we will show that $x[n_1,n_2] = cy[n_1,n_2]$ for some c. Since we know that Re $\{X(\omega_1,\omega_2)\}$ takes on positive and negative values, there must be some region of the (ω_1,ω_2) plane where Re $\{X(\omega_1,\omega_2)\} > 0$ and another region where Re $\{X(\omega_1,\omega_2)\} < 0$. The boundary between these regions is a contour where Re $\{X(\omega_1,\omega_2)\} = 0$, or equivalently, $X_e(\omega_1,\omega_2) = 0$. Thus,

$$X_e(z_1,z_2)\big|_{z_1=e^{j\omega_1},\, z_2=e^{j\omega_2}} = 0 \tag{5}$$

if ω_1 and ω_2 are on this contour. Since $S_x(\omega_1,\omega_2) = S_y(\omega_1,\omega_2)$ for all (ω_1,ω_2), equation (5) also holds for $Y_e(z_1,z_2)$. Thus, we have an infinite set of points where

$$X_e(z_1,z_2) = Y_e(z_1,z_2) = 0 \tag{6}$$

Since $x_e[n_1,n_2]$ is nonzero for positive and negative values of n_1 and n_2, $X_e(z_1,z_2)$ is a polynomial in the variables z_1, z_2, z_1^{-1}, and z_2^{-1}. However, if $x[n_1,n_2]$ and $y[n_1,n_2]$ have finite support, then we can find integers N_1 and N_2 such that $z_1^{N_1} z_2^{N_2} X_e(z_1,z_2)$ and $z_1^{N_1} z_2^{N_2} Y_e(z_1,z_2)$ are polynomials in only z_1 and z_2. Furthermore,

$$z_1^{N_1} z_2^{N_2} X_e(z_1,z_2) = 0 \tag{7}$$
$$z_1^{N_1} z_2^{N_2} Y_e(z_1,z_2) = 0$$

over the contour in the (ω_1,ω_2) plane where Re $\{X(\omega_1,\omega_2)\} = 0$, and therefore, over an infinite set of points. Thus, by Theorem 1, $X_e(z_1,z_2)$ and $Y_e(z_1,z_2)$ must have a common factor. If furthermore, we assume that $X_e(z_1, z_2)$ and $Y_e(z_1, z_2)$ are nonfactorable, then $X_e(z_1, z_2) = cY_e(z_1, z_2)$, $x_e[n_1, n_2] = cy_e[n_1, n_2]$, and $x[n_1, n_2] = cy[n_1, n_2]$.

Instead of constraining the signals to be real and to have support over a nonsymmetric half-plane, we could just as easily permit complex signals or signals symmetric about the origin. We will state one alternate form of Theorem 2 as a corollary since it will be needed later in this paper (see Appendix 1 for proof):

> **Corollary 1.** Let $x[n_1, n_2]$ and $y[n_1, n_2]$ be complex two-dimensional conjugate-symmetric* sequences with finite support with $S_x(\omega_1, \omega_2) \equiv S_y(\omega_1, \omega_2)$. If $X(\omega_1, \omega_2)$ takes on both positive and negative values and $X(z_1, z_2)$ and $Y(z_1, z_2)$ are nonfactorable, then $x[n_1, n_2] = cy[n_1, n_2]$ for some positive constant c.

Note that in this case $X(\omega_1, \omega_2)$ is real and that Corollary 1 directly constrains $X(\omega_1, \omega_2)$ and $X(z_1, z_2)$ rather than the corresponding transforms of the even components.

2.1.2. Extensions

Although Theorem 2 states a particular set of constraints under which a signal is uniquely specified with $S_x(\omega_1, \omega_2)$, a number of different sets of constraints are possible. In this section, we shall introduce three extensions to Theorem 2. The first result imposes most of its constraints on only one sequence, as opposed to imposing a number of constraints on both sequences $x[n_1, n_2]$ and $y[n_1, n_2]$. This result will be convenient to use when discussing algorithms for reconstruction from $S_x(\omega_1, \omega_2)$. The second result generalizes the concept of Fourier transform sign information to include a

*A sequence $x[n_1, n_2]$ is conjugate symmetric if $x[n_1, n_2] = x^*[-n_1, -n_2]$.

broader class of definitions than that given in equation (2). The third result extends Theorem 2 to include sequences with factorable z-transforms by replacing the nonfactorability constraint with a constraint on each factor.

Let us start by considering a case where it would be convenient to have a result which imposes most of its constraints on only one sequence. Suppose a sequence $x[n_1, n_2]$ is known to satisfy the constraints of Theorem 2, and we would like to develop an algorithm to recover $x[n_1, n_2]$ from $S_x(\omega_1, \omega_2)$. It would be convenient to have a set of constraints which guarantee that there are no other sequences $y[n_1, n_2]$ with $S_x(\omega_1, \omega_2) = S_y(\omega_1, \omega_2)$ whether or not $y[n_1, n_2]$ satisfies the constraints of Theorem 2. While it is simple enough to guarantee that a recovered sequence $y[n_1, n_2]$ has the correct region of support, it is extremely difficult to guarantee that $Y_e(z_1, z_2)$ is nonfactorable. Without this restriction, $Y_e(z_1, z_2)$ might contain a real symmetric factor which is positive on the unit surface, and thus we could have $S_x(\omega_1, \omega_2) = S_y(\omega_1, \omega_2)$ but $x[n_1, n_2] \neq cy[n_1, n_2]$. To avoid this problem, we note that since multiplication by an additional factor in the z-domain corresponds to a convolution in the space domain, the sequence $y[n_1, n_2]$ would then have a larger region of support than $x[n_1, n_2]$. Thus, if the exact size of the region of support of $x[n_1, n_2]$ is known, then this information, together with $S_x(\omega_1, \omega_2)$, is sufficient to uniquely specify $x[n_1, n_2]$. Specifically (see Appendix 1 for proof):

> **Theorem 3.** Let $x[n_1, n_2]$ and $y[n_1, n_2]$ be real two-dimensional sequences with support over a finite non-symmetric half-plane contained in $-N \leq n_1, n_2 \leq N$ with $S_x(\omega_1, \omega_2) \equiv S_y(\omega_1, \omega_2)$. If $x[N, N] \neq 0$, $\mathrm{Re}\{X(\omega_1, \omega_2)\}$ takes on both positive and negative values, and $X_e(z_1, z_2)$ is nonfactorable, then $x[n_1, n_2] = cy[n_1, n_2]$ for some positive constant c.

Thus, if a sequence $x[n_1, n_2]$ satisfies the constraints of Theorem 3, then we know that there are no other sequences $y[n_1, n_2]$ with the same region of support and the same Fourier transform sign function. Therefore, in a somewhat more general sense than was possible with Theorem 2, we can say that a sequence satisfying the constraints of Theorem 3 is uniquely specified to within a scale factor with the Fourier transform sign information and the known region of support.

It is also possible to generalize Theorems 2 and 3 to allow unique specification with broader classes of sign information than the $S_x(\omega_1, \omega_2)$ as defined in equation (2). Since $S_x(\omega_1, \omega_2)$ can be viewed as one bit of Fourier transform phase, we could generalize $S_x(\omega_1, \omega_2)$ to allow quantizing the phase in different ways. Specifically, we could define:

$$S_x^{\alpha}(\omega_1, \omega_2) = \begin{cases} 1 & \text{if } \alpha - \frac{\pi}{2} \le \phi_x(\omega_1, \omega_2) \le \alpha + \frac{\pi}{2} \\ -1 & \text{otherwise} \end{cases} \tag{8}$$

or equivalently,

$$S_x^{\alpha}(\omega_1, \omega_2) = \text{sign}\left(\text{Re}\,\{X(\omega_1, \omega_2)\, e^{j\alpha}\}\right) \tag{9}$$

The case $\alpha = 0$ corresponds to the definition of $S_x(\omega_1, \omega_2)$ given in equation (2). Alternatively, since $S_x(\omega_1, \omega_2)$ can be viewed as the zero crossings of the real part of the Fourier transform, we could generalize $S_x(\omega_1, \omega_2)$ to allow crossings of an arbitrary threshold as follows:

$$S_x^{\beta}(\omega_1, \omega_2) = \text{sign}(\text{Re}\,\{X(\omega_1, \omega_2)\} - \beta) \tag{10}$$

To develop a result on unique specification with generalized sign information, we will combine these two ideas and define:

$$S_x^{\alpha\beta}(\omega_1, \omega_2) = \text{sign}(\text{Re}\,\{X(\omega_1, \omega_2) e^{j\alpha}\} - \beta) \tag{11}$$

We can then develop a result similar to Theorem 2 for this definition of sign information (see Appendix 1 for proof):

Theorem 4. Let $x[n_1, n_2]$ and $y[n_1, n_2]$ be real two-dimensional sequences with support over a finite non-symmetric half-plane, with $S_x^{\alpha,\beta}(\omega_1, \omega_2) \equiv S_y^{\alpha,\beta}(\omega_1, \omega_2)$ for any α and β such that $\mathrm{Re}\,\{X(\omega_1, \omega_2)e^{j\alpha}\} - \beta$ takes on both positive and negative values. Also, let:

$$\hat{x}[n_1, n_2] = \frac{x[n_1, n_2]\,e^{j\alpha} + x^*[-n_1, -n_2]\,e^{-j\alpha}}{2} - \beta\,\delta[n_1, n_2] \qquad (12)$$

$$\hat{y}[n_1, n_2] = \frac{y[n_1, n_2]\,e^{j\alpha} + y^*[-n_1, -n_2]\,e^{-j\alpha}}{2} - \beta\,\delta[n_1, n_2]$$

where

$$\delta[n_1, n_2] = \begin{cases} 1 & \text{if } (n_1, n_2) = (0,0) \\ 0 & \text{otherwise} \end{cases}$$

If $\hat{X}(z_1, z_2)$ and $\hat{Y}(z_1, z_2)$ are nonfactorable, then $x[n_1, n_2] = cy[n_1, n_2]$ for $(n_1, n_2) \neq (0,0)$, and $x[0,0]\cos\alpha - \beta = c(y[0,0]\cos\alpha - \beta)$ for some positive constant c.

The ambiguity at $(n_1, n_2) = (0,0)$ is not just a scale factor; it is a scaling of the value with respect to a threshold $\frac{\beta}{\cos\alpha}$. Note that if $\alpha = \frac{k\pi}{2}$ for some odd integer k, then $S_x^{\alpha,\beta}(\omega_1, \omega_2)$ contains no information about $x[0,0]$, and even if $\beta = 0$, $x[0,0]$ cannot be recovered.

A further extension to Theorem 2 involves replacing the nonfactorability constraint with a constraint on each factor. Let us express $X_e(z_1, z_2)$ as a product of real symmetric factors $F_i(z_1, z_2)$. Observe that if $F_i(z_1, z_2) = 0$ for any i, then $X_e(z_1, z_2) = 0$; similarly, if $X_e(z_1, z_2) = 0$, then at least one of the factors $F_i(z_1, z_2)$ must be zero. Thus, if each factor contributes a set of zero crossing contours, each factor will be uniquely specified by its own zero crossing contours, and thus we can develop a set of conditions under which $X_e(z_1, z_2)$ will be uniquely specified by the complete set of zero crossing contours. Specifically, we state (see Appendix 1 for proof):

> **Theorem 5.** Let $x[n_1, n_2]$ and $y[n_1, n_2]$ be real two-dimensional sequences with support over a finite non-symmetric half-plane, with $S_x(\omega_1, \omega_2) \equiv S_y(\omega_1, \omega_2)$. Consider the factorization of $X_e(z_1, z_2)$ and $Y_e(z_1, z_2)$ into real symmetric factors which are irreducible over the set of real symmetric factors. If each of these factors has multiplicity one and takes on both positive and negative values on the unit surface, then $x[n_1, n_2] = cy[n_1, n_2]$ for some positive constant c.

Theorem 5 can be easily modified to permit complex signals or symmetric signals, to impose most of its constraints on one sequence, or to permit different definitions of the Fourier transform sign information $S_x(\omega_1, \omega_2)$ as was done earlier in developing extensions to Theorem 2. This theorem could also be stated in a slightly different manner by considering all possible factorizations of $X_e(z_1, z_2)$ and $Y_e(z_1, z_2)$ rather than one particular factorization. In this case, the requirement would be that every possible factor of $X_e(z_1, z_2)$ and $Y_e(z_1, z_2)$ must take on both positive and negative values on the unit surface. The multiplicity constraint is then unnecessary since if a factor $F(z_1, z_2)$ occurs with multiplicity two (or higher), then there will also be a factor $F^2(z_1, z_2)$ which is nonnegative on the unit surface and violates the constraints of the theorem.

Theorem 5 is related to a result developed in [1] on the unique specification of one-dimensional signals with Fourier transform sign information. The one-dimensional result states constraints on the zeros of the z-transform $X(z)$ which guarantee that $X_e(z)$ has zeros only on the unit circle (and that these zeros are simple zeros) and thus that sign $\{X_e(\omega)\}$ is sufficient to uniquely specify $x[n]$. This result could be stated in a form similar to Theorem 5 by considering the factorization of $X_e(z)$ into real symmetric factors irreducible over the set of real symmetric factors. In this case, the factors will all be second-order and will have either two complex conjugate zeros on the unit circle or two real zeros. Restricting each of these factors to have simple zeros on the unit circle is then equivalent to requiring that each factor take on both positive and negative values on the unit circle, as in Theorem 5. In addition, we require each factor to have multiplicity one so that we can guarantee that there is a sign change in $X_e(\omega)$ corresponding to each zero on the unit circle. The primary difference, then, between the one-dimensional problem and the two-dimensional problem is not in the type of constraints imposed but in the likelihood of a signal satisfying these constraints. While only a small class of one-dimensional signals will satisfy the appropriate constraints, we will show in the next section that a broad class of two-dimensional signals will satisfy these constraints.

2.1.3. Applicability

Having established a set of conditions which guarantee that a signal is uniquely specified by some partial information, it is worthwhile to determine whether or not these conditions are likely to apply to a typical sequence encountered in practice. While it is difficult to answer this question without making some assumptions about the type of application involved, it is still possible to show that a broad class of signals will satisfy the constraints of the results developed earlier. In this discussion, we will refer primarily to Theorem 3 since this is the basic result which states constraints primarily upon the actual signal we are trying to recover; similar results can be easily developed for the other extensions developed in the preceding section.

Let us first note that in many applications, the assumed region of support of a finite-length signal is somewhat arbitrary and thus signals can often be considered to have any desired region of support. However, since the real part of the Fourier transform is modified when the assumed region of support of a sequence is modified, care must be taken to ensure that the data available as $S_x(\omega_1, \omega_2)$ truly corresponds to a sequence with a region of support as specified in Theorem 3. In the remainder of this discussion, we will assume that the region of support constraint is satisfied.

With some informal arguments, we can next show that the probability of a random signal satisfying the remaining constraints of Theorem 3 very rapidly approaches unity as the number of points in the signal increases. First, we note that since "almost all" two-dimensional polynomials are nonfactorable, the nonfactorability constraint is satisfied with probability one [4,12]. Similarly, almost all sequences will have $x[N, N] \neq 0$. Next, consider the condition requiring Re $\{X(\omega_1, \omega_2)\}$ to change sign. If the coefficients of the sequence were random, there would be some small, but finite, chance that the first coefficient, $x[0,0]$, would be greater than the sum of the magnitudes of the others, and thus $S_x(\omega_1, \omega_2) = \text{sign}(x[0,0])$ for all (ω_1, ω_2). Thus, we cannot claim that almost all sequences satisfy the sign-change constraint, but we can argue that the probability of a random first-quadrant sequence satisfying this constraint very rapidly approaches unity as the number of points in the signal increases.

To see this, assume that $x[n_1, n_2]$ is a first quadrant sequence with support $N_1 \times N_2$ and form the one-dimensional sequence

$$a[n_1 + N_1 n_2] = x[n_1, n_2] \tag{13}$$

If $A(z)$ has at least one zero outside the unit circle, then Re $\{A(\omega)\}$ is guaranteed to have at least one sign change over the interval $(0, \pi)$ [1]. Since $A(\omega)$ is a slice of

$X(\omega_1, \omega_2)$, Re $\{X(\omega_1, \omega_2)\}$ must change sign somewhere in the (ω_1, ω_2) plane. If we assume that the zeros of $A(z)$ are equally likely to be inside the unit circle as outside, then the probability of $A(z)$ having at least one zero outside the unit circle is

$$p = 1-(0.5)^{N_1 N_2 - 1} \tag{14}$$

since $a[n]$ is $N_1 N_2$ points long. Since this condition is sufficient but not necessary for Re $\{X(\omega_1, \omega_2)\}$ to change sign, p represents a lower bound on the probability that a random first-quadrant sequence would satisfy the sign-change constraint of Theorem 3. We can also note that p approaches unity very rapidly even for small two-dimensional sequences. For example, for a 3×3 sequence, $p = 0.9961$, and for a 4×4 sequence, $p = 0.999969$. For a 64×64 sequence, $p \approx 1-10^{-1200}$.

Thus, we have shown that the probability of a random first-quadrant signal satisfying the constraints of Theorem 3 very rapidly approaches unity as the number of points in the signal increases. Although this does not guarantee that a particular signal in some particular application will satisfy the constraints of Theorem 3, it does show that the result applies to a broad class of signals.

2.2. Unique Specification with Samples of $S_x(\omega_1, \omega_2)$

Many types of signals or functions can be uniquely specified with a finite set of samples evaluated on a fixed grid. For example, the DFT is a method of sampling the Fourier transform of a sequence and DFT points are known to uniquely specify a finite length sequence. Also, in phase-only reconstruction problems, it has been shown that samples of the phase function will, for the most part, uniquely specify any sequence which is uniquely specified by its complete phase function [2,3,4].

In the problem considered here, however, the information in $S_x(\omega_1, \omega_2)$ is contained in the exact location of the zero crossings and this information is lost when $S_x(\omega_1, \omega_2)$ is sampled. From another point of view, we can say that a finite set of samples of $S_x(\omega_1, \omega_2)$ contains a finite number of *bits* of information and thus cannot be expected to uniquely specify a signal to infinite precision. This is distinctly different from typical sampling problems where each sample is of (theoretically) infinite precision and thus does not contain a finite number of bits of information. (Note, however, that we are referring strictly to theoretical sampling problems; in practical applications signals are generally represented with a finite number of bits, and it may be possible for a signal to be represented to sufficient accuracy with a finite set of samples of $S_x(\omega_1, \omega_2)$.)

In this section, we shall take a somewhat different approach to sampling $S_x(\omega_1, \omega_2)$. Instead of using the values of $S_x(\omega_1, \omega_2)$ over a fixed grid, we shall use the location of points on the zero crossing contours of Re $\{X(\omega_1, \omega_2)\}$ as the "samples". Since these contours consist of an infinite number of points, it is worthwhile to determine if a finite set of such points will uniquely specify a signal. Since Theorem 1 specifies the number of points where two two-dimensional polynomials can both be zero, we can use this theorem to establish that a particular number of arbitrarily-chosen zero crossing points is guaranteed to be sufficient for unique specification. We shall also show that this number of points may not be *necessary* for unique specification; in particular, if the zero crossing points are not chosen arbitrarily but are chosen in some particular way, a smaller set of zero crossing points can be sufficient for unique specification.

Since rectangular regions of support are common in applications, we will develop a result which states requirements in terms of such regions. The result could be easily modified for different regions of support or could be applied directly to a problem involving a different region of support by simply assuming a rectangular region of support large enough to enclose the actual region of support. If reference to a region of support $R(N)$ specifies only that a sequence is zero outside the region $-N \leq n_1, n_2 \leq N$, then we can state:

> **Theorem 6.** Let $x[n_1, n_2]$ and $y[n_1, n_2]$ be two-dimensional sequences with region of support over a nonsymmetric half-plane contained in $R(N)$. If $X_e(z_1, z_2)$ and $Y_e(z_1, z_2)$ are nonfactorable and $\mathrm{Re}\{X(\omega_1, \omega_2)\} = \mathrm{Re}\{Y(\omega_1, \omega_2)\} = 0$ at more than $16N^2$ distinct points, then $x[n_1, n_2] = cy[n_1, n_2]$ for some real constant c.

Proof: Recall that the proof of Theorem 2 requires stating that two polynomials $z_1^N\, z_2^N\, X_e(z_1, z_2)$ and $z_1^N\, z_2^N\, Y_e(z_1, z_2)$ are equal to within a scale factor given that they are both zero at an infinite number of points. In the case of Theorem 6, we know that $z_1^N\, z_2^N\, X_e(z_1, z_2) = z_1^N\, z_2^N\, Y_e(z_1, z_2) = 0$ at more than $16N^2$ points. These polynomials are of degree $4N$ and thus, by Theorem 1, can have at most $16N^2$ common zeros. Thus, $z_1^N\, z_2^N\, X_e(z_1, z_2) \equiv cz_1^N\, z_2^N\, Y_e(z_1, z_2)$ and the theorem follows.

As we did with Theorem 2, we can modify this result slightly so that only one sequence is required to have a nonfactorable z-transform:

> **Theorem 7.** Let $x[n_1, n_2]$ and $y[n_1, n_2]$ be two-dimensional sequences with region of support over a nonsymmetric half-plane contained in $R(N)$. If $X_e(z_1, z_2)$ is nonfactorable, $x[N, N] \neq 0$, and $\mathrm{Re}\{X(\omega_1, \omega_2)\} = \mathrm{Re}\{Y(\omega_1, \omega_2)\} = 0$ at more than $16N^2$ distinct points, then $x[n_1, n_2] = cy[n_1, n_2]$ for some real constant c.

The proof is analogous to the proof of Theorem 3.

Note that for this result, it is necessary to assume the nonfactorability constraint of Theorem 2. If a sequence satisfies the constraints of Theorem 5 but not the constraints of Theorem 2, then a finite set of zero crossings is not guaranteed to uniquely specify the sequence since these zero crossings may all correspond to the same factor. However, a similar result is easily developed by constraining the set of zero crossings to include N_i zero crossings of the i^{th} factor, where each N_i is chosen in accordance with degree of the factor.

Next, note that the number of arbitrarily-chosen zero-crossing points sufficient to uniquely specify a signal is somewhat greater than the number of points in the signal. Specifically, if a sequence $x[n_1, n_2]$ has support over the largest possible nonsymmetric half-plane contained in $R(N)$, then it has $2N^2 + 2N + 1$ distinct points. According to Theorem 6, $x[n_1, n_2]$ is uniquely specified with m zero crossing points if $m > 16N^2$. If $x[n_1, n_2]$ is real, then due to symmetries in the Fourier transform, $x[n_1, n_2]$ is uniquely specified with $m > 8N^2$ points if these points are chosen over an appropriate range of frequencies. This means a sequence consisting of p real points is uniquely specified with approximately $4p$ zero crossing frequencies. If a sequence has support over only one quadrant, then it has approximately N^2 points, and approximately $8N^2$ zero crossing points are sufficient for unique specification.

Although Theorem 6 states that a particular number of zero crossing points is sufficient for unique specification, it does not state that this number of points is *necessary* for unique specification. In particular, as we show next, it is possible to specify a signal consisting of p samples with $p-1$ zero crossing points if the zero crossing points are not chosen arbitrarily but are specifically chosen so that they uniquely specify the signal. To establish this result, note that we can write a set of linear equations of the form:

$$\sum_{n_1} \sum_{n_2} x[n_1, n_2] \cos(\omega_1 n_1 + \omega_2 n_2) = 0 \tag{15}$$

where each equation uses a different pair of frequencies (ω_1, ω_2) on a zero crossing contour, i.e., where the equality is known to hold. If we assume that $x[n_1, n_2]$ satisfies the constraints of Theorem 7, then $x[N, N] \neq 0$, so we can substitute $x[N, N] = 1$ and obtain a non-zero solution. Thus, if $x[n_1, n_2]$ consists of p points, these equations contain $p-1$ unknowns. Although we have not shown that $p-1$ equations of this form are guaranteed to have a unique solution, we know from Theorem 7 that if a sufficient number of equations is used (say, m equations) then these equations are guaranteed to have a unique solution. If we have m equations in $p-1$ unknowns and $m > p-1$, then some of the equations must be dependent and can be eliminated. Thus, it is possible (theoretically) to find $p-1$ independent equations from the set of m equations, and thus the corresponding $p-1$ zero crossing frequencies are sufficient to uniquely specify $x[n_1, n_2]$. This result, however, does not suggest a practical algorithm for choosing the $p-1$ zero crossing points so that these points uniquely specify the signal.

3. Application to Reconstruction from Zero Crossings

As mentioned in the introduction, one promising application for the results presented in section 2 involves reconstructing a two-dimensional signal from its zero crossings, or more generally, from its threshold crossings. Considerable research has been devoted to the problem of reconstructing one-dimensional signals from zero crossings (see Requicha [13] for a survey.) Most of this work has concentrated on identifying types of one-dimensional signals which have a sufficient number of zero crossings for unique specification. The most recent work in this area is by Logan [14], who states a set of conditions under which one-dimensional bandpass signals are uniquely specified by zero crossings. However, no corresponding work has been reported on the problem of reconstructing two-dimensional signals from zero crossings. In addition, the constraints imposed in [14] are distinctly different from the constraints to be imposed in this section.

By interchanging the roles of the signal and transform domains in Theorem 2, that is, by exploiting the duality of the Fourier transform in a straight-forward way, we can develop a new result on the unique specification of two-dimensional signals with zero crossings. Specifically, if a continuous-time signal corresponds to the Fourier transform of a finite-length discrete-time sequence, and if this finite-length sequence satisfies the conditions of any of the theorems developed earlier, then the signal is uniquely specified to within a scale factor by its zero crossings. To illustrate this in more detail, consider a real, band-limited, continuous-time, periodic signal $f(x,y)$ with periods T_1 and T_2 in the x- and y- directions, respectively. With the substitution of variables $\omega_1 = \frac{2\pi x}{T_1}$, $\omega_2 = \frac{2\pi y}{T_2}$, f will be periodic with period 2π in each of the variables ω_1, ω_2 and will correspond to the Fourier transform of some complex conjugate-symmetric finite-length discrete-time sequence $F(n_1, n_2)$:

$$f\left(\frac{\omega_1 T_1}{2\pi}, \frac{\omega_2 T_2}{2\pi}\right) = \sum_{n_1} \sum_{n_2} F(n_1, n_2)\, e^{-j\omega_1 n_1} e^{-j\omega_2 n_2} \tag{16}$$

or in terms of x and y:

$$f(x,y) = \sum_{n_1} \sum_{n_2} F(n_1, n_2)\, e^{-j\frac{2\pi x n_1}{T_1}} e^{-j\frac{2\pi y n_2}{T_2}} \tag{17}$$

From this equation, $F(n_1, n_2)$ can be considered to be coefficients of a Fourier series expansion of $f(x,y)$. By the results developed earlier, $F(n_1, n_2)$ will be uniquely specified by sign $\{f(x,y)\}$ if the z-transform of $F(n_1, n_2)$ is nonfactorable. The z-transform of $F(n_1, n_2)$ can be obtained from the right-hand side of (16) with the substitution of (z_1^{-1}, z_2^{-1}) for $(e^{-j\omega_1}, e^{-j\omega_2})$, (or equivalently, from (17) with the substitution of (z_1^{-1}, z_2^{-1}) for $(e^{-j\frac{2\pi x}{T_1}}, e^{-j\frac{2\pi y}{T_2}})$) and thus represents a complex extension of the signal $f(x,y)$, where $f(x,y)$ is considered to be known over the unit surface in the (z_1, z_2) complex space. Therefore, the z-transform of $F(n_1, n_2)$ will be factorable if and only if $f(x,y)$, expressed as a polynomial in equation (17), is factorable.

Let us state our dual result as a theorem:

Theorem 8. Let $f(x,y)$ and $g(x,y)$ be real, two-dimensional, doubly-periodic, continuous, band-limited functions with sign $f(x,y) =$ sign $g(x,y)$, where $f(x,y)$ takes on both positive and negative values. If $f(x,y)$ and $g(x,y)$ are nonfactorable when expressed as polynomials in the form (17), then $f(x,y) = cg(x,y)$.

Proof: This theorem follows directly from Corollary 1 to Theorem 2. Since $f(x,y)$ and $g(x,y)$ have Fourier Series coefficients which are complex conjugate symmetric, have finite support, and have nonfactorable z-transforms, they satisfy the constraints of Corollary 1. Thus, the Fourier series coefficients must be equal, and $f(x,y) = cg(x,y)$.

Although we have constrained the signals in Theorem 8 to be real, this result is easily extended to include complex analytic signals (in this context, signals with no energy for negative frequencies). In this case, the signals would be uniquely specified by the sign of their real part. However, as one might expect, it is not possible to uniquely specify arbitrary complex signals with independent real and imaginary parts with only the sign of the real part.

Since many two-dimensional signals encountered in practice are not periodic but have finite support, we will next modify Theorem 8 so that it applies to these signals. We will consider the case where $f(x,y)$ is a finite segment of a periodic signal satisfying the constraints of Theorem 8. For example, if $f(x,y)$ represents one period of a band-limited periodic function:

$$\tilde{f}(x,y) = \sum_{n_1}\sum_{n_2} f(x + n_1 T_1, y + n_2 T_2) \tag{18}$$

then it is possible to recover $f(x,y)$ from its zero crossings provided that $\tilde{f}(x,y)$ satisfies the constraints of Theorem 8, even though $f(x,y)$ itself is not band-limited. In general, it is not necessary for the duration of $f(x,y)$ to be equal to one period of the corresponding periodic function. Thus, $f(x,y)$ can represent a finite segment of a variety of different periodic functions. In order for $f(x,y)$ to be uniquely specified by its zero crossings, we only need one periodic function containing $f(x,y)$ to be band-limited. Specifically, let us state (see Appendix 1 for proof):

> **Theorem 9.** Let $f(x,y)$ and $g(x,y)$ be two-dimensional continuous functions defined over the same known region R of finite extent with sign $f(x,y) \equiv$ sign $g(x,y)$, where $f(x,y)$ takes on both positive and negative values. If $f(x,y) = f_p(x,y)$ in R and $g(x,y) = g_p(x,y)$ in R for any periodic, continuous, band-limited functions $f_p(x,y)$ and $g_p(x,y)$ which are nonfactorable when expressed as polynomials in the form (17), then $f(x,y) = cg(x,y)$ for some positive constant c.

Note, however, that Theorems 8 and 9 do not imply that an *arbitrary* (non-periodic) band-limited continuous function, or a finite segment of one, is uniquely specified by its zero crossings. These results would be equivalent to developing a result similar to Theorem 2 for continuous-time signals.

Although Theorems 8 and 9 were developed from Corollary 1 to Theorem 2, the other extensions developed for Theorem 2 can also be useful here. Of particular interest to us is the fact that the definition of sign information can be extended to include crossings of an arbitrary threshold instead of just zero crossings. This result is important since in a number of applications, particularly image processing, signals are constrained to be nonnegative and thus have no zero crossings. Nevertheless, it might be desirable to recover such a signal from knowledge of the points where it crosses some particular level. The same procedure used to establish Theorem 4 can also be used to show that Theorems 8 and 9 can be extended to include reconstruction from $\mathrm{sign}(f(x,y)-\beta)$ provided that β is chosen so that the signal actually crosses the threshold β. In this case, if we have $\mathrm{sign}(f(x,y)-\beta) = \mathrm{sign}(g(x,y)-\beta)$, then $f(x,y) - \beta = c(g(x,y) - \beta)$ for some c.

4. Reconstruction Algorithms

Having established that particular classes of signals are uniquely specified by some partial information, it is of interest to develop algorithms for recovering the original signal from this information. One common approach to developing algorithms for reconstruction from various forms of partial information is to develop an iterative algorithm which alternately impose constraints in the space and frequency domains. Another approach is to express the solution as a set of simultaneous linear equations. In this section, we will discuss each of these methods and present experimental results obtained with each method.

4.1. Iterative Algorithm

The class of iterative algorithms mentioned above can be applied to the problem of reconstruction from Fourier sign information by imposing the correct sign of the real part of the Fourier transform in the frequency domain and imposing the known region of support in the space domain. It can also be applied to the problem of reconstruction from sign $\{f(x,y)\}$ by imposing the correct sign (perhaps with respect to some threshold) in the space domain and the band-limited constraint in the frequency domain. For the sake of clarity, in this section and the one that follows we will primarily refer to the reconstruction from $S_x(\omega_1, \omega_2)$; however, it should be realized that the same algorithms can be applied to reconstruction from sign $\{f(x,y)\}$ by interchanging the roles of the two domains.

Since knowledge of the exact points of discontinuity is necessary for the signal to be uniquely specified (in either problem), the convergence of an iterative algorithm to the correct solution necessarily depends upon the use of the exact zero crossing points. Thus, an algorithm for reconstruction from $S_x(\omega_1, \omega_2)$ which uses a DFT and thus makes use of only those values of $S_x(\omega_1, \omega_2)$ corresponding to DFT points cannot be guaranteed to converge to the correct solution. However, it can be shown that the continuous-frequency version of the algorithm (that is, a similar algorithm imposing the correct value of $S_x(\omega_1, \omega_2)$ for all frequencies and thus using an actual Fourier transform and not a DFT) will converge to the correct sequence. It is also possible to show that the sampled-frequency version of the algorithm will converge to a sequence which satisfies both the space and frequency domain constraints (provided such a solution exists), although the solution is not unique. These results can be developed within the theory of projections onto convex sets, as was used in [15] to establish the convergence of a number of different signal reconstruction algorithms. Specifically, the results developed in [15] apply directly to this problem provided the constraints in each domain are imposed in such a way as to be projections onto convex sets; the details will be presented in Appendix 2.

Once the theoretical properties of a reconstruction algorithm have been determined, it is important to empirically determine the effectiveness of the algorithm in recovering an actual signal. In particular, it is worthwhile to determine if a practical size DFT limits the set of solutions to a sufficiently small set and if convergence (or a good approximation) can be obtained with a practical number of iterations. It is also worthwhile to investigate the effect of using different initial estimates in the iteration. Experimentally, we have found that if the DFT size used is at least 4 times the signal size and an initial estimate is chosen which in some sense resembles the original signal, then reasonable results can be obtained with a small number of iterations. Specifically, we have had good success when using an initial estimate formed from the correct one bit of Fourier transform phase and a Fourier transform magnitude which is the average of a number of unrelated images. Although a large number of iterations are required for the algorithm to converge to a sequence satisfying the time and frequency domain constraints, the improvement of image quality after the first 20 iterations or so is somewhat negligible even if the image at this stage does not satisfy the frequency domain constraints at every point. An example is included in Figure 1, where we show the original image (a) and the image reconstructed from one bit of Fourier transform phase (b). In this example, the original image is 64×64 points, 256×256 DFTs were used, and the results shown were obtained with 25 iterations.

4.2. Linear Equation Method

As mentioned earlier, it is also possible to express the solution to the problem of reconstruction from one bit of phase as a set of linear equations. Thus, another possible reconstruction algorithm would involve solving the set of equations:

$$\sum_{n_1} \sum_{n_2} x[n_1, n_2] \cos(\omega_1 n_1 + \omega_2 n_2) = 0 \tag{19}$$

where each equation uses a different pair of frequencies (ω_1, ω_2) for which the equality is known to hold.

As mentioned earlier, if we assume that $x[n_1, n_2]$ satisfies the constraints of Theorem 7, then $x[N, N] \neq 0$ and we can substitute $x[N, N] = 1$ and obtain a non-zero solution. If the number of equations is chosen in accordance with this corollary, then these equations are guaranteed to have a unique solution. However, this number is significantly higher than the number of equations usually required in practice. Our experience is that for a sequence with p nonzero points, $p-1$ equations in $p-1$ unknowns will generally have a unique solution, although the results are particularly sensitive to numerical errors in the values of the zero crossing frequencies. If the values of ω_1 and ω_2 are obtained to four- or five-digit accuracy and if the number of equations used is only slightly greater than the number of unknowns and a least-squares solution is obtained, results indistinguishable from the original signal can be achieved. We have used this procedure to reconstruct a number of different images of varying sizes which satisfy the constraints of the results developed earlier and have always successfully recovered the original image provided enough equations were used.

An example of results obtained with this method is shown in Figure 2, which shows the original image (a) and the image reconstructed by solving the above equations (b). In this example, the original image is 15×8 (120 points) and 122 equations in 119 unknowns were used. The real parts of the Fourier transforms of these images are shown in Figures 3 (a) and (b). If the image of Figure 3 (a) is considered to be the original image, then Figure 3 (b) represents an image reconstructed from its zero crossings. For comparison, an image showing only the zero crossings of the image in Figure 3 (a) (i.e., Figure 3 (a) quantized to one bit) has been included as Figure 3 (c).

5. Conclusions

In this paper, we have developed conditions under which two-dimensional signals can be reconstructed from Fourier transform sign information. We have also shown that these conditions apply to a broad class of signals by showing that if the coefficients of the signal are random, the probability of a sequence satisfying these conditions very rapidly approaches unity as the number of points in the signal approaches infinity. From the basic result, we have developed a number of extensions to different types of signals, different constraints, and different types of sign information. We have also applied these results to the problem of reconstructing a two-dimensional signal from the zero crossings of the signal itself. In addition, we have discussed some possible algorithms for reconstruction from one bit of Fourier transform phase or from the signal zero crossings. Examples of images reconstructed with these algorithms have also been included.

Appendix 1. Proofs

This appendix contains proofs of some of results presented earlier.

Proof of Corollary 1 to Theorem 2. This proof follows the proof of Theorem 2, with a few modifications. First of all, since the sequences are permitted to be complex, the real part of the Fourier transform corresponds to the conjugate symmetric component of a sequence, i.e., (keeping the notation $x_e[n_1, n_2]$)

$$x_e[n_1, n_2] = \frac{x[n_1, n_2] + x^*[-n_1, -n_2]}{2} \tag{A1}$$

With this substitution, the proof of Theorem 2 is then directly applicable to the case of causal complex signals, with the exception that Im $\{x[0,0]\}$ is not recoverable from $x_e[n_1, n_2]$. For conjugate symmetric signals, we note that $x_e[n_1, n_2] = x[n_1, n_2]$, and thus Theorem 2 and its proof are directly applicable with the substitution of $x[n_1, n_2]$ for $x_e[n_1, n_2]$.

Proof of Theorem 3. Following the proof of Theorem 2, we know that since $S_x(\omega_1, \omega_2) \equiv S_y(\omega_1, \omega_2)$, $X_e(z_1, z_2)$ and $Y_e(z_1, z_2)$ must contain a common factor. Since $X_e(z_1, z_2)$ is nonfactorable, then if $x[n_1, n_2] \neq cy[n_1, n_2]$, $Y_e(z_1, z_2) = X_e(z_1, z_2) F(z_1, z_2)$ for some real symmetric factor $F(z_1, z_2)$. Equivalently, $y_e[n_1, n_2] = x_e[n_1, n_2] * f[n_1, n_2]$, where $f[n_1, n_2]$ is even. Thus, $y_e[n_1, n_2] \neq 0$ for some values $n_1 > N$ or $n_2 > N$, violating the constraints of the theorem. Thus, $x[n_1, n_2] = cy[n_1, n_2]$.

Note that since the convolution involves $x_e[n_1, n_2]$ and not $x[n_1, n_2]$, knowledge that $x[0, 0] \neq 0$ is not sufficient to guarantee uniqueness, but knowledge that $x[N, N] \neq 0$ is sufficient. The information required is the exact size of the region of support of $x_e[n_1, n_2]$, which is the same regardless of the value of $x[0,0]$.

Proof of Theorem 4. Note that with the definitions given in the theorem statement,

$$\hat{X}(\omega_1, \omega_2) = \mathrm{Re}\,\{X(\omega_1, \omega_2)e^{j\alpha}\} - \beta \tag{A2}$$

$$\hat{Y}(\omega_1, \omega_2) = \mathrm{Re}\,\{Y(\omega_1, \omega_2)e^{j\alpha}\} - \beta$$

Then, we note that $S_x^{\alpha\beta}(\omega_1, \omega_2) = S_{\hat{x}}(\omega_1, \omega_2)$, and since $S_x^{\alpha\beta}(\omega_1, \omega_2) = S_y^{\alpha\beta}(\omega_1, \omega_2)$, $S_{\hat{x}}(\omega_1, \omega_2) = S_{\hat{y}}(\omega_1, \omega_2)$. Since $\hat{x}[n_1, n_2]$ and $\hat{y}[n_1, n_2]$ are complex conjugate symmetric sequences and satisfy the constraints of Corollary 1 to Theorem 2, $\hat{x}[n_1, n_2] = c\hat{y}[n_1, n_2]$ for some positive constant c. Then, with some algebra, $x[n_1, n_2] = cy[n_1, n_2]$ for $(n_1, n_2) \neq (0,0)$, and $x[0,0]\cos\alpha - \beta = c(y[0,0]\cos\alpha - \beta)$ for some positive constant c.

Proof of Theorem 5. Recall from the proof of Theorem 2 that if $X_e(z_1, z_2)$ and $Y_e(z_1, z_2)$ have common zero contours on the unit surface then they must have a common factor. We will assume that $x[n_1, n_2] \neq cy[n_1, n_2]$ and attempt to reach a contradiction. For convenience, let us assume that there is some irreducible factor of $X_e(z_1, z_2)$ which is not a factor of $Y_e(z_1, z_2)$. First of all, note that if this factor, denoted $F(z_1, z_2)$, is complex, then $F^*(z_1, z_2)$ will also be a factor of $X_e(z_1, z_2)$ and thus $X_e(z_1, z_2)$ will contain a real symmetric factor $F(z_1, z_2)F^*(z_1, z_2)$ which is nonnegative everywhere, violating the constraints of the theorem. If this factor $F(z_1, z_2)$ is real but nonsymmetric, then $F(z_1^{-1}, z_2^{-1})$ will also be a factor since $X_e(z_1, z_2)$ is symmetric. In this case, $X_e(z_1, z_2)$ will contain a real symmetric factor $F(z_1, z_2)F(z_1^{-1}, z_2^{-1})$, which on the unit surface is equal to $F(\omega_1, \omega_2)F^*(\omega_1, \omega_2)$, a nonnegative function, again violating the constraints of the theorem. Thus, the factor

$F(z_1, z_2)$ must be real and symmetric, and since according to the theorem hypothesis, it has both positive and negative values and has multiplicity one, then we must have $S_x(\omega_1, \omega_2) \neq S_y(\omega_1, \omega_2)$ for some values of (ω_1, ω_2), and we have reached a contradiction. Thus, there cannot be any factor of $X_e(z_1, z_2)$ which is not a factor of $Y_e(z_1, z_2)$ and thus, $x[n_1, n_2] = cy[n_1, n_2]$.

Proof of Theorem 9: First we note that if $f_p(x, y)$ and $g_p(x, y)$ are not simply periodic replications of $f(x, y)$ and $g(x, y)$, then $f_p(x, y)$ and $g_p(x, y)$ may contain zero crossings that cannot be obtained from the zero crossings of $f(x, y)$ and $g(x, y)$. Thus, Theorem 9 does not quite follow directly from Theorem 8. However, as we saw in the proof of Theorem 6, unique specification in terms of zero crossings does not require knowledge of all the zero crossing points; it requires only a finite set of them. In this case, as long as $f(x, y)$ contains both positive and negative values, it will contain at least one zero crossing contour with an infinite number of points. Thus $f_p(x, y) = g_p(x, y) = 0$ at an infinite number of points, and using arguments taken from the proofs of Theorems 2 and 0, $f_p(x, y) = cg_p(x, y)$. Since $f(x, y)$ and $g(x, y)$ are both known to be defined over the same region R and over R, $f(x, y) = f_p(x, y)$ and $g(x, y) = g_p(x, y)$, then $f(x, y) = cg(x, y)$.

Appendix 2. Convergence of Iterative Reconstruction Algorithm

In this appendix, we establish the convergence of our algorithm for reconstruction from one bit of Fourier transform phase by using the theory of projection onto convex sets, using the approach presented in [15] for establishing the convergence of a variety of signal reconstruction algorithms. Specifically, the result we shall be using is as follows*:

> **Theorem A1.** [15,16] Let H be a Hilbert space, G be a composition of projection operators onto closed convex sets, at least one of which is finite-dimensional, and G^* denote the intersection of these sets. If G^* is nonempty, then for all $x \in H$, the sequence $G^n x$ converges to a point in G^*.

The results developed in [15] include showing that a wide variety of constraints often imposed in signal reconstruction algorithms can be imposed in such a way that the transformations will satisfy the constraints of Theorem 1. In fact, the constraints used in our algorithm for reconstruction from $S_x(\omega_1, \omega_2)$ differ from some of those discussed in [15] in only trivial ways. The basic approach shall be repeated here although the mathematical details shall be omitted.

To show that Theorem 1 applies to our iterative algorithm, we must first carefully define the transformations applied in the time (or space) and frequency domains at each iteration so that they can be characterized as projection operators onto closed convex sets. First let us note that although the continuous-frequency variables (ω_1, ω_2) will be used throughout the discussion below, the properties of the transformations in the time and frequency domains apply equally well if the discrete frequencies (k_1, k_2) corresponding to a DFT are used. The only difference is that in the continuous-

*Theorem A1 is a weak form of the results in [15,16] but it is sufficient for our purposes here.

frequency case, the set G^* will contain exactly one sequence (provided the proper constraints are satisfied), whereas in the discrete-frequency case, the set G^* will contain an infinite number of sequences.

Let $h[n_1, n_2]$ denote the sequence we are trying to recover which is known to satisfy the constraints of Corollary 1 to Theorem 2 and is thus uniquely specified by $S_h(\omega_1, \omega_2)$. Let T^* denote the set of sequences which satisfy the time domain constraints:

$$x[n_1, n_2] = 0 \quad \text{for } n_1, n_2 \notin [0, N] \tag{A3}$$
$$x[N, N] = h[N, N]$$

and F^* denote the set of sequences which satisfy the frequency domain constraints:

$$\begin{aligned} \text{Re}\{X(\omega_1, \omega_2)\} \geq 0 \ \text{ if } S_h(\omega_1, \omega_2) = 1 \\ \text{Re}\{X(\omega_1, \omega_2)\} \leq 0 \ \text{ if } S_h(\omega_1, \omega_2) = -1 \end{aligned} \tag{A4}$$

Note that the set T^* is finite-dimensional even if the space H includes infinite-length signals. Also note that the definition of the set F^* is not precisely the same as stating $S_x(\omega_1, \omega_2) = S_h(\omega_1, \omega_2)$; the difference occurs if $\text{Re}\{H(\omega_1, \omega_2)\} < 0$ and $\text{Re}\{X(\omega_1, \omega_2)\} = 0$ at some point. It is necessary to use the definition of F^* given in (4) in order for F^* to be a closed set (a set which includes its limit points.)

Next we will define the operators T and F to be projections onto the sets T^* and F^*. For T and F to be projections, we need:

$$||Tx - x|| \leq ||y - x|| \quad \text{for all } y \in T^* \tag{A5}$$

and

$$||Fx - x|| \leq ||y - x|| \quad \text{for all } y \in F^* \tag{A6}$$

Thus the operator T we need to impose time domain constraints is:

$$T[x(n_1,n_2)] = \begin{cases} x(n_1,n_2) & 0 \le n_1,n_2 \le N \quad (n_1,n_2) \ne (N,N) \\ h(N,N) & (n_1,n_2) = (N,N) \\ 0 & n_1,n_2 > N \end{cases} \tag{A7}$$

or in words, simply substituting the known values of $h[n_1,n_2]$. The operator F we need to impose frequency domain constraints is:

$$F[x(n_1,n_2)] \Leftrightarrow F_f[X(\omega_1,\omega_2)], \tag{A8}$$

where

$$F_f[X(\omega_1,\omega_2)] = \begin{cases} X(\omega_1,\omega_2) & \text{if } S_x(\omega_1,\omega_2) = S_h(\omega_1,\omega_2) \\ \mathrm{Im}\,\{X(\omega_1,\omega_2)\} & \text{otherwise} \end{cases}$$

or in words, keeping the imaginary part constant and setting the real part to zero if its sign is incorrect.

Next, we express our iterative algorithm in the form

$$x_{k+1} = Gx_k \tag{A9}$$

where $G=TF$ is a composition of projection operators. Then, by Theorem 1, the sequence in equation (9) will converge to a point in G^*, that is, a sequence which satisfies the time and frequency domain constraints. Thus, if $h[n_1,n_2]$ satisfies the constraints of Theorem 3, and the iteration imposes the correct $S_h(\omega_1,\omega_2)$ for all frequencies (i.e., actual Fourier transforms are used), then G^* contains exactly one sequence, and the iteration must converge to that sequence. If $S_h(\omega_1,\omega_2)$ is sampled (i.e., a DFT is used), then the iteration must converge to a sequence in G^*, i.e., a sequence which satisfies the time and frequency domain constraints, although this solution is not unique and the solution actually obtained depends on the initial estimate.

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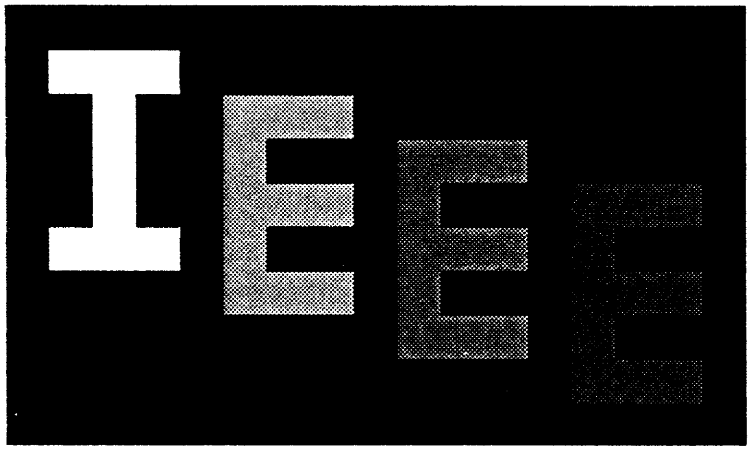
Figure 1. Reconstruction with Iterative Method

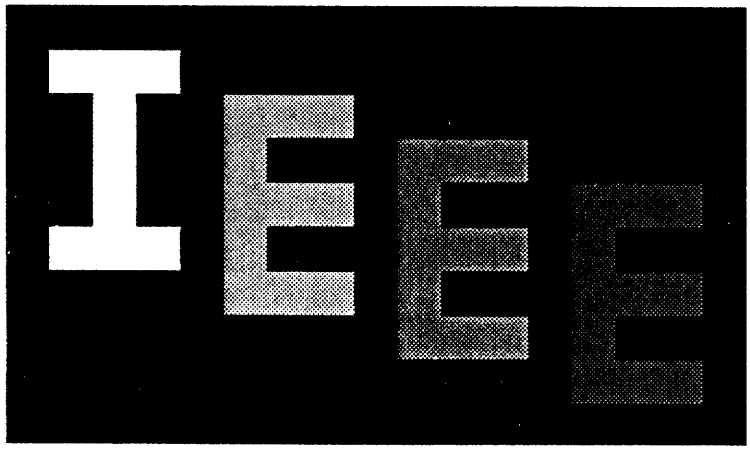
(a) original image

(b) recovered image

Figure 2. Reconstruction with Linear Equation Method

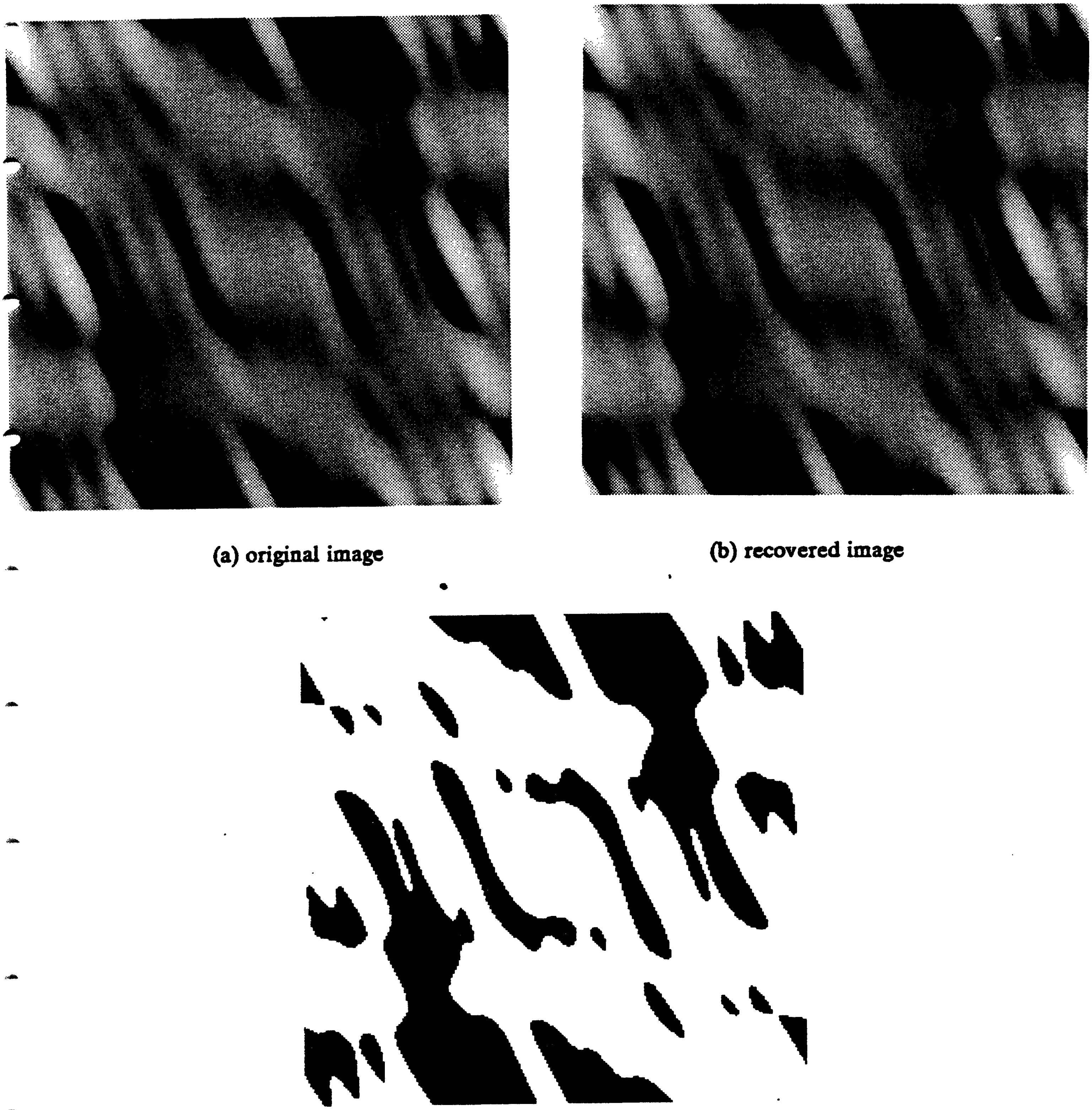


(a) original image



(b) recovered image

Figure 3. Reconstruction from Zero Crossings



(a) original image

(b) recovered image

(c) image showing zero crossings of (a)

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