Signaling and mediation in games with common interests

Ehud Lehrer[†], Dinah Rosenberg[††] and Eran Shmaya[†]

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Abstract

Players who have a common interest are engaged in a game with incomplete information. Before playing they get differential signals that stochastically depend on the actual state of nature. These signal not only provide the players with partial information about the state of nature but also serve as a correlation means.

Different information structures induce different outcomes. An information structure is *better than* another, with respect to a certain solution concept, if the highest solution payoff it induces is at least that induced by the latter structure. This paper fully characterizes when one information structure is better than another with respect to various solution concepts. The solution concepts we refer to differ from each other in the scope of communication allowed between the players. The characterizations are phrased in terms of maps that take signals of one structure and (stochastically) translate them to signals of another structure.

[0†]School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel. e-mails: `lehrer@post.tau.ac.il; gawain@post.tau.ac.il`

[††] LAGA Institut Galilée, Université Paris 13, avenue Jean Baptiste Clément 93430 Villetaneuse, France; and Laboratoire d'Econometrie de l'Ecole Polytechnique, Paris, France. e-mail: `dinah@math.univ-paris13.fr`

1 Introduction

In Bayesian games the amount of information players obtain about the actual payoff matrix crucially affects the outcome. This paper investigates a few aspects of how changes in information may influence the outcome of an interaction.

It has been noticed that one or more players obtaining better information about the state of nature (i.e., the actual game) does not necessarily mean that their payoffs are improved. Hirshleifer (1971) provided the first example that the production of private information, even if costless, can be socially harmful. In the lemon market (see, Akerlof (1970)), providing the seller with private information might render any trade impossible, and thereby reduce social welfare. Neyman (1991) argued that the source of this phenomenon is the fact that any improvement in the information some players obtain about the state of nature generates a change in all other players' knowledge: the latter know that former knows more. Bassan et al. (2003) characterized those games in which getting more information about the state of the world (i.e., about the entire hierarchy of beliefs) always improves all players' payoffs.

The issue of whether or not information has a positive value is related to a broader subject: the comparison between information structures based on the outcomes they induce, not in a particular game, but rather in a large set of interactions.

Blackwell (1953) was the first to compare information structures. He did so in the context of one-player games, that is, in one-person decision problems. An information structure specifies the distribution over signals that the players receive conditional on any particular state of nature. Blackwell defined the notion of one information structure 'being better' than another: an information structure is *better than* another if in *any* decision problem the optimal value associated with the former structure is higher than the optimal value associated with the latter.

Blackwell characterized the information structures that are better than a given one as those that are more informative. An information structure is said to be *more informative* than another if the signals provided by the latter can be reproduced (by using some map, called *garbling*) from the signals provided by the former. This result demonstrates that, in one-player decision problems, information has a positive value: a single agent would always prefer more information to less.

Matters are more intricate in multi-player interactions.
- First, the signal each player obtains contains more than just information about the state of nature. These signals may be correlated across players, and may partially convey what other players know about the game being played, and what they know about what others know about the actual game, etc. A particular information structure affects the outcome of the game by the direct information it provides about the state of nature, by information it provides about other players' information, and by correlations that might exist between the players' signals.
- Second, the outcome of a game also depends on the type of mediation the players may resort to. Nash equilibrium, for instance, needs no mediation device. On the other hand, all sorts of extensions of correlated equilibrium (Aumann (1987), Forges (1993)) to Bayesian games do need some kind of mediation. Each solution concept induces a different set of outcomes, and in turn, results in a different criterion of comparison between information structures.
- Third, for any concept there are, typically, multiple solutions, and comparing between sets of outcomes can be done in more than one reasonable way.
- Finally, there is more than one way to extend the definition of "being more informative than". Each way depends on the extend to which the particular garbling used changes one player's information about other players' information.

In this paper we confine ourselves to games with common interests. There are several reasons for dealing separately with games with common interests. First, games with common interests are closest in nature to one-player decision problems, in the sense that players would like to behave as one player if they could perfectly share their information and coordinate their actions. Second, the results concerning games with common interests pour light on the connections between the existing non-cooperative solution concepts and information structures that provide players with differential signaling in Bayesian games. Finally, unlike the general case, in games with common interests there is *one* outcome which is desired by all players. This outcome can be identified with a common payoff, which can be regarded as the 'value' of the game[1]. Indeed, Blackwell's comparison of information structures is extended in a neat manner to games with common interests.

[1]Zero-sum games own a similar feature: there is a number, the value, that represents the outcome of the game.

We say that one information structure is *better than* another with respect to (w.r.t.) Nash equilibrium if, in any game with identical payoffs, it ensures a Nash equilibrium payoff which dominates any Nash equilibrium payoff of the latter. It turns out that one structure is better than another, if it is derived (a) by adding a public randomizing device; and (b) based on public signal, the players individually garble the information of the latter. This procedure produces a garbling, called *coordinated*. This result implies, in particular, that under the more informative structure each player knows no more than under the less informative structure about what the other player knows. However, the converse is not true.

We extend this result to solution concepts different from Nash equilibrium. We refer to solution concepts that require some sort of communication between the players, such as correlated equilibrium, agent-normal-form correlated equilibrium, Bayesian solution and communication equilibrium (see Forges, 1993). These equilibrium concepts differ from one another in the amount of players' private information the communication means is allowed to use[2]. We provide a complete characterization of information structures that are better than a given one in games with common interest w.r.t. each of these concepts. All the characterizations are stated in terms of maps, called garblings, that enable the players to reconstruct the signals of the inferior information structure from the signals of the superior one. The garblings allowed vary across solution concepts. The restriction on these garblings reflect the amounts of correlation and communication that each solution concept allows between the players.

It turns out that one information structure is better than another w.r.t. correlated and agent-normal-form correlated equilibria precisely when this is

[2]In a *correlated equilibrium* a mediator chooses a pair of strategies (of the Bayesian game), one for each player, and tells each player his selected strategy.In an *agent-normal-form correlated equilibrium* any player has many agents, one for each of his signals. A mediator chooses randomly an action for each player's agent and tells a player only the action selected for the specific signal he received.A *Bayesian solution* is described in terms of a larger state space than that containing the states of nature. In any state of the world each player knows a set of states that contains the actual one. This knowledge represented by a partition of the state space (in the spirit of Aumann ,1987), captures all he knows about the world, including his own private signal and action. Subject to this knowledge, each player plays his best response. In *communication equilibrium* players report to the mediator about their signals and then the mediator chooses randomly a recommended action for each player.

true w.r.t. Nash equilibrium. The question of when one information structure is better than another w.r.t. to the Bayesian solution is answered by means of a special kind of garbling, called non-communicating that translates one information structure to another without providing the players more information about each others' information or about the state of nature: a structure in better than another w.r.t. the Bayesian solution if the former is a garbled version of the latter using non-communicating garbling. A structure in better than another w.r.t. to communication equilibrium if the former is a garbled version of the latter.

These results provide a complete link between the amount of communication involved in the various solution concepts and the kind of information structures that improve upon a given structure in terms of the outcomes induced.

The paper is organized as follows. The next section defines the model of Bayesian games and information structures. Comparison of information structure w.r.t. Nash equilibrium is provided in Section 3, where we introduce the notion of 'being better than', and characterize when an information structure is better than another w.r.t. Nash equilibrium. Section 4 is devoted to the extensions of this result to other solution concepts. We review the various concepts and we characterize when an information structure is better than another w.r.t. each one. The proofs are given in Section 5. Section 6 contains a few final comments.

2 Information structures and Games

Two players participate in a Bayesian game. A *state of nature* k is randomly drawn from a set K according to a known distribution p. The players are not directly informed of the realized state. Rather, each player receives a stochastic signal that depends of k. The signals that the players receive are typically correlated.

Formally, an *information structure* consists of two finite sets of signals, S, T, and a function[3] $\sigma : K \to \Delta(S \times T)$ that assigns to every state of nature a joint distribution over signals. When the realized state is k, player 1 obtains the signal s and player 2 obtains the signal t with probability $\sigma(k)[s,t]$, which we usually denote $\sigma(s,t|k)$. Information structures will be referred to as triples of the kind (S,T,σ) and will be denoted by $\mathcal{I}$.

[3] $\Delta(D)$ denotes the set of distributions over a set D.

Upon receiving a signal, a player takes an action and receives a payoff that depends on both players' actions and on the state of nature. Formally, let A be player 1's set of actions and B be that of player 2. Note that these sets are common to all states. If the state is k, player 1 plays a and player 2 plays b, then the payoff player i receives $r^i_k(a,b)$. A *strategy* x of player 1 assigns a mixed action to every signal in S . When player 1 plays according to strategy x, the action $a \in A$ is played with probability $x(a|s)$ if he observes the signal s. A strategy y of player 2 is defined in a similar manner.

The expected payoff of player i when the strategy profile (x,y) is played is therefore,

$$r^i(x,y) = \sum_{k \in K} p(k) \sum_{(s,t) \in S \times T} \sigma(s,t|k) \sum_{(a,b) \in A \times B} x(a|s) y(b|t) r^i_k(a,b).$$

We focus in this paper on the class of games with common interests.

Definition 2.1 *The game is with* common interests *if for any* $k \in K$, $r^1_k = r^2_k$.

3 Comparison of information structures and equilibria

3.1 Better information structures w.r.t. Nash equilibrium

This section is devoted to the comparison of information structures as far as *Nash equilibrium* is concerned. The strategy profile (x,y) is a Nash equilibrium if no player has an incentive to deviate. That is, $r^1(x,y) \geq r^1(x',y)$ and $r^2(x,y) \geq r^2(x,y')$ for all strategies x' and y'.

In two player zero-sum games, there is a unique Nash equilibrium payoff, the value. From the maximizer's point of view, an information structure can be said to be *better than* another in the class of zero-sum games if it induces a higher value in any game in this class. However simple to describe, the question of characterizing the information structures that are better than a given one in the class of zero-sum games is still open.

It is well known that, in complete information games with common interests, there is a Nash equilibrium that Pareto dominates all the entries of the payoff matrix. A Bayesian game can also be formulated as a complete information normal form game. This means that in a Bayesian game with common interests there is a Nash equilibrium payoff that dominates all other feasible payoffs. Therefore, in games with common interests, although there is typically more than one Nash equilibrium payoff, one is clearly more appealing than the others.

Since the set of Nash equilibria, and the set of Nash equilibrium payoffs of a game are not singletons, there are *a priori* various ways to extend the notion of 'being better than'. The existence of a Pareto dominating equilibrium enables us to choose the following from among these possible notions.

Definition 3.1 *An information structure $\mathcal{I}$ is* better than *$\mathcal{I}'$ w.r.t. Nash equilibrium in common interest games, if for any common interest game every Nash-equilibrium-payoff under $\mathcal{I}'$ is Pareto dominated by some Nash-equilibrium-payoff under $\mathcal{I}$.*

An example of two such structures is provided in Example 3.7.

Our first result characterizes when one structure is better than another w.r.t. Nash equilibrium in common-interest games. The characterization hinges upon the notion of garbling, used first by Blackwell (1953).

3.2 Garbling of information

Let $\mathcal{I} = (S, T, \sigma)$ be an information structure. Suppose that a joint signal (s, t) in $S \times T$ is produced (i.e., is randomly selected according to σ). However, instead of sending signals to the players, a pair of new signals, say (s', t'), is randomly selected from new sets of signals, say S' and T', according to a distribution $q(s, t)$. Players 1 and 2 are then informed of s' and t' respectively. This procedure generates a new information structure, $\mathcal{I}' = (S', T', \sigma')$, which is said to be a garbled version of $\mathcal{I}$. Formally,

Definition 3.2 *Let $\mathcal{I} = (S, T, \sigma)$ and $\mathcal{I}' = (S', T', \sigma')$ be two information structures. $\mathcal{I}'$ is a* garbled version *of $\mathcal{I}$ if there is a map q from $S \times T$ to $\Delta(S' \times T')$ [4] such that the distribution induced by the composition $q \circ \sigma$*

[4] We still denote by q the linear extension of q to a function from $\Delta(S \times T)$ to $\Delta(S' \times T')$.

coincides with[5] σ'.
The map q *is called a* garbling *that transforms* $\mathcal{I}$ *to* $\mathcal{I}'$.

In the one-player case, Blackwell (1953) defined $\mathcal{I}$ as being *more informative* than $\mathcal{I}'$ if there exists a garbling that transforms $\mathcal{I}$ into $\mathcal{I}'$. He then proved that $\mathcal{I}$ is better than $\mathcal{I}'$ iff $\mathcal{I}$ is more informative than $\mathcal{I}'$.

Let us now move to two-player games. Imagine a fictitious agent who knows the signals received by both players. Note that if $\mathcal{I}'$ is a garbled version of $\mathcal{I}$, then this agent would be better informed, in the sense of Blackwell, getting the signal through $\mathcal{I}$, than through $\mathcal{I}'$.

While in a one-player decision problem the signal may only convey some information about the actual state, in games things are much more involved. A signal contains not only information about k, but also about the other player's information about k, and about the other's information about his own information about k and so forth. Moreover, the signals the players receive may be correlated, which may, in turn, enrich the set of possible outcomes. This explains why only specific garblings will be used to characterize the information structures that are better than a given one.

Definition 3.3 . *(i) A garbling* q *is said to be* independent *if there are maps* $q_1 : S \to \Delta(S')$ *and* $q_2 : T \to \Delta(T')$ *such that for every* s, t, s', t',

$$q(s', t'|s, t) = q_1(s'|s) \cdot q_2(t'|t).$$

(ii) A garbling q *is* coordinated *if it is in the convex hull of independent garblings.*

Note that independent garbling can be implemented without any mediation or communication between the players (every player manipulates his signal independently of the other) and a coordinated garbling can be implemented by a public signaling which is independent of the players' signals.

[5]That is, for every k and every $(s', t') \in S' \times T'$, $\sigma'(s', t'|k) = \sum_{(s,t)\in S\times T} \sigma(s, t|k) q(s't'|s, t)$, where $q(s', t'|s, t)$ is the probability that the output signals will be s', t' given the input signals s, t.

Example 3.4 *Let* $S = T = S' = T' = \{0, 1\}$. A garbling will be denoted as a $(S \times T) \times (S' \times T')$ matrix. Let

$$q_1 = \begin{pmatrix} 4/9 & 2/9 & 2/9 & 1/9 \\ 2/9 & 4/9 & 1/9 & 2/9 \\ 2/9 & 1/9 & 4/9 & 2/9 \\ 1/9 & 2/9 & 2/9 & 4/9 \end{pmatrix} \qquad q_2 = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}.$$

The garbling q_1 takes a joint signal in $S \times T$ and garbles it to a joint signal in $S' \times T'$. For instance, the signal $(0, 1)$ is garbled by q_1 to $(0, 0)$ with probability $2/9$, to $(0, 1)$ with probability $4/9$, to $(1, 0)$ with probability $1/9$, and to $(1, 1)$ with probability $2/9$. q_1 is the product of two garblings of player 1 and player 2, both are given by the matrix $\begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}$. The garbling q_1 is, therefore, independent. On the other hand, q_2 is coordinated but not independent. It can be written as $q_2 = \frac{1}{2}q + \frac{1}{2}q'$ where q and q' are the independent garblings given by

$$q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad q' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

■

3.3 A first characterization

Our first result characterizes when one information structure is better than another w.r.t. Nash equilibrium in the class of games with common interests.

Theorem 3.5 *Let* $\mathcal{I}$ *and* $\mathcal{I}'$ *be two information structures.* $\mathcal{I}$ *is Nash-better than* $\mathcal{I}'$ *in the class of games with common interests if and only if there exists a coordinated garbling that transforms* $\mathcal{I}$ *to* $\mathcal{I}'$*.*

The proof of this theorem is postponed to Section 5.

The intuition of this result is as follows. If each player can (independently) mimic $\mathcal{I}'$ by using the signal he got from $\mathcal{I}$, then in any game with common interests the players can ensure in $\mathcal{I}$ whatever they can in $\mathcal{I}'$. Furthermore, in games with common interests, any correlation that does not depend on payoff-relevant information is worthless in the sense that it may not increase the best Nash equilibrium payoff. Thus, even if the players use a public coordination device (which is independent of their information) in order to choose the particular way in which they independently garble their signals, the highest equilibrium payoff does not increase. Therefore, if $\mathcal{I}'$ is a coordinated garbling of $\mathcal{I}$, then $\mathcal{I}$ is better than $\mathcal{I}'$.

Example 3.6 Here we provide two information structures $\mathcal{I}$ and $\mathcal{I}'$ such that $\mathcal{I}'$ is a garbled version of $\mathcal{I}$ but with a garbling that is non-coordinated. Furthermore, we exhibit a game in which there is a Nash equilibrium payoff with $\mathcal{I}'$ that is strictly higher than any Nash equilibrium payoff with $\mathcal{I}$.

Let $K = \{0,1\}^2$, S and T be equal to $\{0,1\}$ and let $\mathcal{I}$ be the information structure in which player 1 knows the first coordinate of k and player 2 knows the second coordinate. Formally, for each $k = (s,t) \in \{0,1\}^2$, $\sigma(s,t|k) = 1$.

Consider the game with common interests in which the action sets are $A = B = \{0,1\}$ and payoff functions are given by $r_k(a,b) = (-1)^{a+b+st}$ for every $k = (s,t) \in K$. By examining all pairs of pure strategies in this game, one can verify that the maximal payoff achievable in any such pair under $\mathcal{I}$ is $\frac{1}{2}$.

Now consider the garbling,

$$q = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

and denote by $\mathcal{I}'$ the garbled version of $\mathcal{I}$ with this garbling.

If the players are informed through $\mathcal{I}'$, and each player plays his signal, then the payoff is $(1,1)$ (which is the maximal feasible payoff, and therefore an equilibrium). Theorem 3.5 implies that the garbling q is not coordinated. This means, in particular, that although $\mathcal{I}'$ is a garbled version of (S,T,σ), (S,T,σ) is not better than $\mathcal{I}'$ w.r.t. Nash equilibrium.

The game in this example, called CHSH game[6] in Cleve et al. (2004), is well known in the quantum physics literature. It is related to the violation of Bell's inequality by measurements over a pair of particles at a maximally entangled state. (See also Remark 4.6 below.) ■

[6] After Clausner, Horne, Shimony and Holt (1969)

Example 3.7 In this example we show a game with common interests and two information structures $\mathcal{I}$ and $\mathcal{I}'$ such that $\mathcal{I}$ is better than $\mathcal{I}'$. Consider $K = S = T = S' = T' = \{0, 1\}$ with probability $(1/2, 1/2)$ over K and the following common interest game,

$$\underset{k=0}{\begin{pmatrix} 1,1 & 0,0 \\ 0,0 & 1,1 \end{pmatrix}} \qquad \underset{k=1}{\begin{pmatrix} -1,-1 & 0,0 \\ 0,0 & -1,-1 \end{pmatrix}}.$$

Let $\mathcal{I}$ be the information structure described by the following signaling matrices

$$\underset{k=0}{\begin{pmatrix} 1/2 & 1/2 \\ 0 & 0 \end{pmatrix}} \qquad \underset{k=1.}{\begin{pmatrix} 0 & 0 \\ 1/2 & 1/2 \end{pmatrix}}$$

For instance, if $k = 0$, then with probability 1/2 players 1 and 2 receive the signal 0 and with probability 1/2 they receive, respectively, the signals 0 and 1; and if $k = 1$, with probability 1/2 players 1 and 2 receive, respectively, the signals 1 and 0, and with probability 1/2 they receive the signal 1. Thus, in $\mathcal{I}$ player 1 knows k and player 2 knows nothing. With this information structure, if the states are equally probable, then the best equilibrium payoff is 1/2.

Denote by $\mathcal{I}'$ the garbled version of $\mathcal{I}$ with the garbling q of the previous example. The information structure $\mathcal{I}'$ is

$$\underset{k=0}{\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}} \qquad \underset{k=1}{\begin{pmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}}.$$

Although the garbling q is not coordinated (see Example 3.6), $\mathcal{I}'$ is a garbled version of $\mathcal{I}$ with a coordinated garbling. Indeed, denote by $\delta_{s_0,s_1;t}$ the information structure under which, with probability 1, player 1 gets the signal s_0 if $k = 0$, the signal s_1 if $k = 1$, and player 2 gets the signal t (independently of k). Obviously, $\delta_{s_0,s_1;t}$ is a garbled version of $\mathcal{I}$ with an independent garbling. Furthermore, $\mathcal{I}' = \frac{1}{4}\delta_{0,0;0} + \frac{1}{4}\delta_{0,1;0} + \frac{1}{4}\delta_{1,1;1} + \frac{1}{4}\delta_{1,0;1}$. Theorem 3.5 implies that $\mathcal{I}$ is better than $\mathcal{I}'$ w.r.t. Nash equilibrium and therefore, the maximal Nash equilibrium payoff under $\mathcal{I}'$ is not greater than 1/2.

This example demonstrates that $\mathcal{I}'$ may be a garbled version of $\mathcal{I}$ with a coordinated garbling and at the same time a garbled version of $\mathcal{I}$ with a non-coordinated garbling. ∎

4 Comparison of information structures with respect to other solution concepts

In this section we discuss various solution concepts that involve some communication between the players before or after they receive the signals. The following definition is analogous to Definition 3.1.

Definition 4.1 *Let* $\mathcal{I}'$ *be two information structures. Fix an equilibrium concept.* $\mathcal{I}$ *is* better *than* $\mathcal{I}'$ *w.r.t. this equilibrium concept in common interest games if, at any common interest game, every equilibrium-payoff under* $\mathcal{I}'$ *is Pareto dominated by an equilibrium-payoff under* $\mathcal{I}$*.*

We now introduce the solutions concepts we will use in the sequel. They are extensions of the notion of correlated equilibrium. All notions are equivalent to correlated equilibrium when restricted to complete information games. They differ in the way correlation and/or communication between the players is allowed to depend on their information. The three first notions are described through a mediator and the fourth one is a Bayesian model. The equivalence between all approaches is proved for complete information games in Aumann (1987).

In the description of the solution concepts we follow Forges (1993). The reader is referred to her paper for formal definitions and the relation between the various concepts. For each of these concepts we will prove analogous result to Theorem 3.5.

4.1 Solution concepts that require a mediator

All the following equilibria are implemented by a mediator.

In a *normal-form correlated* equilibrium, first, the mediator randomly selects a profile of strategies, (x, y). Second, he tells x to player 1 and y to player 2. The strategies x and y are interpreted as recommendations made by the mediator to the players. Third, each player chooses an action as a function of his information and the recommendation sent by the mediator. The incentive compatible condition is that no player has an incentive to deviate from the recommended strategy. For instance, at equilibrium, when player 1 has been told x, if his signal is s he plays the action a with probability $x(a|s)$.

Note that here the mediator does not know the signals the players received. In other words, the mediator provides the players with a correlation that is independent of their signals.

In an *agent-normal-form* correlated equilibrium, the mediator is assumed to *know* the signals received by the players. It then sends a recommendation, one for each pair (player i, signal s_i), where a recommendation for the pair (player i, signal s_i) is an action of player i who received signal s_i. Note that after receiving signal s_i player i is not aware of the recommendation he would have received upon getting signal s'_i.

More precisely, the mediator then chooses its recommendation in two steps: (i) before knowing the signals of the players it chooses a correlation device, i.e. with probability λ_i it chooses to use the mechanism ε_i in step two; (ii) then after being aware of the signals of the players, the mechanism ε_i chooses independently a recommendation for player 1 with signal s and a recommendation for player 2 with signal t. This solution concept can be interpreted as a normal form correlated equilibrium in the game in which each pair (player i, signal s_i) is considered a separate player.

A *communication equilibrium* is implemented by a mediator to which each player sends his signal (but he can of course lie). Then as a function of reported signals the mediator sends to each player a recommended action. The equilibrium condition states that no player has an incentive to lie nor to deviate from the recommended action. Note that a communication equilibrium may involve a stronger dependence between the signals of both players and the recommended actions than in the previous equilibrium notions, because the mediator is allowed to use both signals in order to choose the recommendations he makes.

4.2 The epistemic approach

The previous notions of equilibrium require a mediator. They differ in the information the mediator has about the private signals of the players. In this section we adopt a different approach introduced in Aumann (1987), used by Forges (1993), and adopted by Bassan et al (2003). It can be viewed as a general approach to express Bayesian rationality.

The epistemic approach is based on a probability space $(\Omega, \mathbb{P})$, where Ω is rich enough to reflect the state of nature, the signals and the actions of players.

The epistemic model is described by a probability space, $(\Omega, \mathbb{P})$, two partitions $\mathfrak{A}_1, \mathfrak{A}_2$ of Ω and a few random variables over Ω: (i) κ takes values in K (i.e., κ is the state of nature), (ii) ς and τ take values in S and T, resp. (i.e., ς and τ are the signals); and (iii) α and β take values in A and B, resp. (i.e., α and β are the actions). The partitions $\mathfrak{A}_1, \mathfrak{A}_2$ represent the information available to player 1 and player 2, respectively.

A *Bayesian solution*[7] under the information structure $\mathcal{I} = (S, T, \sigma)$ is an epistemic model that satisfies the following conditions:

1. The distribution of κ over K is p and for every $k \in K$, the joint distribution of ς, τ given that $\kappa = k$ is $\sigma(k)$. I.e., the joint distribution induced of κ, ς and τ coincides with the distribution that p and σ induce on $K \times S \times T$.

2. ς, α (resp. τ, β) are $\mathfrak{A}_1$-measurable (resp. $\mathfrak{A}_2$-measurable). I.e., each player knows his signal and action.

3. For every k, a, the signal τ of player 2 completely summarizes his information on player 1's signal.

   $$\mathbb{P}(\varsigma = s | \mathfrak{A}_2) = \mathbb{P}(\varsigma = s | \tau).$$

   I.e., the information embedded in $\mathfrak{A}_2$ does not give player 2 more knowledge about s than his signal t.

   Similar condition holds for player 1.

4. For every k, the joint signals of the players completely summarize their joint information on the state of the world:

   $$\mathbb{P}(\kappa = k | \mathfrak{A}_1, \mathfrak{A}_2) = \mathbb{P}(\kappa = k | \varsigma, \tau).$$

5. Incentive compatibility conditions: any deviation of player 1 (resp. 2) from playing α (resp. β) is not profitable. (For a formal expression of this condition (5) the reader is referred to Forges (1993).)

We say that a distribution π over $K \times A \times B$ can be achieved by a Bayesian solution if π is the joint distribution of κ, α, β in some Bayesian solution.

The following example illustrates the idea behind the epistemic approach and the Bayesian solution.

[7] We use this terminology that has been coined by Forges (1993).

Example 4.2 Consider the game with common interests and the information structure $\mathcal{I}$ presented in Example 3.6. Recall that under $\mathcal{I}$, the maximal payoff achievable in any pair of pure strategies is $\frac{1}{2}$, which is therefore also the maximal agent-normal-form correlated equilibrium payoff.

Consider now the probability space $\Omega = S \times T \times S' \times T'$ with the following distribution $\mathbb{P}(s,t,s',t') = \frac{1}{4} \cdot q(s',t'|s,t)$, where q is as in Example 3.7. The knowledge of players and the interrelation between the players' knowledge is captured by this space.

For $\omega = (s,t,s',t') \in \Omega$ let $\kappa(\omega) = (s,t), \sigma(\omega) = s, \tau(\omega) = t, \alpha(\omega) = s', \beta(\omega) = t'$. The variable κ represents the state. Finally, ς is equal to s and τ is equal to t. Suppose that player 1 knows s and s' (this defines $\mathfrak{A}_1$) and player 2 knows t and t' (this defines $\mathfrak{A}_2$). In other words, at the point $\omega = (s,t,s',t')$ the state is (s,t), player 1's signal is s and he plays s' and player 2's signal is t and he plays t'. In particular, player 1 knows one component, s, of the state and player 2 knows the other component, t. One can verify that all the conditions in the definition of a Bayesian solution are satisfied by these items.

Suppose, for instance, that the point realized is $\omega = (0,1,1,1)$. Then, the state is $\kappa(\omega) = 01$; player 1 knows $s = 0$ and his action, $s' = 1$; and player 2 know $t = 1$ and his action, $t' = 1$. From player 1's point of view, given that $s = 0$ and $s' = 1$, the probability of $t' = 1$ is 1, and moreover, the states 00 and 01 are equally likely. Thus, the action 1 is player 1's best response. On the other hand, given player 2's information, with probability $\frac{1}{2}$ the state is 01 and player 1 plays 1 and with probability $\frac{1}{2}$ the state is 11 and player 1 plays 0. To this belief player 2's best response is indeed $t' = 1$. The payoff induced by this Bayesian solution is $(1,1)$.

Forges (1993) claims that the set of agent-normal-form correlated equilibrium distributions coincides with the set of distributions induced by Bayesian solutions. This example shows that this claim is erroneous: the maximal agent-normal-form correlated equilibrium payoff is $\frac{1}{2}$, while with a Bayesian solution the players can get the payoff 1. ■

4.3 Comparison of information structures

Theorem 3.5 extends to all solution concepts with variants related to the different amount of correlation involved in each.

Theorem 4.3 *The information structure $\mathcal{I}$ is better than $\mathcal{I}'$ w.r.t. correlated equilibrium, or w.r.t. agent-normal-form correlated equilibrium in games with common interests if and only if there exists a coordinated garbling that transforms $\mathcal{I}$ to $\mathcal{I}'$.*

The proof of this theorem relies on the fact that the Pareto dominant correlated equilibrium payoff (or agent normal form correlated equilibrium payoff) is in fact a Nash equilibrium payoff. This reflects the fact that in a common interest game the players can directly coordinate on the best pure Nash equilibrium without needing an external correlation device.

Before stating the results concerning the epistemic approach, we need the following definition. We say that a garbling is non-communicating if no information has passed between the players through the garbling. This means that the garbled signal, s' of player 1, does not give him more information about the original signal t of player 2 than he had knowing s.

Definition 4.4 *A garbling q is* non-communicating *if whenever q transforms $\mathcal{I} = (S, T, \sigma)$ to $\mathcal{I}' = (S', T', \sigma')$, for every $s \in S$, $s' \in S'$ and $t \in T$,*

(i) $\sum_{t'} q(s', t'|s, t)$ does not depend on t (i.e., for every $\pi \in \Delta(S \times T)$, $\mathbb{P}(s'|s, t) = \mathbb{P}(s'|s)$, where $\mathbb{P}$ is the probability induced by π, σ and q); and

(ii) For every $t \in T$, $t' \in T'$ and $s' \in S'$, $\sum_{s'} q(s', t'|s, t)$ does not depend on s.

Let (s, t) be a pair of random signals generated according to some distribution $\pi \in \Delta(S \times T)$ and let (s', t') be the random garbling according to q. If q is non-communicating, then the posterior distribution of t given s, s' equals the posterior distribution over t given s. Indeed,

$$\mathbb{P}(t|s, s') = \frac{\mathbb{P}(s, s', t)\mathbb{P}(s, t)}{\mathbb{P}(s, t)\mathbb{P}(s'|s)\mathbb{P}(s)} = \frac{\mathbb{P}(s'|s, t)\mathbb{P}(s, t)}{\mathbb{P}(s'|s)\mathbb{P}(s)} =$$

$$\frac{\mathbb{P}(s'|s)\mathbb{P}(s, t)}{\mathbb{P}(s'|s)\mathbb{P}(s)} = \frac{\mathbb{P}(s, t)}{\mathbb{P}(s)} = \mathbb{P}(t|s).$$

In other words, if a non-communicating garbling is performed by a mediator, although this mediator is allowed to use the information of the players, he is not allowed to give a player more information than he had before about the signal of the other player.

Example 4.5 Let $S = T = S' = T' = \{0, 1\}$. Consider the garbling q of Example 3.6. This garbling is not a coordinated garbling but it is non-communicating. ■

Remark 4.6 In a recent paper Barret et al. (2005) elaborate on non-communicating garblings, which they call non-local correlations. It is a remarkable discovery of quantum theory that some (but not all) non-communicating garblings can be physically implemented by instantaneous physical operations at distant locations. This does not contradict Einstein's dictate that information cannot travel faster than light because in non-communicating garblings no information is passed between the players.

Using game theoretical terminology, we can say that Examples 3.6 and 4.2 show the possibility of improving the outcome of a game with common interests by coordination without communication (that is, without any information exchange between the players). ■

The following theorems provide a complete characterization of the order 'being better' w.r.t. all remaining solution concepts in games with common interests. They establish a strong relation between the various types of garblings and the amount of communication allowed in each concept.

Theorem 4.7 *Let $\mathcal{I}$ and $\mathcal{I}'$ be two information structures. $\mathcal{I}$ is better than $\mathcal{I}'$ w.r.t. Bayesian solution in common interest games if and only if there exists a non-communicating garbling that transforms $\mathcal{I}$ to $\mathcal{I}'$.*

Theorem 4.8 *The information structure $\mathcal{I}$ is better than $\mathcal{I}'$ w.r.t. communication equilibrium in games with common interests if and only if there exists a garbling that transforms $\mathcal{I}$ to $\mathcal{I}'$.*

The intuition underlying Theorem 4.8 is that in games with common interests players have an incentive to share their information. Therefore, they have an incentive to correctly report their signals to the mediator. In addition, the latter is then able to perform any garbling.

5 Proofs

Before proving the announced results we need some notations for strategies and payoffs.

5.1 Strategies

Let S and T be the signals sets of players 1 and 2, respectively. A *global strategy* is a function from $S \times T$ to $\Delta(A \times B)$. That is, a global strategy attaches a distribution over $A \times B$ to every pair of signals (s, t).

If x is a strategy of player 1 and y is a strategy of player 2, then $x \otimes y : S \times T \to \Delta(A \times B)$ denotes the global strategy played if the players play independently of each other. Formally, $x \otimes y(a, b|s, t) = x(a|s)y(b|t)$. Such a strategy is called *independent* global strategy.

Let ε be a global strategy of the form $\varepsilon = \sum_{i \in I} \lambda(i)\varepsilon_i$ with I being a finite set, $\lambda \in \Delta(I)$ and $\varepsilon_i = x_i \otimes y_i$ is an independent global strategy, for any $i \in I$. The global strategy ε is obtained by the players observing first a public signal i, which is randomly selected according to the probability distribution λ, and then independently playing x_i and y_i. Such a strategy is called *coordinated*. Note that the set of coordinated strategies is a convex set whose extreme points are the independent global strategies.

A global strategy ε such that $\sum_a \varepsilon(a, b|s, t)$ is independent of s (i.e., $\sum_a \varepsilon(a, b|s, t) = \mathbb{P}(b|t)$) and $\sum_b \varepsilon(a, b|s, t)$ is independent of t, it is called *non-communicating*.

5.2 Nash, correlated and agent-normal-form-correlated equilibria

Here we prove theorems 3.5 and 4.3. We need the following lemma first.

Lemma 5.1 *In a game with common interests, the maximal possible payoff achievable with global coordinated strategies is a Nash equilibrium payoff. It is also a normal-form correlated equilibrium payoff and an agent-normal-form correlated equilibrium payoff.*

Proof: Since the set of global coordinated strategies is compact, the maximum payoff (which is a linear function) is achieved on this set. Since the set of coordinated strategies is convex, the maximum is achieved at an extreme point, which is an independent strategy, say $\varepsilon^* = x^* \otimes y^*$. Since (x^*, y^*) achieves the maximal payoff possible, no player has a profitable deviation, and is therefore a Nash equilibrium. Since any Nash equilibrium is in particular a normal-form correlated equilibrium and an agent-normal-form correlated equilibrium, the proof is complete. ■

We now turn to prove the result.

Proof of Theorem 3.5: Assume that $\mathcal{I} = (S, T, \sigma)$ is better than $\mathcal{I}' = (S', T', \sigma')$. We prove that $\mathcal{I}'$ is a garbled version of $\mathcal{I}$ with a coordinated garbling. Let $\mathcal{G}$ be the set of maps σ'' from K to $\Delta(S' \times T')$ such that (S', T', σ'') is a garbled version of $\mathcal{I}$ with a coordinated garbling. The set $\mathcal{G}$ is closed and convex in $\mathbb{R}^{K \times S' \times T'}$.

Suppose that σ' does not belong to $\mathcal{G}$. By the separation theorem $\mathcal{G}$ can be separated from $\mathcal{G}$ by a hyperplane: $r \in \mathbb{R}^{K \times S' \times T'}$ (r can be thought of also as $|K|$ functions, $r_k : S' \times T' \to \mathbb{R}$, $k \in K$). The separation by r means that for any coordinated garbling q,

$$\sum_{k \in K} \sum_{\substack{s' \in S' \\ t' \in T'}} \sum_{\substack{s \in S \\ t \in T}} r_k(s', t') \sigma(s, t|k) p(k) q(s', t'|s, t) < \sum_{k \in K} \sum_{\substack{s' \in S' \\ t' \in T'}} r_k(s', t') \sigma'(s', t'|k) p(k). \quad (1)$$

Consider the game with the action sets $A = S'$ and $B = T'$ and the payoff function r. The left-hand side of (1) is the payoff associated with the coordinated strategy q when the information structure is $\mathcal{I}$. The right-hand side is the payoff associated with the independent strategy according to which which every player plays his signal when the information structure is $\mathcal{I}'$.

Lemma 5.1 implies that $\mathcal{I}$ is not better than $\mathcal{I}'$ w.r.t. Nash, normal-form correlated and agent-normal-form equilibria.

As for the converse, assume that $\mathcal{I}'$ is a garbled version of $\mathcal{I}$ with a coordinated garbling q. We prove that $\mathcal{I}$ is better than $\mathcal{I}'$ w.r.t. all three solution concepts.

Let ε be a global strategy in some game with common interests with information structure $\mathcal{I}'$ and action sets A and B. Assume that the payoff function is $(r_k)_{k \in K}$, and that ε is a Nash equilibrium. As a Nash equilibrium, ε is an independent strategy $x \otimes y$. The payoff associated with ε is

$$\sum_{k \in K} \sum_{\substack{s' \in S' \\ t' \in T'}} \sum_{\substack{a \in A \\ b \in B}} p(k) \sigma'(s', t'|k) x(a|s') y(b|t') r_k(a, b). \quad (2)$$

The garbling q is coordinated and therefore in the convex hull of the independent garblings. Thus, there is a finite set J and a probability μ on

J, such that $q(s',t'|s,t) = \sum_{j\in J}\mu_j q^1_j(s'|s)q^2_j(t'|t)$ for every $(s,t)\in S\times T$ and $(s',t')\in S'\times T'$. Using this garbling the payoff in (2) can be rewritten as

$$\sum_{k\in K}\sum_{\substack{s\in S\\ t\in T}}\sum_{\substack{a\in A\\ b\in B}}\sum_{j\in J} p(k)\sigma(s,t|k)\mu_j\left(\sum_{s'\in S'} q^1_j(s'|s)x(a|s')\right)\left(\sum_{t'\in T'} q^2_j(t'|t)y(b|t')\right) r_k(a,b). \quad (3)$$

This is the payoff associated with a certain coordinated strategy in the game with information structure $\mathcal{I}$. By Lemma 5.1, the maximal equilibrium payoff in the game with information structure $\mathcal{I}$ is greater than or equal to that in (3). Therefore, $\mathcal{I}$ is better than $\mathcal{I}'$ w.r.t. Nash equilibrium.

Lemma 5.1 ensures that $\mathcal{I}$ is also better than $\mathcal{I}'$ w.r.t. normal-form correlated and agent-normal-form equilibria. ■

5.3 A Bayesian solution and communication equilibria

We first prove an analog of Lemma 5.1.

Lemma 5.2 *In a game with common interests,*
(i) The maximal payoff achievable with a non-communicating strategy is a Bayesian solution payoff. Moreover, every Bayesian solution payoff is achievable by a non-communicating strategy.
(ii) The maximal payoff achievable with a global strategy is a communication equilibrium payoff.

Proof: Let A and B be the action sets and $\mathcal{I}$ be an information structure. Fix a payoff function r_k and let ε be a non-communicating global strategy that achieves the maximal payoff. We prove that ε is a Bayesian solution.

Define, $\Omega = K\times S\times T\times A\times B$, and the probability $\mathbb{P}$ by $\mathbb{P}(k,s,t,a,b) = p(k)\sigma(s,t|k)\varepsilon(a,b|s,t)$. Let κ, ς, τ, α, β be the projections from Ω to K, S, T, A, B respectively. Then, the joint distribution of $(\kappa,\varsigma,\tau,\alpha,\beta)$ is $\mathbb{P}$. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be the partitions generated, respectively, by (ς,α) and (τ,β). Conditions 1,2,4 of the definition of Bayesian solution in Section 4.2 are obviously satisfied. As for 3, we have

$$\mathbb{P}(\kappa=k,\varsigma=s|\mathfrak{A}_2) = \mathbb{P}(\kappa=k,\varsigma=s|\tau,\beta)$$

but we would like to have

$$\mathbb{P}(\kappa=k,\varsigma=s|\mathfrak{A}_2) = \mathbb{P}(\kappa=k,\varsigma=s|\tau).$$

However,

$$\begin{aligned}\mathbb{P}(\beta=b|\kappa=k,\varsigma=s,\tau=t) &= \sum_{a\in A}\mathbb{P}(\beta=b,\alpha=a|\kappa=k,\varsigma=s,\tau=t)\\ &= \sum_{a\in A}\mathbb{P}(\beta=b,\alpha=a|\varsigma=s,\tau=t)=\sum_{a\in A}\varepsilon(a,b|s,t).\end{aligned}$$

By the definition of a non-communicating strategy, $\sum_{a\in A}\varepsilon(a,b|s,t)=\mathbb{P}(\beta=b|\tau=t)$ and therefore,

$$\mathbb{P}(\beta=b|\kappa=k,\varsigma=s,\tau=t)=\mathbb{P}(\beta=b|\tau=t).$$

Hence,

$$\begin{aligned}\mathbb{P}(\kappa=k,\varsigma=s|\tau=t,\beta=b) &= \tfrac{\mathbb{P}(\kappa=k,\varsigma=s,\tau=t)\mathbb{P}(\beta=b|\kappa=k,\varsigma=s,\tau=t)}{\mathbb{P}(\beta=b|\tau=t)\mathbb{P}(\tau=t)}\\ &= \mathbb{P}(\kappa=k,\varsigma=s|\tau=t),\end{aligned}$$

which is condition 3. As in Lemma 5.1, the equilibrium condition follows from the fact the payoff is maximum.

As for the converse, for a Bayesian solution defined over the space $(\Omega,\mathbb{P})$, consider the garbling q given by $q(a,b|s,t)=\mathbb{P}(\alpha=a,\beta=b|\varsigma=s,\tau=t)$. We claim first that q is non-communicating. Indeed, for every s,a,

$$\sum_b q(a,b|s,t)=\mathbb{P}(\alpha=a|\varsigma=s,\tau=t)=\frac{\mathbb{P}(\alpha=a,\varsigma=s,\tau=t)}{\mathbb{P}(\varsigma=s,\tau=t)}=$$
$$\frac{\mathbb{P}(\alpha=a,\varsigma=s)\cdot\mathbb{P}(\tau=t|\alpha=a,\varsigma=s)}{\mathbb{P}(\varsigma=s)\cdot\mathbb{P}(\tau=t|\varsigma=s)}=\frac{\mathbb{P}(\alpha=a,\varsigma=s)}{\mathbb{P}(\varsigma=s)},$$

independently of t. For the last equality note that by condition 3 of Bayesian solution,

$$\mathbb{P}(\tau=t|\alpha=a,\varsigma=s)=\mathbb{P}(\tau=t|\varsigma=s).$$

Next, we show that the joint distribution of (κ, α, β) is indeed distribution that is induced by applying q over σ. For every k, a, b,

$$\mathbb{P}(\kappa = k, \alpha = a, \beta = b) = \mathbb{P}(\kappa = k) \cdot \mathbb{P}(\alpha = a, \beta = b | \kappa = k) =$$

$$\mathbb{P}(\kappa = k) \sum_{s,t} \mathbb{P}(\varsigma = s, \tau = t | \kappa = k) \cdot \mathbb{P}(\alpha = a, \beta = b | \kappa = k, \varsigma = s, \tau = t) =$$

$$\mathbb{P}(\kappa = k) \sum_{s,t} \mathbb{P}(\varsigma = s, \tau = t | \kappa = k) \frac{\mathbb{P}(\alpha = a, \beta = b |_{\tau=t}^{\varsigma=s}) \mathbb{P}(\kappa = k |_{\beta=b}^{\alpha=a}, {}_{\tau=t}^{\varsigma=s})}{\mathbb{P}(\kappa = k | \varsigma = s, \tau = t)}$$

$$= \mathbb{P}(\kappa = k) \sum_{s,t} \mathbb{P}(\varsigma = s, \tau = t | \kappa = k) \cdot \mathbb{P}(\alpha = a, \beta = b | \varsigma = s, \tau = t) =$$

$$p(k) \sum_{s,t} \sigma(s, t | k) q(a, b | s, t).$$

Note that in the third equality we used Bayes' Theorem and in the fourth we used the fact that by condition 4 of Bayesian solution

$$\mathbb{P}(\kappa = k | \alpha = a, \beta = b, \varsigma = s, \tau = t) = \mathbb{P}(\kappa = k | \varsigma = s, \tau = t).$$

Consider now ε a global strategy in a game with action sets A, B and information structure $\mathcal{I}$, that achieves the maximal payoff. We prove that ε is a communication equilibrium. This global strategy is clearly feasible in the game in which the players are asked to report their signals to a mediator. The mediator chooses a couple (a, b) with probability $\varepsilon(a, b | s, t)$ and recommends player 1 to play a and player 2 to play b. The fact the payoff is maximal implies the equilibrium condition. Indeed, any wrong reports of the players (for instance player 1 reporting $\bar{s}$ when his true signal is s) also induce a payoff that is compatible with some global strategy and therefore smaller than the payoff associated to ε. ■

Proof of Theorems 4.7 and 4.8: The proofs follow closely that in the previous section. We therefore provide only a sketch of it. The details are omitted.

First let us assume that $\mathcal{I}$ is better than $\mathcal{I}'$ for Bayesian (resp. communication) equilibrium. Let $\mathcal{G}'$ (resp. $\mathcal{G}''$) denote the set of functions σ'' from K to $\Delta(S' \times T')$ such that (S', T', σ'') is a garbled version of $\mathcal{I}$ with a non-communicating garbling (resp. a general garbling). Assume by contradiction that σ' is not in $\mathcal{G}'$ (resp. $\mathcal{G}''$), then there is a separating hyperplane, and as in the previous section this hyperplane defines a game. By Lemma 5.2, this game contradicts the fact that $\mathcal{I}$ is better than $\mathcal{I}'$.

Assume now that $\mathcal{I}'$ is a garbled version of $\mathcal{I}$ with a non-communicating garbling (resp. any garbling). As in the previous section, simple computation shows that any Bayesian equilibrium payoff (resp. communicating equilibrium payoff) in any game with information structure $\mathcal{I}'$ is a feasible payoff in the same game with information structure $\mathcal{I}$. This follows from Lemma 5.2 and the fact that composition of non-communicating garblings is a non-communicating garbling. ∎

6 Final comments

6.1 More than two players

For the sake of simplicity we stated the model and the results in the case of two players. All the results readily extend to more-than-two-player games with common interests.

6.2 Information structures and the hierarchy of beliefs

When one information structure is better than another and vice versa, we say that they are *equivalent*. A natural question arises is whether two equivalent structures w.r.t. Nash equilibrium in games with common interests induce the same hierarchy of beliefs? Ely and Peski (2005) and Dekel, Fudenberg and Morris (2005) give an example that answers this question in the negative. They showed two information structures that induce the same distribution over players' hierarchies of beliefs, and nevertheless, have different sets of Nash equilibria.

In the class of zero-sum games two equivalent structures induce the *same* value in any zero-sum game. Gossner and Mertens (2001) state that two information structures are equivalent in the class of zero-sum games if and only if they induce the same distribution over players' hierarchies of beliefs.

In a companion paper (Lehrer et al. (2006)) we show in particular that one direction of the question posed above is true. That is, if two information structures are equivalent in games with common interests, then they induce the same distribution over players' hierarchies of beliefs.

6.3 Games with a common objective

Bassan et al. (2003) dealt with a setting similar to that discussed in Section 6.2. An information structure under which player i is endowed with $\mathcal{S}_i$ is said to be *more informative* than an information structure under which player i is endowed with $\mathcal{S}'_i$, if $\mathcal{S}_i$ refines $\mathcal{S}'_i$ for every i. They characterized the games in which a more informative information structure in this sense entails a higher Nash equilibrium payoff for all players.

A game with information structure $\mathcal{I}$ has a *common objective*, if there is a unique Pareto optimal feasible payoff in the induced normal form game. Bassan et al. (2003) prove that the games for which more informative information structures entails a higher Nash equilibrium payoffs for all players are the games that have a common objective.

For given payoff functions, whether or not the Bayesian game has a common objective depends on the information structure. The result of Bassan et al. (2003) implies that $\mathcal{I}$ is better than $\mathcal{I}'$ w.r.t. Nash equilibrium in the class of games that have a common objective with both information structures, $\mathcal{I}$ and $\mathcal{I}'$, if and only if $\mathcal{I}'$ is a garbled version of $\mathcal{I}$ with a coordinated garbling.

Our results extend to all solution concepts in the class of games that have a common objective with both information structures $\mathcal{I}$ and $\mathcal{I}'$. However, it is not clear how to characterize these games.

6.4 The results cannot be extended to general games – an example

The following example is adapted from Forges (1993).

Example 6.1 Let the state space consist of two states: $K = \{1, 2\}$. In the first structure, player 1 knows k and player 2 knows nothing. In the second structure, player 1 knows k and receives in addition the signal s and player 2 receives the signal t, both are independent of k. The signals s and t have the following joint distribution: If $k = 1$ then s, t are identical and randomly chosen from $1, 2, ..., n$. If $k = 2$ then s, t are independent and are chosen randomly according to the uniform distribution from $1, 2, ...n$. The two structures are equivalent in the class of games with common interests. That is, they induce the same Pareto dominant equilibrium payoff. This means that each is a garbled version of the other with a coordinated garbling.

We now provide an example of a game without common interests where these two structures do not induce the same Nash equilibrium payoffs.

Consider the Bayesian game in which $A = \{1, 2, ..., n\}$, $B = \{b_1, b_2\}$, and the two states are equally likely. Suppose that the payoffs are determined only by the state and player 2's action as follows:

$$\begin{array}{cc} b_1 & b_2 \\ (1,2 & 0,0) \\ \multicolumn{2}{c}{k=1} \end{array} \qquad \begin{array}{cc} b_1 & b_2 \\ (1,0 & 0,4) \\ \multicolumn{2}{c}{k=2} \end{array}$$

With the first structure the only Nash and strategic normal form correlated equilibrium payoff is $(0, 2)$. However, in the second structure there is a Nash equilibrium payoff close to $(1/2, 3)$. Such a payoff is obtained by player 1 sending s to player 2, and player 2 playing b_1 if $s = t$ and b_2, otherwise. Thus, although each structure is a garbled version of the other with a coordinated garbling, they induce different Pareto efficient Nash equilibrium payoffs and strategic normal form correlated equilibrium payoffs. ■

Note, however, that the sets of agent-normal-form correlated equilibria under both structures coincide. In a companion paper (Lehrer et al. (2006)) we show that if $\mathcal{I}$ and $\mathcal{I}'$ are two information structures such that each is a garbled version of the other with a coordinated garbling, then they induce the same set of agent-normal-form correlated equilibria.

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