#5

SIGNAL-TO-NOISE RATIOS IN BANDPASS LIMITERS

W. B. DAVENPORT, JR.

LOAN COPY

TECHNICAL REPORT NO. 234

MAY 29, 1952

RESEARCH LABORATORY OF ELECTRONICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MASSACHUSETTS

The research reported in this document was made possible through support extended the Massachusetts Institute of Technology, Research Laboratory of Electronics, jointly by the Army Signal Corps, the Navy Department (Office of Naval Research) and the Air Force (Air Materiel Command), under Signal Corps Contract No. DA36-039 sc-100, Project No. 8-102B-0; Department of the Army Project No. 3-99-10-022; and by the Robert Blair Fellowship of the London (England) County Council.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
RESEARCH LABORATORY OF ELECTRONICS

Technical Report No. 234 May 29, 1952

SIGNAL-TO-NOISE RATIOS IN BANDPASS LIMITERS

W. B. Davenport, Jr.

Abstract

A general analysis is made of the relations between output signal and noise powers and input signal and noise powers for bandpass limiters having odd symmetry to their limiting characteristics. Specific results are given for the case where the limiter has an n-th root characteristic, and they include the ideal symmetrical limiter (or clipper) as a limiting case. This analysis shows that, for the bandpass limiter, the output signal-to-noise power ratio is essentially directly proportional to the input signal-to-noise power ratio for all values of the latter. This result is due to the bandpass characteristics rather than to the symmetrical limiting action.

SIGNAL-TO-NOISE RATIOS IN BANDPASS LIMITERS

I. Introduction

Saturation, or limiting, may often take place in the bandpass amplifier stages of a radio receiver. Sometimes this limiting is inadvertent (as in the case of a much larger than usual input signal which overdrives an amplifier), while sometimes the bandpass stages are deliberately designed to limit (as in the case of most FM receivers). In any case, whether the limiting is inadvertent or deliberate, it is of interest to know quantitatively the action of the bandpass limiter.

The purpose of this report is to determine the relations between the output signal and noise and the input signal and noise for the case of a bandpass limiter for all values of the input signal-to-noise ratio. Previous studies have either considered the effects of noise alone as a limiter input (1); or, when a signal-plus-noise input was considered, either the action of the limiter was studied only in combination with that of a discriminator (2, 3, 4) or results were obtained only for large values of the input signal-to-noise ratio (5).

The system to be considered here consists of a limiter followed by a bandpass filter, as shown in Fig. 1. The input to this system is assumed to consist of an amplitude-modulated sine wave plus a noise wave

$$x(t) \equiv P(t) \cos pt + N(t). \tag{1}$$

The input noise $N(t)$ is assumed to be gaussian in nature and to have a narrow-band spectrum centered in the vicinity of the signal carrier frequency $p/2\pi$. The spectrum of the limiter output $y(t)$ will consist of signal and noise terms centered on the angular frequencies $\pm mp$, where $m = 0, 1, 2, \ldots$. The bandpass filter is assumed to have an ideal rectangular passband transfer characteristic which is centered on the fundamental angular frequency p. The filter passband is assumed to be wide enough to pass all of the limiter output spectrum centered about $\pm p$, but narrow enough to reject those parts of the spectrum centered on $\pm mp$ (where $m \neq 1$).

In the analysis to follow, we will obtain expressions for the autocorrelation function at the input to the bandpass filter, as well as expressions for the signal power and noise power at the output of the bandpass filter. From these expressions we will be able to determine the relation between the output signal-to-noise power ratio and the input signal-to-noise power ratio for all values of that input ratio.

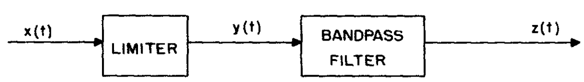

Fig. 1

Block diagram
of the bandpass limiter.

II. General Analysis

Let us first consider the problem of determining the autocorrelation function of the limiter output y(t). This function is defined as the statistical average of y(t) y(t+τ), i.e.

$$R_y(\tau) \equiv \langle y(t)\, y(t+\tau) \rangle_{av} \tag{2}$$

and has been shown by Wiener (6) to be the Fourier transform of the spectral density.

Rice (7) and Middleton (8) have shown that if the output of a nonlinear device may be expressed as a unique function of its input

$$y(t) = g\left[x(t)\right] \tag{3}$$

and if the input to that device is an amplitude-modulated sine wave plus gaussian noise, as in Eq. 1, then the autocorrelation function of the output of the nonlinear device may be expressed as

$$R_y(\tau) = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{\epsilon_m}{k!} \langle h_{mk}(t)\, h_{mk}(t+\tau) \rangle_{av}\, R_N^k(\tau) \cos mp\tau \tag{4}$$

where the ϵ_m are the Neumann numbers

$$\epsilon_0 = 1 \qquad \epsilon_m = 2 \quad \text{for } m \geq 1 \tag{5}$$

where $R_N(\tau)$ is the auto correlation function of the input noise

$$R_N(\tau) \equiv \langle N(t)\, N(t+\tau) \rangle_{av} \tag{6}$$

and where the function $h_{mk}(t)$ is defined by

$$h_{mk}(t) \equiv \frac{j^{m+k}}{2\pi} \int_{-\infty}^{\infty} f(ju)\, e^{-\left[\sigma^2(N) u^2\right]/2}\, u^k\, J_m\left[uP(t)\right] du. \tag{7}$$

In this defining equation for $h_{mk}(t)$, $\sigma^2(N)$ is the variance (mean square about the mean) of the input noise, J_m is the m-th order Bessel function of the first kind, and f(ju) is the Fourier transform of the transfer characteristic of the nonlinear device

$$f(ju) \equiv \int_{-\infty}^{\infty} g(x)\, e^{-jux}\, dx. \tag{8}$$

In our present study we wish to specify that the limiter transfer characteristic g(x) be a nondecreasing odd function of its argument

$$g(x) \equiv \begin{cases} g_+(x) & \text{for } x > 0 \\ -g_+(-x) & \text{for } x < 0. \end{cases} \tag{9}$$

If we try to use this form of $g(x)$ in Eqs. 7 and 8 directly, certain difficulties arise. Because of the required form of $g(x)$, the Fourier transform in Eq. 8 does not exist. This difficulty may be circumvented, however, by defining the bilateral Laplace transform $f(w)$ (refs. 9, 10) as

$$f(w) \equiv f_+(w) + f_-(w) \tag{10}$$

where w is the complex variable

$$w = v + ju \tag{11}$$

and where $f_+(w)$ and $f_-(w)$ are the unilateral Laplace transforms

$$f_+(w) \equiv \int_0^\infty g(x)e^{-wx}\,dx = \int_0^\infty g_+(x)e^{-wx}\,dx \tag{12}$$

and

$$f_-(w) \equiv \int_{-\infty}^0 g(x)e^{-wx}\,dx. \tag{13}$$

Because of the odd character of $g(x)$, it follows that

$$f_-(w) = -f_+(-w). \tag{14}$$

The unilateral Laplace transforms $f_+(w)$ and $f_-(w)$ have different regions of convergence in the w-plane, and, as we shall see later, each may have a singularity at the origin of that plane. Because of these different regions of convergence for $f_+(w)$ and $f_-(w)$, we must in general employ two inversion integrals with separate integration contours in order to return to $g(x)$ from $f(w)$. For this reason, the single integral expression 7 for the function $h_{mk}(t)$ must be replaced by the sum of two contour integrals in this study

$$h_{mk}(t) = \frac{1}{2\pi j}\int_{C+} f_+(w)e^{\left[\sigma^2(N)w^2\right]/2}\, w^k J_m\left[wP(t)\right]\,dw + \frac{1}{2\pi j}\int_{C-} f_-(w)e^{\left[\sigma^2(N)w^2\right]/2}\, w^k J_m\left[wP(t)\right]\,dw \tag{15}$$

where C+ is the contour along the imaginary axis of the w-plane with a possible indentation to the right of the origin, and C- is the contour along the imaginary axis with a possible indentation to the left of the origin.

If we now use the relation 14 between $f_+(w)$ and $f_-(w)$, we obtain

$$h_{mk}(t) = \left[1 - (-1)^{m+k}\right] \frac{1}{2\pi j} \int_{C+} f_+(w) e^{\left[\sigma^2(N)w^2\right]/2} w^k J_m \left[wP(t)\right] dw \quad (16)$$

and therefore

$$h_{mk}(t) = \begin{cases} 0 & \text{for } m + k \text{ even} \\ 2 \cdot \frac{1}{2\pi j} \int_{C+} f_+(w) e^{\left[\sigma^2(N)w^2\right]} w^k J_m \left[wP(t)\right] dw & \text{for } m + k \text{ odd.} \end{cases} \quad (17)$$

Thus we see that, because of our assumed odd symmetry for the limiter transfer characteristic, the functions $h_{mk}(t)$ vanish whenever the sum of the indices $m + k$ is even. Using this extended definition for $h_{mk}(t)$, we may now use Eq. 4 to determine the autocorrelation function of the limiter output.

From this point on, we will for convenience assume that the input signal is unmodulated. That is, we will assume that

$$P(t) = P. \quad (18)$$

In this case, the functions $h_{mk}(t)$ are not functions of t. Therefore

$$\left\langle h_{mk}(t)\, h_{mk}(t+\tau) \right\rangle_{av} = h_{mk}^2. \quad (19)$$

The expression Eq. 4 for the limiter output autocorrelation function then simplifies to

$$R_y(\tau) = \sum_{\substack{m=0 \\ (m+k \text{ odd})}}^{\infty} \sum_{k=0}^{\infty} \frac{\epsilon_m h_{mk}^2}{k!} R_N^k(\tau) \cos mp\tau \quad (20)$$

where the coefficients h_{mk} are determined from Eq. 17. It is convenient to expand Eq. 20 as follows

$$R_y(\tau) = 2 \sum_{\substack{m=1 \\ (m \text{ odd})}}^{\infty} h_{m0}^2 \cos p\tau + \sum_{\substack{k=1 \\ (k \text{ odd})}}^{\infty} \frac{h_{0k}^2}{k!} R_N^k(\tau)$$

$$+ 2 \sum_{\substack{m=1 \\ (m+k \text{ odd})}}^{\infty} \sum_{k=1}^{\infty} \frac{h_{mk}^2}{k!} R_N^k(\tau) \cos mp\tau. \quad (21)$$

The first set of terms (sum over m) is periodic and consists of the signal output terms representing the interaction of the input signal with itself (S × S terms). The remaining terms are the limiter output noise terms. The second set (sum over k) represents the interaction of the input noise with itself (N × N terms), while the last set (sum over m and k) represents the interaction of the input signal and noise ($S \times N$ terms).

The spectral density of the limiter output is simply the Fourier transform of the autocorrelation function $R_y(\tau)$, i.e.

$$G_y(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_y(\tau)\, e^{-j\omega\tau}\, d\tau \tag{22}$$

where $G_y(\omega)$ is the so-called "two-sided" spectral density containing both positive and negative frequencies. The Fourier transform of Eq. 21 may be written as

$$G_y(\omega) = \sum_{\substack{m=1 \\ (m\ \text{odd})}}^{\infty} h_{m0}^2 \cdot 2\pi \left[\delta(\omega - mp) + \delta(\omega + mp)\right] + \sum_{\substack{k=1 \\ (k\ \text{odd})}}^{\infty} \frac{h_{0k}^2}{k!}\, {}_kG_N(\omega)$$

$$+ \sum_{\substack{m=1 \\ (m+k\ \text{odd})}}^{\infty} \sum_{k=1}^{\infty} \frac{h_{mk}^2}{k!} \left[{}_kG_N(\omega - mp) + {}_kG_N(\omega + mp)\right] \tag{23}$$

where $\delta(\omega)$ is the unit impulse function (10) (Dirac delta function), and where ${}_kG_N(\omega)$ is the Fourier transform of $R_N^k(\omega)$. Successive applications of the convolution theorem (10) shows that ${}_kG_N(\omega)$ may be expressed as the (k-1) fold convolution of $G_N(\omega)$ with itself.

$${}_kG_N(\omega) = \frac{1}{(2\pi)^{k-1}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} G_N(\omega_{k-1})\, G_N(\omega_{k-2} - \omega_{k-1}) \cdots G_N(\omega - \omega_1)\, d\omega_{k-1} \cdots d\omega_1 \tag{24}$$

where $G_N(\omega)$ is the spectral density of the input noise.

Now that we have obtained expressions for the autocorrelation function and the spectral density of the limiter output, we are in a position to consider the output of the bandpass filter. Because of this bandpass filter, not all of the terms in Eqs. 21 or 22 will appear at the system output. Of the various terms, only those in the vicinity of $\pm p$ will appear at the filter output. The output signal autocorrelation function is therefore given by

$$R_{So}(\tau) = 2h_{10}^2 \cos mp\tau \tag{25}$$

and the signal output spectral density is

$$G_{So}(\omega) = h_{10}^2 \cdot 2\pi \left[\delta(\omega - p) + \delta(\omega + p)\right]. \tag{26}$$

The output signal power may be obtained by setting τ equal to zero in Eq. 25

$$S_o \equiv R_{So}(0) = 2h_{10}^2. \tag{27}$$

The noise terms in the filter output may also be obtained by picking out those terms in Eqs. 21 and 23 which contribute only to the spectral region in the vicinity of $\pm p$. In order to facilitate this determination, let us consider plots of ${}_kG_N(\omega)$ for several values of the index k as shown in Fig. 2. In this figure we have indicated the relative magnitudes of the various spectral contributions. We will consider the $(N \times N)$ terms and the $(S \times N)$ terms separately.

Examination of Eqs. 21 and 23 shows that only those $(N \times N)$ terms corresponding to odd values of k appear. From Fig. 2, we then see that all of these terms contribute to the noise output from the bandpass filter. We may determine the filter output noise power due to these terms by setting τ equal to zero in the appropriate terms in Eq. 21, and then multiplying each term in the resultant series by a factor representing the fraction of that term that appears at the filter output. From such a process, we find that the filter output noise power due to the interaction of the input noise with itself is given by

$$N_{o(N\times N)} = 1 \cdot \frac{h_{01}^2}{1!} R_N(0) + \frac{3}{4} \cdot \frac{h_{03}^2}{3!} R_N^3(0) + \frac{5}{8} \cdot \frac{h_{05}^2}{5!} R_N^5(0) + \frac{35}{64} \cdot \frac{h_{07}^2}{7!} R_N^7(0) + \dots \tag{28}$$

or upon cancelling

$$N_{o(N\times N)} = h_{01}^2 R_N(0) + \frac{1}{8} h_{03}^2 R_N^3(0) + \frac{1}{192} h_{05}^2 R_N^5(0) + \frac{1}{9216} h_{07}^2 R_N^7(0) + \dots . \tag{29}$$

Plots of the spectra of the various $(S \times N)$ noise terms at the limiter output may easily be constructed from the plots of Fig. 2 by translating each plot in that figure by an amount $\pm mp$ along the ω axis. A study of Fig. 2 shows that there is a maximum value of the shift mp which will allow a given term to contribute to the spectral region in the vicinity of p. From such a study, we see that for a given k, the only significant values of m are those in the range $(1, k+1)$, such that $(m+k)$ is odd. Therefore, the only terms in Eq. 21 that can contribute to the filter output noise are those given by

$$R'_{y(S\times N)}(\tau) = 2 \sum_{\substack{m=1 \\ (m+k \text{ odd})}}^{k+1} \sum_{k=1}^{\infty} \frac{h_{mk}^2}{k!} R_N^k(\tau) \cos mp\tau . \tag{30}$$

We may then find the filter output noise power due to the $(S \times N)$ terms by setting τ equal to zero in Eq. 30 and multiplying each term by an appropriate factor corresponding to the fraction of that term appearing at the filter output. The result is

$$N_{o(S\times N)} = \frac{1}{2}\cdot 2\,\frac{h_{21}^2}{1!}\,R_N(0) + \left(\frac{3}{4}\cdot 2\,\frac{h_{12}^2}{2!} + \frac{1}{4}\cdot 2\,\frac{h_{32}^2}{2!}\right) R_N^2(0)$$

$$+ \left(\frac{1}{2}\cdot 2\,\frac{h_{23}^2}{3!} + \frac{1}{8}\cdot 2\,\frac{h_{43}^2}{3!}\right) R_N^3(0)$$

$$+ \left(\frac{5}{8}\cdot 2\,\frac{h_{14}^2}{4!} + \frac{5}{16}\cdot 2\,\frac{h_{34}^2}{4!} + \frac{1}{16}\cdot 2\,\frac{h_{54}^2}{4!}\right) R_N^4(0) + \ldots \qquad (31)$$

or upon cancelling

$$N_{o(S\times N)} = h_{21}^2\,R_N(0) + \left(\frac{3}{4}\,h_{12}^2 + \frac{1}{4}\,h_{32}^2\right) R_N^2(0)$$

$$+ \left(\frac{1}{6}\,h_{23}^2 + \frac{1}{24}\,h_{43}^2\right) R_N^3(0)$$

$$+ \left(\frac{5}{96}\,h_{14}^2 + \frac{5}{192}\,h_{34}^2 + \frac{1}{192}\,h_{54}^2\right) R_N^4(0) + \ldots\,. \qquad (32)$$

The total filter output noise power is then given by the sum of the $(N \times N)$ terms and the $(S \times N)$ terms

$$N_o = N_{o(N\times N)} + N_{o(S\times N)}. \qquad (33)$$

The filter output signal-to-noise power ratio is defined as the ratio of the filter output signal power S_o to the filter output noise power N_o

$$\left(\frac{S}{N}\right)_o \equiv \frac{S_o}{N_o} \qquad (34)$$

where S_o is given by Eq. 27 and N_o is given by Eq. 32. In order to proceed further, we will have to assume a specific form for the limiter transfer characteristic $g(x)$. From this transfer characteristic, we may then determine the coefficients h_{mk} and substitute the result in the above expressions.

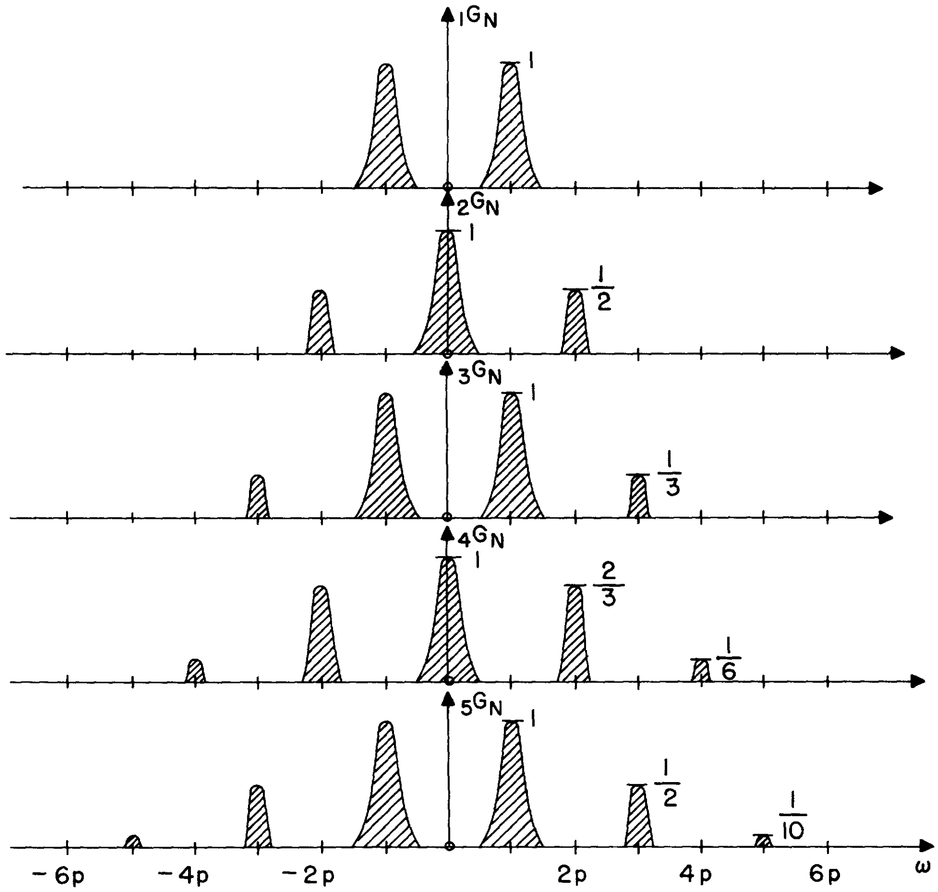

Fig. 2
Plots of the spectral densities ${}_kG_N(\omega)$, showing the relative values of the various spectral contributions.

III. The Rooting Limiter

A. General Results

A convenient form of limiter transfer characteristic is that where the limiter output is proportional to the n-th root of its input. We then have

$$g_+(x) \equiv \begin{cases} \alpha x^{1/n} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases} \tag{35}$$

where α is a scaling constant. Plots of g(x) for several values of n are given in Fig. 3. As may be seen from that figure, the case (n=1) corresponds to the linear amplifier, while the case (n=∞) corresponds to the ideal symmetrical limiter.

The Fourier transform $f_+(w)$ of this transfer characteristic is

$$f_+(w) = \alpha \int_0^\infty x^{1/n} e^{-wx}\, dx. \tag{36}$$

By a change of variable (t = wx) this transform is readily determined to be

$$f_+(w) = \alpha \frac{\Gamma\left(1 + \frac{1}{n}\right)}{w^{1+(1/n)}} \tag{37}$$

where Γ is the usual Gamma function. In general, this transform has a branch point at the origin of the w-plane, and the contour C+ must consequently have an indentation to the right of the origin.

The coefficient h_{mk} is given by substitution of Eq. 37 in Eq. 17 so that

$$h_{mk} = \begin{cases} 0 & \text{for } m + k \text{ even} \\ 2\alpha\Gamma\left(1 + \frac{1}{n}\right)\cdot\frac{1}{2\pi j}\int_{C+} e^{\left[\sigma^2(N)w^2\right]/2}\, w^{k-1-(1/n)}\, J_m(Pw)\, dw & \text{for } m + k \text{ odd.} \end{cases} \tag{38}$$

This contour integral is essentially the same as that required in the study of the ν-th law detector (if $1/n$ is replaced by ν). Paralleling the evaluation of the corresponding integral for the ν-th law detector (6, 7), one can readily determine that the evaluation of Eq. 38 is

$$h_{mk} = \begin{cases} 0 & \text{for } m + k \text{ even} \\ \dfrac{\alpha\Gamma\left(1 + \frac{1}{n}\right)\left[\dfrac{P^2}{2\sigma^2(N)}\right]^{m/2} {}_1F_1\left[\dfrac{m + k - \frac{1}{n}}{2}; m + 1; -\dfrac{P^2}{2\sigma^2(N)}\right]}{\left[\dfrac{\sigma^2(N)}{2}\right]^{\left[k-(1/n)\right]/2}\Gamma(m+1)\Gamma\left(1 - \dfrac{m + k - \frac{1}{n}}{2}\right)} & \text{for } m + k \text{ odd} \end{cases} \tag{39}$$

where ${}_1F_1$ is the confluent hypergeometric function.

Reference to section II, shows that the coefficient h_{mk} always occurs in combination with $R_N(\tau)$ in the form $h_{mk}^2\, R_N^k(\tau)$. Now

$$R_N^k(0) = \sigma^{2k}(N) \tag{40}$$

as the input noise has a bandpass spectrum. Then by using this result, and the fact that the input signal-to-noise power ratio is given by

$$\left(\frac{S}{N}\right)_i \equiv \frac{S_i}{N_i} = \frac{\frac{P^2}{2}}{\sigma^2(N)} \tag{41}$$

we may write

$$h_{mk}^2\, R_N^k(0) = \begin{cases} 0 & \text{for } m + k \text{ even} \\[2ex] \dfrac{\alpha^2 2^{k-(1/n)}\, \Gamma^2\left(1 + \frac{1}{n}\right) \sigma^{2/n}(N) \left(\frac{S}{N}\right)_i^m \, {}_1F_1^2\left[\frac{m + k - \frac{1}{n}}{2}; m + 1; -\left(\frac{S}{N}\right)_i\right]}{\Gamma^2(m+1)\, \Gamma^2\left(1 - \frac{m + k - \frac{1}{n}}{2}\right)} & \text{for } m + k \text{ odd.} \end{cases} \tag{42}$$

This result may now be substituted in the expressions 27, 29 and 32 in order to obtain the output signal and noise powers.

A partial check on our results may be obtained by considering the case (n=1). Now

$$\Gamma^2\left(1 - \frac{m + k - 1}{2}\right) = \infty \tag{43}$$

for (m+k) odd and equal to or greater than 3. Therefore, all of the terms in Eq. 42 vanish except those corresponding to (m=1, k=0) and (m=0, k=1) which become in this case

$$h_{10}^2 = \alpha^2 \frac{P^2}{4} \tag{44}$$

and

$$h_{01}^2\, R_N(0) = \alpha^2 \sigma^2(N). \tag{45}$$

Therefore, from Eq. 27 the output signal power is

$$S_o = \alpha^2 \frac{P^2}{2} = \alpha^2 S_i \tag{46}$$

and from Eqs. 29, 32, and 33 the output noise power is

$$N_o = \alpha^2 \sigma^2(N) = \alpha^2 N_i. \tag{47}$$

These results are comforting in view of the fact that our "limiter" here is a linear amplifier with a power gain α^2.

B. The Ideal Symmetrical Limiter

As may be seen from Fig. 3, the ideal symmetrical limiter may be represented as a limiting case (for n=∞) of the rooting limiter. Substitution of this limiting value of n in Eq. 42 gives

$$h_{mk}^2 R_N^k(0) = \begin{cases} 0 & \text{for } m+k \text{ even} \\ \dfrac{a^2 2^k \left(\frac{S}{N}\right)_i^m {}_1F_1^2\left[\frac{m+k}{2}; m+1; -\left(\frac{S}{N}\right)_i\right]}{\Gamma^2(m+1)\,\Gamma^2\left(1-\frac{m+k}{2}\right)} & \text{for } m+k \text{ odd.} \end{cases} \quad (48)$$

If we use this result in our previously obtained expression 27 for the output signal power, we obtain

$$S_o = \frac{2a^2}{\pi} \cdot \left(\frac{S}{N}\right)_i {}_1F_1^2\left[\frac{1}{2}; 2; -\left(\frac{S}{N}\right)_i\right]. \quad (49)$$

This output signal power has been plotted in Fig. 4 as a function of the input signal-to-noise power ratio $(S/N)_i$.

An expression for the output noise power may be obtained by using Eq. 48 in our previously determined expressions 29, 32, and 33. The result is

$$N_o = \frac{2a^2}{\pi} \left\{ \begin{aligned} & {}_1F_1^2\left[\tfrac{1}{2}; 1; -\left(\tfrac{S}{N}\right)_i\right] + \tfrac{1}{8}\, {}_1F_1^2\left[\tfrac{3}{2}; 1; -\left(\tfrac{S}{N}\right)_i\right] \\ & + \tfrac{3}{64}\, {}_1F_1^2\left[\tfrac{5}{2}; 1; -\left(\tfrac{S}{N}\right)_i\right] + \tfrac{75}{3072}\, {}_1F_1^2\left[\tfrac{7}{2}; 1; -\left(\tfrac{S}{N}\right)_i\right] + \dots \\ & + \tfrac{1}{16}\left(\tfrac{S}{N}\right)_i^2 {}_1F_1^2\left[\tfrac{3}{2}; 3; -\left(\tfrac{S}{N}\right)_i\right] + \tfrac{3}{8}\left(\tfrac{S}{N}\right)_i {}_1F_1^2\left[\tfrac{3}{2}; 2; -\left(\tfrac{S}{N}\right)_i\right] \\ & + \tfrac{1}{128}\left(\tfrac{S}{N}\right)_i^3 {}_1F_1^2\left[\tfrac{5}{2}; 4; -\left(\tfrac{S}{N}\right)_i\right] + \tfrac{3}{32}\left(\tfrac{S}{N}\right)_i^2 {}_1F_1^2\left[\tfrac{5}{2}; 3; -\left(\tfrac{S}{N}\right)_i\right] \\ & + \tfrac{25}{24,576}\left(\tfrac{S}{N}\right)_i^4 {}_1F_1^2\left[\tfrac{7}{2}; 5; -\left(\tfrac{S}{N}\right)_i\right] + \dots \end{aligned} \right\} \quad (50)$$

A plot of this output noise power is given in Fig. 4 as a function of $(S/N)_i$.

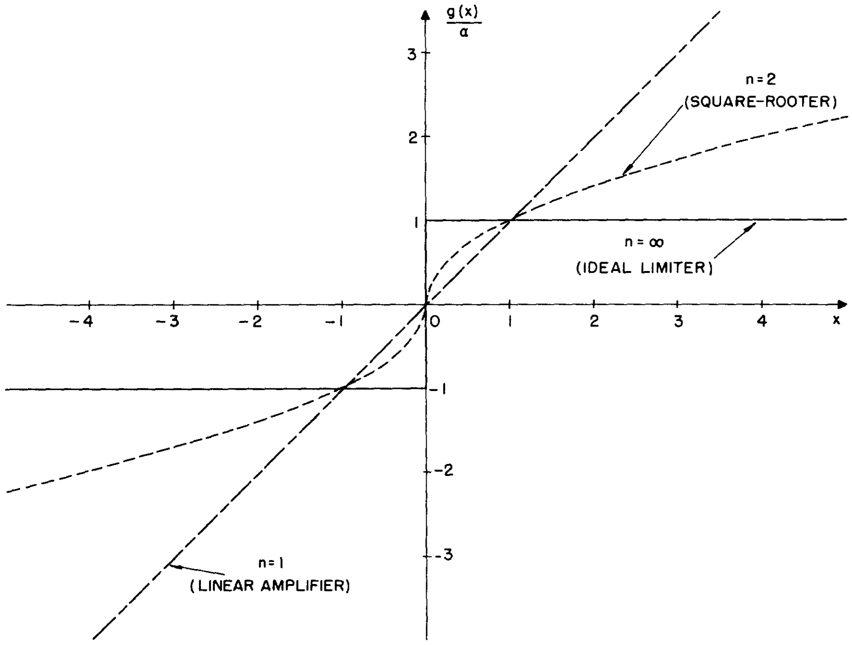

Fig. 3

Plots of the transfer characteristics of several cases of the rooting limiter.

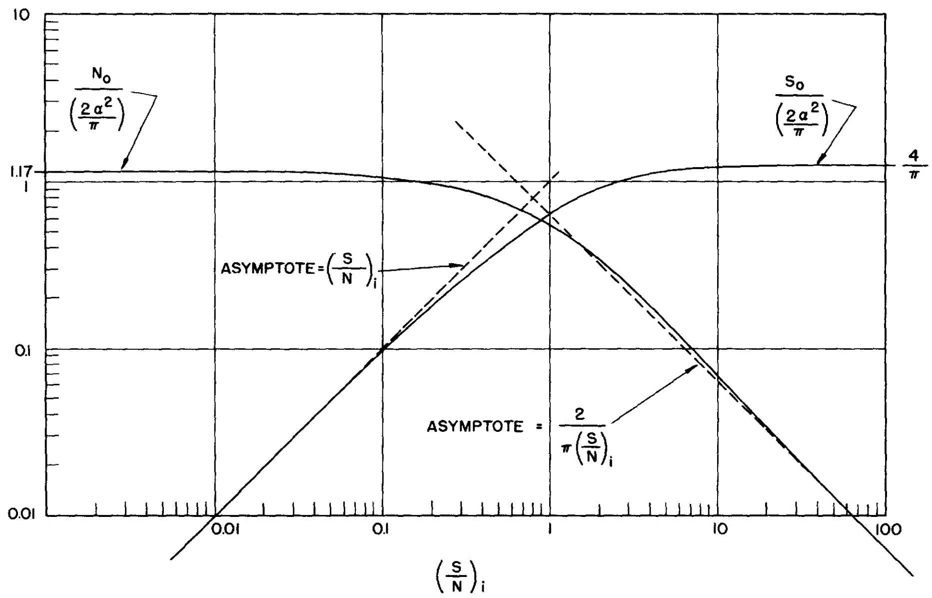

Fig. 4

Output signal power S_o and output noise power N_o plotted as functions of the input signal-to-noise power ratio $(S/N)_i$ for the case of the ideal, symmetrical, bandpass limiter.

Even though we have in Fig. 4 plots of S_o and N_o as functions of $(S/N)_i$, it is desirable also to obtain approximate expressions for these powers which are valid in the regions of very large (or very small) values of the input signal-to-noise power ratio.

The power series expansion of the confluent hypergeometric function ${}_1F_1[a;c;-z]$ about the origin is (11)

$$ {}_1F_1\left[a;c;-z\right] = 1 - \frac{a}{c}\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} - \ldots \tag{51} $$

and hence

$$ {}_1F_1\left[a;c;-z\right] \rightarrow 1 \quad \text{as } z \rightarrow 0. \tag{52} $$

Using the limiting result 52 in Eq. 49, we obtain for the output signal power

$$ S_o \approx \frac{2a^2}{\pi}\left(\frac{S}{N}\right)_i \qquad \text{for}\left(\frac{S}{N}\right)_i \rightarrow 0. \tag{53} $$

Referring to the numerical calculations used to obtain Fig. 4, we see that the error involved in Eq. 53 is less than ten percent when values of $(S/N)_i$ are less than about two-tenths.

Using the limiting result 52 in Eq. 50, we obtain for the output noise power

$$ N_o \approx \frac{2a^2}{\pi}\left[1 + \frac{1}{8} + \frac{3}{64} + \frac{75}{3072} + \ldots\right] $$

or

$$ N_o \approx \frac{2a^2}{\pi}(1.17) \qquad \text{for}\left(\frac{S}{N}\right)_i \rightarrow 0. \tag{54} $$

A comparison of Eq. 54 with Eq. 50 shows that in the region of small $(S/N)_i$, the dominant output noise is that due to direct feedthrough of the noise input to the limiter.

From Eqs. 53 and 54 we obtain

$$ \left(\frac{S}{N}\right)_o \approx 0.85\left(\frac{S}{N}\right)_i \qquad \text{for}\left(\frac{S}{N}\right)_i \rightarrow 0. \tag{55} $$

Thus we see that, for the ideal, symmetrical, bandpass limiter, the output signal-to-noise power ratio is directly proportional to the input signal-to-noise power ratio in the region of very small $(S/N)_i$. This result differs radically from the familiar square-law behavior of detectors (7, 8) in the region of small $(S/N)_i$. The present result is due primarily to the fact that the output terms of interest in the present case are those in the vicinity of the input frequencies, while in the detector case, the output terms of interest are those in the vicinity of zero frequency.

An expression for the confluent hypergeometric function ${}_1F_1[a;c;-z]$ valid for large values of z is (11)

$$ {}_1F_1\left[a;c;-z\right] \approx \frac{\Gamma(c)}{\Gamma(c-a)z^a}\left[1+\frac{a(a-c+1)}{z}+\dots\right] \tag{56} $$

and hence

$$ {}_1F_1\left[a;c;-z\right] \approx \frac{\Gamma(c)}{\Gamma(c-a)z^a} \quad \text{for } z \to \infty. \tag{57} $$

If we now use the limiting result 57 in Eq. 49, we obtain for the output signal power

$$ S_o \approx \frac{2a^2}{\pi}\left(\frac{4}{\pi}\right) \quad \text{for } \left(\frac{S}{N}\right)_i \to \infty. \tag{58} $$

Application of the limiting expression 57 to Eq. 50 gives for the output noise power

$$ N_o \approx \frac{2a^2}{\pi}\left[\begin{array}{l} \dfrac{1}{\pi\left(\frac{S}{N}\right)_i} + \dfrac{1}{32\pi\left(\frac{S}{N}\right)_i^3} + \dfrac{27}{1024\pi\left(\frac{S}{N}\right)_i^5} + \dots \\ \\ + \dfrac{1}{\pi\left(\frac{S}{N}\right)_i} + \dfrac{3}{2\pi\left(\frac{S}{N}\right)_i^2} + \dfrac{261}{96\pi\left(\frac{S}{N}\right)_i^3} + \dots \end{array}\right] \tag{59} $$

and hence we obtain

$$ N_o \approx \frac{2a^2}{\pi}\cdot\frac{2}{\pi\left(\frac{S}{N}\right)_i} \quad \text{for } \left(\frac{S}{N}\right)_i \to \infty. \tag{60} $$

A comparison of Eq. 59 with Eq. 50 shows that the dominant noise output terms in this case are (a) the direct feedthrough noise term, and (b) the noise resulting from the interaction of the input noise with the second harmonic of the input sine-wave signal.

The output signal-to-noise power ratio becomes

$$ \left(\frac{S}{N}\right)_o \approx 2\left(\frac{S}{N}\right)_i \quad \text{for } \left(\frac{S}{N}\right)_i \to \infty. \tag{61} $$

Thus, for very large values of the input signal-to-noise power ratio, we find that the output signal-to-noise power ratio is twice that of the input. This is the result obtained by Tucker (5).

From Eqs. 55 and 61, we see that the output signal-to-noise power ratio is directly proportional to $(S/N)_i$ in both the large and small value regions of $(S/N)_i$. In order to determine the behavior of $(S/N)_o$ for intermediate values of $(S/N)_i$, the ratio of $(S/N)_o$ to $(S/N)_i$ was calculated from the data used to plot Fig. 4, and the results were plotted to form Fig. 5. From this plot, we see that in the case of the ideal symmetrical, band-pass limiter, the output signal-to-noise power ratio is essentially linearly proportional

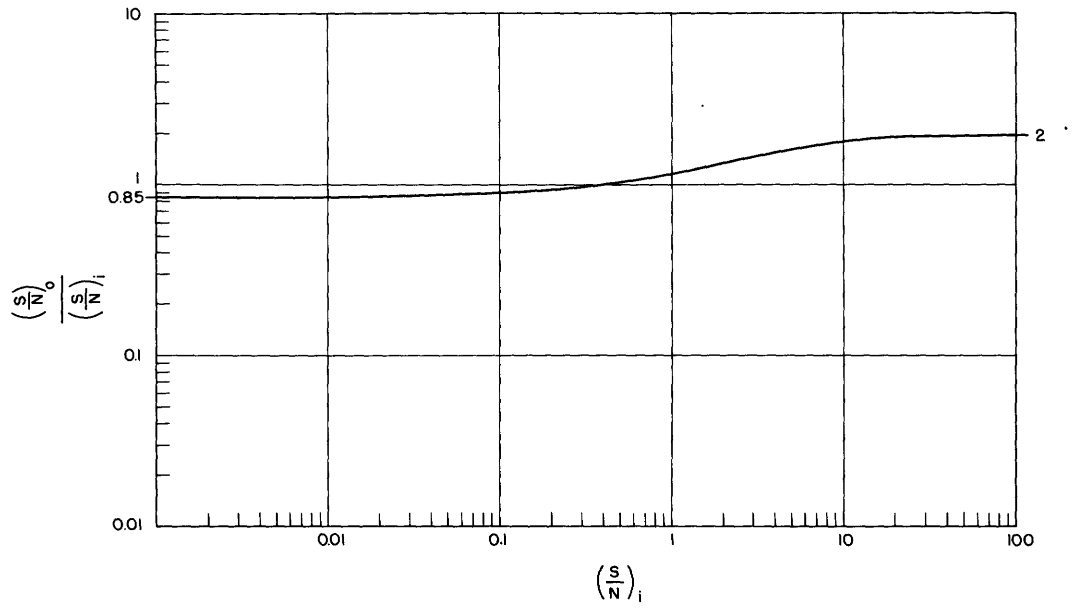

Fig. 5
The ratio of output signal-to-noise power ratio to the input signal-to-noise power ratio as a function of the latter, for the case of the ideal, symmetrical, bandpass limiter.

to the input signal-to-noise power ratio for all values of the latter. A review of this analysis shows that this behavior is due primarily to the action of the bandpass filter following the limiter rather than to the symmetry of the limiter.

C. The Square-Rooter

Having considered the limiting cases (n=1) and (n=∞) of the rooting limiter, we find it desirable to consider a case corresponding to some intermediate value of n. A case of some interest is the one in which (n=2). In this case, the limiter output is proportional to the square root of its input. This characteristic is plotted in Fig. 3.

Substitution of (n=2) in the general expression Eq. 42 gives

$$h_{mk}^2 R_N^k(0) = \begin{cases} 0 & \text{for } m+k \text{ even} \\ \dfrac{a^2\pi}{4}\dfrac{\sigma(N)}{\sqrt{2}}\left(\dfrac{S}{N}\right)_i^m \dfrac{2^k\, {}_1F_1^2\left[\dfrac{m+k-\frac{1}{2}}{2}; m+1; -\left(\dfrac{S}{N}\right)_i\right]}{\Gamma^2(m+1)\,\Gamma^2\left(1-\dfrac{m+k-\frac{1}{2}}{2}\right)} & \text{for } m+k \text{ odd} \end{cases} \quad (62)$$

for the case of the square-rooter.

Substitution of Eq. 62 into Eq. 27 gives

$$S_o = \frac{a^2}{4\pi} \Gamma^2\left(\frac{1}{4}\right) \frac{\sigma(N)}{\sqrt{2}} \left(\frac{S}{N}\right)_i \; {}_1F_1^2\left[\frac{1}{4}; 2; -\left(\frac{S}{N}\right)_i\right] \tag{63}$$

as an expression for the output signal power from the bandpass square-rooter.

Substitution of Eq. 62 into Eqs. 29, 32, and 33 gives

$$N_o = \frac{a^2}{4\pi} \Gamma^2\left(\frac{1}{4}\right) \frac{\sigma(N)}{\sqrt{2}} \left\{ \begin{aligned} & {}_1F_1^2\left[\frac{1}{4}; 1; -\left(\frac{S}{N}\right)_i\right] + \frac{1}{32}\, {}_1F_1^2\left[\frac{5}{4}; 1; -\left(\frac{S}{N}\right)_i\right] \\ & + \frac{25}{3072}\, {}_1F_1^2\left[\frac{9}{4}; 1; -\left(\frac{S}{N}\right)_i\right] + \dots \\ & + \frac{1}{64}\left(\frac{S}{N}\right)_i^2 {}_1F_1^2\left[\frac{5}{4}; 3; -\left(\frac{S}{N}\right)_i\right] \\ & + \frac{3}{32}\left(\frac{S}{N}\right)_i {}_1F_1^2\left[\frac{5}{4}; 2; -\left(\frac{S}{N}\right)_i\right] \\ & + \frac{25}{4608}\left(\frac{S}{N}\right)_i^3 {}_1F_1^2\left[\frac{9}{4}; 4; -\left(\frac{S}{N}\right)_i\right] \\ & + \frac{25}{1536}\left(\frac{S}{N}\right)_i^2 {}_1F_1^2\left[\frac{9}{4}; 3; -\left(\frac{S}{N}\right)_i\right] \\ & + \frac{125}{524,288}\left(\frac{S}{N}\right)_i^4 {}_1F_1^2\left[\frac{13}{4}; 5; -\left(\frac{S}{N}\right)_i\right] + \dots \end{aligned} \right\} \tag{64}$$

as an expression for the output noise power from the bandpass square-rooter.

Let us now obtain limiting expressions for the output signal and noise powers valid in the region of large (or small) values of input signal-to-noise power ratio. Substitution of the small $(S/N)_i$ limiting expression 52 for the confluent hypergeometric function in Eq. 63 gives for the output signal power

$$S_o \approx \frac{a^2}{4\pi} \Gamma^2\left(\frac{1}{4}\right) \frac{\sigma(N)}{\sqrt{2}} \left(\frac{S}{N}\right)_i \qquad \text{for } \left(\frac{S}{N}\right)_i \to 0. \tag{65}$$

Substitution of the expression 52 in Eq. 64 gives for the output noise power

$$N_o \approx \frac{a^2}{4\pi} \Gamma^2\left(\frac{1}{4}\right) \frac{\sigma(N)}{\sqrt{2}} \left[1 + \frac{1}{32} + \frac{25}{3072} + \ldots\right]$$

or

$$N_o \approx \frac{a^2}{4\pi} \Gamma^2\left(\frac{1}{4}\right) \frac{\sigma(N)}{\sqrt{2}} (1.04) \qquad \text{for} \left(\frac{S}{N}\right)_i \rightarrow 0. \tag{66}$$

From these expressions, we obtain

$$\left(\frac{S}{N}\right)_o \approx 0.96\left(\frac{S}{N}\right)_i \qquad \text{for} \left(\frac{S}{N}\right)_i \rightarrow 0. \tag{67}$$

Thus, in the region of very small values of the input signal-to-noise power ratio, the output signal-to-noise power ratio is equal to 96 percent of the input ratio. Substitution of the large $(S/N)_i$ limiting expression 57 for the confluent hypergeometric function in Eq. 63 gives for the output signal power

$$S_o \approx \frac{a^2}{4\pi} \left(\frac{8}{9\pi^2}\right) \Gamma^2\left(\frac{1}{4}\right) \frac{\sigma(N)}{\sqrt{2}} \left(\frac{S}{N}\right)_i^{1/2} \qquad \text{for} \left(\frac{S}{N}\right)_i \rightarrow \infty. \tag{68}$$

Substitution of Eq. 57 in Eq. 64 gives for the output noise power

$$N_o \approx \frac{a^2}{4\pi} \left(\frac{5}{9\pi^2}\right) \Gamma^2\left(\frac{1}{4}\right) \frac{\sigma(N)}{\sqrt{2}} \left(\frac{S}{N}\right)_i^{-(1/2)} \qquad \text{for} \left(\frac{S}{N}\right)_i \rightarrow \infty. \tag{69}$$

From these expressions, we obtain

$$\left(\frac{S}{N}\right)_o \approx 1.60\left(\frac{S}{N}\right)_i \qquad \text{for} \left(\frac{S}{N}\right)_i \rightarrow \infty. \tag{70}$$

Thus, in the region of very large values of the input signal-to-noise power ratio, the output signal-to-noise power ratio is again found to be directly proportional to the input ratio. We see then, as in our previous cases, that the output signal-to-noise power ratio is essentially directly proportional to the input signal-to-noise power ratio for the case of the bandpass square-rooter.

IV. Conclusions

We have presented here an analysis of the relation between the output signal-to-noise power ratio and the input signal-to-noise power ratio for the case of a bandpass limiter. This analysis shows that, for this type of system, the output signal-to-noise power ratio is essentially directly proportional to the input signal-to-noise power ratio for all values of the latter. This type of behavior is due primarily to the fact that, in the systems studied here, the system output has been filtered so as to contain only those frequency components in the immediate vicinity of the input frequencies.

References

1. D. Middleton: J. Appl. Phys. 17, 778, 1946
2. N. M. Blackman: J. Appl. Phys. 20, 38, 1949
3. D. Middleton: J. Appl. Phys. 20, 334, 1949
4. D. Middleton: Quart. Appl. Math. 7, 129, 1949 and 8, 59, 1950
5. D. G. Tucker: Wireless Engineer, 29, 128, 1952
6. N. Wiener: Acta Math. 55, 117, 1930
7. S. O. Rice: Bell System Tech. J. 24, 46, 1945
8. D. Middleton: Quart. Appl. Math. 5, 445, 1948
9. E. C. Titchmarsh: Fourier Integrals, Clarendon Press, Oxford, 1948, second edition
10. B. van der Pol, H. Bremmer: Operational Calculus, Cambridge University Press, 1950
11. W. Magnus, F. Oberhettinger: Formulas and Theorems for the Special Functions of Mathematical Physics, Chelsea, New York, 1949