NASA/TM—2013–217919

Simple Analytic Expressions for the Magnetic Field of a Circular Current Loop

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February 2001

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Preface

In the late 1990s research was being performed at the Kennedy Space Center to develop *in situ* resource utilization technology for Mars. One study topic was the generation and storage of liquid oxygen (LOX) obtained from the atmosphere or regolith, but the transfer of this commodity was of concern. Mechanical LOX pumps were deemed potentially too heavy and unreliable for an autonomous mission to Mars, and alternatives were sought. One option was to use the paramagnetic property of LOX, which is significant enough that small quantities of LOX can be suspended against earth gravity with a rare earth magnet. With this application in mind, a small, internally funded project was initiated at the Kennedy Space Center in 2000 to study the use of pulsed magnetic fields to pump LOX.

Proof-of-concept testing verified the LOX pumping predictions and resulted in a journal publication [1]. Numerous small coils were fabricated on cryogenic flow lines and used to produce intense, short-duration magnetic fields resulting in dramatic motion of the LOX. In addition, effort was expended on modeling the paramagnetic forces in the LOX, which required modeling the magnetic field generated by the coils and the coil inductance, allowing the current versus time to be predicted and compared to experiment.

While we were modeling the coil magnetic field, using *Mathematica*™, we came upon a set of simple analytical expressions for the magnetic field and its spatial derivatives in Cartesian, cylindrical, and spherical coordinates generated by a simple, infinitesimally thin current loop. We wrote a short manuscript documenting these expressions, but did not proceed with publication. Instead, this manuscript entered the public domain through two different routes. It appeared first on the NASA Technical Reports Server (NTRS) at the following link:

http://ntrs.nasa.gov/archive/nasa/casi.ntrs.nasa.gov/20010038494_2001057024.pdf

and then later on the Open Channel Foundation, an organization that publishes software from academic and research institutions. The link for this second site is given below.

http://www.openchannelfoundation.org/project/view_docs.php?group_id=288

Through this link, our manuscript was downloaded multiple times and cited seven times as of the writing of this NASA Technical Memorandum, but only by title and author list and sometimes by web site URL. We, the authors, have decided that there should be a clear, long-term citation for this work, yet we consider it too late to seek formal publication. So we have decided to issue a NASA Technical Memorandum, with the same title and authors as the original manuscript, allowing it to be located by the search engines, and providing a reliable, long-term source and citation for this work. The original manuscript, reformatted, but not in any other way altered, appears on the subsequent pages of this memorandum.

We, the authors, would like to express our gratitude to J.M. Griffith, who found a typo in Equation (10) of the original manuscript. This memorandum contains the corrected equation.

[1] Youngquist, Robert C., Immer, Christopher D., Lane, John E., and Simpson, James C., "Dynamics of a Finite Liquid Oxygen Column in a Pulsed Magnetic Field," *IEEE Transactions on Magnetics,* **39**(4), July 4, 2003, pp. 2068–2073.

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1 INTRODUCTION

Analytic expressions for the magnetic induction (magnetic flux density, **B**) of a simple planar circular current loop have been published in Cartesian and cylindrical coordinates [1,2], and are also known implicitly in spherical coordinates [3]. In this paper, we present explicit analytic expressions for **B** and its spatial derivatives in Cartesian, cylindrical, and spherical coordinates for a filamentary current loop. These results were obtained with extensive use of *Mathematica*™ and are exact throughout all space outside of the conductor. The field expressions reduce to the well-known limiting cases and satisfy $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$ outside the conductor.

These results are general and applicable to any model using filamentary circular current loops. Solenoids of arbitrary size may be easily modeled by approximating the total magnetic induction as the sum of those for the individual loops. The inclusion of the spatial derivatives expands their utility to magnetohydrodynamics where the derivatives are required.

The equations can be coded into any high-level programming language. It is necessary to numerically evaluate complete elliptic integrals of the first and second kind, but this capability is now available with most programming packages.

2 SPHERICAL COORDINATES

We start with spherical coordinates because this is the system usually used in the standard texts. The Cartesian and cylindrical results in Sections 3 and 4 were derived from the spherical coordinate results.

The current loop has radius a, is located in the x-y plane, is centered at the origin, and carries a current I which is positive as shown (Figure 1.). It is assumed that the cross section of the conductor is negligible.

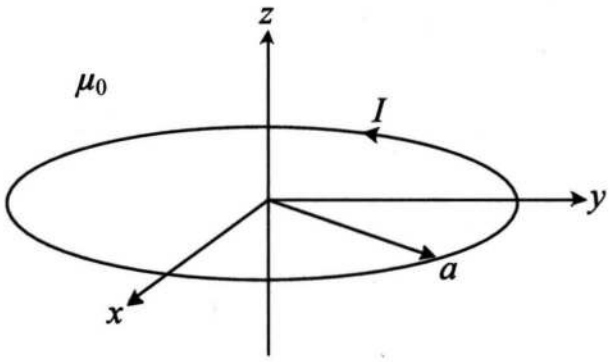

Figure 1. Circular current loop geometry.

The *vector potential*, **A,** of the loop is given by [3]:

$$A_\varphi(r,\theta) = \frac{\mu_0 Ia}{4\pi}\int_0^{2\pi} \frac{\cos\varphi'\, d\varphi'}{\sqrt{a^2+r^2-2ar\sin\theta\cos\varphi'}} = \frac{\mu_0}{4\pi}\frac{4Ia}{\sqrt{a^2+r^2+2ar\sin\theta}}\left[\frac{(2-k^2)K(k^2)-2E(k^2)}{k^2}\right], \tag{1}$$

where r, θ, and φ are the usual spherical coordinates, and the argument of the elliptic integrals is

$$k^2 = \frac{4ar\sin\theta}{a^2+r^2+2ar\sin\theta}. \tag{2}$$

Note that we use k^2 for the argument of the elliptic integrals. This choice is consistent with the convention of Abramowitz and Stegun [4] where $m = k^2$.

For a static field with constant current, the **B** components in spherical coordinates are [3]:

$$B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, A_\varphi) \tag{3}$$

$$B_\theta = -\frac{1}{r}\frac{\partial}{\partial r}(r\, A_\varphi) \tag{4}$$

$$B_\varphi = 0. \tag{5}$$

Analytic expression for the field components and their derivatives in spherical coordinates are given below. For simplicity we use the following substitutions: $\alpha^2 \equiv a^2 + r^2 - 2ar\sin\theta$, $\beta^2 \equiv a^2 + r^2 + 2ar\sin\theta$, $k^2 \equiv 1 - \alpha^2/\beta^2$, and $C \equiv \mu_0 I/\pi$. We note that if desired, further simplifications are possible using various substitutions and groupings.

Field Components:

$$B_r = \frac{Ca^2\cos\theta}{\alpha^2\beta}E(k^2) \tag{6}$$

$$B_\theta = \frac{C}{2\alpha^2\beta\sin\theta}\left[(r^2+a^2\cos 2\theta)E(k^2)-\alpha^2K(k^2)\right] \tag{7}$$

Spatial Derivatives of the Field Components:

$$\frac{\partial B_r}{\partial r} = \frac{Ca^2\cos\theta}{2r\alpha^4\beta^3}\left\{\left[a^4-7r^4-6a^2r^2\cos 2\theta\right]E\left(k^2\right)+\left[a^2\left(r^2-a^2\right)\right]K\left(k^2\right)\right\} \tag{8}$$

$$\frac{\partial B_r}{\partial \theta} = \frac{-Ca^2}{4\sin\theta\alpha^4\beta^3}\Big\{\Big[a^4 - 7a^2r^2 + r^4 + \cos 2\theta\left(-3a^4 + 2a^2r^2 - 3r^4\right) + a^2r^2\cos 4\theta\Big]E\left(k^2\right) + \Big[2\alpha^2\left(a^2 + r^2\right)\cos^2\theta\Big]K\left(k^2\right)\Big\} \tag{9}$$

$$\frac{\partial B_\theta}{\partial r} = \frac{-C}{4\alpha^4\beta^3 r\sin\theta}\Big\{\Big[a^6 - 3a^4r^2 + a^2r^4 + 2r^6 + a^2(3r^2 - a^2)(a^2 + r^2)\cos 2\theta + 3a^4r^2\cos 4\theta\Big]E\left(k^2\right) + \Big[\alpha^2\left(-a^4 + a^2r^2 - 2r^4 + a^2(a^2 - 3r^2)\cos 2\theta\right)\Big]K\left(k^2\right)\Big\} \tag{10}$$

$$\frac{\partial B_\theta}{\partial \theta} = \frac{-C\cos\theta}{4\alpha^4\beta^3\sin^2\theta}\Big\{\Big[5a^6 + 3a^4r^2 - 3a^2r^4 + 2r^6 + (-3a^6 + 2a^4r^2 + 9a^2r^4)\cos 2\theta + a^4r^2\cos 4\theta\Big]E(k^2) - \Big[3a^6 + 2a^4r^2 + a^2r^4 + 2r^6 + a^2(5r^2 - a^2)(a^2 + r^2)\cos 2\theta + (-7a^5r + 7a^3r^3 - 4ar^5)\sin\theta + a^3r(a^2 - 5r^2)\sin 3\theta\Big]K(k^2)\Big\} \tag{11}$$

3 CARTESIAN COORDINATES

The field components and their derivatives in Cartesian coordinates are given below. These are easier to use when rotations or translations are needed and obviate the need to transform the basis vectors. The following substitutions are used for simplicity: $\rho^2 \equiv x^2 + y^2$, $r^2 \equiv x^2 + y^2 + z^2$, $\alpha^2 \equiv a^2 + r^2 - 2a\rho$, $\beta^2 \equiv a^2 + r^2 + 2a\rho$, $k^2 \equiv 1 - \alpha^2/\beta^2$, $\gamma \equiv x^2 - y^2$, and $C \equiv \mu_0 I/\pi$. Note that ρ and r are non-negative.

Field Components:

$$B_x = \frac{C\,x\,z}{2\alpha^2\beta\rho^2}\Big[(a^2 + r^2)E(k^2) - \alpha^2K(k^2)\Big] \tag{12}$$

$$B_y = \frac{C\,y\,z}{2\alpha^2\beta\rho^2}\Big[(a^2 + r^2)E(k^2) - \alpha^2K(k^2)\Big] = \frac{y}{x}B_x \tag{13}$$

$$B_z = \frac{C}{2\alpha^2\beta}\left[(a^2 - r^2)E(k^2) + \alpha^2 K(k^2)\right] \tag{14}$$

Spatial Derivatives of the Field Components:

$$\begin{aligned}\frac{\partial B_x}{\partial x} = \frac{C\,z}{2\alpha^4\beta^3\rho^4}\Big\{\Big[a^4\left(-\gamma(3z^2 + a^2) + \rho^2(8x^2 - y^2)\right) - \\ a^2\left(\rho^4(5x^2 + y^2) - 2\rho^2 z^2(2x^2 + y^2) + 3z^4\gamma\right) - \\ r^4\left(2x^4 + \gamma(y^2 + z^2)\right)\Big]E\left(k^2\right) + \\ \Big[a^2\left(\gamma(a^2 + 2z^2) - \rho^2(3x^2 - 2y^2)\right) + \\ r^2\left(2x^4 + \gamma(y^2 + z^2)\right)\Big]\alpha^2 K\left(k^2\right)\Big\}\end{aligned} \tag{15}$$

$$\begin{aligned}\frac{\partial B_x}{\partial y} = \frac{C\,x\,y\,z}{2\alpha^4\beta^3\rho^4}\Big\{\Big[3a^4(3\rho^2 - 2z^2) - r^4(2r^2 + \rho^2) - \\ 2a^6 - 2a^2(2\rho^4 - \rho^2 z^2 + 3z^4)\Big]E\left(k^2\right) + \\ \Big[r^2(2r^2 + \rho^2) - a^2(5\rho^2 - 4z^2) + 2a^4\Big]\alpha^2 K\left(k^2\right)\Big\}\end{aligned} \tag{16}$$

$$\begin{aligned}\frac{\partial B_x}{\partial z} = \frac{C\,x}{2\alpha^4\beta^3\rho^2}\Big\{\Big[(\rho^2 - a^2)^2(\rho^2 + a^2) + \\ 2z^2(a^4 - 6a^2\rho^2 + \rho^4) + z^4(a^2 + \rho^2)\Big]E\left(k^2\right) - \\ \Big[(\rho^2 - a^2)^2 + z^2(\rho^2 + a^2)\Big]\alpha^2 K\left(k^2\right)\Big\}\end{aligned} \tag{17}$$

$$\frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y} \tag{18}$$

$$\begin{aligned}\frac{\partial B_y}{\partial y} = \frac{C\,z}{2\alpha^4\beta^3\rho^4}\Big\{\Big[a^4\left(\gamma(3z^2 + a^2) + \rho^2(8y^2 - x^2)\right) - \\ a^2\left(\rho^4(5y^2 + x^2) - 2\rho^2 z^2(2y^2 + x^2) - 3z^4\gamma\right) - \\ r^4\left(2y^4 - \gamma(x^2 + z^2)\right)\Big]E\left(k^2\right) + \\ \Big[a^2\left(-\gamma(a^2 + 2z^2) - \rho^2(3y^2 - 2x^2)\right) + \\ r^2\left(2y^4 - \gamma(x^2 + z^2)\right)\Big]\alpha^2 K\left(k^2\right)\Big\}\end{aligned} \tag{19}$$

$$\frac{\partial B_y}{\partial z} = \frac{y}{x}\frac{\partial B_x}{\partial z} \tag{20}$$

$$\frac{\partial B_z}{\partial x} = \frac{\partial B_x}{\partial z} \tag{21}$$

$$\frac{\partial B_z}{\partial y} = \frac{\partial B_y}{\partial z} \tag{22}$$

$$\frac{\partial B_z}{\partial z} = \frac{C\,z}{2\alpha^4\beta^3}\left\{\left[6a^2(\rho^2 - z^2) - 7a^4 + r^4\right]E(k^2) + \alpha^2\left[a^2 - r^2\right]K(k^2)\right. \tag{23}$$

4 CYLINDRICAL COORDINATES

The following substitutions are used for simplicity: $\alpha^2 \equiv a^2 + \rho^2 + z^2 - 2a\rho$, $\beta^2 \equiv a^2 + \rho^2 + z^2 + 2a\rho$, $k^2 \equiv 1 - \alpha^2/\beta^2$, and $C \equiv \mu_0 I/\pi$.

Field Components:

$$B_\rho = \frac{C\,z}{2\alpha^2\beta\rho}\left[(a^2 + \rho^2 + z^2)E(k^2) - \alpha^2 K(k^2)\right] \tag{24}$$

$$B_z = \frac{C}{2\alpha^2\beta}\left[(a^2 - \rho^2 - z^2)E(k^2) + \alpha^2 K(k^2)\right] \tag{25}$$

Spatial Derivatives of the Field Components:

$$\frac{\partial B_\rho}{\partial \rho} = \frac{-Cz}{2\rho^2\alpha^4\beta^3}\left\{\left[a^6 + (\rho^2 + z^2)^2(2\rho^2 + z^2) + a^4(3z^2 - 8\rho^2) + a^2(5\rho^4 - 4\rho^2 z^2 + 3z^4)\right]E\left(k^2\right) - \alpha^2\left[a^4 - 3a^2\rho^2 + 2\rho^4 + (2a^2 + 3\rho^2)z^2 + z^4\right]K\left(k^2\right)\right\} \tag{26}$$

$$\frac{\partial B_\rho}{\partial z} = \frac{C}{2\rho\alpha^4\beta^3}\left\{\left[(a^2 + \rho^2)(z^4 + (a^2 - \rho^2)^2) + 2z^2(a^4 - 6a^2\rho^2 + \rho^4)\right]E(k^2) - \alpha^2\left[(a^2 - \rho^2)^2 + (a^2 + \rho^2)z^2\right]K(k^2)\right\} \tag{27}$$

$$\frac{\partial B_z}{\partial z} = \frac{Cz}{2\alpha^4\beta^3}\left\{\left[\left(6a^2(\rho^2 - z^2) - 7a^4 + (\rho^2 + z^2)^2\right]E(k^2) + \alpha^2\left[a^2 - \rho^2 - z^2\right]K\left(k^2\right)\right\} \tag{28}$$

$$\frac{\partial B_z}{\partial \rho} = \frac{\partial B_\rho}{\partial z} \tag{29}$$

5 LIMITING CASES

Several special limiting cases are given for completeness. We have confirmed that our results given above do indeed converge to these formulas. We also give additional expressions for B_x and B_y near the axis that may prove useful.

Along the axis of the loop:

$$B_z = \frac{\mu_0 I a^2}{2(a^2 + z^2)^{3/2}} \tag{30}$$

Near the axis of the loop ($x,y<<a$):

$$B_x = \frac{3a^2 \mu_0 I x z}{4(a^2 + z^2)^{5/2}} \tag{31}$$

$$B_y = \frac{3a^2 \mu_0 I y z}{4(a^2 + z^2)^{5/2}} \tag{32}$$

Far from the loop ($r>>a$):

$$B_r = \frac{\mu_0}{2\pi}(I\pi a^2)\frac{\cos\theta}{r^3} \tag{33}$$

$$B_\theta = \frac{\mu_0}{4\pi}(I\pi a^2)\frac{\sin\theta}{r^3} \tag{34}$$

6 CONCLUSIONS

We have presented simple, closed-form algebraic formulas for the magnetic induction and its spatial derivatives of a filamentary current loop that are exact everywhere in space outside the conductor. Although these formulas are exact, they do require the numerical evaluation of elliptic integrals.

Solenoids with circular cross sections of arbitrary size and configuration can be modeled by simply summing the contributions of each individual loop.

There are, of course, other ways to obtain **B** for the basic circular current loop. For example, series expansions are available [3] or numerical integration via a finite element approach can be performed. However, these suffer from limitations such as truncating the series expansions after some tolerance is reached or accepting some graininess when using a discrete grid. Our approach has neither of these limitations and yields results are that exact up to the limitations of the numerical arithmetic and the elliptic integral routines.

The inclusion of the spatial derivatives allows convective derivatives to be found exactly and may prove useful for magnetohydrodynamics applications.

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