from which we can find

$$\rho_{VC^2}\sigma_V\sigma_{C^2} = \frac{2n(1-m)}{m(n+2)(m+n)}$$

and

$$\rho^2_{UC^2} \cong \rho^2_{VC^2} = \frac{(n+3)(n+4)(m+n-1)}{(n+1)(m+n+1)(m+n+2)}.$$

If $n/m = \gamma$ (a fixed constant) and n is large

$$\rho^2 \cong \frac{n}{n+m}.$$

ρ^2 will be near 1 when n is much larger than m. This corresponds, in computing C^2, to dividing the smaller sample into subgroups by the larger. In this case U and C^2 give essentially the same information. When m and n are more nearly equal the two criteria are quite different. For $n > m$, C^2 has fewer possible values than for $n < m$, and is therefore a more sensitive test when $n < m$.

While it is doubtful that this test is biased for large samples, this question will not be considered in the present note.

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SIGNIFICANCE TEST FOR SPHERICITY OF A NORMAL n-VARIATE DISTRIBUTION

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1. Introduction. This note is concerned with testing the hypothesis that a sample from a normal n-variate population is in fact from a population for which the variances are all equal and the correlations are all zero. A population having this symmetry will be called "spherical." Under a linear orthogonal transformation of variates, a spherical population remains spherical, and consequently the features of a sample which furnish information relevant to this hypothesis must be invariant under such transformations.

A situation for which this test is indicated arises when the sample consists of N n-dimensional vectors, for which the variates are the n components along coordinate axes known to be mutually perpendicular, but having an orientation which is, a priori at least, quite arbitrary. A specific application for two dimensions, treated elsewhere [1], may be mentioned. Each of N days furnishes a sine and a cosine Fourier coefficient for a given periodicity, and these, when plotted as ordinate and abcissa, yield a somewhat elliptical cloud of N points. The sine and cosine functions are orthogonal, and their variances have equal expectancies for a random series. The arbitrary nature of the orientation of axes appears here as the arbitrary choice of phase, or origin of time. Of the five ellipses studied, three could easily have come from circular populations (random), and two showed highly significant ellipticity.

2. Likelihood ratio criterion for sphericity. The method of Neyman and Pearson [2] will be used to derive a test criterion which seems entirely suitable. Let Ω be the class of all normal n-variate populations, and let ω be the subclass of all normal n-variate populations satisfying the hypothesis of "sphericity." The likelihood ratio criterion is obtained by taking the ratio of the maximum of the likelihood for variation of all population parameters specifying ω, to the maximum of the likelihood for variation of all population parameters specifying Ω. That is,

$$\lambda_s = \frac{P(\omega \max)}{P(\Omega \max)}. \tag{1}$$

For the set Ω, the probability law for a single observation of the n variates may be written:

$$P = K \mid a_{ij} \mid^{-\frac{1}{2}} e^{-\frac{1}{2}\sum_{i,j} c_{ij}(x_i - a_i)(x_j - a_j)} \qquad (i, j = 1, 2 \cdots n), \tag{2}$$

where c_{ij} is an element of the matrix $\| a_{ij} \|^{-1}$, the a_{ij} being variances and covariances, a_i is the mean value of the variate x_i in the population, and K is a constant the value of which does not concern us here. Then a sample of N from Ω has the probability,

$$P = K^N \mid a_{ij} \mid^{-\frac{1}{2}N} e^{-\frac{1}{2}\sum_{i,j} c_{ij} \sum_{\alpha=1}^{N} (x_{i\alpha} - a_i)(x_{j\alpha} - a_j)}. \tag{3}$$

Letting

$$\sum_{\alpha=1}^{N} x_{i\alpha} = N\bar{x}_i \quad \text{and} \quad \sum_{\alpha=1}^{N} (x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j) = N s_{ij}, \tag{4}$$

differentiating the logarithm of P with respect to the parameters a_i and a_{ij}, and setting these derivatives equal to zero, the maximum likelihood estimates,

$$\hat{a}_i = \bar{x}_i ; \qquad \hat{a}_{ij} = s_{ij}, \tag{5}$$

are obtained. Substituting these values in equation (3) we find that the maximum value of the likelihood is

$$P(\Omega \max) = K^N \mid s_{ij} \mid^{-\frac{1}{2}N} e^{-\frac{1}{2}N}. \tag{6}$$

The derivation of $P(\omega \max)$ proceeds upon similar lines, but is simpler, for the probability law for the set ω is obtained from (3) by setting

$$c_{ij} = c\delta_{ij}, \tag{7}$$

where c is any positive constant, and $\delta_{ij} = 0$ if $i \neq j$ and 1 if $i = j$. The result is found to be

$$P(\omega \max) = K^N (s_0)^{-\frac{1}{2}Nn} e^{-\frac{1}{2}N} \tag{8}$$

where s_0 is defined by

$$ns_0 = \sum_{i=1}^{n} s_{ii}. \tag{9}$$

The likelihood ratio criterion is therefore

$$\lambda_s = \left[\frac{|s_{ij}|^{\frac{1}{2}}}{(s_0)^{\frac{1}{2}n}} \right]^N. \tag{10}$$

It will be convenient to designate the Nth root of this statistic as L_{sn}, where the second subscript indicates the number of variates:

$$L_{sn} \equiv \frac{|s_{ij}|^{\frac{1}{2}}}{s_0^{\frac{1}{2}n}}. \tag{11}$$

3. The moments of the distribution of L_{sn} when the population is spherical. The distribution of L_{sn} cannot be easily obtained in explicit form for a general n, but the moments of L_{sn} when the hypothesis tested is true are easily found.

Note first that L_{sn} may be resolved into two factors which are, when the population is spherical, statistically independent:

$$L_{sn} = \frac{(s_1 s_2 s_3 s_4 \cdots s_n)^{\frac{1}{2}}}{s_0^{\frac{1}{2}n}} \cdot |r_{ij}|^{\frac{1}{2}}. \tag{12}$$

The first factor is just the one appropriate for testing the equality of the n variances when the orientation of the coordinate axes is fixed in advance, while the second factor is the square root of the determinant of correlation coefficients. The moments of the distributions of these two statistics are known [3], and since the two are independent (for zero correlation in the population), we may write:

$$M_h(L_{sn}) = M_h(A) M_h(B), \tag{13}$$

where A and B are used to indicate the two factors, and M_h indicates the hth moment. The moments are given by

$$M_h(L_{sn}) = \prod_{i=1}^{n} \left[\frac{\Gamma\frac{1}{2}(N - i + h)}{\Gamma\frac{1}{2}(N - i)} \right] n^{\frac{1}{2}nh} \frac{\Gamma\frac{1}{2}(n(N-1))}{\Gamma\frac{1}{2}(n(N - 1 + h))}. \tag{14}$$

4. Significance test for $n = 2$. For $n = 1$, $M_h(L_{s1}) = 1$ for any h, as it should, since L_{s1} is then identically 1, and the concept of sphericity is meaningless. For $n = 2$, the expression (14) reduces to,

$$M_h(L_{s2}) = \frac{\Gamma(N - 2 + h)\Gamma(N - 1)}{\Gamma(N - 1 + h)\Gamma(N - 2)} = \frac{N - 2}{N - 2 + h} \tag{15}$$

and the distribution is thus found to be

$$D(L_{s2}) = (N - 2)L_{s2}^{N-3}\, dL_{s2}\,. \tag{16}$$

Thus for $n = 2$, the significance of the value of L_{s2} obtained from a given sample of N points in a plane is simply

$$P(L_{s2} < L'_{s2}) = L'^{N-2}_{s2}. \tag{17}$$

These results for $n = 2$ were obtained by another method in [1].

5. Significance test for $n = 3$. For $n = 3$ and higher values of n, no simple expression for the distribution seems obtainable. In this case it appears reasonable to fit a Pearson curve of the type,

$$y = Kx^{p-1}(1 - x)^{q-1}, \tag{18}$$

by adjusting p and q so as to obtain agreement with the first two moments of the actual distribution. The calculations were carried out for L_{s3}^2 rather than L_{s3} itself, to simplify the moment expressions. The first moment of L_{s3}^2 is the second moment of L_{s3}, and is given as a function of N by the equation,

$$\mu_1(N) = \frac{(3N - 6)(3N - 9)}{(3N - 2)(3N - 1)}. \tag{19}$$

Recurrence relations, similar to those noted by Lengyel [4] in carrying out a similar task, hold for the moments of L_{s3}^2; hence,

$$\mu_2(N) = \mu_1(N)\mu_1(N + 2). \tag{20}$$

Explicit solution of the equations for p and q in terms of N is possible:

$$p = \frac{(9N + 5)(N - 2)(N - 3)}{2(9N^2 - 8N - 15)}, \tag{21}$$

$$q = \frac{2(9N - 13)(9N + 5)}{9(9N^2 - 8N - 15)}. \tag{22}$$

For values of $N > 30$, acceptable approximations to p and q are obtained by carrying out the division indicated in (21) and (22):

$$p = \tfrac{1}{2}(N - 4) + 2/9 + 70/81(N + 1) \cdots, \tag{23}$$

$$q = 2 + \frac{140}{9(3N - 2)^2} \cdots. \tag{24}$$

The values of p and q are given in Table I so that those desiring other than the standard significance levels may readily enter the Pearson tables.

For N a multiple of 4 from 8 to 48, and a multiple of 10 from 50 to 100, the significance levels were taken from the Incomplete Beta-Function Tables, using adequate interpolation. The final Table I was then prepared by filling in the skeleton table by interpolation with respect to N.

From the results of Wilks [5] it follows that $-2N \log_e L_{sn}$ is, for large N,

TABLE I

5%, 1%, and 0.1% levels of significance for the 3-dimensional sphericity criterion, $L_{s3}^2 = \lambda^{2/N}$, and the values of p and q for the Pearson Type I curves used in calculating these levels

N	5%	1%	0.1%	p	q
8	0.172	0.083	0.030	2.3239	2.0312
10	.278	.165	.080	3.3044	2.0194
12	.366	.243	.139	4.2911	2.0131
14	.436	.312	.197	5.2816	2.0095
16	.494	.372	.252	6.2744	2.0072
18	.541	.423	.301	7.2688	2.0057
20	.580	.466	.346	8.2642	2.0046
22	.614	.504	.386	9.2605	2.0038
24	.642	.538	.422	10.2574	2.0032
26	.667	.567	.454	11.2548	2.0027
28	.689	.593	.483	12.2526	2.0023
30	.708	.616	.510	13.2506	2.0020
32	.724	.637	.534	14.2488	2.0018
34	.739	.655	.555	15.2473	2.0016
36	.753	.672	.575	16.2458	2.0014
38	.765	.687	.594	17.2447	2.0012
40	.776	.701	.610	18.2435	2.0011
42	.786	.714	.626	19.2425	2.0010
44	.795	.726	.640	20.2416	2.0009
46	.804	.736	.653	21.2408	2.0008
48	.811	.746	.665	22.2400	2.0008
50	.819	.756	677	23.2394	2.0007
55	.834	.776	.703	*	*
60	.848	.793	.725	28.2365	2.0005
65	.859	.808	.744	*	*
70	.869	.821	.760	33.2345	2.0004
75	.877	.832	.775	*	*
80	.885	.842	.788	38.2328	2.0003
85	.891	.851	.799	*	*
90	.897	.859	.809	43.2317	2.0002
95	.902	.866	.819	*	*
100	.907	.872	.827	48.2308	2.0002

* No values for p and q were calculated for these values of N; the levels were obtained by interpolation (see text).

distributed approximately like χ^2 with $n(n - 1)/2$ degrees of freedom. However, equation (24) above suggests that for large N one may get a very good approximation (for $n = 3$) by setting $q = 2$; the significance test for $n = 3$ then becomes,

$$P(L_{s3} < L'_{s3}) = \tfrac{1}{2}L'^{N-4}_{s3}[(N - 2) - (N - 4)L'^{2}_{s3}]. \tag{25}$$

Probably similar approximations can be found for other values of n. It is a pleasure to acknowledge the helpful comments and advice which I received from Mr. A. M. Mood of Princeton. Recognition is also due Mr. Wallace Brey, a student assistant under the National Youth Administration, who aided in the computations.

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A SIMPLE SAMPLING EXPERIMENT ON CONFIDENCE INTERVALS

BY S. KULLBACK AND A. FRANKEL

1. Introduction. In order to illustrate some of the notions of the theory of confidence or fiducial limits in connection with a course in Statistical Inference at the George Washington University, we had the class carry out certain simple experiments, following a suggestion in one of Neyman's papers on Statistical Estimation [1]. In the belief that the experimental data may be of interest to others, we present the results herein.

2. The problem. We consider the problem of estimating the range θ of a rectangular population defined by $p(x, \theta)\, dx = dx/\theta$, $0 \leqq x \leqq \theta$ and in particular, for simplicity, we limit ourselves to samples of two and four. We consider three possible approaches to the problem, viz., by using (a) the sample range (b) the sample average or total (c) the larger (largest) sample value. Let us consider each in turn.

(*a*) *Sample range.* Wilks [2] has shown that for samples of n and confidence coefficient $1 - \alpha$, the confidence or fiducial limits for the population range θ are given by r and r/ψ_α, where r is the sample range and ψ_α is determined by

$$\psi_\alpha^{n-1}[n - (n - 1)\psi_\alpha] = \alpha. \tag{1}$$

For $n = 2, \alpha = 0.19$ and $n = 4, \alpha = 0.1792$, (1) yields $\psi_\alpha = 0.1$ and $\psi_\alpha = 0.4$ respectively. Accordingly, for samples of two with confidence coefficient