



## **Similarity analysis for laminar natural convection flow**

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### **Abstract**

This paper deals with a theoretical study of laminar natural convection over a semi-infinite vertical plate. It is assumed that the plate is maintained at a given concentration in some chemical species and convection is induced by diffusion into and chemical reaction with the ambient fluid. The fluid is assumed to be viscous and the induced fluid flow, steady. A similarity transform to one variable is possible in the absence of chemical reaction. However, when the chemical reaction takes place between the plate and the ambient fluid, a similarity solution is not available. Thus to obtain an analytical solution of the problem, perturbation expansions about an additional similarity variable which is dependent on the reaction rate must be employed. It has been found that the Schmidt number  $Sc$  and the reaction order  $n$  are the two fundamental parameters of the problem.

### **1 Introduction**

Boundary value problems involving the principles of heat and mass diffusion in a fluid medium, where the results are directly influenced by the presence of fluid motion may in general be termed convection processes. If the motion of the fluid is determined by boundary conditions specified externally to the system, the process is called forced convection. Otherwise, if the fluid velocities are caused by the effects of gravity force, i.e. by the interaction of a body force with variable density arising from heat or mass diffusion, then the process is called natural convection. The phenomena of natural

convection can be observed in the atmosphere, in bodies of water, adjacent to domestic heating radiators or over semi-heated fields and roads. In this study we shall be concerned with the natural convection process.

One classical problem of natural convection flow has been stated by Jacob [4] and Kays [5]. The problem concerns the flow between two vertical parallel plates held at different temperatures and placed in a fluid of uniform density and viscosity.

A problem of steady state natural convection induced by chemical diffusion from a vertical plate has been reported by Levich [7]. The plate, at zero concentration of a chemical species  $A$ , but containing some catalytic substances is placed in a fluid solution of  $A$  at concentration  $C_0 > 0$ . When the plate comes into contact with the solution, an heterogeneous chemical reaction takes place on the plate. Changes in concentration imply density gradients while in the presence of gravitational field, induce natural convection flow near the plate.

The principles of transport phenomena are described in a book by Bird [2], and that of chemical kinetics by Aris [1]. The methods of similarity and perturbation are available in the references including the authors Hansen [3] and VanDyke [11]. Recently a few studies on the similar solutions for natural convection flow have been performed by Kulkarni et al [6] and Shu and Wilks [9]. Interested readers are referred to their works.

An interesting extension of the natural convection problem is the study of effects caused by the inclusion of general chemical reaction of order  $n$ . In particular, this article is concerned with the study of the steady state behavior of the same binary system composed of a semi-infinite plate and ambient fluid, each initially at different concentrations of a given species, but between which a homogeneous irreversible isothermal chemical reaction of order  $n$  is assumed to take place.

## 2 Uniform concentration at the plate

We consider that a vertical plate composed of species  $A$  has been immersed in a fluid at rest with uniform density and viscosity. We will assume that the concentration of  $A$  in the solution at the plate surface is maintained (by some suitable external means) at a uniform concentration  $C_0$ . Concentration differences between the plate and the ambient fluid will induce convection-diffusional flow and we assume that steady-state conditions have been attained. During this process, the species  $A$  and the fluid  $B$  will react chemically with each other. The primary objective in this study is to show the effect of this reaction upon the mass transfer and the natural convection flow which ensues.

## 2.1 Perturbation method

The differential equations which describe the steady-state behavior of the system described above can be written as (see Bird [2])

$$\psi_Y \psi_{YX} - \psi_X \psi_{YY} = \nu \psi_{YYY} + g\beta^*(C - C_\infty), \quad (1)$$

$$\psi_Y C_X - \psi_X C_Y = DC_{YY} - k(C - C_\infty)^n, \quad (2)$$

with boundary conditions

$$\left. \begin{array}{l} Y = 0; \\ \psi_Y = \psi_X = 0, \\ C = C_0, \end{array} \right\} \quad (3)$$

$$\left. \begin{array}{l} Y \rightarrow \infty \\ \psi_Y = \psi_X = 0 \\ C = C_\infty \end{array} \right\} \quad (4)$$

Introducing the transformations

$$c = \frac{C - C_\infty}{C_0 - C_\infty}, \quad (5)$$

$$\eta(X, Y) = \frac{Y}{X} \left\{ \frac{Gr_x}{4} \right\}^{1/4} \quad (6)$$

$$\psi(X, Y) = 4\nu \left\{ \frac{Gr_x}{4} \right\}^{1/4} f(\eta) \quad (7)$$

where

$$Gr_x = \frac{g\beta^* X^3 (C_0 - C_\infty)}{\nu^2} \quad (8)$$

which may be called the local Grashof number, equations (1) and (2) reduce to

$$\frac{\partial^3 f}{\partial \eta^3} + c + 3f \frac{\partial^2 f}{\partial \eta^2} - 2 \left( \frac{\partial f}{\partial \eta} \right)^2 = 0 \quad (9)$$

$$\frac{1}{Sc} \frac{\partial^2 c}{\partial \eta^2} + 3f \frac{\partial c}{\partial \eta} - \epsilon(X) c^n = 0, \quad (10)$$

where

$$\epsilon(X) = \frac{2k(C_0 - C_\infty)^{n-3/2}}{\sqrt{g\beta^*}} X^{1/2} \quad (11)$$

and  $Sc = \nu/D$ , which is called the Schmidt number. Use of the transformations (5), (6) and (7) reveals that the partial differential equations (9) and (10) nearly reduce to ordinary differential equations in the variable  $\eta$  excepting for one coefficient  $\epsilon(X)$  which remains dependent on  $X$ . This coefficient may be termed the local reaction-rate number and may be considered arbitrarily small depending on the slowness of the reaction. Hence for  $\epsilon = 0$ , the similarity analysis is complete and equations (9) and (10) may be assumed to admit a solution  $c = c_0(\eta)$  and  $f = f_0(\eta)$ . In the case  $\epsilon \neq 0$ , if the reaction-rate is such that

$$\frac{k^2(C_0 - C_\infty)^{2n-3}}{g\beta^*} = 0(\xi)$$

for some  $\xi > 0$ , a solution may be possible by perturbation-expansion about  $\epsilon(X)$  of order  $\xi$ , provided attention is confined to the region downstream of the leading edge given by  $0 < X \leq \xi$ .

We therefore rewrite the transforms as follows:

$$c(X, Y) = c(\eta, \epsilon) \quad (12)$$

$$\psi(X, Y) = 4\nu \left\{ \frac{Gr_x}{4} \right\}^{1/4} f(\eta, \epsilon) \quad (13)$$

where

$$\eta(X, Y) = \frac{Y}{X} \left\{ \frac{Gr_x}{4} \right\}^{1/4} \quad (14)$$

$$\text{and } \epsilon(X) = \frac{2k(C_0 - C_\infty)^{(n-3/2)}}{\sqrt{g\beta^*}} X^{1/2} \quad (15)$$

Introducing these transformations into equations (9) and (10) results in

$$f_{\eta\eta\eta} + c + 3ff_{\eta\eta} - 2f_\eta^2 + 2\epsilon\{f_\epsilon f_{\eta\eta} - f_\eta f_{\eta\epsilon}\} = 0 \quad (16)$$

$$\frac{c_{\eta\eta}}{Sc} + 3fc_\eta + \epsilon\{2f_\epsilon c_\eta - 2f_\eta c_\epsilon - c^n\} = 0 \quad (17)$$

and the following perturbation scheme about  $\epsilon$  may be employed:

$$f(\eta, \epsilon) = f_0(\eta) + \epsilon f_1(\eta) + \epsilon^2 f_2(\eta) + \dots \quad (18)$$

$$c(\eta, \epsilon) = c_0(\eta) + \epsilon c_1(\eta) + \epsilon^2 c_2(\eta) + \dots \quad (19)$$

Substituting (18) and (19) into (16) and (17) results in the following sets of ordinary differential equations for each order of  $\epsilon$ :

The zeroth-order approximation:

$$f_0''' + c_0 + 3f_0 f_0'' - 2f_0'^2 = 0 \quad (20)$$

$$c_0'' + 3Sc f_0 c_0' = 0 \quad (21)$$

The first-order approximation:

$$f_1''' + c_1 + 5f_1f_0'' + 3f_0f_1'' - 6f_0'f_1' = 0 \quad (22)$$

$$c_1'' + Sc\{5f_1c_0' + 3f_0c_1' - 2c_1f_0' - c_0^2\} = 0 \quad (23)$$

The boundary conditions may be written as

$$\text{at } \eta = 0 : \left. \begin{array}{lcl} f_r(0) & = & 0 \\ f_r'(0) & = & 0 \\ c_0(0) & = & 1 \\ c_{r+1}(0) & = & 0 \end{array} \right\} \quad (24)$$

$$\begin{array}{lcl} f_r'(\infty) & = & c_r(\infty) = 0 \quad \text{when } \eta \rightarrow \infty \\ r & = & 0, 1, 2, 3, \dots m. \end{array} \quad (25)$$

Here the prime denotes differentiation with respect to  $\eta$ . The zeroth-order approximations describe the stream function and the concentration distribution without chemical reaction, whereas the higher order approximations describe these distributions when chemical reaction is present in the system.

### 3 Distributed concentration along the plate for various reaction-kinetics

This section consists of similarity solutions which may be found when the initial concentration along the surface of the plate is assumed to obey some algebraic law with respect to the order of chemical reaction. In the previous section, we have presented an analysis giving rise to perturbation-type similarity solutions.

Consider the following transformation:

$$\text{Similarity variable: } \eta(X, Y) = Yb(X) \quad (26)$$

$$\text{Stream function: } \psi(X, Y) = \nu a(X)f(\eta) \quad (27)$$

$$\text{dimensionless concentration: } c(X, Y) = \frac{C - C_\infty}{C_0 - C_\infty} \quad (28)$$

$$\text{and } e(X) = C_0 - C_\infty \quad (29)$$

where  $C_\infty$  is a constant concentration at infinity.

Introducing these transformations into equations (1) and (2), one obtains the following ordinary differential equations:

$$f''' + \frac{g\beta^*e}{ab^3\nu^2}c + \frac{a_X}{b}ff'' - \left(\frac{ab_X}{b^2} + \frac{a_X}{b}\right)f'^2 = 0 \quad (30)$$

$$\frac{c''}{Sc} + \frac{a_X}{b}fc' - \frac{a}{b}\frac{e_X}{e}f'c - \frac{ke^{n-1}}{\nu b^2}c^n = 0 \quad (31)$$

provided,

$$\frac{a_X}{b} = C_1, \quad \frac{ab_X}{b^2} = C_2, \quad \frac{ae_X}{be} = C_3, \quad \frac{g\beta^*e}{ab^3\nu^2} = C_4, \quad \frac{ke^{n-1}}{\nu b^2} = C_5 \quad (32)$$

where  $C_1, C_2, C_3, C_4$  and  $C_5$  are constants, and sub  $X$  means differentiation with respect to  $X$ . The relations (32) reveal that the transformations (26) to (28) must be of the following forms:

$$\begin{aligned} \eta(X, Y) &= \frac{Y}{X} \left[ \frac{Gr_x}{6-4n} \right]^{1/4} \\ \psi(X, Y) &= \nu(6-4n) \left[ \frac{Gr_x}{6-4n} \right]^{1/4} f(\eta) \\ c(X, Y) &= \frac{C - C_\infty}{C_0 - C_\infty} \end{aligned} \quad (33)$$

and  $C_0 - C_\infty$  = initial distribution function.

$$= NX^{\frac{1}{3-2n}} \quad (34)$$

where  $Gr_x$  = local Grashof numbers

$$= \frac{g\beta^*X^3(C_0 - C_\infty)}{\nu^2} \quad (35)$$

and  $N$  is a dimensional quantity.

The equations (30) and (31) become:

$$f''' + c + (5-3n)ff'' - (4-2n)f'^2 = \theta \quad (36)$$

$$\frac{c''}{Sc} + (5-3n)fc' - 2f'c - pc^n = 0 \quad (37)$$

where  $p = \frac{kN^{n-1}}{\sqrt{\frac{N\theta\beta^*}{6-4n}}}$ , a dimensionless rate number.

The boundary conditions are:

$$\left. \begin{aligned} f(0) &= 0 \\ f'(0) &= 0 \\ c(0) &= 1 \end{aligned} \right\} \quad \eta = 0 \quad (38)$$

$$\eta \rightarrow \infty \quad \left. \begin{aligned} f'(\infty) &= 0 \\ c(\infty) &= 0 \end{aligned} \right\} \quad (39)$$

It has been observed that the initial distribution (34) can produce valid ordinary differential equations (36) and (37) except for  $n = 1, \frac{3}{2}$  and  $\frac{5}{3}$ . From physical considerations, the analysis for the case  $n = \frac{5}{3}$  is of little importance, because a reaction with this order is very rare.



## 4 Discussion of results

Numerical solutions for the dimensionless concentration and velocity profiles are depicted in graphical forms. Fig. 1 shows the effect of order of reaction  $n$ , as it varies from  $n = 0, 1, 2$  at fixed  $Sc = 0.01$ , on the concentration distribution at the station  $x = 0.5$ . From these results it is evident that within the range  $(0, 2)$ , an increase of  $n$  (which effectively increases the sensitivity of the species depletion rate with change in concentration) tends to expand the diffusion-convection layer away from the plate by a factor from two to three times, with a general decrease in concentration gradients. Hence one may expect a larger, while less distinct, diffusion layer for larger orders  $n$ . Fig. 2 shows similar effects of varying  $n$  on the velocity profiles at the same station, emphasizing the expansion of the actual convection-layer in concert with the diffusion-layer when  $n$  is increased.

The results of the numerical solutions for concentration and velocity profiles for the distributed concentration along the plate at  $Sc$  from 0.01 to 1000,  $n = 0$  and  $p = 1$  are given in Fig. 3 and Fig. 4 respectively. These profiles are drawn against the dimensionless similarity variable  $\eta$ . The boundary layer region decreases in thickness as the Schmidt number increases; thus for high Schmidt number fluids, mass diffusion takes place within a thin layer. This agrees with the results discussed by many previous researchers including Ostrach [8] and Sparrow and Gregg [10] in connection with heat transfer problems.

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## References

1. Aris, R., *Introduction to the Analysis of Chemical Reactors*, Prentice Hall, Inc., Englewood Cliffs, N.J., 1965.
2. Bird, R.B., Theory of diffusion, In *Advances in Chemical Engineering*, Vol. 1, Academic Press, New York, 1956.
3. Hansen, A.G., *Similarity Analyses of Boundary Value Problems in Engineering*, Prentice-Hall, Englewood Cliffs, N.J., 1964.
4. Jakob, M. *Heat Transfer*, John Wiley and Sons, New York, Vol. 1, 1949.



5. Kays, W.M. *Convection Heat and Mass Transfer*, McGraw-Hill, New York, 1966.
6. Kulkarni, A. K., Jacobs, H. R. and Hwang, J. J. Similarity solution for natural convection flow over an isothermal vertical wall immersed in thermally stratified medium, *Int. J. Heat Mass Transfer*, 1987, 30(4), 691-698.
7. Levich, V.G., *Physico-Chemical Hydrodynamics*, Prentice-Hall, Inc., Englewood Cliffs, N.J. 1962.
8. Ostrach, S. An analysis of laminar free convection flow and heat transfer about a flat plate parallel to the direction of the generating body force, NACA Rep. IIII, 1953.
9. Shu, J. J. and Wilks, G. Mixed convection laminar film condensation on a semi-infinite vertical plate, *J. Fluid Mech.*, 1995, 300, 207-229.
10. Sparrow, E.M. and Gregg, J.L. Similar solutions for free convection from a non isothermal vertical plate, *Trans. ASME*, 1958, 78, 435-440.
11. Van Dyke, M. *Perturbation Methods in Fluid Mechanics*, Academic Press, New York, 1960.

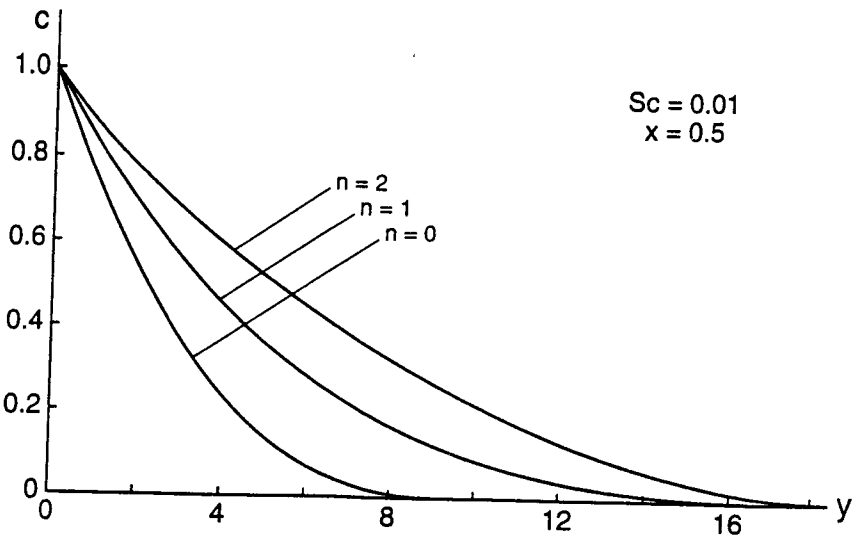


Figure 1: Numerical solutions of concentrations for uniform concentration at the plate

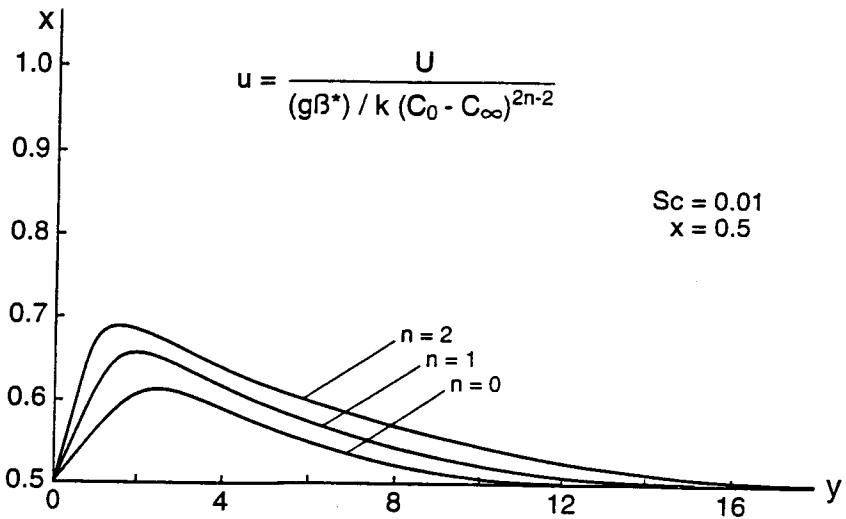


Figure 2: Numerical solutions of velocity profiles for uniform concentration at the plate

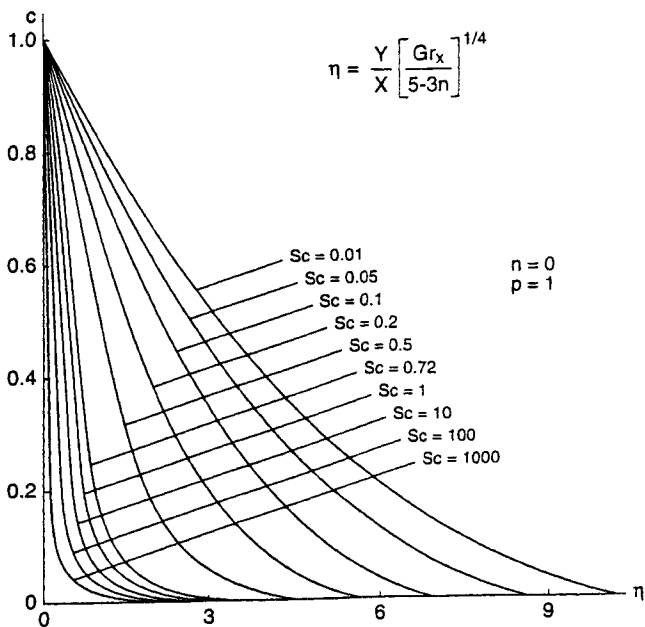


Figure 3: Numerical solutions of concentrations for the distributed concentration at the plate

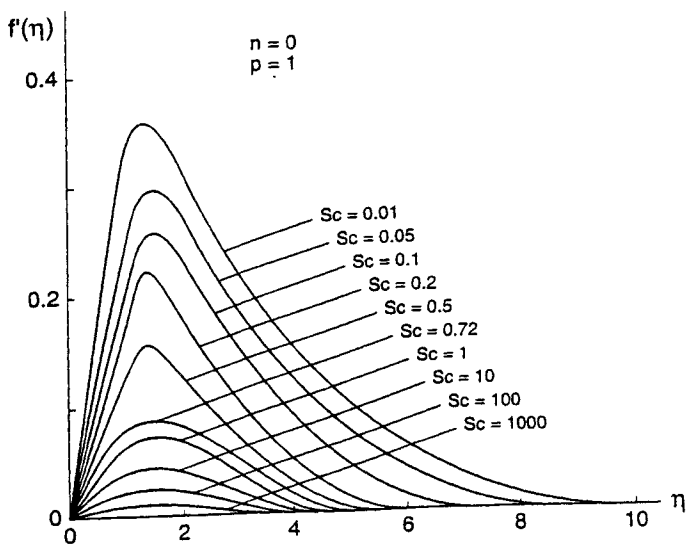


Figure 4: Numerical solutions of velocity profiles for the distributed concentration at the plate