

Simple C^* -Algebras Generated by Isometries

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Abstract. We consider the C^* -algebra \mathcal{O}_n generated by $n \geq 2$ isometries S_1, \dots, S_n on an infinite-dimensional Hilbert space, with the property that $S_1S_1^* + \dots + S_nS_n^* = \mathbf{1}$. It turns out that \mathcal{O}_n has the structure of a crossed product of a finite simple C^* -algebra \mathcal{F} by a single endomorphism scaling the trace of \mathcal{F} by $1/n$. Thus, \mathcal{O}_n is a separable C^* -algebra sharing many of the properties of a factor of type III_λ with $\lambda = 1/n$. As a consequence we show that \mathcal{O}_n is simple and that its isomorphism type does not depend on the choice of S_1, \dots, S_n .

A C^* -algebra is simple if it contains no non-trivial closed two-sided ideals. We call a simple C^* -algebra with unit infinite if it contains an element X such that $X^*X = \mathbf{1}$ and $XX^* \neq \mathbf{1}$. While non-separable algebras of this type are well known (e.g. the Calkin algebra or type III factors on a separable Hilbert space) there is to my knowledge no explicit example of a separable simple infinite C^* -algebra. The existence of such algebras was proved by Dixmier in [9, 2.1] by the following argument. Let S_1, S_2 be two isometries ($S_i^*S_i = \mathbf{1}$, $i = 1, 2$) on an infinite-dimensional Hilbert space \mathcal{H} such that $S_1S_1^* + S_2S_2^* = \mathbf{1}$. Since the C^* -algebra $C^*(S_1, S_2)$ generated by S_1 and S_2 has a unit, it contains a maximal proper two-sided ideal \mathcal{J} . The quotient $C^*(S_1, S_2)/\mathcal{J}$ is separable, simple and infinite. One of the results of the present paper is that $C^*(S_1, S_2)$ itself is already simple (thus answering the question of Dixmier to this effect). More generally, we study the C^* -algebra generated by $n \geq 2$ isometries S_1, \dots, S_n satisfying $\sum_{i=1}^n S_iS_i^* = \mathbf{1}$ (this condition implies in particular that the range projections $S_iS_i^*$ are pairwise orthogonal). We include the case $n = \infty$. We note incidentally that J. Roberts, motivated by investigations on superselection sectors, has studied closed linear spaces generated by isometries with this property [15]. These spaces are in fact Hilbert spaces and $C^*(S_1, \dots, S_n)$ is from this point of view the C^* -algebra generated by a Hilbert space.

We construct a faithful conditional expectation of $C^*(S_1, \dots, S_n)$ onto a C^* -subalgebra \mathcal{F} and show that $C^*(S_1, \dots, S_n)$ is the crossed product of \mathcal{F} by a single endomorphism Φ (in a sense to be made precise in Section 2). If n is finite, then \mathcal{F} is a

UHF-algebra in the sense of Glimm [12] of type n^∞ and Φ scales the trace of \mathcal{F} by $1/n$. Thus we have here the C^* -analogue of a factor of type III_λ with $\lambda = 1/n$ (cf. [6]). We use this description of $C^*(S_1, \dots, S_n)$ to show that the isomorphism class of $C^*(S_1, \dots, S_n)$ does not depend on the choice of S_1, \dots, S_n —that is, if $\hat{S}_1, \dots, \hat{S}_n$ is a second family of isometries satisfying $\sum_{i=1}^n \hat{S}_i \hat{S}_i^* = \mathbf{1}$ then $C^*(\hat{S}_1, \dots, \hat{S}_n)$ is canonically isomorphic to $C^*(S_1, \dots, S_n)$. We denote in the following (the isomorphism class of) $C^*(S_1, \dots, S_n)$ by \mathcal{O}_n .

It is then easy to see that \mathcal{O}_n is simple. What is more, \mathcal{O}_n is simple in a very strong sense—for every $0 \neq X \in \mathcal{O}_n$ there are $A, B \in \mathcal{O}_n$ such that $AXB = \mathbf{1}$. Among infinite simple C^* -algebras the algebras \mathcal{O}_n play a universal role comparable to that which UHF-algebras play among antiliminary C^* -algebras. Any simple infinite C^* -algebra \mathcal{A} with unit $\mathbf{1}$ contains, given $n = 2, 3, \dots, \infty$, a C^* -subalgebra \mathcal{A}_n with $\mathbf{1} \in \mathcal{A}_n$ such that a quotient of \mathcal{A}_n is isomorphic to \mathcal{O}_n . For $n = \infty$ the subalgebra \mathcal{A}_∞ can even be chosen in such a way that \mathcal{A}_∞ itself is isomorphic to \mathcal{O}_∞ .

Since the algebras \mathcal{O}_n represent quite a new type of C^* -algebras they give rise to a number of counterexamples. From the representation as a crossed product it becomes clear by the recent results in [7], [4] that \mathcal{O}_n is nuclear. On the other hand we show that \mathcal{O}_n can not be an inductive limit of C^* -algebras of type I. This answers to the negative a question which arose naturally in the recent development of the theory of nuclear C^* -algebras (cf. [3]). J. Rosenberg after reading this article showed that \mathcal{O}_n is even amenable [16]. Since \mathcal{O}_n is clearly not strongly amenable this solves a problem of Johnson [13, 10.2].

C^* -algebras generated by isometries have been studied before by various authors. Curiously enough, it usually turns out that the isomorphism class of these C^* -algebras does not depend on the choice of the isometries—but only on their algebraic relations. The difference between the present paper and investigations such as [2, 5, 11] lies in the fact that the isometries considered here are in every respect non-commutative.

We remark further that O. Bratteli has recently shown that the crossed product of the CAR-algebra by a gauge automorphism is simple [1]. However, these automorphisms do not scale the trace, so the algebras obtained are finite.

1. The Algebras \mathcal{O}_n

In the following we fix $n = 2, 3, \dots, \infty$ and a (finite or infinite) sequence $\{S_i\}_{i=1}^n$ of isometries (i.e. $S_i^* S_i = \mathbf{1}$) on a Hilbert space \mathcal{H} . If n is finite we assume that $\sum_{i=1}^n S_i S_i^* = \mathbf{1}$. If n is infinite we assume that $\sum_{i=1}^r S_i S_i^* \leq \mathbf{1}$ for every $r \in \mathbb{N}$. We are going to determine the structure of the C^* -algebra $C^*(S_1, \dots, S_n)$ (we use this notation also if n is infinite) generated by $\{S_i\}_{i=1}^n$.

1.1. Given $k \in \mathbb{N}$, let W_k^n be the set of all k -tuples (j_1, \dots, j_k) , with $j_i \in \{1, \dots, n\}$ ($i = 1, \dots, k$) if n is finite, or $j_i \in \mathbb{N}$ if n is infinite. Further let $W_0^n = \{\mathbf{0}\}$ and $W_\infty^n = \bigcup_{k=0}^\infty W_k^n$.

We write $S_0 = \mathbf{1}$ and, given $\alpha = (j_1, \dots, j_k) \in W_k^n$, we denote by S_α the isometry $S_\alpha = S_{j_1} S_{j_2} \dots S_{j_k}$. Let $\ell(\alpha) = k$ be the length of α and $\ell(\mathbf{0}) = 0$.

1.2. With this notation we have the following lemma.

Lemma. a) Let $\mu, v \in W_\infty^n$ and $\ell(\mu) = \ell(v)$. Then $S_\mu^* S_v = \delta_{\mu v} \mathbf{1}$.

b) Let $\mu, v \in W_\infty^n$ and let P, Q be the range projections of S_μ, S_v , respectively. Suppose $S_\mu^* S_v \neq 0$.

If $\ell(\mu) = \ell(v)$ then $S_\mu = S_v$ and $P = Q$.

If $\ell(\mu) < \ell(v)$ then $S_v = S_\mu S_{\mu'}$ with $\mu' \in W_{\ell(v)-\ell(\mu)}^n$ and $P > Q$.

If $\ell(\mu) > \ell(v)$ then $S_\mu = S_v S_{\nu'}$ with $\nu' \in W_{\ell(\mu)-\ell(v)}^n$ and $P < Q$.

Proof. a) follows easily from the relation $S_i^* S_j = \delta_{ij} \mathbf{1}$.

b) The first assertion follows immediately from a). To prove the second assertion write $S_v = S_\alpha S_{\mu'}$ where $\ell(\alpha) = \ell(\mu)$ and $\ell(\mu') = \ell(v) - \ell(\mu)$. By a) we have $S_\mu^* S_\alpha S_{\mu'} = \delta_{\mu\alpha} S_{\mu'}$, whence $\alpha = \mu$. Finally $Q = S_v S_v^* = S_\alpha (S_{\mu'} S_{\mu'}^*) S_\alpha^* < S_\alpha S_\alpha^* = P$.

1.3. Lemma. Let $M \neq 0$ be a word in $\{S_i\} \cup \{S_i^*\}$. Then there are two unique elements $\mu, v \in W_\infty^n$ such that $M = S_\mu S_v^*$.

Proof. Let $M = X_1 \dots X_r$, where $X_j \in \{S_i\} \cup \{S_i^*\}$ ($j = 1, \dots, r$). In this expression we may cancel out every term of the form $X_i X_{i+1}$ with $X_i X_{i+1} = \mathbf{1}$. After finitely many such eliminations we get an expression for M in lowest terms $M = Y_1 \dots Y_s$ where $Y_i Y_{i+1} \neq \mathbf{1}$ ($i = 1, \dots, s-1$). Since $S_i^* S_j = \delta_{ij} \mathbf{1}$ and $M \neq 0$, the Y_i must satisfy the following

$$Y_j \in \{S_i\} \Rightarrow Y_{j-1} \in \{S_i\} \quad (j = 2, \dots, s).$$

Thus, if j_0 is the largest number between 0 and s such that $Y_{j_0} \in \{S_i\}$, we have $Y_j \in \{S_i\}$ for $1 \leq j \leq j_0$ and $Y_j \in \{S_i^*\}$ for $j_0 + 1 \leq j \leq s$. This shows that there are $\mu, v \in W_\infty^n$ such that $M = S_\mu S_v^*$. Assume that $\alpha, \beta \in W_\infty^n$ are such that $M = S_\alpha S_\beta^*$. Then obviously $S_\mu^* S_\alpha \neq 0$ (since $M^* M \neq 0$) and $S_\mu S_\mu^* = M M^* = S_\alpha S_\alpha^*$. Thus the range projections of S_μ and S_α coincide and according to Lemma 1.2b) we get $S_\mu = S_\alpha$. The same argument applied to M^* shows $S_v = S_\beta$.

1.4. Let $\mathcal{F}_0^n = \mathbb{C}\mathbf{1}$ and let \mathcal{F}_k^n be the C^* -algebra generated by the set $\{S_\mu S_v^* | \mu, v \in W_k^n\}$. We denote by \mathcal{M}_r the star algebra of $r \times r$ complex matrices and by \mathcal{K} the algebra of compact operators on an infinite dimensional separable Hilbert space.

Proposition. If n is finite then \mathcal{F}_k^n is star isomorphic to \mathcal{M}_{n^k} and $\mathcal{F}_k^n \subset \mathcal{F}_{k+1}^n$ ($k = 0, 1, 2, \dots$). If n is infinite then \mathcal{F}_k^n is star isomorphic to \mathcal{K} for all $k > 0$.

Proof. According to 1.2a), for $\mu, \mu', v, v' \in W_k^n$, we have

$$(S_\mu S_v^*)(S_{\mu'} S_{v'}^*) = \delta_{v\mu'} S_\mu S_{v'}^*.$$

Since also $(S_\mu S_v^*)^* = S_v S_\mu^*$ this shows that $\{S_\mu S_v^* | \mu, v \in W_k^n\}$ is a self-adjoint system of matrix units generating \mathcal{F}_k^n . If n is finite, then

$$S_\mu S_v^* = \sum_{i=1}^n S_\mu S_i S_i^* S_v^*$$

is in \mathcal{F}_{k+1}^n since each summand on the right hand side is in \mathcal{F}_{k+1}^n .

1.5. Let \mathcal{F}^n be the C^* -algebra generated by the union of all \mathcal{F}_k^n ($k = 0, 1, 2, \dots$). Proposition 1.4 shows that \mathcal{F}^n is a UHF-algebra of type n^∞ , if n is finite. If n is infinite \mathcal{F}^∞ is not a UHF-algebra but an AF-algebra.

1.6. We are now going to describe the algebra \mathcal{P} generated algebraically by $\{S_i\}_{i=1}^n$ and $\{S_i^*\}_{i=1}^n$. We take and fix one of the S_i , say S_1 . To emphasize the special role of

S_1 , we will write V for S_1 and V^{-1} for S_1^* . Let $M = S_\mu S_v^*$ be a word in $\{S_i\}$ and $\{S_i^*\}$. Let $r = \ell(\mu)$, $s = \ell(v)$ and $k = r - s$.

If $k > 0$ set $\hat{M} = S_\mu S_v^* S_1^{*k}$. Then $\hat{M} \in \mathcal{F}_r^n$ and $M = \hat{M} V^k$.

If $k < 0$ set $\tilde{M} = S_1^{-k} S_\mu S_v^*$. Then $\tilde{M} \in \mathcal{F}_s^n$ and $M = V^k \tilde{M}$.

If $k = 0$ then $M \in \mathcal{F}_r^n = \mathcal{F}_s^n$.

Since any $A \in \mathcal{P}$ is a linear combination of words, A can be written in the form

$$A = \sum_{i=-N}^{-1} V^i A_i + A_0 + \sum_{i=1}^N A_i V^i$$

where the A_i are in \mathcal{F}^n . We write $A_i = F_i(A)$.

1.7. Proposition. *The elements $A_i = F_i(A)$ are uniquely determined by the construction described above (they do not depend on the special representation of A as a linear combination of words). We have $\|F_i(A)\| \leq \|A\|$.*

For the proof of this proposition we first need a lemma. Let n be finite and let $\{\varepsilon_i\}_{i \in \mathbb{N}}$ with $\varepsilon_i \in \{1, \dots, n\}$ be a sequence which is aperiodic in the sense that there is no $i_0 > 0$ such that $\{\varepsilon_i\}_{i \geq i_0}$ becomes periodic. Given $r \in \mathbb{N}$, write $U_r = S_{\varepsilon_1} \dots S_{\varepsilon_r}$ and $P_r = U_r U_r^*$.

1.8. Lemma. *Let M_1, \dots, M_m be words in S_1, \dots, S_n and S_1^*, \dots, S_n^* and let k be a natural number. Suppose that each M_i has the form $M_i = S_\mu S_v^*$ where $\ell(\mu) \neq \ell(v)$. Then there is $r \in \mathbb{N}$ such that*

$$P_r S_\alpha^* M_i S_\beta P_r = 0$$

for $i = 1, \dots, m$ and for all $\alpha, \beta \in W_k^n$.

Proof. If $M_i = S_\mu S_v^*$ where $\ell(\mu) \neq \ell(v)$, then $S_\alpha^* M_i S_\beta = 0$ or we have after cancellation $S_\alpha^* M_i S_\beta = S_\gamma^* S_\delta^*$ in lowest terms where $\ell(\gamma) - \ell(\delta) = \ell(\mu) - \ell(v)$ (cf. 1.3). This shows that $S_\alpha^* M_i S_\beta$ also satisfies the hypothesis on M_i of the Lemma for any $\alpha, \beta \in W_k^n$. Thus it suffices to show that for any finite collection $M_1, \dots, M_{m'}$ of words of the form $M_i = S_{\mu_i} S_{v_i}^*$ with $\ell(\mu_i) \neq \ell(v_i)$, there is $r \in \mathbb{N}$ such that $P_r M_i P_r = 0$ ($i = 1, \dots, m'$). It suffices to prove this for the case $m' = 1$.

Let $\ell(\mu_1) = p$ and $\ell(v_1) = q$. Then, for $r > p, q$, the expression $L_r = U^{*r} M_1 U^r$ can be non-zero only if $S_{\mu_1} = U_p$ and $S_{v_1} = U_q$ (1.2b)). Thus $L_r = S_{\varepsilon_r}^* \dots S_{\varepsilon_{p+1}}^* S_{\varepsilon_{q+1}} \dots S_{\varepsilon_r}$. But then L_r must be zero for sufficiently large r since by assumption $p \neq q$ and since $\{\varepsilon_i\}$ is aperiodic.

Proof of Proposition 1.7. Since for $i \geq 0$, by construction $F_{i+1}(A) = F_i(AV^*)$ and for $i \leq 0$, $F_{i-1}(A) = F_i(VA)$, it suffices to prove the assertions for $F_0(A)$.

We consider first the case that n is finite. Choose an aperiodic sequence $\{\varepsilon_i\}$ as in the preceding lemma. Let k be so large that $F_0(A)$ is in \mathcal{F}_k^n . Using Lemma 1.8 we find $r \in \mathbb{N}, r > k$ such that $P_r S_\alpha^* V^j A_j S_\beta P_r = 0$ for $j = -N, \dots, -1$ and $P_r S_\alpha^* A_j V^j S_\beta P_r = 0$ for $j = 1, \dots, N$ and for all $\alpha, \beta \in W_k^n$. We set

$$Q := \sum_{\alpha \in W_k^n} S_\alpha P_r S_\alpha^*.$$

Then $Q V^j A_j Q = 0$ for $j = -N, \dots, -1$ and $Q A_j V^j Q = 0$ for $j = 1, \dots, N$. On the other hand Q commutes with every $X \in \mathcal{F}_k^n$ and $X \mapsto QXQ$ is an isomorphism of \mathcal{F}_k^n onto

$Q\mathcal{F}_k^n Q$. In fact, $QS_\alpha S_\beta^* = S_\alpha S_\beta^* Q = S_\alpha P_r S_\beta^*$ and the set $\{S_\alpha P_r S_\beta^* | \alpha, \beta \in W_k^n\}$ is a self-adjoint system of matrix units generating $Q\mathcal{F}_k^n Q$. Thus

$$\|F_0(A)\| = \|QF_0(A)Q\| = \|QAQ\| \leq \|A\|.$$

Consider now the case $n = \infty$. There is a finite subset \mathbb{I} of \mathbb{N} such that A is a linear combination of words in S_i, S_i^* ($i \in \mathbb{I}$). We assume that $C^*(S_1, S_2, \dots)$ is represented on Hilbert space and choose an isometry \hat{S} such that $\hat{S}^* \hat{S} = \mathbf{1}$ and

$$\hat{S}\hat{S}^* = P = \mathbf{1} - \sum_{i \in \mathbb{I}} S_i S_i^*.$$

We may assume that $1 \in \mathbb{I}$ and define $\hat{F}_i(X)$ for X in the star algebra $\tilde{\mathcal{P}}$ generated algebraically by $S_i, i \in \mathbb{I}$ and \hat{S} , as above with respect to $V = S_1$. Then $\hat{F}_0(A) = F_0(A)$ since A is an expression in S_i, S_i^* only. We know already from above that there is a projection Q in $\tilde{\mathcal{P}}$ such that $QAQ = Q\hat{F}_0(A)Q$ and $\|Q\hat{F}_0(A)Q\| = \|\hat{F}_0(A)\|$. Hence

$$\|F_0(A)\| = \|\hat{F}_0(A)\| = \|Q\hat{F}_0(A)Q\| = \|QAQ\| \leq \|A\|.$$

Since in the finite and in the infinite case the mapping $F_0(A) \mapsto QF_0(A)Q$ is an isomorphism, we finally see that $F_0(A)$ is uniquely determined by $QF_0(A)Q$, hence by A .

1.9. Suppose that $\{\hat{S}_i\}_{i=1}^n$ is a second family of isometries satisfying $\sum_{i=1}^n \hat{S}_i \hat{S}_i^* = \mathbf{1}$ and let $\hat{\mathcal{P}}$ be the star algebra generated algebraically by this family. It follows from 1.4 that $\mathcal{F}^n \cap \mathcal{P}$ and $\mathcal{F}^n \cap \hat{\mathcal{P}}$ are algebraically isomorphic. Since these algebras are inductive limits of finite-dimensional C^* -algebras, they carry a unique C^* -norm. We may therefore identify \mathcal{F}^n and $\hat{\mathcal{P}}$. With this identification, if $A \in \mathcal{P}$ and \hat{A} is the corresponding linear combination of words in $\hat{\mathcal{P}}$, then $F_i(A) = F_i(\hat{A})$ for all $i \in \mathbb{Z}$. In particular, $A = 0$ if and only if $\hat{A} = 0$. This shows that \mathcal{P} and $\hat{\mathcal{P}}$ are algebraically star isomorphic. We equip \mathcal{P} with the largest C^* -norm

$$\|X\|_0 = \sup \{\|\varrho(X)\| \mid \varrho \text{ is a star representation of } \mathcal{P} \text{ on a separable Hilbert space}\}.$$

Let \mathcal{L} be the $\|\cdot\|_0$ -completion of \mathcal{P} . Since $\|\cdot\|_0$ is a C^* -norm which majorizes the initial norm on \mathcal{P} , the C^* -algebra $C^*(S_1, \dots, S_n)$ is a quotient of \mathcal{L} . We shall show that $\mathcal{L} \cong C^*(S_1, \dots, S_n)$. This will imply

$$C^*(S_1, \dots, S_n) \cong \mathcal{L} \cong \hat{\mathcal{L}} \cong C^*(\hat{S}_1, \dots, \hat{S}_n)$$

1.10. The mappings $F_i : \mathcal{P} \rightarrow \mathcal{F}^n$ ($i \in \mathbb{Z}$) extend according to Proposition 1.7 to normdecreasing linear mappings $F_i : C^*(S_1, \dots, S_n) \rightarrow \mathcal{F}^n$ and $F_i : \mathcal{L} \rightarrow \mathcal{F}^n$ (the use of the same notation for both mappings will not cause confusion). F_0 is a conditional expectation [17, p. 101].

Proposition. Let $X \in \mathcal{L}$. If $F_i(X) = 0$ for all $i \in \mathbb{Z}$, then $X = 0$.

Proof. We use an argument which appears in [14, 1.2.5]. Let \mathcal{L} be faithfully represented on \mathcal{H} . By definition of the norm on \mathcal{L} the mapping $\varrho_\lambda : S_i \mapsto \lambda S_i$ ($i = 1, \dots, n$) extends, for every $\lambda \in \mathbb{C}$ with modulus 1 to a continuous star representation ϱ_λ of \mathcal{L} on \mathcal{H} . Note that $\varrho_\lambda(X) = X$ for every $X \in \mathcal{F}^n$.

Given $\xi, \eta \in \mathcal{H}$ with $\|\xi\| = \|\eta\| = 1$, let f be the function on the unit circle \mathbb{T} in \mathbb{C} which is defined by

$$f(\lambda) = (\varrho_\lambda(X) \xi | \eta) \quad (\lambda \in \mathbb{T}).$$

Let $\{A_k\}$ be a sequence in \mathcal{P} which converges in \mathcal{L} to X . Consider the functions

$$h_k(\lambda) = (\varrho_\lambda(A_k) \xi | \eta) \quad (\lambda \in \mathbb{T}).$$

Since $\|\varrho_\lambda(X) - \varrho_\lambda(A_k)\|_0 \leq \|X - A_k\|_0$, the functions h_k tend to f uniformly on \mathbb{T} . We have

$$\begin{aligned} h_k(\lambda) &= \sum_{i=-N}^{-1} (\lambda^i V^i F_i(A_k) \xi | \eta) \\ &\quad + (F_0(A_k) \xi | \eta) + \sum_{i=1}^N (F_i(A_k) \lambda^i V^i \xi | \eta) = \sum_{i=-N}^N a_{ik} \lambda^i. \end{aligned}$$

The i -th Fourier-coefficient a_{ik} of h_k converges to the i -th Fourier-coefficient f_i of f as $k \rightarrow \infty$.

But $\lim_{k \rightarrow \infty} |a_{ik}| \leq \lim_{k \rightarrow \infty} \|F_i(A_k)\|_0 = 0$ by assumption for all $i \in \mathbb{Z}$ so that $f = 0$ and $X = 0$, since ξ, η were arbitrary.

Remark 1. The idea of the proof of 1.10 really consists in interpreting $F_i(X)$ as i -th Fourier coefficient of the function $\lambda \mapsto \varrho_\lambda(X)$ ($\lambda \in \mathbb{T}$). In fact, the equation $F_i(X) = \int_{\mathbb{T}} \varrho_\lambda(X) \lambda^{-i} d\lambda$ holds for every $X \in \mathcal{L}$.

Remark 2. Let $A_k \in \mathcal{P}$ converge to $X \in \mathcal{L}$. Since

$$F_0(X^* X) = \lim_{k \rightarrow \infty} \left[\sum_{i<0} F_i(A_k)^* F_i(A_k) + F_0(A_k)^* F_0(A_k) + \sum_{i>0} V^{-i} F_i(A_k)^* F_i(A_k) V^i \right]$$

we see from the proposition that F_0 is faithful in \mathcal{L} .

This fact and Proposition 1.10 itself could have been derived in a slightly different approach from the general theory of crossed products [18]. We preferred the proof given above because it is very elementary and fits exactly into the framework of this paper.

1.11. Proposition. \mathcal{L} is canonically isomorphic to $C^*(S_1, \dots, S_n)$.

Proof. The identity mapping $\pi: \mathcal{P} \rightarrow \mathcal{P}$ extends to a continuous star homomorphism π of \mathcal{L} onto $C^*(S_1, \dots, S_n)$. We show that π is injective. We obviously have $F_i \circ \pi = \pi \circ F_i$ [after identification of \mathcal{F}^n and $\pi^{-1}(\mathcal{F}^n)$]. If $\pi(X) = 0$ then $F_i(\pi(X)) = 0$ whence $\pi(F_i(X)) = F_i(X) = 0$ for all $i \in \mathbb{Z}$.

1.12. Theorem. If $\{\hat{S}_i\}_{i=1}^n$ is a second family of isometries satisfying $\sum_{i=1}^n \hat{S}_i \hat{S}_i^* = \mathbf{1}$ (or $\sum_{i=1}^r \hat{S}_i \hat{S}_i^* \leq \mathbf{1}$ for every $r \in \mathbb{N}$, if $n = \infty$), then $C^*(\hat{S}_1, \dots, \hat{S}_n)$ is canonically isomorphic to $C^*(S_1, \dots, S_n)$ (i.e. the map $\hat{S}_i \rightarrow S_i$ extends to an isomorphism from $C^*(\hat{S}_1, \dots, \hat{S}_n)$ onto $C^*(S_1, \dots, S_n)$).

Proof. This follows from 1.9 and 1.11. Note that in 1.9 all isomorphisms are canonical.

In view of this it makes sense to write \mathcal{O}_n for $C^*(S_1, \dots, S_n)$ since the isomorphism class of \mathcal{O}_n does not depend on the choice of $\{S_i\}_{i=1}^n$. We remark that Theorem 1.12 also shows that \mathcal{O}_n is simple. In fact, let \mathcal{J} be a maximal ideal in $\mathcal{O}_n = C^*(S_1, \dots, S_n)$ and $\pi: \mathcal{O}_n \rightarrow \mathcal{O}_n/\mathcal{J}$ the canonical projection mapping. Then, by Theorem 1.12, the simple C^* -algebra $\mathcal{O}_n/\mathcal{J} = C^*(\pi(S_1), \dots, \pi(S_n))$ is isomorphic to \mathcal{O}_n . But we are now going to show that \mathcal{O}_n has a property which is much stronger than simplicity (in [8] we raised the question if every infinite simple C^* -algebra with unit has this property).

1.13. Theorem. *Let n be finite and let X be a non-zero element of \mathcal{O}_n . Then there are $A, B \in \mathcal{O}_n$ such that $AXB = 1$.*

Proof. By 1.10 we have $F_0(X^*X) \neq 0$. Without loss of generality assume that $\|F_0(X^*X)\| = 1$. Let $Y \in \mathcal{P}$ be a positive element such that $\|X^*X - Y\| < \varepsilon \leq 1/4$. Then $\|F_0(Y)\| \geq 1 - \varepsilon$ (1.7). In the proof of Proposition 1.7 we constructed a projection $Q \in \mathcal{F}^n \cap \mathcal{P}$ such that $\|QF_0(Y)Q\| = \|F_0(Y)\|$ and $QYQ = QF_0(Y)Q$. Let k be so large that $QF_0(Y)Q$ is in \mathcal{F}_k^n . Since \mathcal{F}_k^n is a finite-dimensional C^* -algebra, QYQ has the form $QYQ = \sum_{i=1}^s \lambda_i R_i$ where R_i are minimal projections in \mathcal{F}_k^n and λ_i are positive real numbers. There is i_0 , $1 \leq i_0 \leq s$ such that $\lambda_{i_0} \geq 1 - \varepsilon$ and there is a partial isometry U in \mathcal{F}_k^n such that $U^*U = R_{i_0}$ and $UU^* = S_1^k S_1^{*k}$ (note that $S_1^k S_1^{*k}$ is a minimal projection in \mathcal{F}_k^n). Then with $A = S_1^{*k} U Q$ we have $AYA^* = \lambda_{i_0} 1$ and

$$\|AX^*XA^* - 1\| \leq \|AX^*XA^* - AYA^*\| + \|AYA^* - 1\| \leq 2\varepsilon$$

(since $\|A\| = 1$ and $1 - \varepsilon \leq \lambda_{i_0} \leq 1 + \varepsilon$). This shows that AX^*XA^* is invertible and we are done.

Remark. If in the situation of the preceding theorem $X \geq 0$ and $\|F_0(X)\| = 1$, then it is obvious from the proof given above that A and B can be chosen such that $\|A\|, \|B\| \leq 1 + \varepsilon$, for any given $\varepsilon > 0$. (Moreover A, B can be chosen such that $B = A^*$.) We will use this in Section 3 where we will prove a version of Theorem 1.13 for \mathcal{O}_∞ . A different proof of 1.13 for the case $n = \infty$ could also be given using methods similar (but more complicated) to those employed in the proof above.

2. Representation of \mathcal{O}_n as a Crossed Product

2.1. Let $n \geq 2$ be finite and let $j \in \mathbb{Z}$. Then \mathcal{F}^n can be represented as an infinite tensor product [17, 1.23.11]

$$\mathcal{F}^n = \bigotimes_{i=j}^{\infty} \mathcal{N}_i = \mathcal{A}_j \quad \text{where} \quad \mathcal{N}_i \cong \mathcal{M}_n \quad \text{for all } i.$$

Define embeddings

$$\mathcal{A}_0 \hookrightarrow \mathcal{A}_{-1} \hookrightarrow \mathcal{A}_{-2} \hookrightarrow \dots$$

by $\mathcal{A}_j \ni X \mapsto e_{11} \otimes X \in \mathcal{A}_{j-1} = \mathcal{M}_n \otimes \mathcal{A}_j$, where $\{e_{ij} | i, j = 1, \dots, n\}$ denotes a self-adjoint system of matrix units in \mathcal{M}_n . If we take the C^* -inductive limit [17, 1.23] of this sequence we get a C^* -algebra \mathcal{C}_n isomorphic to $\mathcal{K} \otimes \mathcal{F}^n$. We may, of course,

continue the above sequence of embeddings to positive integers

$$\dots \hookrightarrow \mathcal{A}_2 \hookrightarrow \mathcal{A}_1 \hookrightarrow \mathcal{A}_0 \hookrightarrow \mathcal{A}_{-1} \hookrightarrow \dots$$

in the same way by $\mathcal{A}_j \ni X \mapsto e_{11} \otimes X \in \mathcal{A}_{j-1}$ ($j \in \mathbb{Z}$). Since all \mathcal{A}_j are isomorphic we may consider the automorphism Φ of \mathcal{C}_n which is induced by the shift to the left, mapping an element in \mathcal{A}_j to the corresponding element in \mathcal{A}_{j+1} . One may express the action of Φ somewhat informally by $\Phi(X) = e_{11} \otimes X \in e_{11} \otimes \mathcal{A}_j \cong \mathcal{A}_j$ for $X \in \mathcal{A}_{j-1}$.

Let the crossed product $C^*(\mathcal{C}_n, \Phi)$ be faithfully represented on the Hilbert space \mathcal{H} . Then there is a unitary U on \mathcal{H} such that $\Phi(X) = UXU^*(X \in \mathcal{C}_n)$ and $C^*(\mathcal{C}_n, \Phi)$ is the closure of the set of finite sums of the form $A = \sum_{i=-N}^N X_i U^i$ ($X_i \in \mathcal{C}_n$). With $\tilde{X}_i = U^{-i} X_i U^i$ this expression becomes

$$A = \sum_{i<0} U^i \tilde{X}_i + X_0 + \sum_{i>0} X_i U^i \quad (\tilde{X}_i, X_i \in \mathcal{C}_n).$$

Let P be the unit of $\mathcal{A}_0 \subset C^*(\mathcal{C}_n, \Phi)$. Since $UPU^* = e_{11} \otimes P \in \mathcal{A}_0 = \mathcal{M}_n \otimes \mathcal{A}_1$ we have $UP = PUP$ and $PX_i U^i P = (PX_i P)(UP)^i$ for $i > 0$ and $P U^i \tilde{X}_i P = (UP)^{* - i} P \tilde{X}_i P$ for $i < 0$. With $V = UP$ we get

$$PAP = \sum_{i<0} V^i P \tilde{X}_i P + PX_0 P + \sum_{i>0} PX_i P V^i.$$

Thus $\mathcal{E}_n = PC^*(\mathcal{C}_n, \Phi)P$ is generated by $\mathcal{A}_0 = P\mathcal{C}_n P$ together with V .

Let $S_i = (e_{ii} \otimes P)V$ ($i = 1, \dots, n$). Then $S_i^* S_i = P$ and $\sum_{i=1}^n S_i S_i^* = P$. Further \mathcal{A}_0 is generated by all elements of the form $S_\mu S_\nu^*$ where $\mu, \nu \in W_\infty^n$ and $\ell(\mu) = \ell(\nu)$. In fact, if $\mu = (j_1, \dots, j_k)$ and $\nu = (i_1, \dots, i_k)$, then $S_\mu S_\nu^* = e_{j_1 i_1} \otimes e_{j_2 i_2} \otimes \dots \otimes e_{j_k i_k} \otimes P \in \mathcal{A}_0 = \mathcal{M}_n \otimes \dots \otimes \mathcal{M}_n \otimes \mathcal{A}_k$. Hence $\mathcal{E}_n = C^*(S_1, \dots, S_n) \cong \mathcal{O}_n$.

Let P_k be the unit of \mathcal{A}_k ($k \leq 0$). Then $C^*(\mathcal{C}_n, \Phi)$ is the inductive limit of $P_k C^*(\mathcal{C}_n, \Phi) P_k$ ($k \rightarrow -\infty$). It is not hard to see that $P_{k-1} C^*(\mathcal{C}_n, \Phi) P_{k-1}$ is generated by $P_k C^*(\mathcal{C}_n, \Phi) P_k$ together with $\{e_{ij} \otimes P_k \mid 1 \leq i, j \leq n\} \subset \mathcal{A}_{k-1}$ and that, consequently, $C^*(\mathcal{C}_n, \Phi)$ is isomorphic to $\mathcal{K} \otimes \mathcal{O}_n$.

2.2. Let now $n = \infty$. For $j \in \mathbb{N}$ let \mathcal{A}_j be the C^* -subalgebra of \mathcal{O}_∞ defined by $\mathcal{A}_j = S_1^j \mathcal{F}^\infty S_1^{*j}$. Then $\mathcal{A}_{j-1} \cong \mathbb{C}1 \oplus (\mathcal{K} \otimes \mathcal{A}_j)$. On the other hand we also have $\mathcal{A}_i \cong \mathcal{A}_0 = \mathcal{F}^\infty$ for all $i \in \mathbb{N}$. Define \mathcal{A}_j for negative j inductively by $\mathcal{A}_{j-1} = \mathbb{C}1 \oplus (\mathcal{K} \otimes \mathcal{A}_j)$. We fix a minimal projection R in \mathcal{K} and consider the sequence of embeddings

$$\mathcal{A}_0 \hookrightarrow \mathcal{A}_{-1} \hookrightarrow \mathcal{A}_{-2} \hookrightarrow \dots$$

defined by $\mathcal{A}_j \ni X \mapsto R \otimes X \in \mathcal{K} \otimes \mathcal{A}_j \subset \mathcal{A}_{j-1}$. Let \mathcal{C}_∞ be the inductive limit of this sequence. Clearly \mathcal{C}_∞ is an AF -algebra. If as above we let Φ be the automorphism of \mathcal{C}_∞ which is induced by the shift to the left on the above sequence (continued to positive integers) then $\mathcal{O}_\infty \cong PC^*(\mathcal{C}_\infty, \Phi)P$ where P is the unit of $\mathcal{A}_0 \subset C^*(\mathcal{C}_\infty, \Phi)$.

2.3. We have seen that \mathcal{O}_n ($n = 2, \dots, \infty$) is isomorphic to the crossed product of an AF -algebra by a single automorphism, cut down by a projection. By recent results of Connes [7, 6.8, 6.5, Theorem 6] and Choi and Effros [4, Corollary 3.2] this

proves that \mathcal{O}_n is nuclear. I am indebted to A. Connes and S. Sakai who called my attention to this fact. We show now that \mathcal{O}_n can not be obtained as an inductive limit of type I C^* -algebras.

Proposition. *Let n be finite and let S_1, \dots, S_n be isometries on a Hilbert space \mathcal{H} satisfying $\sum_{i=1}^n S_i S_i^* = P \leq \mathbf{1}$. Suppose that $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a C^* -algebra containing elements A_1, \dots, A_n such that $\|A_i - S_i\| < \varepsilon$. If ε is sufficiently (depending on n) small then there are $\tilde{A}_1, \dots, \tilde{A}_n \in \mathcal{A}$ such that $\tilde{A}_i^* \tilde{A}_i = \mathbf{1}$ and $\sum_{i=1}^n \tilde{A}_i \tilde{A}_i^* \leq \mathbf{1}$. If $P = \mathbf{1}$ then $\tilde{A}_1, \dots, \tilde{A}_n$ can be chosen such that the sum of the range projections of \tilde{A}_i equals $\mathbf{1}$.*

Proof. Let $\varepsilon < 1/10$. We have

$$\|A_i^* A_i - \mathbf{1}\| \leq \|A_i^* A_i - A_i^* S_i\| + \|A_i^* S_i - S_i^* S_i\| \leq (1 + \varepsilon)\varepsilon + \varepsilon < 3\varepsilon.$$

Hence $A_i^* A_i$ is invertible and

$$\|A_i - A_i(A_i^* A_i)^{-\frac{1}{2}}\| \leq \|A_i\| \|1 - A_i^* A_i^{-\frac{1}{2}}\| < (1 + \varepsilon) 3\varepsilon < 4\varepsilon.$$

Now $V_i = A_i(A_i^* A_i)^{-\frac{1}{2}}$ is an isometry and

$$\|V_i V_i^* - S_i S_i^*\| \leq \|V_i V_i^* - S_i V_i^*\| + \|S_i V_i^* - S_i S_i^*\| < 5\varepsilon + 5\varepsilon = 10\varepsilon.$$

Further

$$\begin{aligned} \|(V_i V_i^*)(V_j V_j^*)\| &\leq \|(S_i S_i^*)(S_j S_j^*)\| + \|(S_i S_i^* - V_i V_i^*) S_j S_j^*\| \\ &\quad + \|V_i V_i^*(V_j V_j^* - S_j S_j^*)\| < 20\varepsilon \quad \text{for } i \neq j. \end{aligned}$$

Given $\delta > 0$, by [12, 1.7], if ε is sufficiently small there is a family of pairwise orthogonal projections E_1, \dots, E_n in \mathcal{A} such that $\|E_i - V_i V_i^*\| < \delta$. Then $\|E_i V_i - V_i\| < \delta$. Thus $V_i^* E_i V_i$ is invertible for small δ and the elements $\tilde{A}_i = (E_i V_i)(V_i^* E_i V_i)^{-\frac{1}{2}}$ are isometries. Moreover the elements $\tilde{A}_i \tilde{A}_i^* = E_i$ are pairwise orthogonal projections and $Q = \sum_{i=1}^n \tilde{A}_i \tilde{A}_i^*$ is a projection such that

$$\|Q - P\| = \left\| \sum_{i=1}^n (E_i - S_i S_i^*) \right\| \leq n(\delta + 10\varepsilon).$$

In particular $Q = \mathbf{1}$ if $P = \mathbf{1}$ and ε and δ are sufficiently small.

Corollary 1. *Let \mathcal{A} be a C^* -subalgebra of \mathcal{O}_n (n finite) containing elements A_1, \dots, A_n such that $\|A_i - S_i\| < \varepsilon$. If ε is sufficiently (depending on n) small then any such \mathcal{A} must contain a C^* -subalgebra which is isomorphic to \mathcal{O}_n .*

Corollary 2. *An infinite simple C^* -algebra \mathcal{B} with unit can not be an inductive limit of type I C^* -algebras.*

Proof. By [8, 2.2] \mathcal{B} contains isometries V_1, V_2 such that $V_1 V_1^* + V_2 V_2^* \leq \mathbf{1}$. Let \mathcal{A} be a C^* -subalgebra of \mathcal{B} containing elements A_1, A_2 such that $\|A_j - V_j\| < \varepsilon$. If ε is sufficiently small, then \mathcal{A} contains isometries \tilde{A}_1, \tilde{A}_2 such that $\tilde{A}_1 \tilde{A}_1^* + \tilde{A}_2 \tilde{A}_2^* \leq \mathbf{1}$. Since a quotient of $C^*(\tilde{A}_1, \tilde{A}_2)$ is isomorphic to \mathcal{O}_2 (3.1) and \mathcal{O}_2 is clearly not of Type I, \mathcal{A} can not be of type I.

2.4. As \mathcal{O}_n is simple, so is $\mathcal{K} \otimes \mathcal{O}_n$. But $\mathcal{K} \otimes \mathcal{O}_n$ is even algebraically simple (i.e. has no non-trivial not necessarily closed two-sided ideals). This follows from the following general theorem.

Theorem. *Let \mathcal{A} be a simple C^* -algebra with unit. Then $\mathcal{K} \otimes \mathcal{A}$ is algebraically simple if and only if there is $k \in \mathbb{N}$ such that $\mathcal{M}_k \otimes \mathcal{A}$ is infinite.*

Proof. “Only if part”. We use the notation of [8]. Assume that $\mathcal{M}_k \otimes \mathcal{A}$ is finite and let P be a projection of dimension r and Q a projection of dimension 1 in \mathcal{M}_k . Then $(P \otimes \mathbf{1}/Q \otimes \mathbf{1}) = r$ in $\mathcal{M}_k \otimes \mathcal{A}$. In fact, we have $a = (P \otimes \mathbf{1}/Q \otimes \mathbf{1}) \leq r$. On the other hand $a < r$ would imply $(P \otimes \mathbf{1}/R \otimes \mathbf{1}) = 1$ for any projection $R \leq P$ of dimension a in \mathcal{M}_k . Since $P \otimes \mathbf{1}$ is a finite projection in $\mathcal{M}_k \otimes \mathcal{A}$ [8, 2.4], this is impossible [8, 2.1]. Assume now that $\mathcal{M}_k \otimes \mathcal{A}$ is finite for any $k \in \mathbb{N}$. If P is a projection of dimension r and Q a projection of dimension 1 in \mathcal{K} then $(P \otimes \mathbf{1}/Q \otimes \mathbf{1})$ in $\mathcal{K} \otimes \mathcal{A}$ equals $(P \otimes \mathbf{1}/Q \otimes \mathbf{1})$ in $(P \otimes \mathbf{1})(\mathcal{K} \otimes \mathcal{A})(P \otimes \mathbf{1}) \cong \mathcal{M}_r \otimes \mathcal{A}$ hence equals r (we may assume $Q \leq P$). Let P_1, P_2, \dots be a sequence of one-dimensional orthogonal projections in \mathcal{K} and let $H = \sum_{i=1}^{\infty} \lambda_i P_i$ where $\lambda_i > 0$ and $\lambda_i \rightarrow 0$.

Then for any $r \in \mathbb{N}$ and for any one-dimensional projection Q in \mathcal{K} we have

$$H \gtrapprox \sum_{i=1}^r P_i = A_r \quad \text{and} \quad (H \otimes \mathbf{1}/Q \otimes \mathbf{1}) \geq (A_r \otimes \mathbf{1}/Q \otimes \mathbf{1}) = r.$$

This shows that the ideal generated algebraically by $Q \otimes \mathbf{1}$ in $\mathcal{K} \otimes \mathcal{A}$ does not contain $H \otimes \mathbf{1}$.

“If part”. The proof is essentially contained already in [10, 3.1.4]. We have only to combine Dixmier’s argument with [8, 2.2]. We may assume that \mathcal{A} itself is infinite. Let E_1, E_2, \dots be a sequence of pairwise orthogonal one-dimensional projections in \mathcal{K} such that the sequence $\{H_k\}_{k=1}^{\infty}$, defined by $H_k = \sum_{i=1}^k E_i$, is an approximate identity for \mathcal{K} . It is easy to see that $H_k \otimes \mathbf{1}$ is an approximate identity for $\mathcal{K} \otimes \mathcal{A}$ (it is enough to check this for the algebraic tensor product of \mathcal{K} and \mathcal{A}).

Let \mathcal{J} be a non-zero ideal of $\mathcal{K} \otimes \mathcal{A}$. If $X \neq 0$ is in \mathcal{J} then there is k such that $(H_k \otimes \mathbf{1})X(H_k \otimes \mathbf{1}) \neq 0$ hence there are $i, j, 1 \leq i, j \leq k$ such that $(E_i \otimes \mathbf{1})X(E_j \otimes \mathbf{1}) \neq 0$. If $E_{ij} \in \mathcal{K}$ is a partial isometry with support projection E_j and range projection E_i then $(E_i \otimes \mathbf{1})X(E_{ij} \otimes \mathbf{1})^*$ is in \mathcal{J} and is non-zero. Thus $\mathcal{J} \cap E_i \otimes \mathcal{A}$ is non-zero, hence equals $E_i \otimes \mathcal{A}$ since $\mathcal{A} \cong E_i \otimes \mathcal{A}$ is algebraically simple.

From [8, 2.2] using induction we get the existence of infinitely many pairwise orthogonal projections F_i and elements V_i in \mathcal{A} such that $V_i^* V_i = \mathbf{1}$ and $V_i V_i^* = F_i$ ($i = 1, 2, \dots$). We have $E_1 \otimes F_i \sim E_1 \otimes \mathbf{1} \sim E_i \otimes \mathbf{1}$ in $\mathcal{K} \otimes \mathcal{A}$. Let U_i be a partial isometry in $\mathcal{K} \otimes \mathcal{A}$ with range projection $E_1 \otimes F_i$ and support projection $E_i \otimes \mathbf{1}$. With $G_k = \sum_{i=1}^k F_i$ and $Y_k = \sum_{i=1}^k U_i$ we have $Y_k Y_k^* = E_1 \otimes G_k$ and $Y_k^* Y_k = H_k \otimes \mathbf{1}$.

To complete the proof it is enough to show that any positive element X of $\mathcal{K} \otimes \mathcal{A}$ is in \mathcal{J} . Since $(H_k \otimes \mathbf{1})X^{\frac{1}{2}}$ is a Cauchy sequence also $Y_k X^{\frac{1}{2}}$ is a Cauchy sequence converging to an element Y of $\mathcal{K} \otimes \mathcal{A}$. Since $(E_1 \otimes \mathbf{1})Y = Y$ and $E_1 \otimes \mathbf{1} \in \mathcal{J}$ we have $Y, Y^* \in \mathcal{J}$. Therefore $Y^* Y = X$ is in \mathcal{J} .

Remark. Let $A, B \in \mathcal{K} \otimes \mathcal{O}_n$ and $B \neq 0$. There are $i, j \in \mathbb{N}$ such that $(E_i \otimes \mathbf{1})B(E_j \otimes \mathbf{1}) \neq 0$. Let $C = (E_{1,i} \otimes \mathbf{1})(E_i \otimes \mathbf{1})B(E_j \otimes \mathbf{1})(E_{j,1} \otimes \mathbf{1})$ (E_{ij} = partial isometry in \mathcal{K} with support projection E_j and range projection E_i). Then $C \neq 0$ and $C \in E_1 \otimes \mathcal{O}_n$. There are F, G in \mathcal{O}_n such that $(E_1 \otimes F)C(E_1 \otimes G) = E_1 \otimes \mathbf{1}$ (1.13, 3.4).

Further there are X_1, \dots, X_r and Y_1, \dots, Y_r in $\mathcal{K} \otimes \mathcal{O}_n$ such that $A = \sum_{i=1}^r X_i(E_1 \otimes \mathbf{1})Y_i$ (the ideal generated by $E_1 \otimes \mathbf{1}$ in $\mathcal{K} \otimes \mathcal{O}_n$ consists exactly of all finite sums of this form). Let V_1, \dots, V_r be isometries in \mathcal{O}_n such that $V_1V_1^*, \dots, V_rV_r^*$ are pairwise orthogonal projections in \mathcal{O}_n . Then

$$A = \left(\sum_{i=1}^r X_i(E_1 \otimes V_i^*) \right) (E_1 \otimes \mathbf{1}) \left(\sum_{i=1}^r (E_1 \otimes V_i) Y_i \right).$$

Together this shows that there are $X, Y \in \mathcal{K} \otimes \mathcal{O}_n$ such that $A = XBY$.

3. Extensions of \mathcal{O}_n

3.1. Proposition. Let V_1, \dots, V_n be isometries on a Hilbert space \mathcal{H} such that $\sum_{i=1}^n V_i V_i^* \leq \mathbf{1}$ (n finite). Then the projection $P = \mathbf{1} - \sum_{i=1}^n V_i V_i^*$ generates a closed two-sided ideal \mathcal{J} in $C^*(V_1, \dots, V_n)$ which is isomorphic to \mathcal{K} and contains P as a minimal projection. The quotient $C^*(V_1, \dots, V_n)/\mathcal{J}$ is isomorphic to \mathcal{O}_n .

Proof. Define, given $\mu \in W_\infty^n$, an isometry V_μ in the same way S_μ was defined in Section 1. The closure of the set \mathcal{J} of all linear combinations of elements of the form $V_\mu P V_\nu^*$ ($\mu, \nu \in W_\infty^n$) is clearly a two-sided ideal in $C^*(V_1, \dots, V_n)$. On the other hand \mathcal{J} is contained in every two-sided ideal containing P .

Consider the product $X = (V_\mu P V_\nu^*)(V_\alpha P V_\beta^*)$ ($\mu, \nu, \alpha, \beta \in W_\infty^n$). After cancellation we have $V_\nu^* V_\alpha = V_\gamma V_\delta^*$ ($\gamma, \delta \in W_\infty^n$) in lowest terms (1.3). But $P V_\gamma V_\delta^* P \neq 0$ if and only if $V_\gamma V_\delta^* = \mathbf{1}$, since $P V_i = 0$ ($i = 1, \dots, n$). Thus $X \neq 0$ if and only if $P V_\nu^* V_\alpha P \neq 0$ if and only if $\nu = \alpha$ (1.2). Hence

$$(V_\mu P V_\nu^*)(V_\alpha P V_\beta^*) = \delta_{\nu\alpha} V_\mu P V_\beta^*$$

and

$$(V_\mu P V_\nu^*)^* = V_\nu P V_\mu^*.$$

In other words the set $\{V_\mu P V_\nu^* | \mu, \nu \in W_\infty^n\}$ is a self-adjoint system of matrix units generating \mathcal{J} . Therefore \mathcal{J} can be mapped isomorphically onto a dense star subalgebra of \mathcal{K} which is an inductive limit of finite-dimensional C^* -algebras, hence carries a unique C^* -norm. This mapping must be isometric and extends to an isomorphism of $\mathcal{J} = \bar{\mathcal{J}}$ onto \mathcal{K} .

Remark 1. It seems to be interesting to study more general extensions of \mathcal{O}_n by the compacts.

Remark 2. In the situation of the proposition, given i ($1 \leq i \leq n$) and $\mu, \nu \in W_\infty^n$, there is $k \in \mathbb{N}$ such that $V_i^{*k} V_\mu P V_\nu^* V_i^k = 0$. This shows that $V_i^{*k} A V_i^k$ tends to zero as $k \rightarrow \infty$ for each $A \in \mathcal{J}$.

3.2. Let \mathcal{A} be a simple C^* -algebra with unit. It follows by induction from [8, 2.2] that \mathcal{A} contains a sequence V_1, V_2, \dots of isometries satisfying $\sum_{i=1}^k V_i V_i^* \leq \mathbf{1}$ for every $k \in \mathbb{N}$. We know already from Section 1 that $C^*(V_1, V_2, \dots) \cong \mathcal{O}_\infty$. From 3.1 we see that $C^*(V_1, \dots, V_n)$ ($n \geq 2$ finite) contains a closed two-sided ideal \mathcal{J} such that $C^*(V_1, \dots, V_n)/\mathcal{J} \cong \mathcal{O}_n$. Therefore \mathcal{O}_∞ is contained (with the same unit) in \mathcal{A} and \mathcal{O}_n is for any finite $n \geq 2$ contained up to quotients in \mathcal{A} .

3.3. Consider $\mathcal{O}_2 = C^*(S_1, S_2)$. We put $\hat{S}_1 = S_1^2$, $\hat{S}_2 = S_1 S_2$, and $\hat{S}_3 = S_2$. Then $\hat{S}_i^* \hat{S}_i = \mathbf{1}$ and $\sum_{i=1}^3 \hat{S}_i \hat{S}_i^* = \mathbf{1}$ so that $\mathcal{O}_3 \cong C^*(\hat{S}_1, \hat{S}_2, \hat{S}_3) \subset \mathcal{O}_2$. By induction we get the following chain of inclusions

$$\mathcal{O}_2 \supset \mathcal{O}_3 \supset \mathcal{O}_4 \supset \dots \supset \mathcal{O}_\infty.$$

3.4. We use 3.1 to prove a version of 1.13 for \mathcal{O}_∞ .

Theorem. *Let X be a non-zero element of \mathcal{O}_∞ . Then there are $A, B \in \mathcal{O}_\infty$ such that $AXB = \mathbf{1}$.*

Proof. We may assume that $X \geq 0$ and $\|F_0(X)\| = 1$. Let Y be a positive element of the star algebra generated algebraically by S_1, S_2, \dots such that $\|X - Y\| < \varepsilon < 1/4$. Without loss of generality we may assume that $\|F_0(Y)\| = 1$.

There is a finite subset \mathbb{I} of \mathbb{N} such that Y is a linear combination of words in S_i, S_i^* ($i \in \mathbb{I}$). We assume that \mathcal{O}_∞ is represented on the Hilbert space \mathcal{H} and choose an isometry \hat{S} on \mathcal{H} such that $\hat{S} \hat{S}^* = \mathbf{1} - \sum_{i \in \mathbb{I}} S_i S_i^*$. Further we fix $i_0 \in \mathbb{N}$ such that $i_0 \notin \mathbb{I}$. We consider the C^* -algebras \mathcal{A}_1 , generated by S_i ($i \in \mathbb{I}$) together with \hat{S} , and \mathcal{A}_2 , generated by S_i ($i \in \mathbb{I}$) together with S_{i_0} . The projection $P = \mathbf{1} - \sum_{i \in \mathbb{I}} S_i S_i^* - S_{i_0} S_{i_0}^*$ generates a non-trivial closed two-sided ideal \mathcal{J} in \mathcal{A}_2 (3.1) and $\mathcal{A}_2/\mathcal{J}$ is canonically isomorphic to \mathcal{A}_1 (1.12).

We may assume that $i \in \mathbb{I}$ and define \tilde{F}_i in \mathcal{A}_1 with respect to S_1 and \tilde{F}_i in $\mathcal{A}_2/\mathcal{J}$ with respect to $\varrho(S_1)$ (where $\varrho: \mathcal{A}_2 \rightarrow \mathcal{A}_2/\mathcal{J}$ is the canonical mapping) in the same way in which F_i was defined in Section 1. Then $\tilde{F}_0(Y) = F_0(Y)$ since Y is an expression in S_i, S_i^* ($i \in \mathbb{I}$) only. Therefore

$$\|\tilde{F}_0(\varrho(Y))\| = \|\tilde{F}_0(Y)\| = \|F_0(Y)\| = 1.$$

By the remark in 1.13 there are $A, B \in \mathcal{A}_2/\mathcal{J}$ such that $A\varrho(Y)B = \mathbf{1}$ and $\|A\|, \|B\| < 1 + \varepsilon$. Then A, B can be lifted to elements \tilde{A}, \tilde{B} in \mathcal{A}_2 such that $\|\tilde{A}\|, \|\tilde{B}\| < 1 + 2\varepsilon$. We have $\tilde{A}Y\tilde{B} = \mathbf{1} + K$ with $K \in \mathcal{J}$. By Remark 2 in 3.1 we get $S_i^{*k}(\tilde{A}Y\tilde{B})S_i^k \rightarrow \mathbf{1}$ as $k \rightarrow \infty$ for each $i \in \mathbb{I}$. Since

$$\|S_i^{*k}(\tilde{A}X\tilde{B})S_i^k - S_i^{*k}(\tilde{A}Y\tilde{B})S_i^k\| < (1 + 2\varepsilon)^2 \varepsilon < 1$$

this shows that $S_i^{*k}(\tilde{A}X\tilde{B})S_i^k$ is invertible for sufficiently large k .

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