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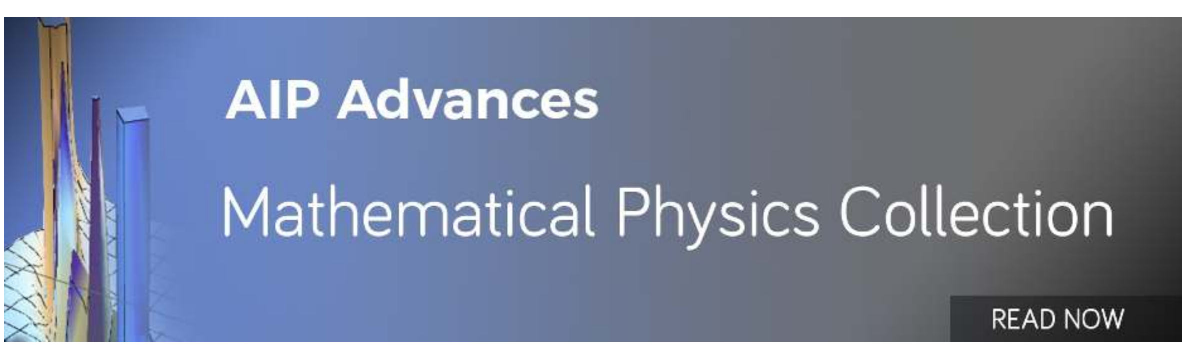

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Similarity solution for flow over an unsteady nonlinearly stretching rotating disk

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The unsteady laminar flow of an incompressible viscous fluid over a nonlinearly stretching rotating disk is investigated. The axisymmetric three-dimensional boundary layer equations are reduced into self-similar form with the help of new similarity transformation. The resulting coupled nonlinear equations are solved numerically using shooting method coupled with Range-Kutta 6 (RK-6). An exact analytical solution for the large stretching parameter is also presented. Some interesting observations are made while interpreting the results physically. Dual solutions are obtained due to the presence of unsteadiness parameter for the nonlinear stretching of the rotating disk. The analytical results reveal that for large stretching parameter the azimuthal velocity becomes negligible and the flow behaviors turn into steady state, which is the most surprising observation of the paper. These results are also verified numerically by solving original self similar equations using shooting method. [http://dx.doi.org/10.1063/1.4917459]

I. INTRODUCTION

An elegant exact analytical solution of the viscous fluid flow over a stretching surface was first presented by Crane.[2] Since then so much has been said on the stretching phenomena that it is hardly possible to mention all the later developments. The subject of our investigation being stretching of rotating disk, it suffices to give a brief review of the stretching phenomenon for rotating disks.

The study of flow field due to rotating disk has found many applications in different fields of engineering and industry. A number of real processes are undertaken using disk rotation such as fans, turbines, centrifugal pumps, rotors, viscometers, spinning disk reactors and other rotating bodies etc. The analytical study of rotating disk goes back to the celebrated paper by Von Karman[1] while initiating the study of incompressible viscous fluid over an infinite plane disk rotating with a uniform angular velocity. This work was extended to the three-dimensional case by Wang.[3] The exact solutions for the steady flow over a rotating disk and for the heat and mass transfer over a permeable rotating disk were presented by Fang[4] and Turkyilmazoglu.[5] Fang and Zhang[6] studied the flow between two stretching disks. Recently, Asghar *et al.*[7] derived new similarity transformations for the flow and heat transfer for the nonlinearly stretching rotating disc using group theoretic methods. They presented the exact analytical solutions of the nonlinear equations. All these studies were undertaken for steady flow over a rotating disk.

The steady flows are important for the fully developed flow and are investigated to understand the behavior and physics of fluid. However, there are situations in which it is highly desirable to introduce time evolution through some sort of unsteadiness. To emphasize further, we can say that

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there is no actual flow situation, natural or artificial, which does not involve some unsteadiness. Some of the likely technological and engineering unsteady processes are nuclear fuel element cooling passages, electric resistive heaters, processing of materials geophysical and biological flows and heat exchangers. Further, startup and shutdown operations and changes in steady-state power levels give rise to the transient conditions in the surface velocity and flow of fluid. A flow becomes unsteady either due to the impulsive change in the free stream velocity, surface velocity, and sudden change in wall temperature or due to the time dependent variation. In this paper unsteadiness in flow is caused by impulsive change in radially stretching velocity.

The unsteady flow over a rotating disk was first studied by Watson and Wang.[8] In this, the angular velocity of the disk was set inversely proportional to time. The solution was shown to exist only for a deceleration of the rotating disk. The decelerating disk problem was further extended to a porous disk with mass transfer effects.[9] Recently, Fang and Tao[10] presented the numerical solution of unsteady flow over a rotating stretchable disk. In this, the linear stretching with deceleration is taken in the radial direction. However, the non-linear stretching with deceleration is not discussed in the literature.

The aim of the present study is to investigate the effects of unsteady nonlinear stretching for the rotating disk. This paper is an extension of a recent paper by the authors[7] where the steady nonlinear stretching of the rotating disk is investigated introducing new similarity transformations. The similarity transformations are now derived for the unsteady case (not available in the literature) that converts the original Navier Stokes equations into a system of self similar ordinary differential equations. The numerical solution of the nonlinear ordinary differential equations is presented using shooting method coupled with RK-6. Effects of unsteady parameter and the stretching parameter are discussed physically. A closed form exact analytical solution for large stretching parameter is also presented. The telling points of this study are (a) The derivation of the similarity transformation that converts the nonlinear PDE into self similar ODE (b) Existence of two solution branches instead of one, as reported for the steady flow. (c) The effects of the stretching and unsteady parameters on the fluid flow. (d) A rare exact analytical solution for large stretching parameter is obtained, showing that the unsteadiness parameter and the azimuthathal velocity are of no significance in the large stretching parameter limit.

II. THE FORMULATION

Consider a three dimensional laminar unsteady flow of an incompressible fluid over a stretchable rotating disk, which has angular speed varying with time. The disk is stretching in radial direction with velocity $u_w(r,\ t)$. Using boundary layer approximations, the governing Navier Stokes equations for an axisymmetric flow in cylindrical coordinates are given by

$$\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{\partial w}{\partial z}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left\{\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}-\frac{u}{r^2}\right\}, \tag{2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}=\nu\left\{\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial z^2}-\frac{v}{r^2}\right\}, \tag{3}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left\{\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right\}, \tag{4}$$

$$\begin{aligned} &z=0; \quad u=u_w(r,t),\ \ v=\frac{\Omega r^n}{1-ct},\ \ w=0,\\ &z\to\infty; \quad u=0\,,\ \ v=0. \end{aligned} \tag{5}$$

In the above equations u, v and w are the components of velocity in r, θ and z directions, ρ is the fluid density, p is the pressure and Ω is a constant angular velocity of the disk.

The form of stretching velocity is given as

$$u_w = \frac{\alpha\,\Omega\, r^n}{1-ct}. \tag{6}$$

Where, the nonzero constant α is disk stretching parameter. The governing equations (1)-(5) are the partial differential equations and can be transformed into ordinary differential equations using similarity transformations. In this study, our intention is to study the effects of physical quantities on unsteady flow in the boundary layer region. The boundary layer form of Eqs. (1)–(5) reveals that the pressure term is constant.

The methodology of deriving similarity transformation for steady flow over a rotating disk is illustrated in Ref. 7, however we also refer[11–15] to understand the technique of finding similarity transformations. The following similarity transformations reduce the governing equations into self similar form:

$$\begin{gathered} \eta = \sqrt{\frac{\Omega}{\upsilon\,(1-ct)}}\,\frac{z}{r^{(1-n)/2}},\;\; \psi = \sqrt{\frac{\Omega\upsilon}{(1-ct)}}\,r^{(3+n)/2} f(\eta), \\ v = \frac{\Omega\, r^n}{1-ct}\, g(\eta), \end{gathered} \tag{7}$$

where η is the similarity variable and the stream function $\psi(x,\,y)$ is defined as

$$ru = \frac{\partial\psi}{\partial z}\,,\; rw = -\frac{\partial\psi}{\partial r}. \tag{8}$$

Using Eqs. (7), the boundary layer problem (1)-(5) takes the self similar form:

$$f''' + \frac{3+n}{2} ff'' - nf'^2 + g^2 - A\left\{f' + \frac{\eta}{2}f''\right\} = 0, \tag{9}$$

$$g'' + \frac{3+n}{2} fg' - (n+1)f'g - A\left\{g + \frac{\eta}{2}g'\right\} \;= 0, \tag{10}$$

$$\begin{aligned} &f'(0) = \alpha,\;\; f(0) = 0,\, g(0) = 1, \\ &f'(\infty) = 0,\;\; g(\infty) = 0. \end{aligned} \tag{11}$$

Where $A = \frac{c}{\Omega}$ is the unsteady parameter and corresponds to disk acceleration for $A > 0$ or disk deceleration ($A < 0$). The beauty of similarity transformations (7) is that for $n = 1$ we get the similarity equations for linearly unsteady stretching[10] and for $A = 0$ we arrive at the similarity equations for steady nonlinear stretching.[7]

III. EXACT ANALYTICAL SOLUTION

We first present an exact analytical solution for large stretching parameter. The boundary value problem (9)–(11) can be much simplified by defining the following transformations:

$$\eta = \xi/\sqrt{\alpha},\;\; f(\eta) = \sqrt{\alpha}F(\xi),\;\; g(\eta) = \sqrt{\alpha}G(\xi). \tag{12}$$

Using Eqs. (12) in Eqs. (9) and (10) we get the following system of equations:

$$\alpha^2\left[F''' + \frac{3+n}{2}FF'' - nF'^2\right] + \alpha\left[G^2 - A\left\{F' + \frac{\eta}{2}F''\right\}\right] = 0, \tag{13}$$

$$\alpha\sqrt{\alpha}\left[G'' + \frac{3+n}{2}FG' - (n+1)F'G\right] + \sqrt{\alpha}\left[A\left\{G + \frac{\eta}{2}G'\right\}\right] = 0. \tag{14}$$

Now considering the terms upto order α for large α , Eqs. (13) and (14) reduce to

$$F'' + \frac{3+n}{2}FF'' - nF'^2 = 0, \tag{15}$$

$$G'' + \frac{3+n}{2}FG' - (n+1)F'G = 0, \tag{16}$$

and the transformed boundary conditions are

$$\begin{aligned} &F'(0) = 1, \quad F(0) = 0, G(0) = 1/\sqrt{\alpha}, \ \Theta(0) = 1 \\ &F'(\infty) = 0, \ G(\infty) = 0, \ \Theta(\infty) = 0 \end{aligned} \tag{17}$$

The exact analytical solution of the boundary value problem (15)-(17), for $n = 3$, can be easily expressed as:

$$F(\eta) = \sqrt{\frac{1}{3}}\left(1 - e^{-\sqrt{3}\eta}\right), \tag{18}$$

$$F''(0) = -\sqrt{3}, \tag{19}$$

$$G(\xi) = \frac{\varsigma e^{-\varsigma} M\left(\frac{7}{3}, 2, \varsigma\right)}{e^{-1/\sqrt{\alpha}} M\left(\frac{7}{3}, 2, \frac{1}{\sqrt{\alpha}}\right)}. \tag{20}$$

where $\varsigma = e^{-\sqrt{3}\eta}$ and $M(a, b, \varsigma)$ is Kummer function. From backward substitution and taking the first derivative then putting $\xi = 0$, the rotational skin friction is given by:

$$G'(0) = -\frac{8}{\sqrt{3\alpha}}\left(\frac{M\left(\frac{1}{3}, 2, 1\right) - 10\, M\left(-\frac{2}{3}, 2, 1\right)}{3\, M\left(\frac{1}{3}, 2, 1\right) - 25\, M\left(-\frac{2}{3}, 2, 1\right)}\right) = \frac{-1.937049}{\sqrt{\alpha}} \tag{21}$$

It is of interest to note that for large stretching parameter the transformed Eqs. (15)-(17) are the same as for steady radially stretching.[7] Thus the effects of unsteadiness on the physical quantities for large stretching parameter are ignorable. The physical reason is: for large stretching the evolution between steady to unsteady regime is negligibly small. We strengthen our arguments by comparing exact analytical solutions with the numerical solutions of the original self similar equations i.e. Eqs. (9) and (10). Using transformations (12) in Eqs. (18) and (21) the exact analytical solution for $n = 3$ in terms of the original self similar variables is given by:

$$f''(0) = -\alpha\sqrt{3\alpha} \tag{22}$$

$$G'(0) = -1.937049\sqrt{\alpha} \tag{23}$$

In the next section we will give comparison of analytical results of the reduced equations (without unsteadiness effects) with the numerical results of Eqs. (9)-(11) (with unsteadiness effects).

IV. RESULTS AND DISCUSSIONS

Numerical results of the boundary value problem (9)–(11) are obtained through finite difference scheme; namely, shooting method using Range Kutta 6. It is observed that dual solutions exist for various values of physical parameter because of the unsteadiness parameter. These solutions, named as the "upper solution branch" and the "lower solution branch", are evident from table and figures. Generally, it is true that the lower solution branch is not physically feasible because it indicates negative azimuthal velocity.

In Table I, we compare $f''(0)$ and $g'(0)$, for different combinations of stretching and the unsteady parameters, with Fang and Tao.[10] An excellent agreement is found between the two which ensures the accuracy of results obtained here.

Before going further to look for the effects of the physical parameters on the field quantities, it will be useful to discuss the case of the large stretching parameter, solved analytically in the previous section. We observe that some terms do not contribute in the large stretching parameter as observed from transformed equations (15) and (16). Physically, these terms represent the azimuthal velocity and the unsteadiness of the flow. To verify the correctness of the transformations and the consequences thereof, we solve the original self similar equations numerically using shooting method. Tables II and III show a comparison of the exact analytical solution of the transformed equations (azimuthal velocity and unsteadiness are absent), for $f''(0)$ and $g'(0)$, with the numerical

TABLE I. Comparison of values of $f''(0)$ and $g'(0)$ with Fang and Tao[10] for different values of unsteady parameter A and stretching parameter α when $n = 1.0$.

		$\alpha = 0.0$				$\alpha = 2.0$			
		$f''(0)$		$g'(0)$		$f''(0)$		$g'(0)$	
A	Branch	Self	Ref. 10	Self	Ref. 10	Self	Ref. 10	Self	Ref. 10
−0.1	U	0.5307	0.5308	−0.5789	−0.5789	−3.1178	−3.1178	−2.053	−2.053
	L	0.5259	0.5288	−0.5789	−0.5602	−3.2235	−3.2235	−2.0022	−2.0022
−0.2	U	0.5514	0.5515	0.5416	−0.5416	−3.0784	−3.0784	−2.0373	−2.0373
	L	0.5406	0.5409	−0.5077	−0.508	−3.2328	−3.2328	−1.9687	−1.9687
−0.5	U	0.6138	0.6143	−0.4285	−0.4284	−2.9601	−2.9601	−1.9901	−1.9901
	L	0.5504	0.5516	−0.3446	−0.3452	−3.2463	−3.2463	−1.9036	−1.9036
−1	U	0.7186	0.7198	−0.2368	−0.2366	−2.7622	−2.7622	−1.9111	−1.9111
	L	0.478	0.4821	−0.0823	−0.0832	−3.2798	−3.2798	−1.8307	−1.8307
−2	U	0.9283	0.9315	0.1543	0.155	−2.3641	−2.3641	−1.7523	−1.7523
	L	−0.023	−0.0099	−0.0447	−0.0298	−3.4124	−3.4124	−1.7479	−1.7479
−5	U	1.552	1.5627	1.358	1.3609	−1.1549	−1.1549	−1.2701	−1.2701
	L	−0.838	−0.8192	−1.734	−1.7106	−4.1829	−4.1829	−1.7931	−1.7931
−10	U	2.5748	2.6008	3.4055	3.4139	0.8935	0.8935	−0.4532	−0.4532
	L	−1.802	−1.7634	−4.0816	−4.0429	−6.065	−6.065	−2.3432	−2.3432
−20	U	4.5867	4.6464	7.5574	7.5796	5.0627	5.0627	1.2108	1.2108
	L	−3.682	−3.6023	−8.6078	−8.5402	−10.504	−10.504	−3.9896	−3.9896

TABLE II. Comparison of numerical solution for $f''(0)$ with exact analytical solution (22) for different α for $n = 3$ taking unsteadiness parameter $A = -0.5$ and $A = -5.0$.

	$A = -0.5$			$A = -5.0$		
α	Anal	Numer	Abs. % Error	Anal	Numer	Abs. % Error
0.01	−0.00173	0.45158	100.4	−0.1937	−0.5066	61.76
0.05	−0.01936	0.43846	104.4	−0.43314	−0.63831	32.14
0.1	−0.05477	0.37134	114.7	−0.61255	−0.7266	15.7
0.5	−0.61237	−0.3019	102.9	−1.3697	−1.32376	3.47
1	−1.73205	−1.4595	18.67	−1.93705	−1.87811	3.138
5	−19.3649	−19.572	1.056	−4.33137	−4.21875	2.669
10	−54.7723	−55.527	1.359	−6.12549	−5.99336	2.205
50	−612.372	−611.65	0.118	−13.697	−13.6843	0.093
100	−1732.05	−1731	0.058	−19.3705	−19.3609	0.049
500	−19364.9	−19363	0.011	−43.3137	−43.3019	0.027
1000	−54772.3	−54769	0.006	−61.2549	−61.2301	0.04
5000	−612372	−612366	0.001	−136.97	−136.755	0.157

solution of the original equations (azimuthal velocity and unsteadiness present), for large stretching parameter. This comparison is made for small and large unsteady parameter by setting $A = -0.5$ to $A = -5.0$. It is observed that the percentage error decreases drastically by increasing the values of stretching parameter, ultimately reaching the exact numerical solutions for large stretching parameter. This reveals a phenomenal physical conclusion; that is, for the large radially stretching disk the azimuthal velocity is negligible and the flow behavior is steady. This is the first such observation, and largely depends upon the strength of the analytical solutions.

Figs. 1–6 exhibit the velocity profiles for different values of controlling parameters in radial, vertical and azimuthal directions. Effects of power-law stretching index n on velocity profiles are shown in Figs. 1 and 2. With an increase of n the velocity profile decreases in the upper branch solution. However a crossover exists in the radial and vertical components of velocities in the lower

TABLE III. Comparison of numerical solution for $g'(0)$ with exact analytical solution (23) for different α for $n = 3$ taking unsteadiness parameter $A = -0.5$ and $A = -5.0$.

	A = -0.5			*A* = -5.0		
α	Anal	Numer	Abs. % Error	Anal	Numer	Abs. % Error
0.01	−0.00173	1.10848	100.2	−0.1937	0.990314	119.6
0.05	−0.01936	1.06311	101.8	−0.43314	0.863854	150.1
0.1	−0.05477	1.00516	105.4	−0.61255	0.711107	186.1
0.5	−0.61237	−1.0766	43.12	−1.3697	−1.58214	13.43
1	−1.73205	−2.095	17.32	−1.93705	−1.87967	3.052
5	−19.3649	−19.556	0.979	−4.33137	−4.02388	7.642
10	−54.7723	−55.322	0.994	−6.12549	−5.80787	5.469
50	−612.372	−621.34	1.443	−13.697	−13.333	2.73
100	−1732.05	−1757.6	1.454	−19.3705	−18.9495	2.222
500	−19364.9	−19343	0.114	−43.3137	−43.2663	0.11
1000	−54772.3	−54741	0.057	−61.2549	−61.2049	0.082
5000	−612372	−612304	0.011	−136.97	−136.744	0.165

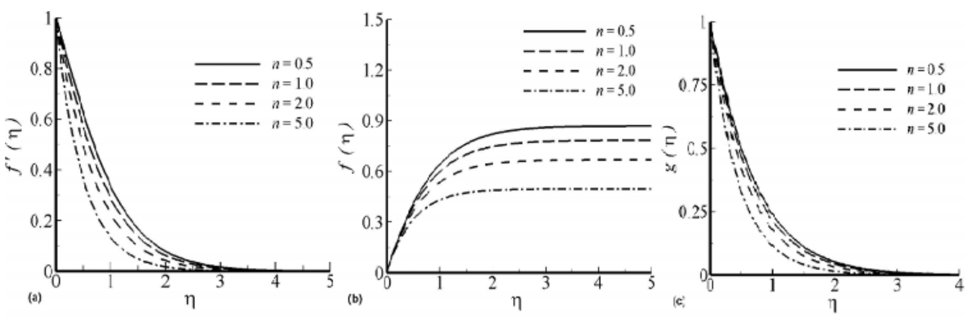

FIG. 1. Effects of power-law stretching index $n = 0.5$, 1.0, 2.0, 5.0 on velocity profile for $\alpha = 1.0$, $A = -0.1$ (Upper Branch Solution).

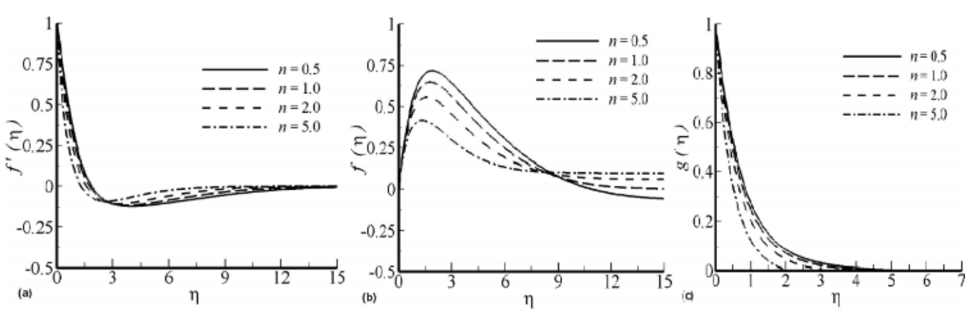

FIG. 2. Effects of power-law stretching index $n = 0.5$, 1.0, 2.0, 5.0 on velocity profile for $\alpha = 1.0$, $A = -0.1$ (Lower Branch Solution).

branch solution. Physically it can be explained as: increasing n the disk accelerates rapidly in radial direction and the fluid is moved towards the disk to decrease the boundary layer thickness and the velocity.

The cross over are observed for an increase in absolute value of unsteadiness parameter in all the components of velocities in the upper branch solutions and in radial and azimuthal velocities in the lower branch solution (Figs. 3 and 4).

In the upper branch solution all the velocity components first increase and then decrease. The behavior in the lower branch solution is different in that the. Radial and azimuthal velocities firstly

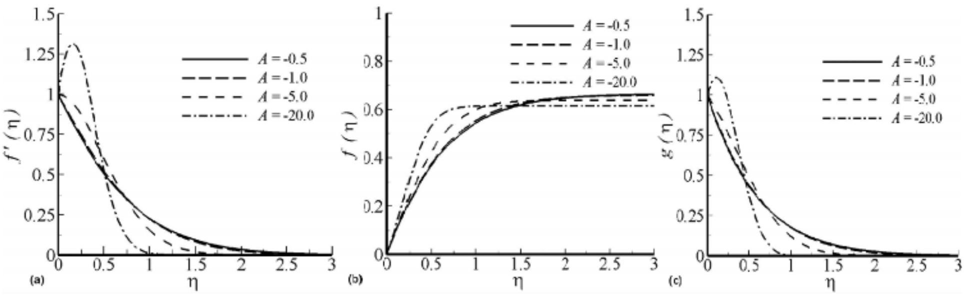

FIG. 3. Effects of unsteady parameter A = −0.5, −1.0, −5.0, −10.0 on velocity profile for $n = 2.0$, $\alpha = 1.0$ (Upper Branch Solution).

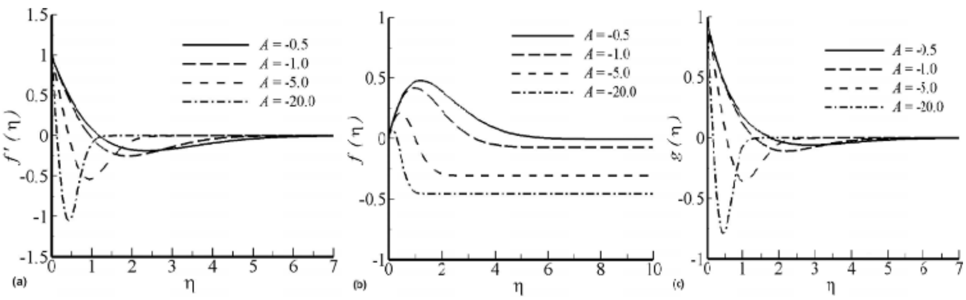

FIG. 4. Effects of unsteady parameter A = −0.5, −1.0, −5.0, −10.0 on velocity profile for $n = 2.0$, $\alpha = 1.0$ (Lower Branch Solution).

decrease then increase while the vertical velocity decreases throughout the boundary layer. All the velocity components have a common characteristic; that is, increasing |A| (that is, fast deceleration) the boundary layer becomes thinner. However upper branch solution is different from lower branch solution in the sense that flows are in opposite directions for all the velocity profiles.

The effect of disk stretching parameter and the disk rotation on the various components of velocities is shown in Figs 5 and 6. The vertical and the radial velocities increase with the increase of disk stretching parameter. Away from the surface, the radial velocity decreases by increasing disk stretching parameter for both lower and upper branch solution, although it is is more prominent in lower branch solution.

It is further observed that the effect of disk stretching parameter α is responsible to decrease the azimuthal velocity for both the upper and lower branch solutions. This is because increasing disk stretching parameter, at unsteadiness parameter $\alpha = 0.1$, the disk is stretched with slow deceleration, resulting in increase of radial velocity. Consequently, this causes the fluid to flow towards the disk at fist and later away from the disk in the far field decreasing the azimuthal velocity.

Effects of power-law stretching index n, unsteadiness parameter A and stretching parameter α on the radial and azimuthal skin friction are listed in Table IV. It is observed that both the values of $f''(0)$ and $g'(0)$ are monotonically decreasing with the increasing values of n, A and α, for the upper branch solution. However, for the lower branch solution the behavior is different as compared to upper branch solution. The effects of n and α are the same as in the case of upper branch solution, i.e., $f''(0)$ and $g'(0)$ decrease with an increase in these parameters. Values of $f''(0)$ increase with increase in A but the values of $g'(0)$ are quite diverse in nature for the lower solution branch. With the increase in $|A|g'(0)$ first increases (for $|A| < 0$) and then decreases (for $|A| > 1$).

To sum up, the flow of viscous fluid over a nonlinearly stretching rotating disk is studied in this paper. Two solution branches are observed due to the unsteadiness parameter. The upper branch solution is normally reported in literature, but the lower branch solution is found to be rightfully valid mathematically.

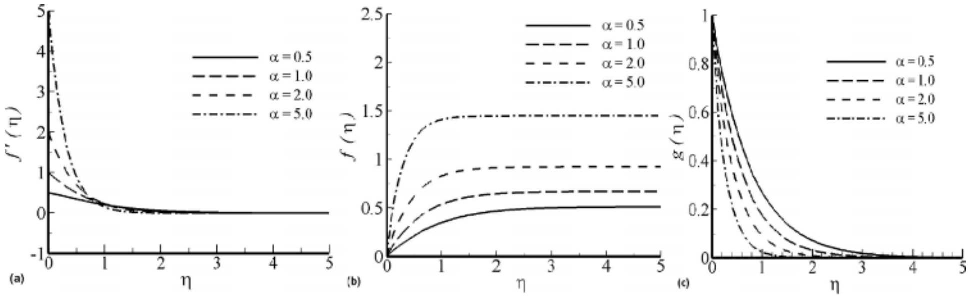

FIG. 5. Effects of stretching parameter $\alpha = 0.0$, 0.5, 1.0, 2.0 on velocity profile for $n = 2.0$, $A = -1.0$ (Upper Branch Solution).

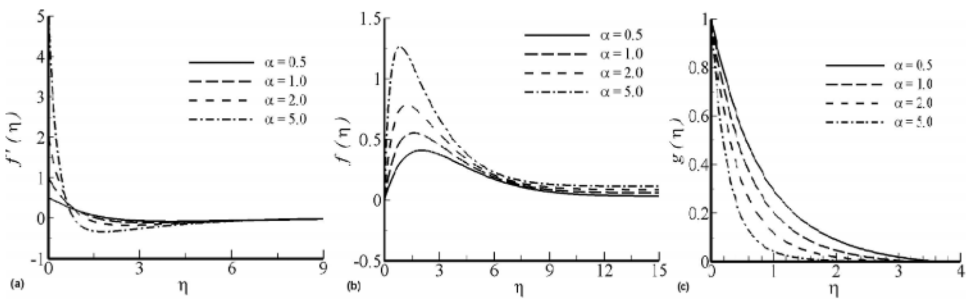

FIG. 6. Effects of stretching parameter $\alpha = 0.0$, 0.5, 1.0, 2.0 on velocity profile for $n = 2.0$, $A = -1.0$ (Lower Branch Solution).

TABLE IV. Variation of $f''(0)$ and $g'(0)$ for various values of power-law index n, unsteady parameter A and stretching parameter α.

N	A	α	Branch	$f''(0)$	$g'(0)$
0.5	−0.1	1.0	U	−0.70413	−1.31841
			L	−0.75747	−1.27094
1.0			U	−0.91897	−1.46559
			L	−0.96838	−1.42057
2.0			U	−1.26365	−1.72291
			L	−1.30881	−1.68409
5.0			U	−1.99594	−2.33066
			L	−2.03673	−2.3044
2.0	−0.5		U	−1.16716	−1.64719
			L	−1.30971	−1.60623
	−1.0		U	−1.04581	−1.55197
			L	−1.33298	−1.54166
	−5.0		U	−0.05564	−0.77489
			L	−2.05285	−1.70949
	−20.0		U	3.788	1.96352
			L	−6.28732	−4.80627
	−0.1	0.5	U	−0.24698	−1.2578
			L	−0.26796	−1.22853
		1.0	U	−1.26365	−1.72291
			L	−1.30881	−1.68409
		2.0	U	−4.01199	−2.41893
			L	−4.11297	−2.36846
		5.0	U	−16.3911	−3.82322
			L	−16.6913	−3.75867

V. CONCLUSION

The main focus of this study is to find unsteady flow for nonlinear stretching of the rotating disk. The dual solutions for the unsteady flow for the linear stretching[10] have been recounted for the nonlinear stretching also. Firstly, we develop the important mathematically tool of similarity transformation for the existing problem to convert the partial differential equation into consistent self similar ordinary differential equation. The resulting nonlinear differential equation is solved numerically to find the solution for all values of the stretching parameter. To add beauty, reliance and reliability to the analysis, we present an exact analytical solution for large stretching parameter. Besides, the perfect agreement with the numerical solution for large stretching parameter limit, the analysis reveals important and interesting physical revelations which would have been obscure in the numerical results. The most phenomenal observation is that for large stretching limit the azimuthal velocity and the unsteadiness has no significant consequences. This study has thus been of equal importance in the engineering processes through the numerical results and providing physical insight through analytical results. The study has thus far more meaning than its practical applications only.

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