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SIGNED WORDS AND PERMUTATIONS, III; THE MACMAHON VERFAHREN

Dominique Foata and Guo-Niu Han

Glory to Viennot,
Wizard of Bordeaux,
A prince in Physics,
In Mathematics,
Combinatorics,
Even in Graphics,
Sure, in Viennotics.
He builds bijections,
Top calculations.
No one can beat him.
Only verbatim
Can we follow him.
He has admirers,
Who have got down here.
They all celebrate
Such a happy fate.
Sixty years have gone,
He still is our don.

Dedicated to Xavier.
Lucelle, April 2005.

Abstract. The MacMahon Verfahren is fully exploited to derive a five-variable generating polynomial over signed permutations, where the parameters are: "neg" (number of negative values), "fdes" (flag descent number), "fmaj" (flag major index), "ifdes" and "ifmaj" (fdes and fmaj for the inverse permutation); then a three-variable generating polynomial over signed words with given letter multiplicities, where the parameters are "neg", "fmaj", and "fdes".

1. Introduction

To paraphrase Leo Carlitz [Ca56], the present paper could have been entitled "Expansions of certain products," as we want to expand the product

$$K_\infty(u) := \prod_{i\ge0,j\ge0} \frac{1}{(1-uZ_{ij}q_1^iq_2^j)}, \tag{1.1}$$

in its infinite version, and

$$K_{r,s}(u) := \prod_{0\le i\le r,\,0\le j\le s} \frac{1}{(1-uZ_{ij}q_1^iq_2^j)}, \tag{1.2}$$

in its graded version, where

$$Z_{ij} := \begin{cases} Z, & \text{if } i \text{ and } j \text{ are both odd;}\\ 1, & \text{if } i \text{ and } j \text{ are both even;}\\ 0, & \text{if } i \text{ and } j \text{ have different parity.}\end{cases}$$

The second pair under study, which depends on r variables $u_1, \ldots, u_r$, reads

$$L_\infty(u_1,\ldots,u_r) := \prod_{1\le i\le r} \frac{1}{(u_i;q^2)_\infty}\frac{1}{(u_iqZ;q^2)_\infty}, \tag{1.3}$$

$$L_s(u_1,\ldots,u_r) := \prod_{1\le i\le r} \frac{1}{(u_i;q^2)_{\lfloor s/2\rfloor+1}}\frac{1}{(u_iqZ;q^2)_{\lfloor (s+1)/2\rfloor}}. \tag{1.4}$$

In those expressions we have used the usual notations for the q-*ascending factorial*

$$(a;q)_n := \begin{cases} 1, & \text{if } n=0;\\ (1-a)(1-aq)\ \ldots\ (1-aq^{n-1}), & \text{if } n\ge 1; \end{cases} \tag{1.5}$$

in its *finite* form and

$$(a;q)_\infty := \lim_n (a;q)_n = \prod_{n\ge 0}(1-aq^n); \tag{1.6}$$

in its *infinite* form.

All those products will be the basic ingredients for deriving the distributions of various statistics attached to signed permutations and words. By *signed word* we understand a word $w=x_1x_2\ldots x_m$, whose letters are integers, positive or negative. If $\mathbf{m}=(m_1,m_2,\ldots,m_r)$ is a sequence of nonnegative integers such that $m_1+m_2+\cdots+m_r=m$, let $B_{\mathbf{m}}$ be the set of rearrangements $w=x_1x_2\ldots x_m$ of the sequence $1^{m_1}2^{m_2}\ldots r^{m_r}$, with the convention that some letters i $(1\le i\le r)$ may be replaced by their opposite values $-i$. For typographical reasons we shall use the notation $\bar{i} := -i$ in the sequel. When $m_1=m_2=\cdots=m_r=1$, the class $B_{\mathbf{m}}$ is simply the hyperoctahedral group B_m (see [Bo68], p. 252-253) of the signed permutations of order m $(m=r)$.

Using the χ-notation that maps each statement A onto the value $\chi(A)=1$ or 0 depending on whether A is true or not, we recall that the usual *inversion number*, $\operatorname{inv} w$, of each signed word $w=x_1x_2\ldots x_n$ is defined by

$$\operatorname{inv} w := \sum_{1\le j\le n}\sum_{i<j}\chi(x_i>x_j).$$

It also makes sense to introduce

$$\overline{\operatorname{inv}}\, w := \sum_{1\le j\le n}\ \sum_{i<j}\chi(\overline{x}_i>x_j),$$

and define the *flag-inversion number* of w by

$$\operatorname{finv} w := \operatorname{inv} w + \overline{\operatorname{inv}}\, w + \operatorname{neg} w,$$

where $\operatorname{neg} w := \sum_{1\le j\le n}\chi(x_j<0)$. As noted in our previous paper [FoHa05a], the flag-inversion number coincides with the traditional *length function* ℓ, when applied to each signed permutation.

The *flag-major index* “fmaj” and the *flag descent number* “fdes”, which were introduced by Adin and Roichman [AR01] for signed permutations, are also valid for signed words. They read

$$\begin{aligned} \operatorname{fmaj} w &:= 2 \operatorname{maj} w + \operatorname{neg} w; \\ \operatorname{fdes} w &:= 2 \operatorname{des} w + \chi(x_1 < 0); \end{aligned}$$

where $\operatorname{maj} w := \sum_j j\,\chi(x_j > x_{j+1})$ denotes the usual *major index* of w and $\operatorname{des} w$ the *number of descents* $\operatorname{des} w := \sum_j \chi(x_j > x_{j+1})$. Finally, for each signed *permutation* w let w^{-1} denote the inverse of w and define $\operatorname{ifdes} w := \operatorname{fdes} w^{-1}$ and $\operatorname{ifmaj} := \operatorname{fmaj} w^{-1}$.

Notations. In the sequel B_n (resp. $B_{\mathbf{m}}$) designates the hyperoctahedral group of order n (resp. the set of signed words of multiplicity $\mathbf{m} = (m_1, m_2, \ldots, m_r)$), as defined above. Each generating polynomial for B_n (resp. for $B_{\mathbf{m}}$) by some k-variable statistic will be denoted by $B_n(t_1, \ldots, t_k)$ (resp. $B_{\mathbf{m}}(t_1, \ldots, t_k)$). When the variable t_i is missing in the latter expression, it means that the variable t_i is given the value 1.

The main two results of this paper corresponding to the two pairs of products earlier introduced can be stated as follows.

Theorem 1.1. *Let*

$$B_n(t_1, t_2, q_1, q_2, Z) := \sum_{w \in B_n} t_1^{\operatorname{fdes} w} t_2^{\operatorname{ifdes} w} q_1^{\operatorname{fmaj} w} q_2^{\operatorname{ifmaj} w} Z^{\operatorname{neg} w} \tag{1.7}$$

be the generating polynomial for the group B_n by the five-variable statistic (fdes, ifdes, fmaj, ifmaj, neg). *Then,*

$$\sum_{n \ge 0} \frac{u^n}{(t_1^2; q_1^2)_{n+1} (t_2^2; q_2^2)_{n+1}} (1 + t_1)(1 + t_2) B_n(t_1, t_2, q_1, q_2, Z) = \sum_{r \ge 0, s \ge 0} t_1^r t_2^s\, K_{r,s}(u), \tag{1.8}$$

where $K_{r,s}(u)$ is defined in (1.2).

Theorem 1.2. *For each sequence $\mathbf{m} = (m_1, \ldots, m_r)$ let*

$$B_{\mathbf{m}}(t, q, Z) := \sum_{w \in B_{\mathbf{m}}} t^{\operatorname{fdes} w} q^{\operatorname{fmaj} w} Z^{\operatorname{neg} w} \tag{1.9}$$

be the generating polynomial for the class $B_{\mathbf{m}}$ of signed words by the three-variable statistic (fdes, fmaj, neg). *Then*

$$\sum_{\mathbf{m}} (1 + t) B_{\mathbf{m}}(t, q, Z) \frac{u_1^{m_1} \cdots u_r^{m_r}}{(t^2; q^2)_{1 + |\mathbf{m}|}} = \sum_{s \ge 0} t^s L_s(u_1, \ldots, u_r), \tag{1.10}$$

where $|\mathbf{m}| := m_1 + \cdots + m_r$ and $L_s(u_1, \ldots, u_r)$ is defined in (1.4).

It is worth noticing that Reiner [Re93a] has calculated the generating polynomial for B_n by another 5-variable statistic involving each signed permutation and its inverse. The bibasic series he has used are normalized by products of the form $(t_1;q_1)_{n+1}(t_2;q_2)_n$ instead of $(t_1^2;q_1^2)_{n+1}(t_2^2;q_2^2)_{n+1}$.

Using the properties of the fundamental transformation on signed words described in our first paper [FoHa05a] we obtain the following specialization of Theorem 1.1. Let

$$ {}^{\mathrm{finv}}B_n(t,q) := \sum_{w\in B_n} t^{\mathrm{fdes}\,w} q^{\mathrm{finv}\,w} \tag{1.11} $$

be the generating polynomial for the group B_n by the pair (fdes, finv). Then, the q-factorial generating function for the polynomials ${}^{\mathrm{finv}}B_n(t,q)$ $(n\ge 0)$ has the following form:

$$ \sum_{n\ge 0} \frac{v^n}{(q^2;q^2)_n}\, {}^{\mathrm{finv}}B_n(t,q) = \frac{1-t}{-t^2+(v(1-t^2);q)_\infty}\big(t+(v(1-t^2)q;q^2)_\infty\big). \tag{1.12} $$

From identity (1.12) we deduce the generating function for the polynomials ${}^{\mathrm{dess}}B_n(t,q) := \sum_{w\in B_n} t^{\mathrm{dess}\,w} q^{\mathrm{finv}\,w}$, where "dess" is the traditional number of descents for signed permutations defined by

$$ \mathrm{dess}\,w := \mathrm{des}\,w + \chi(x_1<0) \quad \text{instead of} \quad \mathrm{fdes}\,w := 2\,\mathrm{des}\,w + \chi(x_1<0) $$

and recover Reiner's identity [Re93a]

$$ \sum_{n\ge 0} \frac{u^n}{(q^2;q^2)_n}\, {}^{\mathrm{dess}}B_n(t,q) = \frac{1-t}{1-t\,e_q(u(1-t))}\, \frac{1}{(v(1-t);q^2)_\infty}, \tag{1.13} $$

where $e_q(v(1-t^2)) = 1/(v(1-t^2);q)_\infty$ is the traditional q-exponential. This is done in section 4.

As in our second paper [FoHa05b] we make use of the *MacMahon Verfahren* technique to prove the two theorems, which consists of transferring the topology of the signed permutations or words measured by the various statistics, "fdes", "fmaj", to a set of pairs of matrices with integral entries in the case of Theorem 1.1 and a set of plain words in the case of Theorem 1.2 for which the calculation of the associated statistic is easy. Each time there is then a combinatorial bijection between signed permutations (resp. words) and pairs of matrices (resp. plain words) that has the adequate properties. This is the content of Theorem 3.1 and Theorem 4.1.

In all our results we have tried to include the variable Z that takes the number "neg" of *negative* letters of each signed permutation or word into account. This allows us to re-obtain the classical results on the symmetric group and sets of words.

In the next section we recall the *MacMahon Verfahren* technique, which was developed in our previous paper [FoHa05b] for signed permutations. Notice that Reiner [Re93a, Re93b, Re93c, Re95a, Re95b], extending Stanley's [St72] (P,ω)-partition approach, has successfully developed a (P,ω)-partition theory for the combinatorial study of the hyperoctahedral group, which could have been used in this paper. Section 3 contains the proof of Theorem 1.1, whose specializations are given in Section 4. We end the paper with the proof of Theorem 1.2 and its specializations. Noticeably, the generating polynomial for the class $B_{\mathbf{m}}$ of signed *words* by the two-variable statistic $(\mathrm{fdes}, \mathrm{fmaj})$ is completely explicit, in the sense that we derive the factorial generating function for those polynomials and also a recurrence relation, while only the generating function given by (1.10) has a simple form when the variable Z is kept.

2. The MacMahon Verfahren

Let $\mathbb{N}^n$ (resp. $\mathrm{NIW}(n)$) be the set of words (resp. nonincreasing words) of length n, whose letters are nonnegative integers. As done in [FoHa05b] the MacMahon Verfahren consists of mapping each pair $(b,w) \in \mathrm{NIW}(n) \times B_n$ onto a word $c \in \mathbb{N}^n$ as follows. Write the signed permutation w as the linear word $w = x_1x_2\ldots x_n$, where x_k is the image of the integer k $(1 \le k \le n)$. For each $k = 1, 2, \ldots, n$ let z_k be the number of descents in the right factor $x_kx_{k+1}\ldots x_n$ and ϵ_k be equal to 0 or 1 depending on whether x_k is positive or negative. Next, form the words $z(w) := z_1z_2\ldots z_n$ and $\epsilon(w) := \epsilon_1\epsilon_2\ldots\epsilon_n$.

Now, take a nonincreasing word $b = b_1b_2\ldots b_n$ and define $a_k := b_k + z_k$, $c'_k := 2a_k + \epsilon_k$ $(1 \le k \le n)$, then $a(b,w) := a_1a_2\ldots a_n$ and $c'(b,w) := c'_1c'_2\ldots c'_n$. Finally, form the two-row matrix $\begin{pmatrix} c'_1 & c'_2 & \ldots & c'_n \\ |x_1| & |x_2| & \ldots & |x_n| \end{pmatrix}$. Its bottom row is a permutation of $1\,2\ldots n$; rearrange the columns in such a way that the bottom row is precisely $1\,2\ldots n$. Then the word $c(b,w) = c_1c_2\ldots c_n$ which corresponds to the pair (b,w) is defined to be the top row in the resulting matrix.

Example. Start with the pair (b,w) below and calculate all the necessary ingredients:

$$\begin{array}{rcl} b &=& 4\ 2\ 2\ 1\ 1\ 0\ 0 \\ w &=& 3\ \bar{5}\ \bar{1}\ 6\ 7\ \bar{4}\ 2 \\ z(w) &=& 2\ 1\ 1\ 1\ 1\ 0\ 0 \\ \epsilon(w) &=& 0\ 1\ 1\ 0\ 0\ 1\ 0 \\ a(b,w) &=& 6\ 3\ 3\ 2\ 2\ 0\ 0 \\ c'(b,w) &=& 12\ 7\ 7\ 4\ 4\ 1\ 0 \\ c(b,w) &=& 7\ 0\ 12\ 1\ 7\ 4\ 4 \end{array}$$

For each $c = c_1 \ldots c_n \in \mathbb{N}^n$ and let $\operatorname{tot} c := c_1 + \cdots + c_n$, $\max c := \max\{c_1, \ldots, c_n\}$ and let $\operatorname{odd} c$ denote the number of *odd* letters in c . The proof of the following theorem can be found in [FoHa05b, Theorem 4.1].

Theorem 2.1. *For each nonnegative integer s the above mapping is a bijection $(b, w) \mapsto c(b, w)$ of the set of pairs*

$$(b, w) = (b_1 b_2 \ldots b_n,\ x_1 x_2 \ldots x_n) \in \mathrm{NIW}(n) \times B_n$$

such that $2b_1 + \operatorname{fdes} w = s$ onto the set of words $c = c_1 c_2 \ldots c_n \in \mathbb{N}^n$ such that $\max c = s$. Moreover,

$$\begin{gathered} 2b_1 + \operatorname{fdes} w = \max c(b, w); \quad 2\operatorname{tot} b + \operatorname{fmaj} w = \operatorname{tot} c(b, w); \\ \operatorname{neg} w = \operatorname{odd} c(b, w). \end{gathered} \tag{2.1}$$

We end this section by quoting the following classical identity, where b_1 is the first letter of b.

$$\frac{1}{(u;q)_{n+1}} = \sum_{s \ge 0} u^s \sum_{b \in \mathrm{NIW}(n),\, b_1 \le s} q^{\operatorname{tot} b}. \tag{2.2}$$

3. Proof of Theorem 1.1

We apply the MacMahon Verfahren just described to the two pairs (b, w) and (β, w^{-1}), where b and β are two nonincreasing words. The pair (b, w) (resp. (β, w^{-1})) is mapped onto a word $c := c(b, w) = c_1 c_2 \ldots c_n$ (resp. $C := c(\beta, w^{-1}) = C_1 C_2 \ldots C_n$) of length n, with the properties

$$2b_1 + \operatorname{fdes} w = \max c; \quad 2\operatorname{tot} b + \operatorname{fmaj} w = \operatorname{tot} c; \tag{3.1}$$

$$2\beta_1 + \operatorname{ifdes} w = \max C; \quad 2\operatorname{tot} \beta + \operatorname{ifmaj} w = \operatorname{tot} C. \tag{3.2}$$

There remains to investigate the relation between the two words c and C before pursuing the calculation. Form the two-row matrix

$$\begin{pmatrix} c' \\ C \end{pmatrix} = \begin{pmatrix} c'_1 & c'_2 & \ldots & c'_n \\ C_1 & C_2 & \ldots & C_n \end{pmatrix},$$

where $c' = c'_1 c'_2 \ldots c'_n$ designates the nonincreasing rearrangement of the word c. The following lemma is readily verified.

Lemma. *The following properties hold:*
(i) *for each $i = 1, 2, \ldots, n$ the letters c'_i and C_i are of the same parity;*
(ii) *if $c'_i = c'_{i+1}$ is even (resp. is odd), then $C_i \ge C_{i+1}$ (resp. $C_i \le C_{i+1}$).*
(iii) *if $\binom{c'}{C}$ is a two-row matrix of length n having properties* (i) *and* (ii), *there exists one and only one signed permutation w and two nonincreasing words b, β satisfying* (3.1) *and* (3.2).

Example. We take the same example for w as in the previous section, but we also calculate $C = c(\beta, w^{-1})$.

$$\begin{array}{llll}
b & = 4\ 2\ 2\ 1\ 1\ 0\ 0 & \beta & = 3\ 2\ 1\ 1\ 0\ 0\ 0 \\
w & = 3\ \overline{5}\ \overline{1}\ 6\ 7\ \overline{4}\ 2 & w^{-1} & = \overline{3}\ 7\ 1\ \overline{6}\ \overline{2}\ 4\ 5 \\
z(w) & = 2\ 1\ 1\ 1\ 1\ 0\ 0 & z(w^{-1}) & = 2\ 2\ 1\ 0\ 0\ 0\ 0 \\
\epsilon(w) & = 0\ 1\ 1\ 0\ 0\ 1\ 0 & \epsilon(w^{-1}) & = 1\ 0\ 0\ 1\ 1\ 0\ 0 \\
a(b,w) & = 6\ 3\ 3\ 2\ 2\ 0\ 0 & a(\beta, w^{-1}) & = 5\ 4\ 2\ 1\ 0\ 0\ 0 \\
c' := c'(b,w) & = 12\ 7\ 7\ 4\ 4\ 1\ 0 & C' := c'(\beta, w^{-1}) & = 11\ 8\ 4\ 3\ 1\ 0\ 0 \\
c := c(b,w) & = 7\ 0\ 12\ 1\ 7\ 4\ 4 & C := c(\beta, w^{-1}) & = 4\ 1\ 11\ 0\ 0\ 3\ 8
\end{array}$$

$$\begin{pmatrix} c' \\ C \end{pmatrix} = \begin{pmatrix} 12\ 7\ 7\ 4\ 4\ 1\ 0 \\ 4\ 1\ 11\ 0\ 0\ 3\ 8 \end{pmatrix}.$$

Let $(i_1 < i_1 < \cdots < i_r)$ (resp. $(j_1 < j_2 < \cdots < j_s)$) be the increasing sequence of the integers i (resp. the integers j) such that c'_i is *even* (resp. c'_j is *odd*). Define ("e" for "even" and "o" for "odd")

$$2d^e = \begin{pmatrix} 2f^e \\ 2g^e \end{pmatrix} := \begin{pmatrix} c'_{i_1} & c'_{i_2} & \dots & c'_{i_r} \\ C_{i_1} & C_{i_2} & \dots & C_{i_r} \end{pmatrix};$$

$$2d^o + 1 = \begin{pmatrix} 2f^o + 1 \\ 2g^o + 1 \end{pmatrix} := \begin{pmatrix} c'_{j_1} & c'_{j_2} & \dots & c'_{j_s} \\ C_{j_1} & C_{j_2} & \dots & C_{j_s} \end{pmatrix};$$

so that the two two-row matrices

$$d^e = \begin{pmatrix} f^e \\ g^e \end{pmatrix} := \begin{pmatrix} c'_{i_1}/2 & c'_{i_2}/2 & \dots & c'_{i_r}/2 \\ C_{i_1}/2 & C_{i_2}/2 & \dots & C_{i_r}/2 \end{pmatrix},$$

$$d^o = \begin{pmatrix} f^o \\ g^o \end{pmatrix} := \begin{pmatrix} (c'_{j_1} - 1)/2 & (c'_{j_2} - 1)/2 & \dots & (c'_{j_s} - 1)/2 \\ (C_{j_1} - 1)/2 & (C_{j_2} - 1)/2 & \dots & (C_{j_s} - 1)/2 \end{pmatrix},$$

may be regarded as another expression for the two-row matrix $\binom{c'}{C}$.

Define the integers $i_{\max}$ and $j_{\max}$ by:

$$i_{\max} := \begin{cases} (\max c - 1)/2, & \text{if } \max c \text{ is odd;} \\ \max c/2, & \text{if } \max c \text{ is even;} \end{cases}$$

$$j_{\max} := \begin{cases} (\max C - 1)/2, & \text{if } \max C \text{ is odd;} \\ \max C/2, & \text{if } \max C \text{ is even.} \end{cases}$$

Then, to the pair d^e, d^o there corresponds a pair of unique *finite* matrices $D^e = (d^e_{ij})$, $D^o = (d^o_{ij})$ $(0 \le i \le i_{\max}, 0 \le j \le j_{\max})$ (and no other pair of smaller dimensions), where d^e_{ij} (resp. d^o_{ij}) is equal to the number of the two-letters in d^e (resp. in d^o) that are equal to $\binom{i}{j}$.

On the other hand, with $|f|$ designating the length of the word f,

$$\begin{aligned}\operatorname{tot} c &= \operatorname{tot} 2f^e + \operatorname{tot}(2f^o+1) = 2\operatorname{tot} f^e + 2\operatorname{tot} f^o + |f^o|\\ &= 2\sum_{i,j} i\, d^e_{ij} + 2\sum_{i,j} i\, d^o_{ij} + \sum_{i,j} d^o_{ij};\\ \operatorname{tot} C &= \operatorname{tot} 2g^e + \operatorname{tot}(2g^o+1) = 2\operatorname{tot} g^e + 2\operatorname{tot} g^o + |g^o|\\ &= 2\sum_{i,j} j\, d^e_{ij} + 2\sum_{i,j} j\, d^o_{ij} + \sum_{i,j} d^o_{ij}.\end{aligned}$$

The following proposition follows from Theorem 2.1.

Proposition 3.1. *For each pair of nonnegative integers (r,s) the map $(b,\beta,w) \mapsto (D^e, D^o)$ is a bijection of the triples (b,β,w) such that $2b_1 + \operatorname{fdes} w \le r$, $2\beta_1 + \operatorname{ifdes} w \le s$ onto the pairs of matrices $D^e = (d^e_{i,j})$, $D^o = (d^o_{i,j})$ $(0 \le i \le r, 0 \le j \le s)$. Moreover,*

$$2\operatorname{tot} b + \operatorname{fmaj} w = 2\sum_{i,j} i\, d^e_{ij} + 2\sum_{i,j} i\, d^o_{ij} + \sum_{i,j} d^o_{ij}; \tag{3.3}$$

$$2\operatorname{tot} \beta + \operatorname{ifmaj} w = 2\sum_{i,j} j\, d^e_{ij} + 2\sum_{i,j} j\, d^o_{ij} + \sum_{i,j} d^o_{ij}; \tag{3.4}$$

$$\operatorname{neg} w = \sum_{i,j} d^o_{ij}. \tag{3.5}$$

$$|w| = \sum_{i,j} d^e_{ij} + \sum_{i,j} d^o_{ij}. \tag{3.6}$$

Again work with the same example as above. To the two-row matrix

$$\begin{pmatrix} c' \\ C \end{pmatrix} = \begin{pmatrix} 12 & 7 & 7 & 4 & 4 & 1 & 0 \\ 4 & 1 & 11 & 0 & 0 & 3 & 8 \end{pmatrix}$$

there corresponds the pair

$$d^e = \begin{pmatrix} 6 & 2 & 2 & 0 \\ 2 & 0 & 0 & 4 \end{pmatrix}, \quad d^o = \begin{pmatrix} 3 & 3 & 0 \\ 0 & 5 & 1 \end{pmatrix}.$$

As $\max c = 12$ is even (resp. $\max C = 11$ is odd), we have $i_{\max} = 6$, $j_{\max} = 5$ and

$$D^e = \begin{array}{c} \\ 0\\1\\2\\3\\4\\5\\6 \end{array} \begin{array}{c} \begin{array}{cccccc} 0&1&2&3&4&5 \end{array} \\ \begin{pmatrix} .&.&.&.&1&.\\ .&.&.&.&.&.\\ 2&.&.&.&.&.\\ .&.&.&.&.&.\\ .&.&.&.&.&.\\ .&.&.&.&.&.\\ .&.&1&.&.&. \end{pmatrix} \end{array}; \quad D^o = \begin{array}{c} \\ 0\\1\\2\\3\\4\\5\\6 \end{array} \begin{array}{c} \begin{array}{cccccc} 0&1&2&3&4&5 \end{array} \\ \begin{pmatrix} .&1&.&.&&.\\ .&.&.&.&.&.\\ .&.&.&.&.&.\\ 1&.&.&.&.&1\\ .&.&.&.&.&.\\ .&.&.&.&.&.\\ .&.&.&.&.&. \end{pmatrix} \end{array}.$$

Also verify that $2\operatorname{tot} b + \operatorname{fmaj} w = 35 = 2\times(2+2+6) + 2\times(3+3) + 3$; $2\operatorname{tot}\beta + \operatorname{ifmaj} w = 27 = 2\times(2+4) + 2\times(1+5) + 3$ and $\operatorname{neg} w = 3$.

In the following summations b and β run over the set of nonincreasing words of length n. By using identity (2.2) we have

$$\begin{aligned}
&\sum_{n\ge 0}\frac{u^n}{(t_1^2;q_1^2)_{n+1}(t_2^2;q_2^2)_{n+1}}(1+t_1)(1+t_2)B_n(t_1,t_2,q_1,q_2,Z)\\
&=\sum_{n\ge 0}u^n(1+t_1)(1+t_2)B_n(t_1,t_2,q_1,q_2,Z)\sum_{\substack{r'\ge 0,\, s'\ge 0\\ b,\beta,b_1\le r',\,\beta_1\le s'}} t_1^{2r'}t_2^{2s'}q_1^{2\,\mathrm{tot}\,b}q_2^{2\,\mathrm{tot}\,\beta}\\
&=\sum_{n,r',s'}u^n(t_1^{2r'}+t_1^{2r'+1})(t_2^{2s'}+t_2^{2s'+1})\\
&\qquad\times\sum_{\substack{w\in B_n\\ b,\beta,b_1\le r',\,\beta_1\le s'}} t_1^{\mathrm{fdes}\,w}t_2^{\mathrm{ifdes}\,w}q_1^{\mathrm{fmaj}\,w+2\,\mathrm{tot}\,b}q_2^{\mathrm{ifmaj}\,w+2\,\mathrm{tot}\,\beta}Z^{\mathrm{neg}\,w}\\
&=\sum_{n,r',s'}u^n t_1^{r'}t_2^{s'}\sum_{\substack{w\in B_n\\ b,\beta,2b_1\le r',\,2\beta_1\le s'}} t_1^{\mathrm{fdes}\,w}t_2^{\mathrm{ifdes}\,w}q_1^{\mathrm{fmaj}\,w+2\,\mathrm{tot}\,b}q_2^{\mathrm{ifmaj}\,w+2\,\mathrm{tot}\,\beta}Z^{\mathrm{neg}\,w}\\
&=\sum_{n,r,s}u^n t_1^r t_2^s\sum_{\substack{w\in B_n,b,\beta\\ 2b_1+\mathrm{fdes}\,w\le r,\,2\beta_1+\mathrm{ifdes}\,w\le s}} q_1^{\mathrm{fmaj}\,w+2\,\mathrm{tot}\,b}q_2^{\mathrm{ifmaj}\,w+2\,\mathrm{tot}\,\beta}Z^{\mathrm{neg}\,w}.
\end{aligned}$$

We can continue the calculation by using (3.3), (3.4) and (3.5), the inner summation being over matrices D^e, D^o of dimensions $(r+1)\times(s+1)$ with $\sum_{i,j}d_{ij}^e+\sum_{i,j}d_{ij}^o=n$, that is,

$$\begin{aligned}
&=\sum_{n,r,s}u^n t_1^r t_2^s\sum_{D^e,D^o}q_1^{2\Sigma\, id_{ij}^e+2\Sigma\, id_{ij}^o+\Sigma\, d_{ij}^o}q_2^{2\Sigma\, jd_{ij}^e+2\Sigma\, jd_{ij}^o+\Sigma\, d_{ij}^o}Z^{\Sigma\, d_{ij}^o}\\
&=\sum_{r,s}t_1^r t_2^s\sum_{D^e}u^{\Sigma\, d_{ij}^e}q_1^{\Sigma\,(2i)d_{ij}^e}q_2^{\Sigma\,(2j)d_{ij}^e}\sum_{D^o}u^{\Sigma\, d_{ij}^o}q_1^{\Sigma\,(2i+1)d_{ij}^o}q_2^{\Sigma\,(2j+1)d_{ij}^o}Z^{\Sigma\, d_{ij}^o}\\
&=\sum_{r,s}t_1^r t_2^s\sum_{D^e}\prod_{i,j}(uq_1^{2i}q_2^{2j})^{d_{ij}^e}\sum_{D^o}\prod_{i,j}(uZq_1^{2i+1}q_2^{2j+1})^{d_{ij}^o}\quad(0\le i\le r,0\le j\le s)\\
&=\sum_{r,s}t_1^r t_2^s\frac{1}{\prod_{i,j}(1-uq_1^{2i}q_2^{2j})}\frac{1}{\prod_{i,j}(1-uZq_1^{2i+1}q_2^{2j+1})}\quad(0\le i\le r,0\le j\le s)\\
&=\sum_{r\ge 0,s\ge 0}t_1^r t_2^s\frac{1}{\prod_{0\le i\le r,0\le j\le s}(1-uZ_{ij}q_1^iq_2^j)}.\quad\square
\end{aligned}$$

4. Specializations and Flag-inversion number

Define

$$(u;q_1,q_2)_{r+1,s+1}=\prod_{\substack{0\le i\le r\\ 0\le j\le s}}(1-uq_1^iq_2^j);\quad (u;q_1,q_2)_{\infty,\infty}=\prod_{i,j\ge 0}(1-uq_1^iq_2^j).$$

When $Z:=0$ and t_1, t_2, q_1, q_2 are replaced by their square roots, identity (1.8) becomes

$$\sum_{n\ge0}\frac{u^n}{(t_1;q_1)_{n+1}(t_2;q_2)_{n+1}}A_n(t_1,t_2,q_1,q_2)=\sum_{r\ge0,s\ge0}\frac{t_1^r t_2^s}{(u;q_1,q_2)_{r+1,s+1}},\tag{4.1}$$

where $A_n(t_1,t_2,q_1,q_2)$ is the generating polynomial for the symmetric group $\mathfrak{S}_n$ by the quadruple (des, ides, maj, imaj), an identity first derived by Garsia and Gessel [GaGe78]. Other approaches can be found in [Ra79], [DeFo85].

Remember that when a variable is missing in $B_n(t_1,t_2,q_1,q_2,Z)$ it means that the variable has been given the value 1. Multiply both sides of (1.8) by $(1-t_2)$ and let $t_2=1$. We get:

$$\begin{aligned}\sum_{n\ge0}\frac{u^n}{(t_1^2;q_1^2)_{n+1}(q_2^2;q_2^2)_n}&(1+t_1)B_n(t_1,q_1,q_2,Z)\\ &=\sum_{r\ge0}t_1^r\frac{1}{\prod\limits_{0\le i\le r,j\ge0}(1-uZ_{ij}q_1^iq_2^j)}.\end{aligned}\tag{4.2}$$

Again multiply both sides by $(1-t_1)$ and let $t_1=1$:

$$\sum_{n\ge0}\frac{u^n}{(q_1^2;q_1^2)_n(q_2^2;q_2^2)_n}B_n(q_1,q_2,Z)=\frac{1}{\prod\limits_{i\ge0,j\ge0}(1-uZ_{ij}q_1^iq_2^j)}.\tag{4.3}$$

Also the (classical) generating function for the polynomials $A_n(q_1,q_2)$ can be derived from identity (4.3) and reads:

$$\sum_{n\ge0}\frac{u^n}{(q_1;q_1)_n(q_2;q_2)_n}A_n(q_1,q_2)=\frac{1}{(u;q_1,q_2)_{\infty,\infty}}.$$

With $q_1=1$ the denominator of the fraction on the right side of identity (4.2) becomes

$$\begin{cases}(u;q_2^2)_\infty^{(r/2)+1}(uq_2Z;q_2^2)_\infty^{r/2}, & \text{if } r \text{ is even;}\\ (u;q_2^2)_\infty^{(r+1)/2}(uq_2Z;q_2^2)_\infty^{(r+1)/2}, & \text{if } r \text{ is odd.}\end{cases}$$

Hence,

$$\begin{aligned}&\sum_{n\ge0}\frac{u^n}{(1-t_1^2)^{n+1}(q_2^2;q_2^2)_n}(1+t_1)B_n(t_1,q_2,Z)\\ &=\sum_{s\ge0}\frac{t_1^{2s+1}}{((u;q_2^2)_\infty(uq_2Z;q_2^2)_\infty)^{s+1}}+\sum_{s\ge0}\frac{t_1^{2s}}{((u;q_2^2)_\infty(uq_2Z;q_2^2)_\infty)^{s}}\frac{1}{(u;q_2^2)_\infty}.\\ &=\frac{1}{-t_1^2+(u;q_2^2)_\infty(uq_2Z;q_2^2)_\infty}\bigl(t_1+(uq_2Z;q_2^2)_\infty\bigr).\end{aligned}$$

Now replace u by $v(1-t_1^2)$. This implies the following result stated as a theorem.

Theorem 4.1. *Let $B_n(t_1, q_2, Z)$ be the generating polynomial for the group B_n by the triple* (fdes, ifmaj, neg). *Then,*

$$\sum_{n\geq 0} \frac{v^n}{(q_2^2;q_2^2)_n} B_n(t_1,q_2,Z) = \frac{1-t_1}{-t_1^2+(v(1-t_1^2);q_2^2)_\infty\,(v(1-t_1^2)q_2Z;q_2^2)_\infty}\bigl(t_1+(v(1-t_1^2)q_2Z;q_2^2)_\infty\bigr). \tag{4.4}$$

Several consequences are drawn from Theorem 4.1. First, when $Z=0$ and when t_1, q_2 are replaced by their square roots, we get

$$\sum_{n\geq 0} \frac{v^n}{(q_2;q_2)_n} A_n(t_1,q_2) = \frac{1-t_1}{-t_1+(v(1-t_1);q_2)_\infty}, \tag{4.5}$$

where $A_n(t_1,q_2)$ is the generating polynomial for the group $\mathfrak{S}_n$ by the pair (des, imaj), but also by the pair (des, inv) (see [St76], [FoHa97]).

Let $\mathbf{i}\,w := w^{-1}$ denote the inverse of the signed permutation w. At this stage we have to remember that the bijection Ψ of B_n onto itself that we have constructed in our first paper [FoHa05a] preserves the *inverse ligne of route* [FoHa05a, Theorem 1.2], so that the chain

$$\begin{matrix} w & \xrightarrow{\ \mathbf{i}\ } & w_1 & \xrightarrow{\ \Psi\ } & w_2 & \xrightarrow{\ \mathbf{i}\ } & w_3 \\ \begin{pmatrix}\text{fdes}\\ \text{ifmaj}\end{pmatrix} & & \begin{pmatrix}\text{ifdes}\\ \text{fmaj}\end{pmatrix} & & \begin{pmatrix}\text{ifdes}\\ \text{finv}\end{pmatrix} & & \begin{pmatrix}\text{fdes}\\ \text{finv}\end{pmatrix}\end{matrix}$$

shows that the four pairs (fdes, ifmaj), (ifdes, fmaj), (ifdes, finv) and (fdes, finv) are all *equidistributed* over B_n. Therefore,

$$\sum_w t^{\mathrm{fdes}\,w} q^{\mathrm{ifmaj}\,w} = \sum_w t^{\mathrm{ifdes}\,w} q^{\mathrm{fmaj}\,w} = \sum_w t^{\mathrm{ifdes}\,w} q^{\mathrm{finv}\,w} = \sum_w t^{\mathrm{fdes}\,w} q^{\mathrm{finv}\,w},$$

where w runs over B_n. The rightmost generating polynomial was designated by ${}^{\mathrm{finv}}B_n(t,q)$ in (1.11). Therefore we can derive a formula for ${}^{\mathrm{finv}}B_n(t,q)$ by using its (fdes, ifmaj) interpretation. We have then ${}^{\mathrm{finv}}B_n(t,q) = B_n(t_1,q_2,Z)$ with $t_1=t$, $q_2=q$ and $Z=1$. Let $Z:=1$ in (4.4); as $(v(1-t_1^2);q_2^2)_\infty\,(v(1-t_1^2)q_2;q_2^2)_\infty = (v(1-t_1^2);q_2)_\infty$, identity (4.4) implies identity (1.12).

To recover Reiner's identity (1.13) we make use of (1.12) by sorting the signed permutations according to the parity of their flag descent numbers: $B_n(t,q) =: B_n'(t^2,q) + t\,B_n''(t^2,q)$, so that ${}^{\mathrm{dess}}B_n(t^2,q) = B_n'(t^2,q) + t^2\,B_n''(t^2,q)$. Hence

$$\sum_{n\geq 0} \frac{v^n}{(q^2;q^2)_n} {}^{\mathrm{dess}}B_n(t^2,q) = \frac{(v(1-t^2)q;q^2)_\infty - t^2}{-t^2+(v(1-t^2);q)_\infty} + t^2\,\frac{1-(v(1-t^2)q;q^2)_\infty}{-t^2+(v(1-t^2);q)_\infty}$$
$$= \frac{1-t^2}{-t^2+(v(1-t^2);q)_\infty}(v(1-t^2)q;q^2)_\infty.$$

As $1/(v(1-t^2);q)_\infty$ can also be expressed as the q-exponential $e_q(v(1-t^2))$, we then get identity (1.13).

5. The MacMahon Verfahren for signed words

Let $w = x_1x_2\ldots x_m$ be a signed *word* belonging to the class $B_{\mathbf{m}}$, where $\mathbf{m} = (m_1, m_2, \ldots, m_r)$ is a sequence of nonnegative integers such that $m_1 + m_2 + \cdots + m_r = m$. Remember that this means that w is a rearrangement of $1^{m_1}2^{m_2}\ldots r^{m_r}$, with the convention that some letters i $(1 \le i \le r)$ may be replaced by their opposite values $\bar{i}$. Again, let $\epsilon := \epsilon(w) := \epsilon_1\epsilon_2\ldots\epsilon_m$ be the binary word defined by $\epsilon_i = 0$ or 1, depending on whether x_i is positive or negative. As before, let $b \in \mathrm{NIW}(m)$.

The MacMahon Verfahren bijection for signed words is constructed as follows. First, compute the word $z = z_1z_2\ldots z_m$, where z_k is equal to the number of descents in the right factor $x_kx_{k+1}\ldots x_m$, as well as the word $\epsilon = \epsilon_1\epsilon_2\ldots\epsilon_m$ mentioned above, so that, as in the case of signed permutations,

$$\mathrm{fmaj}\, w = 2\,\mathrm{tot}\, z + \mathrm{tot}\,\epsilon. \tag{5.1}$$

Next, define $a_k := b_k + z_k$, $c'_k := 2a_k + \epsilon_k$ $(1 \le k \le m)$, then $a := a_1a_2\ldots a_m$ and $c' := c'_1c'_2\ldots c'_m$. Finally, form the two-row matrix $\begin{pmatrix} c' \\ \mathrm{abs}\, w\end{pmatrix} = \begin{pmatrix} c'_1 & c'_2 & \ldots & c'_m \\ |x_1| & |x_2| & \ldots & |x_m|\end{pmatrix}$. Its bottom row is a rearrangement of the word $1^{m_1}2^{m_2}\ \ldots\ r^{m_r}$.

Make the convention that two biletters $\binom{c'_k}{|x_k|}$ and $\binom{c'_l}{|x_l|}$ commute if and only if $|x_k|$ and $|x_l|$ are different and rearrange the biletters of that biword in such a way that the bottom row is precisely $1^{m_1}2^{m_2}\ \ldots\ r^{m_r}$. The top row in the resulting two-row matrix is then the juxtaposition product of r *nonincreasing* words $b^{(1)} = b^{(1)}_1\ldots b^{(1)}_{m_1}$, $b^{(2)} = b^{(2)}_1\ldots b^{(2)}_{m_2}$, $\ldots$, $b^{(r)} = b^{(r)}_1\ldots b^{(r)}_{m_r}$, of lengths $m_1, m_2, \ldots, m_r$, respectively. Moreover,

$$\begin{aligned}\mathrm{tot}\, b^{(1)} + \mathrm{tot}\, b^{(2)} + \cdots + \mathrm{tot}\, b^{(r)} &= \mathrm{tot}\, c' = 2\,\mathrm{tot}\, a + \mathrm{tot}\,\epsilon\\ &= 2\,\mathrm{tot}\, b + 2\,\mathrm{tot}\, z + \mathrm{tot}\,\epsilon\\ &= 2\,\mathrm{tot}\, b + \mathrm{fmaj}\, w.\end{aligned} \tag{5.2}$$

On the other hand,

$$\begin{aligned}2b_1 + \mathrm{fdes}\, w &= 2b_1 + 2z_1 + \epsilon_1 = c'_1\\ &= \max_i b^{(i)}_1\ (1 \le i \le r).\end{aligned} \tag{5.3}$$

As in the case of signed permutations, we can easily see that for each nonnegative integer s the map $(b, w) \mapsto (b^{(1)}, b^{(2)}, \ldots, b^{(r)})$ is a bijection of the set of pairs $(b, w) \in \mathrm{NIW}(m) \times R_{\mathbf{m}}$ such that $2b_1 + \operatorname{fdes} w = s$ onto the set of juxtaposition products $b^{(1)} b^{(2)} \ldots b^{(r)}$ such that $\max_i b_1^{(i)} = s$. The reverse bijection is constructed in the same way as in the case of signed permutations.

Example. Start with the pair (b, w), where w belongs to $B_{\mathbf{m}}$ with $\mathbf{m} = (2, 2, 2, 2, 2, 2)$, $r = 6$, $m = 12$, the negative elements being overlined.

$$\begin{array}{rl}
\mathrm{Id} & = 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10\ \ 11\ \ 12 \\
b & = 4\ \ 4\ \ 4\ \ 4\ \ 4\ \ 4\ \ 4\ \ 2\ \ 2\ \ 1\ \ 1\ \ 1 \\
w & = 1\ \ \overline{5}\ \ \overline{2}\ \ 2\ \ \overline{3}\ \ \overline{3}\ \ \overline{\mathbf{1}}\ \ 4\ \ \overline{6}\ \ \overline{5}\ \ \mathbf{4}\ \ \mathbf{6} \\
z & = 3\ \ 2\ \ 2\ \ 2\ \ 1\ \ 1\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0 \\
\epsilon & = 0\ \ 1\ \ 1\ \ 0\ \ 1\ \ 1\ \ 1\ \ 0\ \ 1\ \ 1\ \ 0\ \ 0 \\
a & = 7\ \ 6\ \ 6\ \ 6\ \ 5\ \ 5\ \ 5\ \ 3\ \ 2\ \ 1\ \ 1\ \ 1 \\
c' & = 14\ \ 13\ \ 13\ \ 12\ \ 11\ \ 11\ \ \mathbf{11}\ \ 6\ \ 5\ \ 3\ \ \mathbf{2}\ \ \mathbf{2} \\
\operatorname{abs} w & = 1\ \ 5\ \ 2\ \ 2\ \ 3\ \ 3\ \ \mathbf{1}\ \ 4\ \ 6\ \ 5\ \ \mathbf{4}\ \ \mathbf{6} \\
b^{(1)} \ldots b^{(r)} & = 14\ \ \mathbf{11} \mid 13\ \ 12 \mid 11\ \ 11 \mid 6\ \ \mathbf{2} \mid 13\ \ 3 \mid 5\ \ \mathbf{2} \\
1^{m_1} \ldots r^{m_r} & = 1\ \ \mathbf{1} \mid 2\ \ 2 \mid 3\ \ 3 \mid 4\ \ \mathbf{4} \mid 5\ \ 5 \mid 6\ \ \mathbf{6}
\end{array}$$

We verify that $2 \operatorname{tot} b + \operatorname{fmaj} w = 2 \operatorname{tot} b + 2 \operatorname{tot} z + \operatorname{tot} \epsilon = 2 \times 35 + 2 \times 13 + 7 = 103 = \operatorname{tot} b^{(1)} + \cdots + \operatorname{tot} b^{(6)}$ and $2b_1 + \operatorname{fdes} w = 2b_1 + 2z_1 + \epsilon_1 = 2 \times 4 + 2 \times 3 + 0 = 14 = c'_1 = \max_i b_1^{(i)}$.

The combinatorial theorem for signed words that corresponds to Theorem 2.1 is now stated.

Theorem 5.1. *For each nonnegative integer s the above mapping is a bijection of the set of pairs $(b, w) = (b_1 b_2 \ldots b_m,\ x_1 x_2 \ldots x_m) \in \mathrm{NIW}(m) \times B_{\mathbf{m}}$ such that $2b_1 + \operatorname{fdes} w = s$ onto the set of juxtaposition products $b^{(1)} \ldots b^{(r)} \in \mathrm{NIW}(m_1) \times \cdots \times \mathrm{NIW}(m_r)$ such that $\max_i b_1^{(i)} = s$. Moreover,* (5.2) *and* (5.3) *hold, together with*

$$\operatorname{neg} w = \operatorname{odd}(b^{(1)} \ldots b^{(r)}) \tag{5.4}$$

Relation (5.4) is obvious, as the negative letters of w are in bijection with the odd letters of the juxtaposition product. Now consider the generating polynomial $B_{\mathbf{m}}(t, q, Z)$, as defined in (1.9). Making use of (2.2) and of the usual q-identities

$$\frac{1}{(u;q)_N} = \sum_{n \ge 0} \begin{bmatrix} N+n-1 \\ n \end{bmatrix}_q u^n;$$

$$\begin{bmatrix} N+n \\ n \end{bmatrix}_q = \sum_{b \in \mathrm{NIW}(N),\, b_1 \le n} q^{\operatorname{tot} b};$$

we have

$$\begin{aligned}\frac{1+t}{(t^2;q^2)_{m+1}}\,B_{\mathbf m}(t,q,Z)&=\sum_{s'\ge 0}(t^{2s'}+t^{2s'+1})\begin{bmatrix}m+s'\\ s'\end{bmatrix}_{q^2}B_{\mathbf m}(t,q,Z)\\ &=\sum_{s'\ge 0}t^{s'}\begin{bmatrix}m+\lfloor s'/2\rfloor\\ \lfloor s'/2\rfloor\end{bmatrix}_{q^2}B_{\mathbf m}(t,q,Z)\\ &=\sum_{s'\ge 0}t^{s'}\sum_{\substack{b\in \mathrm{NIW}(m),\\ 2b_1\le s'}}q^{2\,\mathrm{tot}\,b}\sum_{w\in B_{\mathbf m}}t^{\mathrm{fdes}\,w}q^{\mathrm{fmaj}\,w}Z^{\mathrm{neg}\,w}\\ &=\sum_{s\ge 0}t^{s}\sum_{\substack{b\in \mathrm{NIW}(m),\,w\in B_{\mathbf m}\\ 2b_1+\mathrm{fdes}\,w\le s}}q^{2\,\mathrm{tot}\,b+\mathrm{fmaj}\,w}Z^{\mathrm{neg}\,w}.\end{aligned}$$

But using (5.2), (5.3), (5.4) we can write

$$\frac{1+t}{(t^2;q^2)_{m+1}}\,B_{\mathbf m}(t,q,Z)=\sum_{s\ge 0}t^{s}\sum_{\substack{b^{(1)},\dots,b^{(r)},\\ \max_i b_1^{(i)}\le s}}q^{\mathrm{tot}\,b^{(1)}+\cdots+\mathrm{tot}\,b^{(r)}}Z^{\mathrm{odd}\,b^{(1)}+\cdots+\mathrm{odd}\,b^{(r)}}$$

and

$$\sum_{\mathbf m}(1+t)B_{\mathbf m}(t,q,Z)\frac{u_1^{m_1}\cdots u_r^{m_r}}{(t^2;q^2)_{1+|\mathbf m|}}=\sum_{s\ge 0}t^{s}\prod_{1\le i\le r}\sum_{m_i\ge 0}\sum_{b^{(i)}}u_i^{m_i}q^{\mathrm{tot}\,b^{(i)}}Z^{\mathrm{odd}\,b^{(i)}},$$

using the notation: $|\mathbf m|:=m_1+\cdots+m_r$.

There remains to evaluate $\sum_m\sum_b u^m q^{\mathrm{tot}\,b}Z^{\mathrm{odd}\,b}$, where the second sum is over all nonincreasing words $b=b_1\dots b_m$ of length m such that $b_1\le s$. Let $1\le i_1<\cdots<i_k\le m$ (resp. $1\le j_1<\cdots<j_l\le m$) be the sequence of all integers i (resp. j) such that b_i is even (resp. b_j is odd). Then, b is completely characterized by the pair (b^e,b^o), where $b^e:=(b_{i_1}/2)\dots(b_{i_k}/2)$ and $b^o:=((b_{j_1}-1)/2)\dots((b_{j_l}-1)/2)$. Moreover, $\mathrm{tot}\,b=2\,\mathrm{tot}\,b^e+2\,\mathrm{tot}\,b^o+|b^o|$. Hence,

$$\begin{aligned}\sum_{m\ge 0}\sum_{b,|b|=m,b_1\le s}u^m q^{\mathrm{tot}\,b}Z^{\mathrm{odd}\,b}&=\sum_{b,b_1\le s}u^{|b|}q^{\mathrm{tot}\,b}Z^{\mathrm{odd}\,b}\\ &=\sum_{b^e,\,2b_1^e\le s}u^{|b^e|}q^{2\,\mathrm{tot}\,b^e}\sum_{b^o,\,2b_1^o\le s-1}(uqZ)^{|b^o|}q^{2\,\mathrm{tot}\,b^o}\\ &=\frac{1}{(u;q^2)_{\lfloor s/2\rfloor+1}}\,\frac{1}{(uqZ;q^2)_{\lfloor (s+1)/2\rfloor}}.\end{aligned}$$

This completes the proof of Theorem 1.2.

6. Specializations

As has been seen in this paper a *t-graded* form such as (1.10) has an ungraded (infinite) version, obtained by multiplying the formula by $(1-t)$ and letting $t := 1$. This gives

$$(6.1)\qquad \sum_{\mathbf{m}} B_{\mathbf{m}}(q,Z)\frac{u_1^{m_1}\cdots u_r^{m_r}}{(q^2;q^2)_{|\mathbf{m}|}} = \prod_{1\le i\le r}\frac{1}{(u_i;q^2)_\infty}\frac{1}{(u_iqZ;q^2)_\infty} = \prod_{1\le i\le r} e_{q^2}(u_i)\, e_{q^2}(u_iqZ),$$

where $e_{q^2}(u)$ denotes the usual q-exponential with basis q^2 ([GaRa90], p. 9). Notice that $B_{\mathbf{m}}(q,Z)$ is the generating polynomial for the class $B_{\mathbf{m}}$ by the pair $(\mathrm{fmaj},\mathrm{neg})$, namely

$$(6.2)\qquad B_{\mathbf{m}}(q,Z) = \sum_w q^{\mathrm{fmaj}\,w} Z^{\mathrm{neg}\,w} \quad (w\in B_{\mathbf{m}}).$$

On the other hand, Let

$$(6.3)\qquad {}^{\mathrm{finv}}B_{\mathbf{m}}(q,Z) := \sum_{w\in B_{\mathbf{m}}} q^{\mathrm{finv}\,w} Z^{\mathrm{neg}\,w}$$

be the generating polynomial for the class $B_{\mathbf{m}}$ by the pair $(\mathrm{finv},\mathrm{neg})$. Using a different approach (the derivation is not reproduced in the paper; it is not quite straightforward), we can prove the identity

$$(6.4)\qquad {}^{\mathrm{finv}}B_{\mathbf{m}}(q,Z) = (-Zq;q)_{m_1+\cdots+m_r}\frac{(q;q)_{m_1+\cdots+m_r}}{(q;q)_{m_1}\cdots(q;q)_{m_r}} = (-Zq;q)_{m_1+\cdots+m_r}\begin{bmatrix} m_1+\cdots+m_r\\ m_1,\ldots,m_r\end{bmatrix}_q,$$

using the traditional notation for the q-multinomial coefficient. In general, $B_{\mathbf{m}}(q,Z)\ne {}^{\mathrm{finv}}B_{\mathbf{m}}(q,Z)$. This can be shown by means of a combinatorial argument.

Let $Z := 1$ in (6.1) and make use of the q-binomial theorem (see [An76], p. 17, or [GaRa90], chap. 1), on the one hand, and let $Z := 1$ in (6.4), on the other hand. We get the evaluations

$$(6.5)\qquad B_{\mathbf{m}}(q) = (-q;q)_{m_1+\cdots+m_r}\begin{bmatrix} m_1+\cdots+m_r\\ m_1,\ldots,m_r\end{bmatrix}_q = {}^{\mathrm{finv}}B_{\mathbf{m}}(q).$$

This shows that "fmaj" and "finv" are equidistributed over each class $B_{\mathbf{m}}$, a property proved "bijectively" in our first paper [FoHa05a].

Next, let $q:=1$ in (6.4). We obtain

$$ {}^{\mathrm{finv}}B_{\mathbf{m}}(Z)=(1+Z)^{m_1+\cdots+m_r}\binom{m_1+\cdots+m_r}{m_1,\ldots,m_r}\quad \bigl(=B_{\mathbf{m}}(Z)\bigr), \tag{6.6}$$

an identity which is equivalent to

$$ \sum_{\mathbf{m}} B_{\mathbf{m}}(Z)\frac{u_1^{m_1}\cdots u_r^{m_r}}{|\mathbf{m}|!}=\prod_{1\le i\le r}\exp(u_i)\,\exp(u_iZ). \tag{6.7}$$

Thus, the q-analog of (6.6) yields (6.4) with a combinatorial interpretation in terms of the flag-inversion number "finv," while (6.1) may be interpreted as q-analog of (6.7) with an interpretation in terms of the flag-major index number "fmaj."

Finally, for $Z=0$, formula (1.10) yields the identity

$$\sum_{\mathbf{m}} A_{\mathbf{m}}(t,q)\frac{u_1^{m_1}\cdots u_r^{m_r}}{(t;q)_{1+|\mathbf{m}|}}=\sum_{s\ge0}t^s\prod_{1\le i\le r}\frac{1}{(u_i;q)_{s+1}},$$

where $A_{\mathbf{m}}(t,q)$ is the generating polynomial for the class of the rearrangements of the word $1^{m_1}2^{m_2}\ldots r^{m_r}$ by $(\mathrm{des},\mathrm{maj})$. As done by Rawlings [Ra79], [Ra80], the polynomials $A_{\mathbf{m}}(t,q)$ can also be defined by a recurrence relation involving either the polynomials themselves, or their coefficients.

7. The Signed-Word-Euler-Mahonian polynomials

We end the paper by showing that the polynomials $B_{\mathbf{m}}(t,q)=B_{\mathbf{m}}(t,q,Z)\,|_{Z=1}$ can be calculated not only by their factorial generating function given by (1.10) for $Z:=1$, but also by a *recurrence formula.*

Definition. A sequence $\Bigl(B_{\mathbf{m}}(t,q)=\sum_{k\ge0}t^kB_{\mathbf{m},k}(q)\Bigr)$ $(\mathbf{m}=(m_1,\ldots,m_r);$ $m_1\ge0,\ldots,m_r\ge0))$ of polynomials in two variables t, q, is said to be *signed-word-Euler-Mahonian*, if one of the following four *equivalent* conditions holds:

(1) The (t^2,q^2)-factorial generating function for the polynomials

$$ C_{\mathbf{m}}(t,q):=(1+t)B_{\mathbf{m}}(t,q) \tag{7.1}$$

is given by identity (1.10) when $Z=1$, that is,

$$ \sum_{\mathbf{m}} C_{\mathbf{m}}(t,q)\frac{u_1^{m_1}\cdots u_r^{m_r}}{(t^2;q^2)_{1+|\mathbf{m}|}}=\sum_{s\ge0}t^s\prod_{1\le i\le r}\frac{1}{(u_i;q)_{s+1}}. \tag{7.2}$$

(2) For each multiplicity $\mathbf{m}$ we have:

$$ \frac{C_{\mathbf{m}}(t,q)}{(t^2;q^2)_{1+|\mathbf{m}|}}=\sum_{s\ge0}t^s\begin{bmatrix}m_1+s\\ s\end{bmatrix}_q\cdots\begin{bmatrix}m_r+s\\ s\end{bmatrix}_q. \tag{7.3}$$

Let $\mathbf{m}+1_r := (m_1,\dots,m_{r-1},m_r+1)$.

(3) The following recurrence relation

$$(7.4)\quad (1-q^{m_r+1})B_{\mathbf{m}+1_r}(t,q) = (1-t^2q^{2+2|\mathbf{m}|})B_{\mathbf{m}}(t,q) - q^{m_r+1}(1-t)(1+tq)B_{\mathbf{m}}(tq,q),$$

holds with $B_{(0,\dots,0)}(t,q)=1$.

(4) The following recurrence relation for the coefficients $B_{\mathbf{m},k}(q)$

$$(7.5)\quad (1+q+\dots+q^{m_r})B_{\mathbf{m}+1_r,k}(q) = (1+q+\dots+q^{m_r+k})B_{\mathbf{m},k}(q) + q^{m_r+k}B_{\mathbf{m},k-1}(q) + (q^{m_r+k}+q^{m_r+k+1}+\dots+q^{2|\mathbf{m}|+1})B_{\mathbf{m},k-2}(q),$$

holds with $B_{(0,\dots,0),0}(q)=1$ and $B_{(0,\dots,0),k}(q)=0$ for every $k\neq 0$.

Theorem 7.1. *The conditions* (1), (2), (3) *and* (4) *in the previous definition are equivalent.*

Proof. The proofs of the equivalences [(1) ⇔ (2)] and [(3) ⇔ (4)] are easy and therefore omitted. For proving the equivalence [(1) ⇔ (3)] proceed as follows. Let $C(t,q;u_1,\dots,u_r)$ denote the *right* side of (7.2) and form the *q-difference* $C(t,q;u_1,\dots,u_r)-C(t,q;u_1,\dots,u_{r-1},u_rq)$ applied to the sole variable u_r. We get

$$\begin{aligned}(7.6)\quad & C(t,q;u_1,\dots,u_r)-C(t,q;u_1,\dots,u_{r-1},u_rq)\\ &=\sum_{s\ge0}\frac{t^s}{(u_1;q)_{s+1}\dots(u_r;q)_{s+1}}-\sum_{s\ge0}\frac{t^s}{(u_1;q)_{s+1}\dots(u_rq;q)_{s+1}}\\ &=\sum_{s\ge0}\frac{t^s}{(u_1;q)_{s+1}\dots(u_r;q)_{s+1}}\Big[1-\frac{1-u_r}{1-u_rq^{s+1}}\Big]\\ &=u_r\sum_{s\ge0}\frac{t^s}{(u_1;q)_{s+1}\dots(u_r;q)_{s+1}}\Big[1-q^{s+1}\frac{1-u_r}{1-u_rq^{s+1}}\Big]\\ &=u_r\big(C(t,q;u_1,\dots,u_r)-qC(tq,q;u_1,\dots,u_{r-1},u_rq)\big).\end{aligned}$$

Now, let $C(t,q;u_1,\dots,u_r) := \sum_{\mathbf{m}} C_{\mathbf{m}}(t,q)u_1^{m_1}\cdots u_r^{m_r}/(t^2;q^2)_{1+|\mathbf{m}|}$ and express each term $C(\dots)$ occurring in identity (7.6) as a factorial series in the u_i's. We obtain

$$\begin{aligned}\sum_{\mathbf{m}}&(1-q^{m_r+1})C_{\mathbf{m}+1_r}(t,q)\frac{u_1^{m_1}\cdots u_r^{m_r+1}}{(t^2;q^2)_{2+|\mathbf{m}|}}\\ &=\sum_{\mathbf{m}}(1-t^2q^{2+2|\mathbf{m}|})C_{\mathbf{m}}(t,q)\frac{u_1^{m_1}\cdots u_r^{m_r+1}}{(t^2;q^2)_{2+|\mathbf{m}|}}\\ &\qquad-\sum_{\mathbf{m}}q^{m_r+1}(1-t^2)C_{\mathbf{m}}(tq,q)\frac{u_1^{m_1}\cdots u_r^{m_r+1}}{(t^2;q^2)_{2+|\mathbf{m}|}}.\end{aligned}$$

Taking the coefficients of $u_1^{m_1}\dots u_{r-1}^{m_{r-1}}u_r^{m_r+1}$ yields

$$(1-q^{m_r+1})C_{\mathbf{m}+1_r}(t,q)=(1-t^2q^{2+2|\mathbf{m}|})C_{\mathbf{m}}(t,q)-q^{m_r+1}(1-t^2)C_{\mathbf{m}}(tq,q),$$

which in turn is equivalent to (7.4) in view of (7.1). All the steps of the argument are reversible. ∎

Remark 1. The fact that $B_{\mathbf{m}}(t,q)$ is the generating polynomial for the class $B_{\mathbf{m}}$ by the pair (fdes, fmaj) can also be proved by the insertion technique using (7.5). The argument has been already developed in [ClFo95a, § 6] for ordinary words. Again let $m:=|\mathbf{m}|=m_1+\cdots+m_r$. With each word from $B_{\mathbf{m}+1^r}$ associate (m_r+1) new words obtained by *marking* one and only one letter equal to r or $\overline{r}$. Let $B^*_{\mathbf{m}+1^r}$ denote the class of all those *marked* signed words. If $w^*=x_1\dots x_i^*\dots x_{m+1}$ is such a word, where the i-th letter is marked (accordingly, equal to either r or $\overline{r}$), let mark_i be the number of letters equal to r or $\overline{r}$ in the right factor $x_{i+1}x_{i+2}\dots x_{m+1}$ and define:

$$\mathrm{fmaj}^*\,w^*:=\mathrm{fmaj}\,w+\mathrm{mark}_i\,w^*.$$

On the other hand, let

$$B_{\mathbf{m},k}(q):=\sum_{w\in B_{\mathbf{m}},\,\mathrm{fdes}\,w=k}q^{\mathrm{fmaj}\,w}.$$

Clearly,

$$\sum_{w^*\in B^*_{\mathbf{m}+1^r},\,\mathrm{fdes}\,w=k}q^{\mathrm{fmaj}^*\,w^*}=(1+q+\cdots+q^{m_r})B_{\mathbf{m}+1_r,k}(q).$$

Now each word w from the class $B_{\mathbf{m}}$ gives rise to $2(m+1)$ *distinct* marked signed words of length $(m+1)$, when the marked letter r or $\overline{r}$ is inserted between letters of w, as well as in the beginning of and at the end of the word. As in the case of the signed permutations, we can verify that for each $j=0,1,\dots,2m+1$ there is one and only one marked signed word w^* of length $(m+1)$ derived by insertion such that $\mathrm{fmaj}^*\,w^*=\mathrm{fmaj}\,w+j$.

On the other hand, "fdes" is not modified if r is inserted to the right of w, or if r or $\overline{r}$ is inserted into a descent $x_i>x_{i+1}$. Furthermore, "fdes" increases by one, if $x_1>0$ (resp. $x_1<0$) and $\overline{r}$ (resp. r) is inserted to the left of w. For all the other insertions "fdes" increases by 2.

Hence, all the marked signed words w^* from $B^*_{\mathbf{m}+1^r}$, such that $\mathrm{fdes}\,w^*=k$ are derived by insertion from three sources:

(i) the set $\{w\in B_{\mathbf{m}}:\mathrm{fdes}\,w=k\}$ and the contribution is: $(1+q+\cdots+q^{m_r+k})B_{\mathbf{m},k}(q)$;

(ii) the set $\{w\in B_{\mathbf{m}}:\mathrm{fdes}\,w=k-1\}$ and the contribution is: $q^{m_r+k}B_{\mathbf{m},k-1}(q)$;

(iii) the set $\{w\in B_{\mathbf{m}}:\mathrm{fdes}\,w=k-2\}$ and the contribution is: $(q^{m_r+k}+\cdots+q^{2|\mathbf{m}|+1})B_{\mathbf{m},k-2}(q)$. ∎

Remark 2. Let $\mathbf{m} := 1^n$ and $B_n(t,q) = B_{\mathbf{m}}(t,q)$, so that $B_n(t,q)$ is now the generating polynomial for the set of signed *permutations* of order n. Then (7.3), (7.4) and (7.5) become

$$\frac{(1+t)B_n(t,q)}{(t^2;q^2)_{n+1}} = \sum_{s\ge 0} t^s(1+q+\cdots+q^s)^n \tag{7.7}$$

$$(1-q)B_n(t,q) = (1-t^2q^{2n})B_{n-1}(t,q) - q(1-t)(1+tq)B_{n-1}(tq,q). \tag{7.8}$$

$$\begin{aligned} B_{n,k}(q) &= (1+q+\cdots+q^k)B_{n-1,k}(q) \\ &\quad + q^k B_{n-1,k-1}(q) + (q^k+q^{k+1}+\cdots+q^{2n-1})B_{n-1,k-2}(q). \end{aligned} \tag{7.9}$$

The last three relations have been derived by Brenti et al. [ABR01], Chow and Gessel [ChGe04], Haglund et al. [HLR04].

Concluding remarks. The statistical study of the hyperoctahedral group B_n was initiated by Reiner ([Re93a], [Re93b], [Re93c], [Re95a], [Re95b]). It had been rejuvenated by Adin and Roichman [AR01] with their introduction of the flag-major index, which was shown [ABR01] to be equidistributed with the length function. See also their recent papers on the subject [ABR05], [ReRo05]. Another approach to Theorems 1.1 and 1.2 would be to make use of the Cauchy identity for the Schur functions, as was done in [ClFo95b].

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Dominique Foata
Institut Lothaire
1, rue Murner
F-67000 Strasbourg, France
`foata@math.u-strasbg.fr`

Guo-Niu Han
I.R.M.A. UMR 7501
Université Louis Pasteur et CNRS
7, rue René-Descartes
F-67084 Strasbourg, France
`guoniu@math.u-strasbg.fr`