Simple Closed Analytic Formulas for the Approximation of the Legendre Complete Elliptic Integrals K(*k*) and E(*k*) (and their First Derivatives)

RICHARD SELESCU
Flow Physics Department, Experimental Aerodynamics Compartment, Trisonic Wind Tunnel Laboratory
"Elie Carafoli" National Institute for Aerospace Research – INCAS (under the Aegis of the Romanian Academy)
Bucharest, Sector 6, Bd. Iuliu Maniu, No. 220, Code 061126
ROMANIA

Abstract: Two sets of closed analytic functions are proposed for the approximate calculus of the complete elliptic integrals of the 1st and 2nd kind in the normal form due to Legendre, their expressions having a remarkable simplicity and accuracy. The special usefulness of the newly proposed original formulas consists in that they allow performing the analytic study of variation of the functions in which they appear, using derivatives (they being expressed in terms of elementary functions only, without any special function; this would mean replacing one difficulty by another of the same kind). Comparative tables of the approximate values so obtained and the exact ones, reproduced from special functions tables are given (vs. the elliptic integrals modulus k). It is to be noticed that both sets of formulas are given neither by spline nor by regression functions, but by asymptotic expansions, the identity with the exact functions being accomplished for the left domain's end. As for their simplicity, the formulas in k / k' do not need any mathematical table (are purely algebraic). As for their accuracy, the 2nd set, although more intricate, gives more accurate values than the 1st one and extends itself more closely to the right domain's end. Legendre complete elliptic integrals of the 1st and 2nd kinds are very useful in Signal Processing and especially in the design of Digital Filters (Legendre Digital Filters).

Key-Words: analytic methods; Legendre complete elliptic integrals of the 1st and 2nd kind, K(*k*) and E(*k*); elliptic integral's modulus *k*; elliptic integral's complementary modulus *k'*; tables of Legendre complete elliptic integrals; approximate formulas Digital Signal Processing Filters

Received: June 24, 2020. Revised: October 17, 2020. Accepted: November 26, 2020. Published: December 2 , 2020.

1 Introduction – elliptic integrals

There are many interesting domains in pure and applied mathematics where appear one or both complete elliptic integrals of the 1st and 2nd kind in the normal form due to Legendre. The period of oscillations in a vacuum of the simple pendulum, in the dynamics of a constrained heavy particle, is given by a complete elliptic integral of the 1st kind. The length of an ellipse, in the geometry of plane curves, as well as the lift coefficient of a thin delta wing with subsonic leading edges, in supersonic aerodynamics (small perturbations theory), are given by a complete elliptic integral of the 2nd kind. The following relations define these integrals of the 1st and 2nd kind, respectively

$$\mathrm{K}(k)=\int_0^{\pi/2}(1-k^2\sin^2\varphi)^{-1/2}d\varphi=\int_0^1[(1-t^2)(1-k^2t^2)]^{-1/2}dt;$$

$$\mathrm{E}(k)=\int_0^{\pi/2}(1-k^2\sin^2\varphi)^{1/2}d\varphi=\int_0^1[(1-k^2t^2)(1-t^2)^{-1}]^{1/2}dt;$$

$k=\sin\theta\geq 0$ is called *modulus*. K(*k*), E(*k*) are typical *elliptic* integrals. They do not admit primitive functions (cannot be expressed in terms of elementary functions), being calculated by expanding the integrands into series, integrating term-by-term, and presented vs. $k\in[0,1]$, or vs. $\theta\in[0,\pi/2]$, in some mathematical tables [1] – [6]. Modern mathematics defines an elliptic integral as any function *f* which can be expressed in the form $f(x)=\int_c^x R[t, P(t)^{1/2}]\,dt$; *R* is a rational function of its two arguments; *P* is a polynomial of degrees 3 or 4 with no repeated roots; *c* is a constant. The values given in some special tables allow performing the calculus for a given case (point), but not the analytic study of variation of the functions in which these integrals appear, using the derivatives. In the next chapter two sets (subscripts 0 and 1) of closed analytic functions are given for the approximate calculus of K(*k*) and E(*k*). The method used in this work is *purely analytic*, not needing any numerical procedure, or sophisticated computer programs. There also is a Legendre complete elliptic integral of the 3rd kind. Legendre complete elliptic integrals of the 1st and 2nd kinds are very useful in Signal Processing and especially in the design of Digital Filters (Legendre Digital Filters)

2 The two sets of newly proposed formulas

The *complementary modulus* is $k'=(1-k^2)^{1/2}=\cos\theta$. For the first set the following formulas are proposed:

$$\mathrm{K}_0(k)=\frac{\pi}{\sqrt[4]{1-k^2}}\left(1-\frac{1}{2\sqrt{2}}\sqrt{\frac{1+\sqrt{1-k^2}}{\sqrt[4]{1-k^2}}}\right)=\pi\left(\frac{1}{\sqrt{k'}}-\frac{1}{2\sqrt{2}}\frac{\sqrt{1+k'}}{k'^{3/4}}\right),$$

$$\mathrm{K}_0(\theta)=\frac{\pi}{\cos^{1/2}\theta}\left(1-\frac{1}{2}\frac{\cos(\theta/2)}{\cos^{1/4}\theta}\right)=\pi\left(\frac{1}{\cos^{1/2}\theta}-\frac{1}{2}\frac{\cos(\theta/2)}{\cos^{3/4}\theta}\right).$$

$$\mathrm{E}_0(k)=\frac{\pi}{4}\sqrt[4]{1-k^2}\left(\frac{3}{2}\frac{1+\sqrt{1-k^2}}{\sqrt[4]{1-k^2}}-1\right)=\frac{\pi}{4}\left[\frac{3}{2}(1+k')-\sqrt{k'}\right],$$

$$\mathrm{E}_0(\theta)=\frac{\pi}{4}\cos^{1/2}\theta\left(3\frac{\cos^2(\theta/2)}{\cos^{1/2}\theta}-1\right)=\frac{\pi}{4}\left(3\cos^2\frac{\theta}{2}-\sqrt{\cos\theta}\right).$$

Similarly, for the second set are proposed the formulas:

$$\mathrm{K}_1(k)=\frac{\pi\sqrt{2}}{\sqrt{(1+k')\sqrt{k'}}}\left(1-\frac{\sqrt[4]{2}}{4}\frac{1+\sqrt{k'}}{\sqrt[4]{(1+k')\sqrt{k'}}}\right),$$

$$\mathrm{K}_1(\theta)=\frac{\pi}{\cos(\theta/2)\cos^{1/4}\theta}\left(1-\frac{1}{4}\frac{1+\cos^{1/2}\theta}{\cos^{1/2}(\theta/2)\cos^{1/8}\theta}\right).$$

$$\mathrm{E}_1(k)=\frac{\pi}{4}\left[\frac{3}{2}\left(1+\sqrt{k'}\right)^2-\sqrt{2}\sqrt{1+k'}\sqrt[4]{k'}\right]-k'\cdot\mathrm{K}_1(k),$$

$$\mathrm{E}_1(\theta)=\frac{\pi}{4}\left[\frac{3}{2}\left(1+\sqrt{\cos\theta}\right)^2-2\cos\frac{\theta}{2}\sqrt[4]{\cos\theta}\right]-\cos\theta\cdot\mathrm{K}_1(\theta).$$

Table 1. Values of the functions K (part one)

θ (°)	$k=\sin\theta$	K(k)	$\mathrm{K}_0(k)$	$\mathrm{K}_1(k)$
0	0.00000	1.5708	1.5708	1.5708
1	0.01745	1.5709	1.5709	1.5709
2	0.03490	1.5713	1.5713	1.5713
3	0.05234	1.5719	1.5719	1.5719
4	0.06976	1.5727	1.5727	1.5727
5	0.08716	1.5738	1.5738	1.5738
6	0.10453	1.5751	1.5751	1.5751
7	0.12187	1.5767	1.5767	1.5767
8	0.13917	1.5785	1.5785	1.5785
9	0.15643	1.5805	1.5805	1.5805
10	0.17365	1.5828	1.5828	1.5828
11	0.19081	1.5854	1.5854	1.5854
12	0.20791	1.5882	1.5882	1.5882
13	0.22495	1.5913	1.5913	1.5913
14	0.24192	1.5946	1.5946	1.5946
15	0.25882	1.5981	1.5981	1.5981
16	0.27564	1.6020	1.6020	1.6020
17	0.29237	1.6061	1.6061	1.6061
18	0.30902	1.6105	1.6105	1.6105
19	0.32557	1.6151	1.6151	1.6151
20	0.34202	1.6200	1.6200	1.6200
21	0.35837	1.6252	1.6252	1.6252
22	0.37461	1.6307	1.6307	1.6307
23	0.39073	1.6365	1.6365	1.6365
24	0.40674	1.6426	1.6426	1.6426
25	0.42262	1.6490	1.6490	1.6490
26	0.43837	1.6557	1.6557	1.6557
27	0.45399	1.6627	1.6627	1.6627
28	0.46947	1.6701	1.6701	1.6701
29	0.48481	1.6777	1.6777	1.6777
30	0.50000	1.6858	1.6857	1.6858
31	0.51504	1.6941	1.6941	1.6941
32	0.52992	1.7028	1.7028	1.7028
33	0.54464	1.7119	1.7119	1.7119
34	0.55919	1.7214	1.7214	1.7214
35	0.57358	1.7312	1.7312	1.7312
36	0.58779	1.7415	1.7415	1.7415
37	0.60182	1.7522	1.7522	1.7522
38	0.61566	1.7633	1.7632	1.7633
39	0.62932	1.7748	1.7748	1.7748
40	0.64279	1.7868	1.7867	1.7868
41	0.65606	1.7992	1.7992	1.7992
42	0.66913	1.8122	1.8121	1.8122
43	0.68200	1.8256	1.8256	1.8256
44	0.69466	1.8396	1.8395	1.8396
45	0.70711	1.8541	1.8540	1.8541
46	0.71934	1.8691	1.8691	1.8691
47	0.73135	1.8848	1.8847	1.8848
48	0.74314	1.9011	1.9009	1.9011
49	0.75471	1.9180	1.9178	1.9180
50	0.76604	1.9356	1.9354	1.9356
51	0.77715	1.9539	1.9536	1.9539
52	0.78801	1.9729	1.9726	1.9729
53	0.79864	1.9927	1.9923	1.9927
54	0.80902	2.0133	2.0128	2.0133
55	0.81915	2.0347	2.0341	2.0347
56	0.82904	2.0571	2.0564	2.0571
57	0.83867	2.0804	2.0795	2.0804
58	0.84805	2.1047	2.1037	2.1047
59	0.85717	2.1300	2.1288	2.1300
60	0.86603	2.1565	2.1551	2.1565
61	0.87462	2.1842	2.1825	2.1842
62	0.88295	2.2132	2.2111	2.2132
63	0.89101	2.2435	2.2410	2.2435
64	0.89879	2.2754	2.2723	2.2754
65	0.90631	2.3088	2.3051	2.3088
66	0.91355	2.3439	2.3394	2.3439
67	0.92050	2.3809	2.3754	2.3809
68	0.92718	2.4198	2.4132	2.4198
69	0.93358	2.4610	2.4530	2.4610
70	0.93969	2.5046	2.4948	2.5045
70.5	0.94264	2.5273	2.5165	2.5273
71	0.94552	2.5507	2.5389	2.5507
71.5	0.94832	2.5749		2.5749
72	0.95106	2.5998		2.5998
72.5	0.95372	2.6256		2.6255
73	0.95630	2.6521		2.6521
73.5	0.95882	2.6796		2.6796
74	0.96126	2.7081		2.7081
74.5	0.96363	2.7375		2.7375
75	0.96593	2.7681		2.7680
75.5	0.96815	2.7998		2.7997
76	0.97030	2.8327		2.8326
76.5	0.97237	2.8669		2.8669
77	0.97437	2.9026		2.9025
77.5	0.97630	2.9397		2.9397
78	0.97815	2.9786		2.9785
78.5	0.97992	3.0192		3.0191
79	0.98163	3.0617		3.0616
79.5	0.98325	3.1064		3.1063
80	0.98481	3.1534		3.1533
80.2	0.98541	3.1729		3.1727
80.4	0.98600	3.1928		3.1927
80.6	0.98657	3.2132		3.2130
80.8	0.98714	3.2340		3.2338
81	0.98769	3.2553		3.2551

Table 1. Values of the functions K (part two)

81.2	0.98823	3.2771	3.2769
81.4	0.98876	3.2995	3.2992
81.6	0.98927	3.3223	3.3221
81.8	0.98978	3.3458	3.3455
82	0.99027	3.3699	3.3696
82.2	0.99075	3.3946	3.3942
82.4	0.99122	3.4199	3.4196
82.6	0.99167	3.4460	3.4456
82.8	0.99211	3.4728	3.4724
83	0.99255	3.5004	3.4999
83.2	0.99297	3.5288	3.5283
83.4	0.99337	3.5581	3.5575
83.6	0.99377	3.5884	3.5877
83.8	0.99415	3.6196	3.6188
84	0.99452	3.6519	3.6510
84.2	0.99488	3.6852	3.6843
84.4	0.99523	3.7198	3.7187
84.6	0.99556	3.7557	3.7545
84.8	0.99588	3.7930	3.7916
85	0.99619	3.8317	3.8302
85.2	0.99649	3.8721	3.8704
85.4	0.99678	3.9142	3.9122
85.6	0.99705	3.9583	3.9560
85.8	0.99731	4.0044	4.0018
86	0.99756	4.0528	4.0498
86.2	0.99780	4.1037	4.1003
86.4	0.99803	4.1574	4.1535
86.6	0.99824	4.2142	4.2097
86.8	0.99844	4.2744	4.2692
87	0.99863	4.3387	4.3325
87.2	0.99881	4.4073	4.4001
87.4	0.99897	4.4811	4.4726
87.6	0.99912	4.5609	4.5507
87.8	0.99926	4.6477	4.6354
88	0.99939	4.7427	4.7277
88.2	0.99951	4.8478	4.8293
88.4	0.99961	4.9654	
88.6	0.99970	5.0988	
88.8	0.99978	5.2527	
89	0.99985	5.4329	
89.1	0.99988	5.5402	
89.2	0.99990	5.6579	
89.3	0.99993	5.7914	
89.4	0.99995	5.9455	
89.5	0.99996	6.1278	
89.6	0.99998	6.3509	
89.7	0.99999	6.6385	
89.8	0.99999	7.0440	
89.9	1.00000	7.7371	
90	1.00000	∞	

The values strings in the last two columns of table 1 were canceled when each of the two closed analytic formulas proposed for the approximation of the Legendre complete elliptic integral of the 1st kind K(k) gives too great relative errors (≥ 4 ‰ – also see chapter 3) for being still accepted in the usual mathematical / technical calculus. The same procedure will be applied in case of the next table (no. 2), for the same reason, concerning the accuracy of the values given by each of the other two closed analytic formulas proposed for the approximation of the Legendre complete elliptic integral of the 2nd kind E(k). The accuracy analysis of the two sets of formulas will be performed in the next chapter (no. 3). In chapter 4 some series representations for the exact functions and for both sets of approximation, as well as for their first order derivatives, will be given.

Table 2. Values of the functions E (part one)

θ (°)	$k = \sin\theta$	E(k)	$E_0(k)$	$E_1(k)$
0	0.00000	1.5708	1.5708	1.5708
1	0.01745	1.5707	1.5707	1.5707
2	0.03490	1.5703	1.5703	1.5703
3	0.05234	1.5697	1.5697	1.5697
4	0.06976	1.5689	1.5689	1.5689
5	0.08716	1.5678	1.5678	1.5678
6	0.10453	1.5665	1.5665	1.5665
7	0.12187	1.5649	1.5649	1.5649
8	0.13917	1.5632	1.5632	1.5632
9	0.15643	1.5611	1.5611	1.5611
10	0.17365	1.5589	1.5589	1.5589
11	0.19081	1.5564	1.5564	1.5564
12	0.20791	1.5537	1.5537	1.5537
13	0.22495	1.5507	1.5507	1.5507
14	0.24192	1.5476	1.5476	1.5476
15	0.25882	1.5442	1.5442	1.5442
16	0.27564	1.5405	1.5405	1.5405
17	0.29237	1.5367	1.5367	1.5367
18	0.30902	1.5326	1.5326	1.5326
19	0.32557	1.5283	1.5283	1.5283
20	0.34202	1.5238	1.5238	1.5238
21	0.35837	1.5191	1.5191	1.5191
22	0.37461	1.5141	1.5141	1.5141
23	0.39073	1.5090	1.5090	1.5090
24	0.40674	1.5037	1.5037	1.5037
25	0.42262	1.4981	1.4981	1.4981
26	0.43837	1.4924	1.4924	1.4924
27	0.45399	1.4864	1.4864	1.4864
28	0.46947	1.4803	1.4803	1.4803
29	0.48481	1.4740	1.4740	1.4740
30	0.50000	1.4675	1.4675	1.4675
31	0.51504	1.4608	1.4608	1.4608
32	0.52992	1.4539	1.4539	1.4539
33	0.54464	1.4469	1.4469	1.4469
34	0.55919	1.4397	1.4397	1.4397
35	0.57358	1.4323	1.4323	1.4323
36	0.58779	1.4248	1.4248	1.4248
37	0.60182	1.4171	1.4171	1.4171
38	0.61566	1.4092	1.4093	1.4092
39	0.62932	1.4013	1.4013	1.4013
40	0.64279	1.3931	1.3932	1.3931

Table 2. Values of the functions E (part two)

41	0.65606	1.3849	1.3849	1.3849
42	0.66913	1.3765	1.3765	1.3765
43	0.68200	1.3680	1.3680	1.3680
44	0.69466	1.3594	1.3594	1.3594
45	0.70711	1.3506	1.3507	1.3506
46	0.71934	1.3418	1.3419	1.3418
47	0.73135	1.3329	1.3330	1.3329
48	0.74314	1.3238	1.3239	1.3238
49	0.75471	1.3147	1.3148	1.3147
50	0.76604	1.3055	1.3057	1.3055
51	0.77715	1.2963	1.2964	1.2963
52	0.78801	1.2870	1.2872	1.2870
53	0.79864	1.2776	1.2778	1.2776
54	0.80902	1.2681	1.2684	1.2681
55	0.81915	1.2587	1.2590	1.2587
56	0.82904	1.2492	1.2496	1.2492
57	0.83867	1.2397	1.2401	1.2397
58	0.84805	1.2301	1.2307	1.2301
59	0.85717	1.2206	1.2212	1.2206
60	0.86603	1.2111	1.2118	1.2111
61	0.87462	1.2015	1.2024	1.2015
62	0.88295	1.1920	1.1930	1.1920
63	0.89101	1.1826	1.1838	1.1826
64	0.89879	1.1732	1.1745	1.1732
65	0.90631	1.1638	1.1654	1.1638
66	0.91355	1.1545	1.1564	1.1545
67	0.92050	1.1453	1.1475	1.1453
68	0.92718	1.1362	1.1387	1.1362
69	0.93358	1.1272	1.1301	1.1273
70	0.93969	1.1184	1.1217	1.1184
70.5	0.94264	1.1140	1.1176	1.1140
71	0.94552	1.1096	1.1135	1.1096
71.5	0.94832	1.1053		1.1053
72	0.95106	1.1011		1.1011
72.5	0.95372	1.0968		1.0968
73	0.95630	1.0927		1.0927
73.5	0.95882	1.0885		1.0885
74	0.96126	1.0844		1.0844
74.5	0.96363	1.0804		1.0804
75	0.96593	1.0764		1.0764
75.5	0.96815	1.0725		1.0725
76	0.97030	1.0686		1.0686
76.5	0.97237	1.0648		1.0648
77	0.97437	1.0611		1.0611
77.5	0.97630	1.0574		1.0574
78	0.97815	1.0538		1.0538
78.5	0.97992	1.0502		1.0503
79	0.98163	1.0468		1.0468
79.5	0.98325	1.0434		1.0435
80	0.98481	1.0401		1.0402
80.2	0.98541	1.0388		1.0389
80.4	0.98600	1.0375		1.0376
80.6	0.98657	1.0363		1.0364
80.8	0.98714	1.0350		1.0351
81	0.98769	1.0338		1.0339
81.2	0.98823	1.0326		1.0327
81.4	0.98876	1.0314		1.0315
81.6	0.98927	1.0302		1.0303
81.8	0.98978	1.0290		1.0292
82	0.99027	1.0278		1.0280
82.2	0.99075	1.0267		1.0269
82.4	0.99122	1.0256		1.0258
82.6	0.99167	1.0245		1.0247
82.8	0.99211	1.0234		1.0236
83	0.99255	1.0223		1.0226
83.2	0.99297	1.0213		1.0215
83.4	0.99337	1.0202		1.0205
83.6	0.99377	1.0192		1.0196
83.8	0.99415	1.0182		1.0186
84	0.99452	1.0172		1.0176
84.2	0.99488	1.0163		1.0167
84.4	0.99523	1.0153		1.0158
84.6	0.99556	1.0144		1.0150
84.8	0.99588	1.0135		1.0141
85	0.99619	1.0127		1.0133
85.2	0.99649	1.0118		1.0125
85.4	0.99678	1.0110		1.0118
85.6	0.99705	1.0102		1.0110
85.8	0.99731	1.0094		1.0103
86	0.99756	1.0086		1.0097
86.2	0.99780	1.0079		1.0091
86.4	0.99803	1.0072		1.0085
86.6	0.99824	1.0065		1.0080
86.8	0.99844	1.0059		1.0075
87	0.99863	1.0053		1.0071
87.2	0.99881	1.0047		1.0067
87.4	0.99897	1.0041		1.0064
87.6	0.99912	1.0036		1.0062
87.8	0.99926	1.0031		1.0060
88	0.99939	1.0026		1.0060
88.2	0.99951	1.0021		1.0061
88.4	0.99961	1.0017		
88.6	0.99970	1.0014		
88.8	0.99978	1.0010		
89	0.99985	1.0008		
89.1	0.99988	1.0006		
89.2	0.99990	1.0005		
89.3	0.99993	1.0004		
89.4	0.99995	1.0003		
89.5	0.99996	1.0002		
89.6	0.99998	1.0001		
89.7	0.99999	1.0001		
89.8	0.99999	1.0000		
89.9	1.00000	1.0000		
90	1.00000	1.0000		

In the comparative tables 1 and 2, the 4*D* (four digit) exact values of both Legendre complete elliptic integrals reproduced from special functions tables [6], as well as their 4*D* approximate values obtained by applying the two

DOI: 10.37394/232014.2020.16.19

sets of proposed closed analytic formulas were given (all versus the respective elliptic integrals modulus, $k = \sin\theta$). It is to be noticed that both sets of approximate formulas are not given by spline or regression functions, but by asymptotic expansions, the respective expressions having a remarkable simplicity (see, e.g.: the 2nd form of $E_0(k)$ or $E_0(\theta)$; more, *all newly found formulas in k / k′ do not need any mathematical table*, being purely algebraic) and accuracy (see table 3). The identity with the exact functions is satisfied for the left end $k = 0$ ($\theta = 0°$) of the domain. As one can see, the 2nd set of functions (K_1, E_1), although something more intricate, gives more accurate values than the first one (K_0, E_0) and extends itself more closely to the right end $k = 1$ ($\theta = 90°$) of the domain.

3 The accuracy of the two sets of formulas

Let us define the following relative error functions:

$\varepsilon_{K0}(k) = K_0(k)/K(k) - 1$; $\varepsilon_{K1}(k) = K_1(k)/K(k) - 1$,

for both sets of approximation of the 1st kind integral and

$\varepsilon_{E0}(k) = E_0(k)/E(k) - 1$; $\varepsilon_{E1}(k) = E_1(k)/E(k) - 1$,

for both sets of approximation of the 2nd kind integral. Their values are given in the table 3, being expressed in thousandths (‰). These errors were calculated for the 1st set (K_0 and E_0) only in the field $\theta \in [54°, 71°]$ of the domain, with an increment of 1°, while for the 2nd set (K_1 and E_1) only in the field $\theta \in [84°.8, 88°.2]$, with an increment of 0°.2, like in the above tables 1 and 2.

Table 3. Relative errors ε distribution

θ(°)	$k = \sin\theta$	ε_{K0}(‰)	ε_{K1}(‰)	ε_{E0}(‰)	ε_{E1}(‰)
54	0.80902	− 0.250		+ 0.255	
55	0.81915	− 0.272		+ 0.243	
56	0.82904	− 0.353		+ 0.293	
57	0.83867	− 0.420		+ 0.334	
58	0.84805	− 0.497		+ 0.454	
59	0.85717	− 0.558		+ 0.502	
60	0.86603	− 0.669		+ 0.566	
61	0.87462	− 0.799		+ 0.742	
62	0.88295	− 0.961		+ 0.874	
63	0.89101	− 1.118		+ 0.973	
64	0.89879	− 1.366		+ 1.135	
65	0.90631	− 1.619		+ 1.377	
66	0.91355	− 1.918		+ 1.627	
67	0.92050	− 2.299		+ 1.900	
68	0.92718	− 2.709		+ 2.215	
69	0.93358	− 3.253		+ 2.573	
70	0.93969	− 3.907		+ 2.959	
71	0.94552	− 4.642		+ 3.525	
		-		-	
84.8	0.99588	-	− 0.369	-	+ 0.607
85	0.99619	-	− 0.396	-	+ 0.592
85.2	0.99649	-	− 0.451	-	+ 0.705
85.4	0.99678	-	− 0.500	-	+ 0.748
85.6	0.99705	-	− 0.582	-	+ 0.823
85.8	0.99731	-	− 0.652	-	+ 0.932
86	0.99756	-	− 0.737	-	+ 1.076
86.2	0.99780	-	− 0.832	-	+ 1.160
86.4	0.99803	-	− 0.945	-	+ 1.284
86.6	0.99824	-	− 1.077	-	+ 1.453
86.8	0.99844	-	− 1.214	-	+ 1.571
87	0.99863	-	− 1.421	-	+ 1.743
87.2	0.99881	-	− 1.626	-	+ 1.976
87.4	0.99897	-	− 1.894	-	+ 2.275
87.6	0.99912	-	− 2.234	-	+ 2.553
87.8	0.99926	-	− 2.655	-	+ 2.922
88	0.99939	-	− 3.156	-	+ 3.397
88.2	0.99951	-	− 3.808	-	+ 4.004

The relative errors strings are stopped for values ≥ 4 ‰.

4 Comparative series representations

Expanding into power series, one obtains for the complete elliptic integrals the set of representations below ([5] – [7]):

$$K(k)=\frac{\pi}{2}\left(1+\frac{1}{4}k^2+\frac{9}{64}k^4+\frac{25}{256}k^6+\frac{1225}{16384}k^8+\frac{3969}{65536}k^{10}+\frac{53361}{1048576}k^{12}+\frac{184041}{4194304}k^{14}+\frac{41409225}{1073741824}k^{16}+\ldots\right)$$

$$=\frac{\pi}{2}\left\{1+\sum_{n=1}^{\infty}\left[\frac{1\cdot 3\cdot\ldots(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}\right]^2 k^{2n}\right\}=\frac{\pi}{2}\left\{1+\sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{2^n n!}\right]^2 k^{2n}\right\};$$

$$E(k)=\frac{\pi}{2}\left(1-\frac{1}{4}k^2-\frac{3}{64}k^4-\frac{5}{256}k^6-\frac{175}{16384}k^8-\frac{441}{65536}k^{10}-\frac{4851}{1048576}k^{12}-\frac{14157}{4194304}k^{14}-\frac{2760615}{1073741824}k^{16}-\ldots\right)$$

$$=\frac{\pi}{2}\left\{1-\sum_{n=1}^{\infty}\left[\frac{1\cdot 3\cdot\ldots(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}\right]^2\frac{k^{2n}}{2n-1}\right\}=\frac{\pi}{2}\left\{1-\sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{2^n n!}\right]^2\frac{k^{2n}}{2n-1}\right\}.$$

Proceeding in the same manner, we get for the 1st set (the most inaccurate) of approximate functions the expansions

$$K_0(k)=\frac{\pi}{2}\left(1+\frac{1}{4}k^2+\frac{9}{64}k^4+\frac{25}{256}k^6+\frac{1222}{16384}k^8+\ldots\right);$$

$$E_0(k)=\frac{\pi}{2}\left(1-\frac{1}{4}k^2-\frac{3}{64}k^4-\frac{5}{256}k^6-\frac{172}{16384}k^8-\ldots\right),$$

for the 2nd set being *practically identical with the exact ones*

$$K_1(k)=\frac{\pi}{2}\left(1+\frac{1}{4}k^2+\frac{9}{64}k^4+\frac{25}{256}k^6+\frac{1225}{16384}k^8+\frac{3969}{65536}k^{10}+\frac{53361}{1048576}k^{12}+\frac{184041}{4194304}k^{14}+\frac{41409222}{1073741824}k^{16}+\ldots\right);$$

$$E_1(k)=\frac{\pi}{2}\left(1-\frac{1}{4}k^2-\frac{3}{64}k^4-\frac{5}{256}k^6-\frac{175}{16384}k^8-\frac{441}{65536}k^{10}-\frac{4851}{1048576}k^{12}-\frac{14157}{4194304}k^{14}-\frac{2760606}{1073741824}k^{16}-\ldots\right).$$

The difference with respect to the expansions of the exact functions begins at the terms in k^8 for the 1st set of approximation, and at the terms in k^{16} for the 2nd one. For the 1st order derivatives of the exact functions we get

$$\frac{d\mathrm{K}(k)}{dk}=\frac{\mathrm{E}(k)}{k(1-k^2)}-\frac{\mathrm{K}(k)}{k}=\frac{\pi}{4}k\left(1+\frac{9}{8}k^2+\frac{75}{64}k^4+\frac{1225}{1024}k^6+\frac{19845}{16384}k^8+\frac{160083}{131072}k^{10}+\frac{1288287}{1048576}k^{12}+\frac{41409225}{33554432}k^{14}+\dots\right)$$

$$=\frac{\pi}{4}\sum_{n=1}^{\infty}\left[\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}\right]^2 n k^{2n-1}=\frac{\pi}{4}\sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{2^{n-1}n!}\right]^2 n k^{2n-1};$$

$$\frac{d\mathrm{E}(k)}{dk}=\frac{\mathrm{E}(k)-\mathrm{K}(k)}{k}=-\frac{\pi}{4}k\left(1+\frac{3}{8}k^2+\frac{15}{64}k^4+\frac{175}{1024}k^6+\frac{2205}{16384}k^8+\frac{14553}{131072}k^{10}+\frac{99099}{1048576}k^{12}+\frac{2760615}{33554432}k^{14}+\dots\right)$$

$$=-\frac{\pi}{4}\sum_{n=1}^{\infty}\left[\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}\right]^2\frac{nk^{2n-1}}{2n-1}=-\frac{\pi}{4}\sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{2^{n-1}n!}\right]^2\frac{nk^{2n-1}}{2n-1}.$$

Applying the previous two exact relations and using the four definitions from chapter 2 one gets the expansions

$$\left[\frac{d\mathrm{K}(k)}{dk}\right]_0=\frac{\pi}{4}k\left(1+\frac{9}{8}k^2+\frac{75}{64}k^4+\frac{1225.75}{1024}k^6+\dots\right);$$

$$\left[\frac{d\mathrm{E}(k)}{dk}\right]_0=-\frac{\pi}{4}k\left(1+\frac{3}{8}k^2+\frac{15}{64}k^4+\frac{174.25}{1024}k^6+\dots\right),$$

for the 1st set of approximate functions, and respectively

$$\left[\frac{d\mathrm{K}(k)}{dk}\right]_1=\frac{\pi}{4}k\left(1+\frac{9}{8}k^2+\frac{75}{64}k^4+\frac{1225}{1024}k^6+\frac{19845}{16384}k^8+\frac{160083}{131072}k^{10}+\frac{1288287}{1048576}k^{12}+\frac{41409226.125}{33554432}k^{14}+\dots\right);$$

$$\left[\frac{d\mathrm{E}(k)}{dk}\right]_1=-\frac{\pi}{4}k\left(1+\frac{3}{8}k^2+\frac{15}{64}k^4+\frac{175}{1024}k^6+\frac{2205}{16384}k^8+\frac{14553}{131072}k^{10}+\frac{99099}{1048576}k^{12}+\frac{276061425}{33554432}k^{14}+\dots\right),$$

for the 2nd set of approximate functions.

The difference with respect to the expansions of the 1st order derivatives of the exact functions begins at the terms in k^7 for the 1st set of approximation, and at the terms in k^{15} for the 2nd one, being much smaller than that for the expansions of the respective sets of approximate functions. One can also easily find the analytic expressions and series representations for the 2nd derivatives of all K, $\mathrm{K}_{0,1}$, E, $\mathrm{E}_{0,1}$.

5 Graphic comparison

The variation curves of both Legendre complete elliptic integrals, as well as that of the two sets of newly proposed closed analytic functions are graphically represented in the comparative figures 1 and 2, all versus the angle θ, expressed in sexagesimal degrees and given by $\theta=\sin^{-1}k$. In both figures the exact functions K(k), E(k) were represented by solid (continuous) black lines, the 1st set of approximation $\mathrm{K}_0(k)$, $\mathrm{E}_0(k)$ by dashed black lines, and the 2nd set of approximation $\mathrm{K}_1(k)$, $\mathrm{E}_1(k)$ by solid red lines, resp.

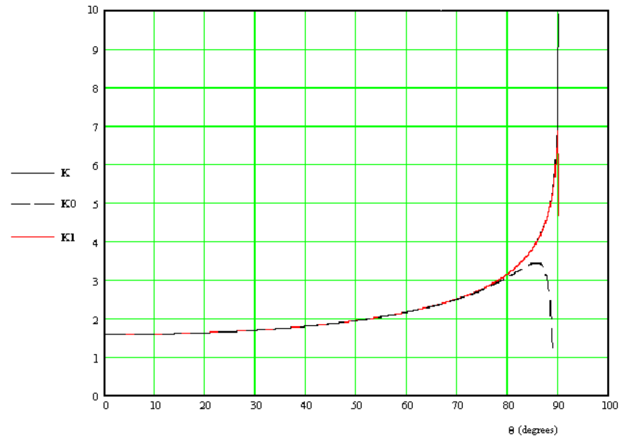

Fig. 1. Comparison of the Legendre complete elliptic integral K(k) with the closed analytic functions $\mathrm{K}_0(k)$, $\mathrm{K}_1(k)$

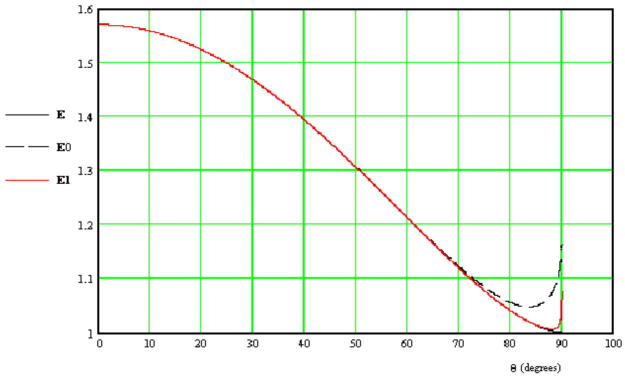

Fig. 2. Comparison of the Legendre complete elliptic integral E(k) with the closed analytic functions $\mathrm{E}_0(k)$, $\mathrm{E}_1(k)$

6 Conclusions

As for their simplicity, the formulas in $k\,/\,k'$ do not need any mathematical table (are purely algebraic). As for their accuracy, in current mathematical / technical applications, it must use the 1st set until $\theta=70°.5$ ($k=0.94264$) only, and for a better accuracy or a greater upper limit of the validity domain, to use the 2nd set, but until $\theta=88°.2$ ($k=0.99951$). Legendre complete elliptic integrals of the 1st and 2nd kinds are very useful in Signal Processing and especially in the design of Digital Filters (Legendre Digital Filters)

7 Notes; other methods; future research

With an appropriate reduction formula, every elliptic integral can be brought into a form that involves integrals over rational functions and the *three Legendre canonical forms* (of the 1st, 2nd & 3rd kind). Without the comparative tables 1 and 2, the errors table becoming so table 1, this work was published previously in a proceedings volume (scientific bulletin), in Romanian [8]. For the first English version of this work see [9]. Approximations for the complete elliptic integrals based on the trapezoidal-type numerical integration formulas discussed in [10], are developed in [11], [12] (a mixed numerical-analytic method). Newer formulas (using Γ function – not an elementary, but a special one, like K & E, even if these formulas are the most accurate) are in [13], [14]; as stated in their abstracts, the works [9], [13] do not have the same goal. Notable *special functions* suitable for applying such an approximate method of calculation (like in [9]) are: Si(x); Ci(x); Ei(x); li(x).

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