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Similarity Solution for Flow of a Micro-Polar Fluid Through a Porous Medium

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Abstract

The equations of two dimensional incompressible steady micro-polar fluid flows through a porous medium are studied. Lie group analysis is employed and the solutions corresponding to the translational symmetry are developed. A boundary value problem is investigated and the results are sketched graphically. The effect on the flow of the porosity coefficient of the porous medium and the micro-polar parameters are observed.

Keywords: Lie group analysis; Similarity solutions; micro-polar fluid; porous medium.

MSC 2010 No.: 53C11, 76S05

1. Introduction

Micro-polar fluids are fluids with microstructure belonging to a class of fluids with non-symmetrical stress tensor referred to as polar fluids. Physically they represent fluids consisting of randomly oriented particles suspended in a viscous medium. The classical theories of continuum mechanics are inadequate to explicate the microscopic manifestations of microscopic events. A new stage in the evolution of fluid dynamic theory is in progress. Eringen (1964) presented the earliest formulation of a general theory of fluid micro-continua taking into account the inertial characteristics of the substructure particles which are allowed to undergo rotation. Lukaszewicz (1999) gave the mathematical theory of equations of micro-polar fluids and applications of these fluids in the theory of lubrication and in the porous space. Several workers in the field have made useful investigations that involve a micro-polar fluid. Seddeek (2003) studied the effect of a magnetic field on the flow of a micro-polar fluid past a continuously moving plate. Youn and Lee (2003) investigated the problem of the oscillatory two-dimensional laminar flow of a viscous incompressible electrically conducting micro-polar fluid over a semi-infinite vertical moving porous plate in the presence of a transverse magnetic field.

The influence of a transverse magnetic field on the motion of an electrically conducting micro-polar fluid through a porous medium in one-dimensional has been studied by Zakaria (2004). Seddeek and Abdelmeguid (2004) studied the hall and ion-slip currents effects on a magneto-micro-polar fluid with combined forced and free convection in boundary layer flow over a horizontal plate. El-Mistikawy (2009) studied the boundary layer flow of a micro-polar fluid due to a linearly stretching sheet in the limit of a vanishing coupling parameter. Mekheimer and Abdelmaboud (2008) studied the influence of a micro-polar fluid on peristaltic transport in an annulus. Also Seddeek et al. (2009) studied the analytical solution for the effect of radiation on flow of a magneto-micro-polar fluid past a continuously moving plate with suction and blowing.

The investigation of exact solutions of nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena [El-Sabbagh and Ali (2005), (2008)]. The symmetry group transformations method is an important method for finding exact solutions of both ordinary and partial differential equations by using the transformations groups (similarity transformations). This was introduced for the first time by Sophus Lie. Lie's theory allows one to find the symmetries (infinitesimal generators of Lie group) of the differential equations, which leave a given family of equations invariant. Lie himself introduced the procedure to find such symmetries. With the symmetries of the differential equations, a solution can be obtained. The symmetry transformation method transforms the given family of equations of n independent variables, say, to another family of equations of $n+1$ independent variables, which can further be solved Attallah et al. (2007) and Ali (2009). The fundamental concepts of this approach can be found in [Bluman and Kumei (1998) and Olver (1995)].

Yurusoy and Pakdemirli (1999) investigated the boundary layer equations of a non-Newtonian fluid model in which the shear stress is an arbitrary function of the velocity gradient. Yurusoy et al. (2001) have obtained the solution for the creeping flow of a second grade fluid by using Lie group analysis method. The two-dimensional equation of motion for the slowly flowing and heat transfer in a second grade fluid in cartesian coordinates neglecting the inertial terms are considered by Yurusoy (2004). Recently, Shahzad et al. (2007) found the analytical solution of a micro-polar fluid by using the similarity transformation method and discussed their results on a problem occurring in geology.

In the Earth there are a large number of problems that can be described by the interaction of a low viscosity fluid (water, oil, gas, magma) in a permeable (and possibly deformable) matrix. Darcy's law is the classic, empirically derived equation for the flux of a low viscosity fluid in a permeable matrix. This equation assumes that flow in the pores or cracks of the medium is essentially laminar and provides the average flux through a representative area that is larger than the pore scale and smaller than the scale of significant permeability variation (if such a scale exists). Various approaches have been used to justify this rule from first principles [e.g. see Dagan (1989)], but it generally seems to work.

With the above discussion in mind, the goal of this paper is to provide an analytical solution for flow of a micro-polar fluid through a porous medium. The translation parameter type symmetry has been taken in to account, and graphics are plotted and discussed. The motivation of this problem is to understand the magma flow in earth through a porous medium.

2. Equations of Motion

The two dimensional equations for an incompressible micro-polar fluid through a porous medium are

$$\frac{\partial \hat{u}}{\partial \hat{x}}+\frac{\partial \hat{v}}{\partial \hat{y}}=0 \ , \tag{1}$$

$$\rho\left(\hat{u}\frac{\partial \hat{u}}{\partial \hat{x}}+\hat{v}\frac{\partial \hat{u}}{\partial \hat{y}}\right)=(\mu+k_1)\left(\frac{\partial^2 \hat{u}}{\partial \hat{x}^2}+\frac{\partial^2 \hat{u}}{\partial \hat{y}^2}\right)+k_1\frac{\partial \hat{\sigma}}{\partial \hat{y}}-\frac{\partial \hat{p}}{\partial \hat{x}}-\frac{\mu}{k}\hat{u}, \tag{2}$$

$$\rho\left(\hat{u}\frac{\partial \hat{v}}{\partial \hat{x}}+\hat{v}\frac{\partial \hat{v}}{\partial \hat{y}}\right)=(\mu+k_1)\left(\frac{\partial^2 \hat{v}}{\partial \hat{x}^2}+\frac{\partial^2 \hat{v}}{\partial \hat{y}^2}\right)-k_1\frac{\partial \hat{\sigma}}{\partial \hat{x}}-\frac{\partial \hat{p}}{\partial \hat{y}}-\frac{\mu}{k}\hat{v}, \tag{3}$$

$$\rho\hat{j}\left(\hat{u}\frac{\partial \hat{\sigma}}{\partial \hat{x}}+\hat{v}\frac{\partial \hat{\sigma}}{\partial \hat{y}}\right)=G_1\left(\frac{\partial^2 \hat{\sigma}}{\partial \hat{x}^2}+\frac{\partial^2 \hat{\sigma}}{\partial \hat{y}^2}\right)+k_1\left(\frac{\partial \hat{v}}{\partial \hat{x}}-\frac{\partial \hat{u}}{\partial \hat{y}}\right)-2k_1\hat{\sigma}, \tag{4}$$

where $\hat{u}$ and $\hat{v}$ are the components of the velocity field in the $\hat{x}$ and $\hat{y}$ direction, $\hat{\sigma}(\hat{x},\hat{y})$ is the micro-rotation component and $\hat{p}(\hat{x},\hat{y})$ is the pressure distribution. Here ρ, μ, k_1, k, G_1 and $\hat{j}$ are mass density, coefficient of viscosity, coupling constant, porosity coefficient, micro-rotation constant and local micro inertia respectively.

By using the following dimensionless parameters

$$u=\frac{\hat{u}}{U}, v=\frac{\hat{v}}{U}, x=\frac{\hat{x}}{L}, y=\frac{\hat{y}}{L}, p=\frac{\hat{p}}{P}, \sigma=\frac{\hat{\sigma}}{\tilde{\sigma}}, \tag{5}$$

equations (1)-(4) reduce to

$$E_1=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \ , \tag{6}$$

$$E_2 = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - (\epsilon_1+\epsilon_2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) - \epsilon_3\frac{\partial \sigma}{\partial y} + \epsilon_4\frac{\partial p}{\partial x} + \epsilon_5 u = 0, \tag{7}$$

$$E_3 = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} - (\epsilon_1+\epsilon_2)\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) + \epsilon_3\frac{\partial \sigma}{\partial x} + \epsilon_4\frac{\partial p}{\partial y} + \epsilon_5 v = 0, \tag{8}$$

$$E_4 = u\frac{\partial \sigma}{\partial x} + v\frac{\partial \sigma}{\partial y} - \epsilon_6\left(\frac{\partial^2 \sigma}{\partial x^2}+\frac{\partial^2 \sigma}{\partial y^2}\right) + \epsilon_7\sigma - \epsilon_8\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) = 0, \tag{9}$$

where

$$\epsilon_1 = \frac{\mu}{\rho L U},\ \ \epsilon_2 = \frac{k_1}{\rho L U}, \epsilon_3 = \frac{k_1\tilde{\sigma}}{\rho U^2},\ \epsilon_4 = \frac{k_1 P}{\rho U^2},$$

$$\epsilon_5 = \frac{\mu L}{k\rho U}, \epsilon_6 = \frac{G_1}{\rho L U J}, \epsilon_7 = \frac{2k_1 L}{\rho U J}, \epsilon_8 = \frac{k_1}{\rho\tilde{\sigma}J}\ .$$

3. Symmetry Analysis

In order to obtain the analytical solution, we apply Lie group theory to equations (6)- (9). For this we write

$$\left.\begin{aligned}
x^* &= x + \epsilon\xi_1(x,y,u,v,p,\sigma) + \mathbf{o}(\epsilon^2),\\
y^* &= y + \epsilon\xi_2(x,y,u,v,p,\sigma) + \mathbf{o}(\epsilon^2),\\
u^* &= u + \epsilon\eta_1(x,y,u,v,p,\sigma) + \mathbf{o}(\epsilon^2),\\
v^* &= v + \epsilon\eta_2(x,y,u,v,p,\sigma) + \mathbf{o}(\epsilon^2),\\
p^* &= p + \epsilon\eta_3(x,y,u,v,p,\sigma) + \mathbf{o}(\epsilon^2),\\
\sigma^* &= \sigma + \epsilon\xi_4(x,y,u,v,p,\sigma) + \mathbf{o}(\epsilon^2),
\end{aligned}\right\} \tag{10}$$

as the infinitesimal Lie point transformations. We have assumed that equations (6)-(9) are invariant under the transformations given in equation (10). The corresponding infinitesimal generator of Lie groups (symmetries) is given by

$$X = \xi_1\frac{\partial}{\partial x} + \xi_2\frac{\partial}{\partial y} + \eta_1\frac{\partial}{\partial u} + \eta_2\frac{\partial}{\partial v} + \eta_3\frac{\partial}{\partial p} + \eta_4\frac{\partial}{\partial \sigma}, \tag{11}$$

where ξ_1, ξ_2, η_1, η_2, η_3 and η_4 are functions of all the independent and dependent variables and are the components of infinitesimals symmetries corresponding to x, y, u, v, p and σ, respectively, to be determined from the invariance conditions:

$$Pr^2 X(E_a)|_{E_a=0} = 0, \qquad a = 1,2,3,4, \tag{12}$$

where $E_a = 0, a = 1,2,3,4$ are the equations (6)-(9) under study and Pr^2 is the second prolongation of the symmetries X. Since our equations are at most of order two, we need second order prolongation of the infinitesimal generator in equation (11). Expanding the system

(12) with the aid of Mathematica and using the original equations (6)-(9) to eliminate u_x, p_x, p_y, $\sigma_{xx}$and setting the coefficients of the system (12) involving u_y, u_{yy}, v_x, v_y, v_{xx}, v_{xy}, v_{yy}, σ_x, σ_y, σ_{xy}, σ_{yy} and various products to zero gives rise to a set of over-determined equations of the components of the symmetries of equations (6)-(9). Solving this set of equations with the aid of Mathematica, we obtain the components of symmetries as the following:

$$\xi_1 = a_1, \xi_2 = a_2, \eta_1 = 0, \eta_2 = 0, \eta_3 = a_3, \eta_4 = 0. \tag{13}$$

Therefore the equations admit a three parameter Lie group of transformations corresponding to the arbitrary constants a_1, a_2 and a_3, that correspond to translations in the x, y and p coordinates, respectively. By considering the translations in the x, y directions, the similarity variable ξ and similarity transformation functions u, v, p and σ are given as

$$\xi = y - m\,x, u = f(\xi), v = g(\xi), p = h(\xi) + M\,x, \sigma = N(\xi), \tag{14}$$

where $m = \frac{a_2}{a_1}$ and $M = \frac{a_3}{a_1}$ is an arbitrary constant. In view of the variables and functions in equation (14), equations (6)-(9) become the following ordinary differential equations:

$$g' - m\,f' = 0 \tag{15}$$

$$(g - m\,f)\,f' = (\epsilon_1 + \epsilon_2)(1 + m^2)f'' + \epsilon_3\,N' + \epsilon_4(m\,h' - M) - \epsilon_5 f, \tag{16}$$

$$(g - m\,f)\,g' = (\epsilon_1 + \epsilon_2)(1 + m^2)g'' + \epsilon_3 m\,N' + \epsilon_4 h' - \epsilon_5\,g, \tag{17}$$

$$(g - m\,f)\,N' = \epsilon_6\,(1 + m^2)N'' + \epsilon_7 N - \epsilon_8\,(m\,g' + f'). \tag{18}$$

Integrating equation (15) yields

$$g = m\,f + C_1\,, \tag{19}$$

where C_1 is an arbitrary constant. Eliminating $h(\xi)$ from equations (16) and (17) and using equation (19) we obtain

$$\begin{aligned} C_1\,(1 + m^2)f' = (\epsilon_1 + \epsilon_2)(1 + m^2)f'' - \epsilon_5(1 + m^2)f \\ -(\epsilon_5\,mC_1 + \epsilon_4\,M\,) + \epsilon_3\,(1 + m^2)N'. \end{aligned} \tag{20}$$

From equations (18) and (19) one can write

$$\mathrm{C}_1\,\mathrm{N}' = \epsilon_6\,(1 + m^2)N'' - \epsilon_7 N - \epsilon_8\,(m\,g' + f'). \tag{21}$$

Eliminating $f(\xi)$ from equations (20) and (21) we have

$$N^{(iv)} - A\,N''' + B\,N'' + C\,N' + D\,N = 0, \tag{22}$$

where

$$\left.\begin{aligned} A &= \frac{C_1[\epsilon_1+\epsilon_2+\epsilon_6]}{(1+m^2)\,\epsilon_6\,(\epsilon_1+\epsilon_2)}, \\ B &= \frac{{C_1}^2-(1+m^2)[\epsilon_7(\epsilon_1+\epsilon_2)+\epsilon_3\epsilon_8+\epsilon_5\epsilon_6]}{(1+m^2)^2\,\epsilon_6\,(\epsilon_1+\epsilon_2)}, \\ C &= \frac{C_1[\epsilon_5+\epsilon_7]}{(1+m^2)^2\,\epsilon_6\,(\epsilon_1+\epsilon_2)}, \\ \mathrm{D} &= \frac{\epsilon_5\epsilon_7}{(1+m^2)^2\,\epsilon_6\,(\epsilon_1+\epsilon_2)}. \end{aligned}\right\} \tag{23}$$

The solution of equation (22) is given by

$$N(\xi) = C_2\,e^{\alpha_1\xi} + C_3\,e^{\alpha_2\xi} + C_4\,e^{\alpha_3\xi} + C_5\,e^{\alpha_4\xi}, \tag{24}$$

where C_2, C_3, C_4 and C_5 are arbitrary constants and $\alpha_i (i = 1,2,3,4)$ are the roots of the following equation:

$$\alpha^4 - A\,\alpha^3 + B\,\alpha^2 + C\,\alpha + D = 0. \tag{25}$$

From equations (21) and (25) the expression for $f(\xi)$ is

$$f(\xi) = \beta_1\,e^{\alpha_1\xi} + \beta_2\,e^{\alpha_2\xi} + \beta_3\,e^{\alpha_3\xi} + \beta_4\,e^{\alpha_4\xi} + C_6\,, \tag{26}$$

where C_6 is an arbitrary constant and $\beta_i (i = 1,2,3,4)$ are given by the following expression:

$$\beta_i = \frac{C_{i+1}\left[\epsilon_6\,(1+m^2)\alpha_i^2 - C_1\alpha_i - \epsilon_7\right]}{\epsilon_8\,(1+m^2)\alpha_i}. \tag{27}$$

From equations (19), (26) we get

$$g(\xi) = m\big(\beta_1\,e^{\alpha_1\xi} + \beta_2\,e^{\alpha_2\xi} + \beta_3\,e^{\alpha_3\xi} + \beta_4\,e^{\alpha_4\xi} + C_6\big) + C_1. \tag{28}$$

From equations (16) and (17) along with equation (19) we get

$$h'(\xi) = \frac{\epsilon_4\,m\,M - C_1\,\epsilon_5}{\epsilon_4(1+m^2)}. \tag{29}$$

By integrating equation (29) we get

$$h(\xi) = \frac{\epsilon_4\,m\,M - C_1\,\epsilon_5}{\epsilon_4(1+m^2)}\,\xi + C_7\,, \tag{30}$$

where C_7 is an arbitrary constant. In the form of the original variables we have

$$u(x,y) = \beta_1\,e^{\alpha_1(y-mx)} + \beta_2\,e^{\alpha_2(y-mx)} + \beta_3\,e^{\alpha_3(y-mx)} + \beta_4\,e^{\alpha_4(y-mx)} + C_6, \tag{31}$$

$$v(x,y) = m\big(\beta_1\, e^{\alpha_1(y-mx)} + \beta_2\, e^{\alpha_2(y-mx)} + \beta_3\, e^{\alpha_3(y-mx)} + \beta_4\, e^{\alpha_4(y-mx)} + C_6\big) + C_1, \tag{32}$$

$$\sigma(x,y) = C_2\, e^{\alpha_1(y-mx)} + C_3\, e^{\alpha_2(y-mx)} + C_4\, e^{\alpha_3(y-mx)} + C_5\, e^{\alpha_4(y-mx)}, \tag{33}$$

$$p(x,y) = \frac{\epsilon_4\, m\, M - C_1\, \epsilon_5}{\epsilon_4(1+m^2)}(y-mx) + Mx + C_7. \tag{34}$$

In the limit as $\epsilon_5 \longrightarrow 0$ our results in Eqs. (31)-(34) are the same as those obtained by Shahzad (2007), where m is the translation parameter. Equations (31)-(34) give the solution of equations (6)-(9) that involve seven unknown constants. For determining the values of these constants we consider a problem that occurs in geology. Consider a magmatic micro-polar fluid in a porous medium and a plate over it. The plate occupies the position $y = 0$. The positive y goes deep into the fluid beneath the plate. The relevant boundary conditions are of the form:

$$\begin{aligned} &u(x,0) = U_0,\;\; u(x,\infty) = 0,\;\; v(x,0) = -V_0,\;\; p(x,\infty) = P_0, \\ &\frac{\partial u}{\partial x}(0,y),\;\; \sigma(x,0) = 0, \quad \sigma(x,\infty) = 0, \end{aligned} \tag{35}$$

where U_0 is the velocity of the plate, V_0 is the magmatic fluid penetrating into the plate and P_0 is the pressure deep in the magmatic fluid. From the first boundary condition in equation (35) we have obtain the following conditions, $C_1 = M = 0$, $m = -\frac{V_0}{U_0}$, , $C_6 = 0$ and $C_7 = P_0$. The second condition in equation (35) gives the following solutions:

$$u(y) = -\frac{U_0}{\delta_2-\delta_1}\left(\delta_1 e^{-\alpha y} - \delta_2 e^{-\beta y}\right), \tag{36}$$

$$v(y) = \frac{V_0}{\delta_2-\delta_1}\left(\delta_1 e^{-\alpha y} - \delta_2 e^{-\beta y}\right), \tag{37}$$

$$\sigma(y) = -\frac{U_0}{\delta_2-\delta_1}\left(e^{-\alpha y} - e^{-\beta y}\right), \tag{38}$$

$$p(y) = P_0, \tag{39}$$

where

$$\delta_1 = \frac{\epsilon_7 U_0^2 - \epsilon_6\left(U_0^2+V_0^2\right)\alpha^2}{\alpha\epsilon_8\left(U_0^2+V_0^2\right)}, \qquad \delta_2 = \frac{\epsilon_7 U_0^2 - \epsilon_6\left(U_0^2+V_0^2\right)\beta^2}{\beta\epsilon_8\left(U_0^2+V_0^2\right)}, \tag{40}$$

and

$$\alpha = \sqrt{\frac{\tilde{B}+\sqrt{\tilde{B}^2-4\tilde{D}}}{2}}\,, \qquad \beta = \sqrt{\frac{\tilde{B}-\sqrt{\tilde{B}^2-4\tilde{D}}}{2}}\,, \tag{41}$$

$$\tilde{B} = \frac{[\epsilon_7(\epsilon_1+\epsilon_2)-\epsilon_3\epsilon_8+\epsilon_5\epsilon_6]U_0^2}{\epsilon_6\,(\epsilon_1+\epsilon_2)(U_0^2+V_0^2)},\ \tilde{\mathrm{D}} = \frac{\epsilon_5\epsilon_7 U_0^2}{\epsilon_6\,(\epsilon_1+\epsilon_2)(U_0^2+V_0^2)}. \tag{42}$$

4. Results and Discussion

To study the behavior of the components of the velocity and angular velocity profiles, curves are drawn for various values of the parameters that describe the flow. Equations (1-12) display the variation of the velocity profiles u and v (the x and y components of the velocity) and the angular velocity profile σ for different values of the porosity parameter ϵ_5, magmatic fluid penetrating parameter V_0 and velocity of the plate U_0. First we consider how varying the porosity parameter ϵ_5 influences, v and it is found from Figures 1 and 2 that the velocity components u and v decrease as ϵ_5 increases. It is evident from Figure 3 that increasing the values of the permeability parameter ϵ_5 increases the maximum value of the angular velocity σ and moves the location of this maximum value near to the surface of magma. Figures (4-12) show the variations of u, v and σ for different values of V_0 where ($V_0 < 0, V_0 > 0$) and U_0. It is observed that the x component of the velocity decreases by increasing the value of V_0 for either $V_0 < 0$, or $V_0 > 0$ but increases by increasing the value of U_0, while the y component of the velocity increases by increasing the value of U_0 and V_0 for either $V_0 < 0$, or $V_0 > 0$, Figures 6 and 12 show that the angular velocity σ increases with increasing values of U_0 and V_0 for either $V_0 < 0$, or $V_0 > 0$. Figures (13-15) show the effect of different values of the micro-rotation parameter ϵ_6, on u, v and σ. Figures 13 and 14 show that, with increasing the value of ϵ_6 the component velocity u increases whenever the component velocity v decreases. Finally, Fig.15 shows the angular velocity σ decreases by increasing ϵ_6 until the half of y, its value increases with increasing ϵ_6 .

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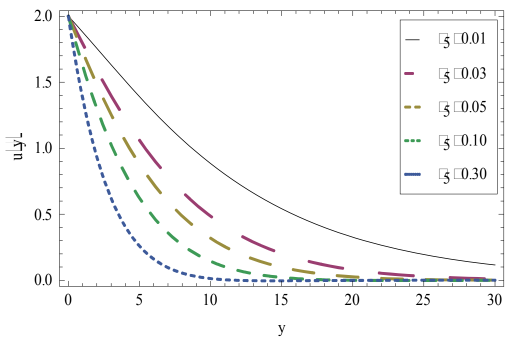

Fig. 1 Variation of the dimensionless velocity u(y) with the values of $\in_5$ ($\in_1 - \in_4$; $\in_7$; $\in_8 = 0.5$; $\in_6 = 2$; $U_0 = V_0 = 2$

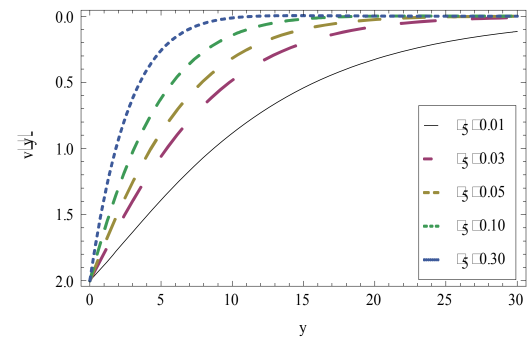

Fig. 2 Variation of the dimensionless velocity v(y) with the values of $\in_5$ ($\in_1 - \in_4$; $\in_7$; $\in_8 = 0.5$; $\in_6 = 2$; $U_0 = V_0 = 2$)

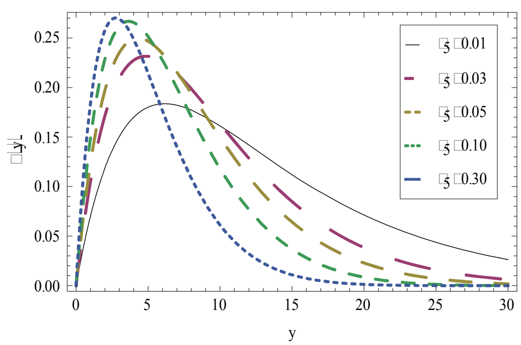

Fig. 3 Variation of the dimensionless angular velocity σ(y) with the values of $\in_5$ ($\in_1 - \in_4$; $\in_7$; $\in_8 = 0.5$; $\in_6 = 2$; $U_0 = V_0 = 2$)

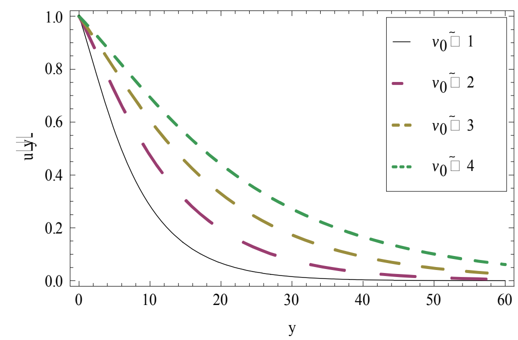

Fig. 4 Variation of the dimensionless velocity u(y) with the values of V_0 ($\in_1 - \in_4$; $\in_7$; $\in_8 = 0.5$; $\in_5$=0.02; $\in_6 = 0.6$; $U_0 = 1$)

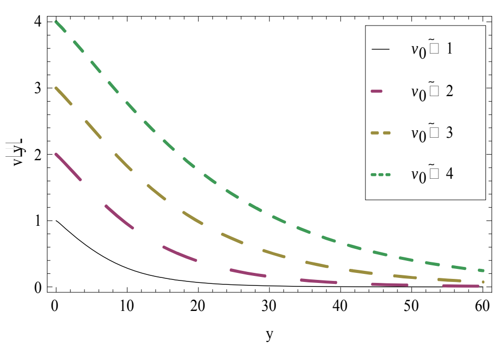

Fig. 5 Variation of the dimensionless velocity v(y) with the values of V_0 ($\in_1 - \in_4$; $\in_7$; $\in_8 = 0.5$; $\in_5$=0.02; $\in_6 = 0.6$; $U_0 = 1$)

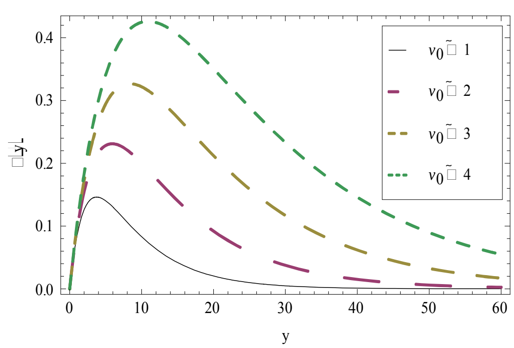

Fig. 6 Variation of the dimensionless angular velocity σ(y) with the values of V_0 ($\in_1 - \in_4$; $\in_7$; $\in_8 = 0.5$; $\in_5$=0.02; $\in_6 = 0.6$; $U_0 = 1$)

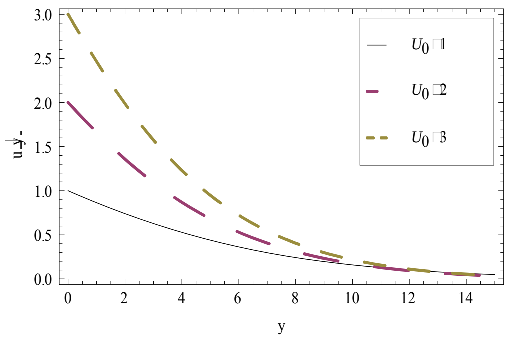

Fig. 7 Variation of the dimensionless velocity u(y) with the values of U_0 ($\epsilon_1 - \epsilon_4$; ϵ_7; $\epsilon_8 = 0.5$; $\epsilon_6 = 2$; $\epsilon_5 = 0.05$; $V_0 = 1$)

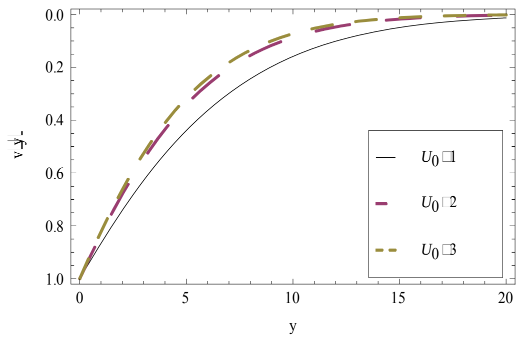

Fig. 8 Variation of the dimensionless velocity v(y) with the values of U_0 ($\epsilon_1 - \epsilon_4$; ϵ_7; $\epsilon_8 = 0.5$; $\epsilon_6 = 2$; $\epsilon_5 = 0.05$; $V_0 = 1$)

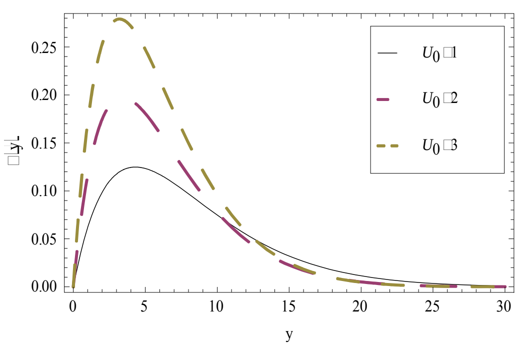

Fig. 9 Variation of the dimensionless angular velocity σ(y) with the values of U_0 ($\epsilon_1 - \epsilon_4$; ϵ_7; $\epsilon_8 = 0.5$; $\epsilon_6 = 2$; $\epsilon_5 = 0.05$; $V_0 = 1$)

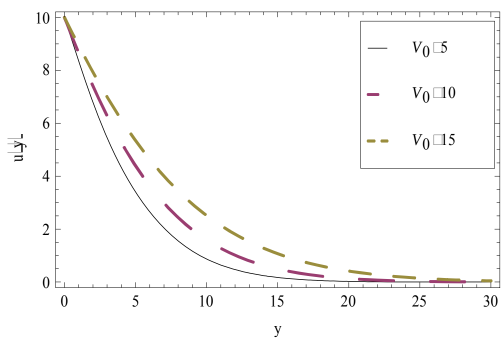

Fig. 10 Variation of the dimensionless velocity u(y) with the values of V_0 ($\epsilon_1 - \epsilon_4$; ϵ_7; $\epsilon_8 = 0.5$; $\epsilon_5 = 0.05$; $\epsilon_6 = 2$; $U_0 = 10$)

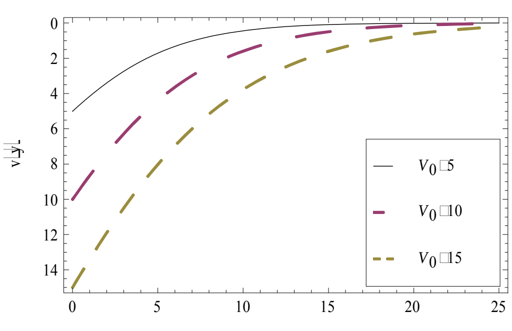

Fig. 11 Variation of the dimensionless velocity v(y) with the values of V_0 ($\epsilon_1 - \epsilon_4$; ϵ_7; $\epsilon_8 = 0.5$; $\epsilon_5 = 0.05$; $\epsilon_6 = 2$; $U_0 = 10$)

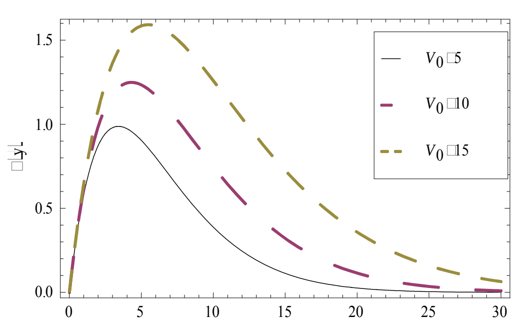

Fig. 12 Variation of the dimensionless angular velocity σ(y) with the values of V_0 ($\epsilon_1 - \epsilon_4$; ϵ_7; $\epsilon_8 = 0.5$; $\epsilon_5 = 0.05$; $\epsilon_6 = 2$; $U_0 = 10$)

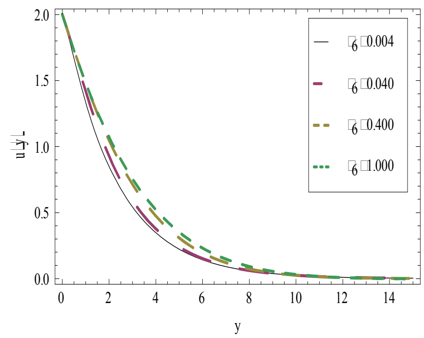

Fig. 13 Variation of the dimensionless velocity u(y) with the values of ϵ_6 ($\epsilon_1 - \epsilon_4$; ϵ_7; $\epsilon_8 = 0.5$; $\epsilon_5 = 0.2$; $U_0 = V_0 = 2$)

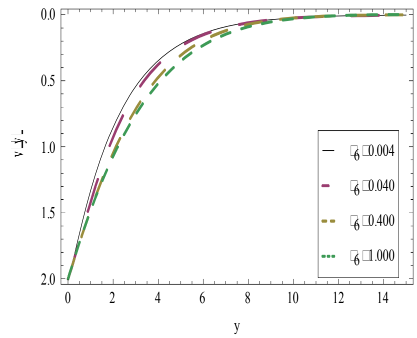

Fig. 14 Variation of the dimensionless velocity v(y) with the values of ϵ_6 ($\epsilon_1 - \epsilon_4$; ϵ_7; $\epsilon_8 = 0.5$; $\epsilon_5 = 0.2$; $U_0 = V_0 = 2$)

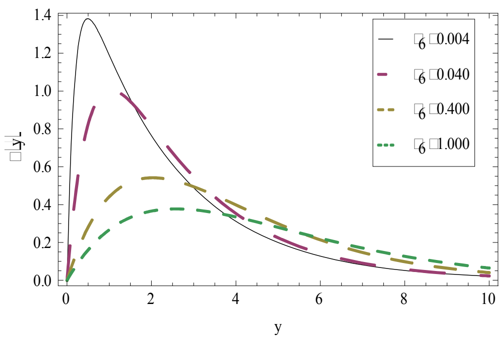

Fig. 15 Variation of the dimensionless angular velocity σ(y) with the values of ϵ_6 ($\epsilon_1 - \epsilon_4$; ϵ_7; $\epsilon_8 = 0.5$; $\epsilon_5 = 0.2$; $U_0 = V_0 = 2$)