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SIMILARITY OF MATRICES OVER FINITE RINGS

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Abstract. It is shown that questions of similarity of certain invertible matrices over a finite ring can be reduced to questions of similarity over finite fields through the application of canonical epimorphisms.

Suprunenko has shown in [3] that two invertible matrices over Z/Z_m whose orders are relatively prime to m are similar if and only if their canonical images are similar over the fields Z/Zp for each prime divisor p of m. An analogous result holds for invertible matrices over any finite commutative ring with identity.

Preliminaries. If R is a finite commutative ring with identity, then R is uniquely a ring direct product of finite local rings [**1**, Theorem 8.7, p. 90]. Suppose that $R=\prod_{i=1}^t R_i$, where R_i is a finite local ring with maximal ideal M_i. Each R_i has cardinality $p_i^{e_i}$ for some prime p and has associated with it a canonical projection,

$$h_i:R_i \to R_i/M_i = \mathrm{GF}(p_i^{f_i}).$$

Setting $k_i=\mathrm{GF}(p_i^{f_i})$ we will say that the finite fields $\{k_i:i=1, 2, \cdots, t\}$ are the *fields associated with R*.

Observe that the decomposition of R carries over to the general linear group of degree n over R yielding $\mathrm{GL}_n(R)\cong\prod_{i=1}^t\mathrm{GL}_n(R_i)$. Furthermore, for each i, the projection h_i induces an epimorphism,

$$h_i:\mathrm{GL}_n(R_i) \to \mathrm{GL}_n(k_i).$$

If $\mathrm{GL}_n(R_i)$ is taken as the group of n by n invertible matrices over R_i, then h_i is simply reduction modulo M_i. Note that the kernel of h_i, K_i, has cardinality a power of p_i and thus is a solvable group.

The following corollary to P. Hall's extension of the Sylow theorems [**2**, Theorem 9.3.1, p. 141] is the key result needed for Theorems 1 and 2.

Observation. Let G be a finite group with solvable normal subgroup K and let $\bar{G}=G/K=\{\bar{g}|g \in G\}$. Let α and β be elements of G with $(|\alpha|, |k|)=1=(|\beta|, |K|)$. Then $\bar{\alpha}\sim\bar{\beta}$ implies $\alpha\sim\beta$.

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PROOF. Since $\bar{\alpha}=\bar{\gamma}^{-1}\bar{\beta}\bar{\gamma}$ for some γ it follows that $\langle\alpha\rangle K=\langle\gamma^{-1}\beta\gamma\rangle K$. By P. Hall's theorem it follows that $\langle\alpha\rangle$ and $\langle\gamma^{-1}\beta\gamma\rangle$ are conjugate in $\langle\alpha\rangle K$. Thus there is a μ in K and $r>0$ such that $\mu^{-1}\gamma^{-1}\beta\gamma\mu=\alpha^r$. Hence $\bar{\alpha}^r=\bar{\gamma}^{-1}\bar{\beta}\bar{\gamma}=\bar{\alpha}$ and, since α and $\bar{\alpha}$ have the same order, $\alpha=\alpha^r$. Therefore $\alpha=(\gamma\mu)^{-1}\beta(\gamma\mu)$ and $\alpha\sim\beta$.

The theorems.

THEOREM 1. *Let R be a finite local ring with maximal ideal M and $R/M=\mathrm{GF}(p^f)=k$. Let α, β be elements of $\mathrm{GL}_n(R)$ with $(|(\alpha)|,p)=1$ and $(|(\beta)|,p)=1$. Then α is similar to β if and only if α is similar to β modulo M.*

PROOF. This follows from the Observation by noting that the kernel, K, of $h:\mathrm{GL}_n(R)\to\mathrm{GL}_n(R/M)$ is solvable with cardinality a power of p.

THEOREM 2. *Let R be a finite commutative ring with identity and let the cardinality of R be m. Two elements α and β of $\mathrm{GL}_n(R)$ satisfying $(|(\alpha)|,m)=(|(\beta)|,m)=1$ are similar if and only if their canonical images over the Galois fields associated with R are similar.*

PROOF. This follows from Theorem 1 directly by means of the sequence of epimorphisms

$$\mathrm{GL}_n(R)=\prod_{i=1}^{t}\mathrm{GL}_n(R_i)\xrightarrow{\pi_i}\mathrm{GL}_n(R_i)\xrightarrow{h_i}\mathrm{GL}_n(k_i).$$

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