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Simple currents, modular invariants and fixed points

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ABSTRACT

We review the use of simple currents in constructing modular invariant partition functions and the problem of resolving their fixed points. We present some new results, in particular regarding fixed-point resolution. Additional empirical evidence is provided in support of our conjecture that fixed points are always related to some conformal field theory. We complete the identification of the fixed-point conformal field theories for all simply laced and most non-simply laced Kac-Moody algebras, for which the fixed point CFT's turn out to be Kac-Moody algebras themselves. For the remaining non-simply laced ones we obtain spectra that appear to correspond to new non-unitary conformal field theories. The fusion rules of the simplest unidentified example are computed.

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1. Introduction

One of the many unsolved problems in conformal field theory is that of finding all positive modular invariants for a given (rational) CFT. The problem is simple enough to formulate. One is given a set of characters $\mathcal{X}_i(\tau)$, which are defined as follows

$$\mathcal{X}_i(\tau) = \mathrm{Tr}\, e^{2\pi i\tau(L_0 - \frac{c}{24})} .$$

Here L_0 is the zero-mode Virasoro generator of some conformal field theory with central charge c. The trace is over all non-null states created by the generators of some algebra acting on a ground state (or a degenerate set of ground states) $|i\rangle$. The algebra contains at least the Virasoro algebra, but may be some extension of it by integral spin currents, such as a Kac-Moody or W-algebra. The characters transform in the following way under the generating transformations of the modular group

$$\mathcal{X}_i(\tau+1) = e^{2\pi i(h_i - c/24)}\mathcal{X}_i(\tau)$$
$$\mathcal{X}_i(-\frac{1}{\tau}) = \sum_j S_{ij}\mathcal{X}_j(\tau),$$

where S is a given unitary and symmetric matrix, and h_i is a set of real numbers. The values of h_i and the matrix S depend on the theory under consideration. They are restricted by the requirement that the unitary matrices S and T (where $T_{ij} = e^{2\pi i(h_i - c/24)}\delta_{ij}$) must form a representation of the modular group, *i.e.* $S^2 = (ST)^3 = C$, where C is the charge conjugation matrix.

In a rational conformal field theory the number of characters is finite. The objective is to find a matrix M_{ij} consisting of positive integers, such that the combination

$$P(\tau) = \sum_{ij} \mathcal{X}_i(\tau) M_{ij} \mathcal{X}_j(\bar{\tau})$$

is invariant under modular transformations. Thus one is looking for a matrix

M consisting of non-negative integers which commutes with S and T. We are implicitly assuming here that the algebras of the holomorphic and anti-holomorphic sectors of the theory are identical, and we will continue to assume that throughout this paper. This is by no means necessary, but unfortunately very little is known at present about "heterotic" modular invariants with different algebras in the two sectors.

For small enough matrices S and T the modular invariants can always be found by brute force, but this is not a very satisfactory solution. One would think that for mathematically simple systems like Kac-Moody algebras it should be possible to find a general group-theoretical classification of the modular invariant partition functions. This may well be true, but so far the answer has not been found, despite intensive efforts. Indeed, the only Kac-Moody algebra for which all positive modular invariant partition functions have been classified at arbitrary level is $SU(2)$. The solutions are labelled by the Dynkin diagrams of the simply laced simple Lie algebras (*i.e.* types A,D and E) [1]. The fact that already for $SU(2)$ the answer turns out to be so interesting makes the problem even more intriguing.

The construction of modular invariant partition functions is the main goal of a method that was developed in [2], and that uses the fusion rules of the chiral algebra. The fusion rules describe which representations of the chiral algebra can appear in tensor products of two representations, and are the conformal field theory equivalent of the Clebsch-Gordan series of group theory. The reason why the fusion rules might be expected to be relevant in this context is Verlinde's formula, which expresses the fusion coefficients in terms of the matrix S [3] (see section 4). The crucial feature of the fusion rules that one should look for is the existence of primary fields whose fusion rules with any other field give a unique answer. We have called such fields "simple currents". These currents play a key rôle in our construction. Roughly speaking, we find a new, non-diagonal modular invariant for each independent simple current. This approach does unfortunately not complete the classification of modular invariants of Kac-Moody algebras, but it has the advantage of not being restricted to Kac-Moody algebras. Roughly speaking, it generalizes the D-series of modular invariants of the $SU(2)$ Kac-Moody algebras to arbitrary conformal field theories.

The purpose of this paper is to explain the use of simple currents in a pedagogical way, and to present some new results. In sections 3-6 we will mainly review results presented in [2], [4] and [5]. Since the appearance of these papers we have gained additional insights in many aspects of the problem, which, we hope, will allow a clearer exposition of the main ideas. In section 3 we will discuss the heuristic arguments behind our method. Although this approach will appeal to those readers with a well-developed intuition for orbifolds, it has several disadvantages: it does not cover all possible cases, it may not be rigorous enough to convince everyone, and it uses a lot more explicit and implicit conformal field theory assumptions than are really required. In section 4 we present a proof that overcomes all of these disadvantages. This proof is essentially the one of [4], but it has been streamlined even more, so that the precise conditions for its validity become much more transparent. Sections 3 and 4 are each more or less self-contained. In section 5 we discuss the application to Kac-Moody algebras, and enumerate their simple currents, as far as they are presently known. In section 6 we deal with the problem of fixed points of simple currents, following essentially [5]. However, in that paper the fixed points of simple currents of Kac-Moody algebras were only partly analyzed. In section 7 we present the complete analysis, and derive the exact maps between the fixed points and the conformal field theory corresponding to them. The latter turn out to be almost always Kac-Moody algebras, but some of the fixed point conformal field theories of C_n and B_n are as yet unidentified and presumably non-unitary theories. The fusion rules of one of these unidentified theories are derived in section 7. Finally in section 8 we show that the formalism discussed in the rest of this paper has applications to many, rather different problems. We begin by recalling a few general facts about modular invariant partition functions.

2. Modular invariant partition functions: Generalities

2.1 Consistency conditions

The conditions on M_{ij} listed so far are necessary for consistency of a conformal field theory, but may not be sufficient. They guarantee consistency (absence of global anomalies) of the zero-point function at genus one, whereas of course we need consistency of arbitrary correlation functions at arbitrary genus. Requiring the matrix elements of M_{ij} to be positive (which is not required by one-loop modular invariance) is undoubtedly necessary for modular invariance at higher loops, but may not be sufficient either. One additional constraint that is certainly necessary is uniqueness of the vacuum, *i.e.* $M_{00} = 1$ (the vacuum state is labelled by $i = 0$). In practice most choices of M that satisfy these conditions are in some way related to a sensible theory. In some cases the correct interpretation may not be straightforward; for example M could be a linear combination of two matrices M_1 and M_2 corresponding to sensible theories. However, we do not know an example of a positive modular invariant with $M_{00} = 1$ that is not in any way related to a sensible theory.

We will therefore focus our attention on the problem of finding solutions for M for a given set of representation matrices S and T of the modular group, which should satisfy just one additional requirement, namely the existence of non-negative integer fusion coefficients.

2.2 The general form of a modular invariant

There are two basic types of modular invariant partition functions. The first type, henceforth referred to as *integer spin invariants* has a general form which is a sum of squares

$$\sum_i N_i |\sum_j m_{ij} \mathcal{X}_{i,j}|^2 \tag{2.1}$$

Here $\mathcal{X}_{i,j}$ is some subset of the characters of the theory, and N_i, m_{ij} are positive integers. It follows from [6] that such an invariant can always be regarded as a diagonal invariant of a larger chiral algebra than was originally considered. The extension of the algebra can be read off from the term in the sum on i that contains the identity character. There can only be one such term, and we choose the labels $i = j = 0$ to represent the identity. Then $N_0 = m_{00} = 1$. The other characters appearing in the same term as the identity must have integral conformal spin, and correspond to the extra currents that extend the algebra.

Integral spin invariants usually look far more regular than the expression given above might suggest. Some rather general emperical features are

1. Some characters of the original theory are absent.
2. In nearly all cases the coefficients m_{ij} are equal to one. However, there are counter-examples (see *e.g.* [2]; in the two counter-examples we know the sum on i consists only of the identity term).
3. Each character of the original theory appears in at most one of the squares. Here as well a counter-example is known [2].
4. If the first square contains N terms, then all other squares contain either N terms as well, or d terms, where d is a divisor of N. The counter-example to the previous point turns out to be an exception here as well.
5. The coefficient N_i of a square with d terms is $\frac{N}{d}$.

If these points are not satisfied one can often obtain such an invariant via one or more intermediate steps. Each step consists of an extension of the algebra by some of the currents appearing in the identity block. It is possible that the invariants obtained in each step do have the generic features listed above, but that the final result does not.

For an integral spin invariant one should always be able to write down a new modular transformation matrix $\tilde{S}$ which describes the transformation of the new characters, *i.e.* the sums appearing inside each of the squares. This is straightforward if all points listed above are satisfied, and if furthermore

all squares have the same number of terms. A more difficult problem is that of finding $\tilde{S}$ if some of the coefficients N_i are not equal to 1. If we want to interpret the partition function as a diagonal invariant of an extended algebra, then all coefficients should be equal to one. There are two possible ways out. One possibility is that the coefficients N_i should in fact be absorbed into the characters they multiply. Since characters must have integer coefficients, this is only possible if N_i is a square. The second, and most common way out is that the multiplicity N_i indicates that there are N_i different representations of the extended algebra that happen to have the same characters, each contributing just once. We will discuss this in more detail in section 6, where a method for constructing the new matrix $\tilde{S}$ in such a situation is given.

Using the matrix $\tilde{S}$ of a new theory one may once again attempt to build a non-trivial modular invariant. The result can of course always be expressed in terms of the original characters, but it may then look rather complicated, and may not have the generic features listed above. Therefore one should always try to extend the chiral algebra in steps which are as small as possible. In fact in all examples we know that appear to violate one of the five points listed above, there is always an interpretation via intermediate invariants, each of which satisfies all five points. It would be interesting to prove this, or to find a counter-example.

Integer spin modular invariants are characterized by having at least one off-diagonal matrix element $M_{i0} \neq 0$. If all such matrix elements vanish, it can be shown [6] that the matrix M_{ij} must have the form Π_{ij}, where Π defines some permutation of the characters. We will call such an invariant an ***automorphism invariant***, because Π defines an automorphism of the fusion rules.

One may encounter modular invariant partition functions with $M_{i0} \neq 0$, which, however, are not exactly of the form (2.1). Such invariants can always be viewed as normal integer spin invariants, twisted by some automorphism Π of the fusion rules of the extended algebra.

2.3 Regular and exceptional modular invariants

None of the above is of any help in actually finding modular invariant partition functions. It only tells us how a solution should be interpreted once it has been found. The present status of systematic searches for modular invariants can be summarized as follows. One may distinguish three regular types of modular invariant partition functions of rational conformal field theories (with identical left- and right-moving algebras):

— The diagonal invariant $M_{ij} = \delta_{ij}$.

— The charge conjugation invariant $M_{ij} = C_{ij}$. This is obviously an automorphism invariant.

— Simple current invariants, to be discussed in more detail in this paper. These invariants may be of integral spin or automorphism type, or a combination of both.

(These classes may overlap, and furthermore one can combine the second and the third one by multiplying the matrices M_{ij}.)

The E-invariants of $SU(2)$ do not belong to any of these types. We may therefore expect that in other conformal field theories there may occasionally be an invariant that cannot be obtained by means of simple currents. It is natural to call solutions that are not of one of the three types listed above "exceptional invariants". Very little is known about these exceptional solutions. There does exist a method to find many of them, at least in Kac-Moody-related conformal field theories, namely by using conformal embeddings of Kac-Moody algebras [7], which have been completely classified in [8]. However, we know now that this does not yield all solutions, not even for Kac-Moody algebras. For example the integer spin invariant found in [2] for F_4 level 6 cannot be obtained in this way.

2.4 Complementary conformal field theories

Consider a conformal theory $\mathcal{C}$ with modular transformation matrices S and T. Then obviously the matrices S^* and T^* also form a representation of the modular group. Furthermore the fusion coefficients associated with S^* via Verlinde's formula are the same as those of S. If there exists a conformal field theory whose characters transform with S^* and T^* then we call such a theory a *complement* of $\mathcal{C}$ and we denote it as $\mathcal{C}^c$. (In addition to the modular transformation matrices one can also write down a set of braid and fusion matrices satisfying the polynomial equations [9], but this still does not guarantee the existence of such a theory. This is discussed in [10], where $\mathcal{C}^c$ is called the "dual" of $\mathcal{C}$. This is an unfortunate choice of terminology, since the word dual has already three other meanings in this context.)

The notion of complementary conformal field theories was introduced in [11] for lattice theories, and can be generalized straightforwardly to arbitrary conformal field theories. The name refers to the fact that the characters $\mathcal{X}_i$ of $\mathcal{C}$ and $\mathcal{Y}_i$ of $\mathcal{C}^c$ can be combined to form a meromorphic modular invariant $\sum_i \mathcal{X}_i(\tau)\mathcal{Y}_i(\tau)$ (which thus depends only on τ, and not on $\bar{\tau}$). The sum can be interpreted as a character of a theory with just one primary field, $S = T = 1$ and $c = 24k, k \in \mathbf{Z}$.

Conversely, given such a meromorphic $c = 24k$ theories one may "split" it in various ways to obtain complementary conformal field theories. Unfortunately, the classification of $c = 24k$ theories is far from complete. A well-known set of examples of such theories are the 24 Niemeier lattices. In addition 17 other theories are presently known for $c = 24$: 15 that can roughly be described as Z_2-twisted Niemeier lattices [12,13], plus two other ones built out of characters of $SO(17)_1 \times (E_8)_2$ and $(F_4)_6 SU(3)_2$, where the subscripts indicate the levels [14]. The complete set is not even known for $c = 24$, and for larger values of c all that is known is that the number of such theories is extremely large.

Examples of complementary conformal field theories are $SU(3)^c = E_6 \times$

$E_8 \times E_8$ or $G_2^c = F_4 \times E_8 \times E_8$, all at level 1. These pairs can be combined to form the $(E_8)^3$ Niemeier lattice. The E_8 factors are not essential; alternatively one could define the complement by allowing a phase $e^{2m\pi i/3}$ in the definition of T, which can be taken into account by additional E_8 factors. With that definition two complementary theories can be combined to a $c = 8k$ theory.

Every Kac-Moody algebra has a complement, since one can always embed it in a sufficiently large even self-dual Euclidean lattice. To see this more explicitly, note first of all that the Kac-Moody algebra $\mathcal{K}$, level k can be embedded "diagonally" in k copies of $\mathcal{K}$ level 1. The level 1 Kac-Moody algebras can then be embedded in an even self-dual lattice in the following way: types A and C in A_{24k} (with k sufficiently large), B and D in D_{24k}, and the exceptional algebras can all be embedded in E_8. Once one has found an embedding, the resulting coset theory defines a complement, although certainly not the simplest one. It follows then that also every coset theory has a complement, and it is a reasonable guess that a complement exists for every unitary conformal field theory. Formally, the definition extends also to non-unitary theories, but here the situation is far less clear.

3. Simple currents: Heuristic arguments

3.1 Orbits

Consider a conformal field theory which we will assume to be rational and unitary (although further generalizations can be considered; see also the next section). The fusion rules of the chiral algebra of such a theory have the form

$$\phi_i \times \phi_j = \sum_k N_{ij}{}^k \phi_k \;, \qquad N_{ij}{}^k \in \mathbf{Z} \text{ and } \geq 0 \;, \qquad (3.1)$$

where the labels i, j, k have a finite range. Suppose now that for some label i the fusion coefficients have the property that $\sum_k N_{ij}{}^k = 1$, for all j. Then we

call ϕ_i a simple current, and we will denote it as J. Our terminology requires perhaps some explanation. The adjective "simple" refers to the fact that for each value of j just one of the fusion coefficients $N_{ij}{}^k$ can be non-vanishing, so that $J \times \phi_j = \phi_k$. The word "current" refers to the rôle played by J in modular invariant partition functions, to be discussed later. For the moment J stands just for a representation of the chiral algebra with a special property, and should not be confused with the currents that generate the chiral algebra.

Every conformal field theory has at least one simple current, namely the identity. Our interest is of course in non-trivial ones. We will see that simple currents do indeed occur in many familiar theories. The presence of simple currents in a conformal field theory has some obvious consequences. First of all it will have a conjugate J^c, so that

$$J \times J^c = \mathbf{1} \;.$$

Remarkably, this is already sufficient to identify a simple current [15]. Suppose that J does not have simple fusion rules with some field ϕ_1, *i.e.* $J \times \phi_1 = \phi_2 + \phi_3 + \ldots$. Multiplying both sides with J^c and using associativity of the fusion rule we find then $\mathbf{1} \times \phi_1 = J^c\phi_2 + J^c\phi_3 + \ldots$, which is obviously in contradiction with the fusion rules of the identity, unless there is just one term on the right-hand side.

Consider now the fusion of J with itself. Obviously, one cannot have $J \times J = J$, since this leads to a contradiction if one multiplies both sides with J^c. Thus either $J \times J = \mathbf{1}$, or the product yields a new primary field J^2, which is obviously (using again associativity) a simple current itself. Continuing this argument one concludes that the powers of J define an *orbit* of simple currents which are all different, until one reaches the identity. Since the number of primary fields is finite, there must be an integer N so that $J^N = \mathbf{1}$. If this is the smallest positive integer with that property we call N the *order* of the simple current.

Similarly on all other primary fields in the theory, there must be a smallest positive integer d so that $J^d\phi = \phi$. By associativity we know that $J^N\phi = \phi$, so that d must be a divisor of N. If a simple current J takes a field ϕ into itself, we call ϕ a fixed point of that current. Thus in the present case, if $d \neq N$, ϕ is a fixed point of the simple current J^d. We see thus that all fields in the theory are organized into orbits, whose length d is either N or a divisor of N (which may be 1).

A very useful concept associated with simple currents is a conserved charge that one can assign to each field, by means of its monodromy with respect to the simple current. We define the charge $Q(\phi)$ of a field ϕ as

$$J(z)\phi(w) \sim (z-w)^{-Q(\phi)}(J \times \phi)(w) \; ,$$

so that

$$Q(\phi) = h(J) + h(\phi) - h(J \times \phi) \bmod 1 \; . \tag{3.2}$$

The definition of Q makes sense only modulo integers because in general one cannot be sure about the presence of the leading pole in operator products. Furthermore we will see in a moment that Q is conserved, but only up to integers. Note that it is crucial for this definition to have simple monodromy, which one automatically has for simple currents. (Simple monodromy means that each term on the right-hand side of the operator product has the same branch cut structure. A sufficient condition for simple monodromy is that J is a simple current. This may not be a necessary condition, but counter-examples are hard to find.)

If $Q(\phi)$ is non-integer, this means that $\phi(0)$ creates a branch cut in $J(z)$. If one inserts a second operator $\phi'(w)$, $|w| < |z|$, then this will create a second branch cut of order $Q(\phi')$, so that the total discontinuity in $J(z)$ around the origin is $Q(\phi) + Q(\phi')$. It follows that all terms in the operator product of ϕ

with ϕ' must have the same charge, so that we have the addition rule

$$Q(\phi\phi') = Q(\phi) + Q(\phi') \bmod 1 \; . \tag{3.3}$$

Correlation functions $< \phi_1\phi_2 \ldots \phi_l >$ can only be non-vanishing if the operator product of all l fields contains the identity. Then, using (3.3), we find $\sum_i Q(\phi_i) = Q(\mathbf{1}) = 0 \bmod 1$, so that the charge is conserved (modulo integers) in all correlation functions.

If we denote the charge with respect to a current J by Q, then obviously the charge associated to J^d is equal to dQ, as usual modulo integers. Since $J^N = \mathbf{1}$ we find then that $NQ = 0 \bmod 1$, so that the charge of any field must have the form $\frac{t}{N}, t \in \mathbf{Z}$. In particular we can define the charge of the current itself as

$$Q(J) = \frac{\tilde{r}}{N} \bmod 1 \; , \tag{3.4}$$

where $\tilde{r}$ is defined modulo N.

Using (3.2) with $\phi = J^{n-1}$ as well as (3.3) we get the following recursion relation for the conformal weights of the currents

$$h(J^n) = h(J^{n-1}) + h(J) - (n-1)Q(J) \bmod 1 \; . \tag{3.5}$$

If we choose $n = N$, and furthermore use $h(J^{N-1}) = h(J^c) = h(J)$ we get

$$2h(J) = (N-1)Q(J) \bmod 1$$

We would like to divide this expression by two to get the conformal spin of the current. Since it is only defined up to integers we should, however, be careful. In any case the following expression is correct

$$h(J) = \frac{\tilde{r}(N-1)}{2N} + \epsilon \bmod 1 ,$$

where ϵ is either 0 or $\frac{1}{2}$. One can now solve (3.5) to obtain an expression for the conformal spin of the other currents

$$h(J^n) = \frac{\tilde{r}n(N-n)}{2N} + n\epsilon \bmod 1 . \tag{3.6}$$

If N is odd the condition $h(J) = h(J^c) = h(J^{N-1})$ relates odd and even powers of J, and can be satisfied by (3.6) only if $\epsilon = 0$. If on the other hand N is even both values of ϵ are allowed. This can be taken into account rather nicely by doubling the range of $\tilde{r}$. Thus we get

$$h(J^n) = \frac{rn(N-n)}{2N} \bmod 1, \quad \tilde{r} = r \bmod N\,, r \text{ defined} \begin{cases} \bmod 2N & \text{if } N \text{ even} \\ \bmod N & \text{if } N \text{ odd} \end{cases}$$

Finally, using (3.2) once more we get the variation of the conformal spin on other orbits

$$h(J^n\phi) = h(\phi) + h(J^n) - nQ(\phi) \bmod 1 . \tag{3.7}$$

3.2 Modular invariant partition functions

We have just seen that in the presence of simple currents the conformal field theory has a discrete symmetry, whose generator is $e^{2\pi i Q}$. Although in general this may not be the symmetry of some target manifold, we can nevertheless try to use it formally in an orbifold-like procedure. Thus the idea is to start with an already modular invariant partition function (*e.g.* a diagonal

one), and remove states that are not invariant under the symmetry, *i.e.* all states with charge not equal to zero. As is well known, modular invariance under S forces us to add new states to the spectrum, corresponding to strings closing only up to a symmetry operation. In our case these "twisted sectors" correspond to string configurations with the property

$$\phi(\sigma{=}\pi) \;=\; e^{2\pi i Q(\phi)}\phi(\sigma{=}0) .$$

After the usual conformal map to the complex plane this corresponds to ϕ having a branch cut $z^{-Q(\phi)}$ around the origin.

A familiar object in orbifold constructions is the "twist field" which creates a twisted sector from the vacuum. In our case this field should have the property of generating a $z^{-Q(\phi)}$ branch cut on all fields when it is inserted at the origin. It is clear that the operator performing this task is precisely the simple current J.

The orbifold recipe tells us thus to include in the spectrum all states generated from the untwisted sector by J and its powers. In general this would, however, violate T-invariance ("level matching"), unless J has integral conformal spin. This problem can be avoided by using instead of the chiral current $J(z)$ the combination $J(z)J^c(\bar{z})$, which does have integral conformal spin. The charge with respect to this current is $Q_L - Q_R^c = Q_L + Q_R$. Note that we might also consider $J(z)J(\bar{z})$, but this combination has an associated charge $Q_L - Q_R$ that is zero for all terms in the diagonal invariant, so that nothing interesting is generated.

The construction of the modular invariant partition functions is now straightforward if one knows the orbits and the charges on the orbits. We illustrate this here with some simple examples. Consider first a current of order 5, with $r = 0 \bmod 5$. Then the current has fractional charge $\frac{r}{5}$, and hence the charges on each orbit take precisely all the values $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$, which appear in some order depending on r. The characters of a theory that has such a current can be organized into orbits

$$\mathcal{X}_0^a, \mathcal{X}_1^a, \mathcal{X}_2^a, \mathcal{X}_3^a, \mathcal{X}_4^a \ . \tag{3.8}$$

Here the superscript a labels different orbits, and the subscript m indicates a charge $\frac{mr}{5}$. Let us now examine the orbifold procedure in detail. We start with the diagonal invariant

$$\sum_a \sum_{m=0}^{4} |\mathcal{X}_m^a|^2 \ . \tag{3.9}$$

Now we project out all terms not invariant under the symmetry, *i.e.* all terms with non-integral total charge. This charge is for each diagonal term $Q_L + Q_R = \frac{2mr}{5}$, and is an integer only if $m = 0$. Thus we are left with

$$\sum_a |\mathcal{X}_0^a|^2 \ .$$

Finally we add the "twisted sectors". They are obtained by acting with $[J(z)J^c(\bar{z})]^j$, $j = 1,2,3,4$. The point is that within each orbit these currents move in opposite directions for left- and right-movers, so that we get

$$\sum_a \left[|\mathcal{X}_0^a|^2 + \mathcal{X}_1^a(\mathcal{X}_4^a)^* + \mathcal{X}_2^a(\mathcal{X}_3^a)^* + \mathcal{X}_3^a(\mathcal{X}_2^a)^* + \mathcal{X}_4^a(\mathcal{X}_1^a)^* \right] \tag{3.10}$$

Note that we can transform this result back to the diagonal partition function by performing an orbifold procedure with the current combination $J(z)J(\bar{z})$.

If r is a multiple of 5 the result is completely different. Now the current has integral charge (and conformal weight), and hence the charge is now constant, $Q(a)$, on each orbit a. This constant value can be $\frac{m}{5}$ mod $1, m = 0, \ldots, 4$, and orbits with all possible values of m occur in a conformal field theory that has such a current. To derive the new theory, we start again with (3.8), but already in the next step we find now a different answer

$$\sum_{a, Q(a)=0} |\mathcal{X}_m^a|^2 \ . \tag{3.11}$$

In this case an entire orbit is either projected out or preserved. It is easy to see that the current $J(z)J^c(\bar{z})$ and its powers now provide the missing off-diagonal terms needed to complete squares. The final result is

$$\sum_{a, Q(a)=0} \Big| \sum_{m=0}^{4} \mathcal{X}_m^a \Big|^2 \ . \tag{3.12}$$

In this case there is no inverse procedure that gets us back to the diagonal invariant. The charge $Q_L - Q_R$ (belonging to the current $J(z)J(\bar{z})$) vanishes for each term in (3.12), and thus nothing is projected out. It may happen that some orbits have length 1 instead of 5 (this is obviously impossible if $r \neq 0$ mod 5, since the charge is then not a constant on an orbit). In that case each power of the current $J(z)J^c(\bar{z})$ yields the same answer, so that a short orbit f contributes a term $5|\mathcal{X}^f|^2$.

In terms of the general classification discussed in section 2 the first example is an automorphism invariant, and the second one an integer spin invariant. The currents of the extended algebra implied by this invariant are of course nothing but the currents J by which it was obtained. Note that also for automorphism invariants obtained with simple currents there is a notion of a "current algebra". In this case, however, the currents have fractional spin, and cannot appear in the spectrum as chiral fields. These fractional spin currents generate a parafermionic algebra, as defined in [16].

It is an amusing exercise to work out other cases, and in particular those for which the order of the current is not prime or even. One will often find invariants of mixed type (automorphisms of integral spin invariants). Some examples are given in [2]. For even order the procedure given above will sometimes not yield anything new, because in that case some currents are equal to their own conjugate. A slightly more general procedure, which does yield new invariants in some of those cases, will be discussed in the next section.

4. Simple currents: an algebraic proof

In this section we will show how all the foregoing results can be derived from a few simple axioms, to be satisfied by the matrices S and T. The derivations in this section do not use conformal fields, conformal blocks, braid group representations or polynomial equations. We will only require the existence of a finite unitary representation of the modular group with an associated fusion algebra that is sensible. We are not requiring that any conformal field theory with such a fusion algebra exists. In particular we are not requiring unitarity of any conformal field theory (which should of course not be confused with unitarity of the modular group representation).

We will make the following assumptions.

1. We are given a set of finite unitary matrices S and T satisfying $(ST)^3 = S^2$ and $S^4 = \mathbf{1}$.
2. There must exist a basis on which simultaneously T is diagonal and S symmetric.
3. In this basis, there must be one special basis element (which will be labelled "0", and called the identity) so that the coefficients [3]

$$N_{ijk} = \sum_n \frac{S_{in} S_{jn} S_{kn}}{S_{0n}} \tag{4.1}$$

are positive integers for all i, j and k. In particular we require that $S_{0n} \neq 0$ for all n.

4. Finally we require that $N_{000} = 1$.

Given such a set of matrices S and T we begin by defining the charge conjugation matrix $S^2 = C$. It follows from point 3 that $C_{ij} = N_{ij0}$ is a symmetric matrix with entries that are non-negative integers. Furthermore it follows from point 1 that $C^2 = \mathbf{1}$. This implies that all entries are either 0 or 1, and that every row and every column has exactly one entry equal to 1. Hence C can indeed be regarded as a charge conjugation matrix, which maps every element of the basis to precisely one other element (or to itself). We will denote the basis elements as (i), and their charge conjugates as $(i)^c$, where $(i)^c = (j)$, with $C_{ij} = 1$. Because $C = (ST)^3 = S^2$, C obviously commutes with S and ST, and hence with T. Thus charge conjugates have the same T eigenvalue.

Because $S^2 = C$ and $SS^\dagger = SS^* = \mathbf{1}$ we see that $S = CS^*$. This implies that $S_{ij} = S^*_{i^c j}$, so that S_{ij} is real if i (or j) is self-conjugate. Point 4 implies that $C_{00} = 1$, so that the identity is self-conjugate, and all matrix elements S_{0i} are real.

We can now define the fusion rules as

$$(i) \times (j) = \sum_k N_{ij}{}^k (k)$$

where $N_{ij}{}^k = \sum_l N_{ijl} C^{lk}$. It is elementary to show that this operation is commutative and associative, that $(0) \times (i) = (i)$ and that $(i) \times (j)$ contains (k) N times if and only if $(k) \times (i)^c$ contains (j) N times (point 4 is essential here). Thus all the usual conditions for a fusion algebra are now satisfied. In the following we will refer to N_{ijk} (three in-going lines) as the "three-point coupling coefficients", and to $N_{ij}{}^k$ (two in-going, one out-going) as the "fusion coefficients".

Let us now assume that there exists a special element (J) with simple fusion rules, *i.e.* $(J)\times(i)=(Ji)$, where (Ji) denotes one of the basis elements. Then

$$N_{Jik}=\sum_n \frac{S_{Jn}S_{in}S^*_{nk}}{S_{0n}}=\delta_{k,Ji}\ .$$

Defining the vector $V_n=S_{Jn}S_{in}/S_{0n}$ we can read this as $\sum_n V_nS^*_{nk}=\delta_{k,Ji}$. Hence V_n is orthogonal to all but one column of the unitary matrix S, and has inner product 1 with the remaining column, Ji. Hence we conclude that $V_n=S_{Ji,n}$, so that

$$S_{Ji,n}=\left(\frac{S_{Jn}}{S_{0n}}\right)S_{in}\ .$$

Using this identity recursively, and applying the analogous identity to the second index, we derive easily

$$S_{J^pi,J^qk}=\left(\frac{S_{JJ}}{S_{0J}}\right)^{pq}\left(\frac{S_{00}}{S_{0J}}\right)^{pq}\left(\frac{S_{Ji}}{S_{0i}}\right)^{q}\left(\frac{S_{Jk}}{S_{0k}}\right)^{p}S_{ik}\ . \tag{4.2}$$

The factors in this expression must satisfy the following identity, which can be derived by choosing $p=N$ (the order of the current) and $q=0$:

$$\left(\frac{S_{Jk}}{S_{0k}}\right)^{N}=1\ .$$

Hence

$$\frac{S_{Jk}}{S_{0k}}=e^{2\pi i\Phi(k)}\ ,\qquad N\Phi(k)\in\mathbf{Z}\ .$$

Note that since S_{J0} is real, $\Phi(0)=0$ mod 1 or $\frac{1}{2}$ mod 1, where the latter value is allowed only if N is even. (Furthermore the latter value is not allowed for

unitary conformal field theories, for which $S_{i0}>0$.) Substituting this into (4.2) we get

$$S_{J^pi,J^qk}=e^{2\pi ip\Phi(k)}e^{2\pi iq\Phi(i)}e^{2\pi ipq[\Phi(J)-\Phi(0)]}S_{ik}\ . \tag{4.3}$$

We can now relate the phases $\Phi(i)$ to the conformal weights. We begin with the relation $(ST)^3=S^2$, and rewrite it as $STS=T^{-1}ST^{-1}$. The matrix T is diagonal, and its eigenvalues can be written as $e^{2\pi it(i)}$. Now consider

$$\sum_l S_{Ji,l}T_{ll}S_{l,J^ck}\ .$$

Because $J^c=J^{N-1}$ the two currents in this expression contribute cancelling phases. Thus we conclude

$$T^{-1}_{Ji,Ji}S_{Ji,J^ck}T^{-1}_{J^ck,J^ck}=T^{-1}_{i,i}S_{i,k}T^{-1}_{k,k}$$

Hence the phase by which these two expressions differ must vanish, *i.e.*

$$t(i)-t(Ji)-\Phi(i)+\Phi(k)-\Phi(J)+\Phi(0)-t(J^ck)+t(k)=0 \bmod 1\ ,$$

provided that $S_{ik}\neq 0$. This condition is certainly satisfied if $k=0$. First of all, for $i=k=0$ we get, using $t(J)=t(J^c)$,

$$\Phi(J)=\Phi(0)-2[t(J)-t(0)] \bmod 1.$$

Substituting this into the previous identity with $k=0$ we get

$$\begin{aligned}\Phi(i)-\Phi(0)&=t(i)+t(J)-t(Ji)-t(0)\\&=h(i)+h(J)-h(Ji)\\&\equiv Q(i)\ ,\end{aligned}$$

where $h(i) = t(i) - t(0)$. This is the definition of the charge introduced in the previous section. We will maintain it even if $\Phi(0) \neq 0$, because, as we will see in a moment, in this way Q is conserved. In terms of the conserved charge Q the expression for S takes the following form

$$S_{J^p i, J^q k} = e^{2\pi i p[Q(k)+\Phi(0)]} e^{2\pi i q[Q(i)+\Phi(0)]} e^{2\pi i pq Q(J)} S_{ik} \ . \tag{4.4}$$

This is exactly the result obtained in [4], apart from the extra $\Phi(0)$-dependent phases, which are absent in unitary theories.

Note that there may be orbits that are shorter than the canonical length N. If f labels such an orbit, so that $J^d f \equiv f$, then (4.4) has to satisfy the condition $S_{J^d f, J^q k} = S_{f, J^q k}$, for all k and q. This leads to two conditions

$$J^d f = f \quad \rightarrow \quad \begin{cases} (1) \quad d\Phi(k) = 0 \bmod 1 \quad \text{if } S_{fk} \neq 0 \\ (2) \quad dQ(J) = 0 \bmod 1 \end{cases} \tag{4.5}$$

We now wish to derive the other properties of the charge that were obtained in the previous section in a more intuitive way. To prove that the charge is conserved in the fusion of two operators, consider

$$N_{i,j,k} = \sum_{\{v,n\}} e^{2\pi i v[Q(i)+Q(j)+Q(k)+2\Phi(0)]} \frac{S_{in} S_{jn} S_{kn}}{S_{0n}} \ . \tag{4.6}$$

Here we have written the summation in (4.1) in orbit basis. The notation $\{v, n\}$ indicates that one should sum over all orbits, choosing in each orbit some representative which is labelled n; then one should sum over the orbit members $J^v n$, where of course the range of v is chosen in such a way that the orbit is covered precisely once. Since $\Phi(0)$ is integer or half-integer we see thus that indeed Q is a conserved charge. Note that the three-point coefficients are invariant if the three ingoing lines i, j, k are replaced by $J^p i$, $J^q j$, $J^s k$, with $p + q + s = 0 \bmod N$.

We have now obtained (3.2) and (3.3) directly from the four basic assumptions made in the beginning of this section and hence we can now derive the expressions for h along each orbit exactly as in section 3.

4.1 Modular invariant partition functions

We are now ready to derive the modular invariant partition functions. We simply use the intuitive approach presented in the previous section to write down a candidate partition function. Then we will show that this candidate is indeed modular invariant.

The proof for $\Phi(0) = 0$ has been given before in [4]. We give it here once more in order to correct a few possibly confusing misprints.

If $\Phi(0) = 0$, an off-diagonal modular invariant partition function can be written down for every current with order N and monodromy parameter r provided that r is even. If N is odd, r is defined modulo N and can always be made even. We will assume here that this has been done. If N is even and r is odd there does not exist a non-trivial modular invariant associated with this current. In orbifold language this is due to a violation of the level-matching condition. Thus in any case we may now assume that r is even.

The following expression is then modular invariant

$$M_{J^p i J^q j} = \sum_{l=1}^{N} \delta_{ij} \delta^{N_j}_{q,p+l} \delta^1 [Q(i) + \left(\frac{2p+l}{2N}\right) r] \ . \tag{4.7}$$

Here $\delta^M(x) = 1$ if $x = 0 \bmod M$ and vanishes otherwise, and $\delta^M_{ab} = 1$ if $a = b \bmod N$ and vanishes otherwise; N_j is the length of the orbit j. This formula can be understood as follows. The first δ-function ensures that there are matrix elements only within the orbits. The sum is over all rotations of one orbit with respect to the second. This sum is over N values of l independent of the orbit length. Hence the second δ-function yields $\frac{N}{N_j}$ contributions for each

p and q. This sum can be viewed as a sum over twisted sectors. The last δ-function restricts the diagonal partition function to the untwisted sector. Note that the argument of this δ-function is, roughly speaking, equal to $\frac{1}{2}[Q(J^p i)+Q(J^q j)]$. The factor $\frac{1}{2}$ makes this intuitive definition imprecise (since Q is defined modulo integers), but (4.7) is unambiguous. This factor $\frac{1}{2}$ is necessary to deal properly with even values of N.

To prove modular invariance, we write the δ-function as follows

$$\delta^1[Q(i)+\left(\frac{2p+l}{2N}\right)r]=\frac{1}{N}\sum_{s=1}^{N}e^{2\pi i s[Q(i)+(\frac{2p+l}{2N})r]}\ . \qquad (4.8)$$

Note that this representation of the δ-function is valid only if r is even; a similar expression for odd r would require us to double the range of p. Using (4.7) and (4.4) we get now

$$(SMS^*)_{J^p i,J^q m}=\sum_{\{v,j\}}\sum_{\{w,k\}}\frac{1}{N}\sum_{l=1}^{N}\sum_{s=1}^{N}e^{2\pi i[vQ(i)+pQ(j)+pvr/N]}$$
$$\times S_{ij}S^*_{km}\delta_{jk}\delta^{N_k}_{w,v+l}e^{2\pi i s[Q(j)+(\frac{2v+l}{2N})r]}e^{-2\pi i[wQ(m)+qQ(k)+wqr/N]}\ .$$

Here the definition of the first two sums is exactly as in (4.6). As a first step we now perform the sum on w and remove the second δ-function. Note that this δ-function contributes precisely once, since w only goes from 1 to N_k. Next we perform the sum on k, removing the other δ-function. Note that the first sum sets $w=v+l \bmod N_k$. However, because of (4.5) this $\bmod N_k$ restriction becomes irrelevant after the sum on k has been performed. The intermediate result is

$$(SMS^*)_{J^p i,J^q m}=\sum_{\{v,j\}}\frac{1}{N}\sum_{l=1}^{N}\sum_{s=1}^{N}e^{2\pi i[vQ(i)+pQ(j)+vpr/N]}$$
$$\times S_{ij}S^*_{jm}e^{2\pi i s[Q(j)+(\frac{2v+l}{2N})r]}e^{-2\pi i[(v+l)Q(m)+qQ(j)+q(v+l)r/N]}\ . \qquad (4.9)$$

Now we use orthogonality of S, which in this basis reads

$$(SS^*)_{J^p i,J^t m}=\sum_{\{v,j\}}e^{2\pi i[vQ(i)+pQ(j)+vpr/N]}$$
$$\times e^{-2\pi i[vQ(m)+tQ(j)+tvr/N]}S_{ij}S^*_{jm}=\delta_{im}\delta^{N_m}_{pt}\ .$$

Substituting this into (4.9) with $t=q-s$ we get

$$(SMS^*)_{J^p i,J^q m}=\frac{1}{N}\sum_{l=1}^{N}\sum_{s=1}^{N}\delta_{im}\delta^{N_m}_{p,q-s}e^{-2\pi i l[Q(m)+qr/N]}e^{-2\pi i s l r/2N}\ . \qquad (4.10)$$

This is equal to $M_{J^p i,J^q m}$ upon interchange of $-l$ and s. The fact that the rôle of s and l is interchanged in the calculation has two important implications. First of all it tells us that even on orbits of length $d<N$ we should sum over all $l, 1\le l\le N$, in order for the ranges of l and p to be the same. Secondly it shows that the proof does not go through for odd values of r, since in the latter case one has to double the range of p.

Consider now invariance under T.

$$(TMT^*)_{J^p i,J^q j}=e^{2\pi i[t_i+t(J^p)-t_0-pQ(i)]}\sum_{l=1}^{N}\delta_{ij}\delta^{N_j}_{q,p+l}\delta^1[Q(i)+\frac{2p+l}{2N}r]$$
$$\times e^{-2\pi i[t_j+t(J^q)-t_0-qQ(j)]}\ . \qquad (4.11)$$

There are many cancellations in the phase factor multiplying the non-vanishing terms in this expression. The remaining phase factor is equal to

$$e^{2\pi i(q-p)[Q(i)+(\frac{p+q}{2N})r]}\ . \qquad (4.12)$$

If one naively substitutes the δ-functions one would get 1, but we have to be a little bit more careful for orbits of length $N_j\equiv d<N$, especially if N

is even. From the second δ-function in (4.11) we get $q = p + l + kd$, where $k \in \mathbf{Z}$. Substituting this into (4.12) and using the third δ-function in (4.11) we find that the remaining phase is $(p-q)\frac{kdr}{2N}$. From (4.5) we conclude that $dQ(J) = \frac{dr}{N}$ mod 1. Furthermore if N is odd we find then $\frac{dr}{2N} \in \mathbf{Z}$, since r is even. Hence for odd N the phase is an integer, and the phase factor (4.12) is equal to one. However, as usual we have some more problems with even N, namely if $\frac{dr}{2N} = \frac{1}{2}$ mod 1 and k and $(p-q)$ are both odd.

To show that this combination cannot occur we first use (3.7) to show that for fixed points f the charge $Q(f)$ satisfies $dQ(f) = h(J^d)$. Now consider the argument of the last δ-function in (4.11) and multiply it by d. To have a non-vanishing matrix element the result should be a multiple of d and therefore it should certainly be an integer. After some straightforward manipulations one finds the condition (using the fact that r is even)

$$[(p+q)-(k+1)d]\frac{dr}{2N} \in \mathbf{Z}$$

It follows that the quantity in square brackets should be an even integer. Hence k and $(p+q)$ (or equivalently $p-q$) cannot be odd simultaneously.

If the phase $\Phi(0)$ is half-integer, the foregoing proof of invariance under S does not go through. One may correct this by replacing $Q(i)$ by $Q(i) + \Phi(0)$ in (4.7), but then invariance under T is violated. It seems thus that in this particular case simple currents do not produce any invariants. We will not dwell on this point very long, since it is rather academic: we do not know any non-unitary conformal field theories with simple currents, and hence certainly none with $\Phi(0) = \frac{1}{2}$.

It is not hard to see that two currents J that generate the same cyclic subgroup of the center yield the same partition function. For unitary CFT's with a center $\mathbf{Z}_N$ generated by a simple current with monodromy parameter r we find thus a different modular invariant for every divisor of N if N is odd, or if N and r are both even, and for every divisor of $\frac{N}{2}$ if N even and

r odd. The latter statement is a consequence of the fact that the current J^2 generating $\mathbf{Z}_{N/2}$ always has an even monodromy parameter.

If the center consists of more $\mathbf{Z}_N$ factors the number of modular invariants is considerably larger, and consists of matrices of the form $M_1(J_1)\dots M_l(J_l)$, where l may be arbitrarily large, and where each J_i is some product of the generating currents of the $\mathbf{Z}_N$ factors. We do not know a simple and systematic classification of all the distinct solutions.

5. Simple currents of Kac-Moody algebras

One of the most important classes of rational CFT's is formed by the Kac-Moody algebras, from which most (all?) known rational and unitary CFT's can be obtained via the coset construction [17]. All information necessary to formulate the problem of finding modular invariants is explicitly available in this case. The different algebras are labelled by a simple Lie-algebra G and a positive integer k, the level. The unitary highest weight representations and their characters are labelled by the representations of the Lie-algebra, restricted to the set $k \le \vec{\lambda}\cdot\vec{\psi}$, where $\vec{\lambda}$ is the highest weight of a representation and $\vec{\psi}$ is the highest weight of the adjoint representation. The conformal weights of the corresponding Kac-Moody representations are given by

$$h(\vec{\lambda}) = \frac{\frac{1}{2}C(\lambda)}{k+g} \tag{5.1}$$

where C denotes the quadratic Casimir eigenvalue, defined in the usual way as

$$C(\Lambda) = (\Lambda, \Lambda + 2\delta)\ , \tag{5.2}$$

where δ is half the sum of the positive roots. In (5.1) g is the dual Coxeter number, which is equal to half the Casimir eigenvalue of the adjoint representation, $g = \frac{1}{2}C(\psi)$. The Virasoro central charge is

$$c(G,k) = \frac{k \dim G}{k+g} \,.$$

The matrix elements of S are given by

$$S(\vec{\lambda},\vec{\mu}) = N \sum_{w} \epsilon(w) \exp\left(-\frac{2\pi i}{k+g}(w(\lambda+\delta),\mu+\delta)\right) \,. \tag{5.3}$$

Here the sum is over all elements w of the Weyl group, and $\epsilon(w)$ is the determinant of w ($\epsilon(w) = +1$ for pure rotations and -1 for reflections). An expression exists for the normalization N, but we will not need it; of course N is in any case fixed by unitarity plus the requirement $S_{00} > 0$.

One way to obtain the fusion rules is to simply substitute (5.3) into Verlinde's formula (4.1). This is rather cumbersome, and more direct methods have been discussed in the literature [18], [19]. For non-trivial cases (*i.e.* not $SU(2)$ and not level 1) these methods are, however, not really practical unless one uses a computer. In that case Verlinde's formula, combined with (5.3), is probably easier to program and certainly more reliable, since any error is likely to yield non-integer fusion coefficients.

To find out whether a primary field of some RCFT is a simple current it is not necessary to examine *all* its fusion rules. As was mentioned already in section 3, a sufficient condition is that the fusion rule with its charge conjugate is simple, in other words that the result yields only the identity and no other primary field [15]. A second simple characterization follows from [20]: a field ϕ_i is a simple current if and only if $S_{0i} = S_{00}$, where "0" denotes the identity. This is a useful criterium for Kac-Moody algebras, as long as a really practical direct computation of the fusion rules is not available.

A large class of simple currents of Kac-Moody algebras can be derived from symmetries of the extended Dynkin diagram. This works as follows. Label

the simple roots of a rank-n Lie-algebra as $1,\ldots,n$. The extended Dynkin diagram has one extra node, labelled "x", corresponding to the highest root. A unitary highest weight representation of the Kac-Moody algebra is denoted by n Dynkin labels $a_1,\ldots,a_n$, to which in the usual way corresponds some highest weight $\vec{\lambda}$. With the extra node x we associate a label $a_x = k - \vec{\lambda}\cdot\vec{\psi}$, which for unitary highest weight representations is a positive integer.

The extended Dynkin diagram can have two kinds of symmetries, those which act trivially on the node x and those that act non-trivially. The first kind is an automorphism of the (unextended) Dynkin diagram, and corresponds to outer automorphisms of the Lie algebra. This kind of transformation does yield non-diagonal partition functions, but rather trivial ones, in which fields are paired with their charge conjugate instead of with themselves (because of triality there are some additional possibilities if the Lie-algebra is D_4). The other kind of automorphism of the extended Dynkin diagram corresponds always to a simple current. The orbits of the simple current are obtained by applying the automorphism to the Dynkin labels $(a_1,\ldots,a_n,a_x)$. This yields a permutation of the labels, after which one can read off the Dynkin labels of the next representation on the orbit from the first n components. Of course this transformation respects the constraint $a_x = k - \vec{\lambda}\cdot\vec{\psi}$, as one can easily check.

The simple currents themselves can be read off from the identity orbit, obtained by using the labels $(0,\ldots,0,k)$ as the starting point. It follows from the calculations of [21] and [22] that the corresponding fields are indeed simple currents, since these authors show that $S_{0i} = S_{00}$. (In these papers relations between many other matrix elements of S are obtained, but they follow immediately from formula (4.4) once we know that they are due to simple currents.)

The Dynkin labels of the simple currents of Kac-Moody algebras that are obtained in this way are thus

A_n	$J_l = (\overbrace{0,\ldots,0,k}^{l},0,\ldots,0)\,, l = 1,\ldots,N$
B_n	$J = (k,0,\ldots,0)$
C_n	$J = (0,\ldots,0,k)$
D_n	$J_v = (k,0,\ldots,0)$
	$J_s = (0,0,\ldots,k,0)$
	$J_c = (0,0,\ldots,0,k)$
E_6	$J_1 = (k,0,\ldots,0)$
	$J_2 = (0,0,\ldots,0,k)$
E_7	$J = (0,0,\ldots,k,0)$

(The simple roots are ordered as *e.g.* in [23].) The fact that these representations have simple fusion rules had been observed before in [24], in a study of four-point functions.

It might seem a plausible guess that this list exhausts the simple currents of Kac-Moody algebras, but this turns out to be incorrect. Only one counterexample is known, namely the primary field $(1,0,0,0,0,0,0,0)$ [the representation (3875)] of E_8 level 2 [25]. It is an interesting question whether there are other examples like this one. For a given Kac-Moody algebra one can always verify the presence of such "exceptional" simple currents by computing S_{0i}/S_{00} for all primary fields. In practice this becomes very difficult for Lie-algebras with a large Weyl group like E_8 (whose Weyl-group has order $192 \times 10!$). However, it can be shown that in unitary conformal field theories $S_{0i}/S_{00} \geq 1$ for all unitary highest weight representations. If we regard this as a continuous function on weight space, it is likely that it can saturate the lower bound only on the corner points of the triangular region containing the highest weights of unitary highest weight representations. These corner points are vectors $\vec{\lambda}_m$ with Dynkin labels $(0,\ldots,0,m,0,\ldots,0)$, so that $\vec{\lambda}_m \cdot \vec{\psi} = k$. Indeed, all simple currents we know do correspond to such corner points. We have computed S_{0i}/S_{00} for the corner points for the Kac-Moody algebra E_8

up to level 9, and did not find any new simple currents. Furthermore the value of S_{0i}/S_{00} on the corner points increases rapidly with the level k, making the presence of simple currents at higher levels very unlikely. We have also investigated all other Kac-Moody algebras for sufficiently small rank and level, with the exception of E_6 and E_7. We know no other examples of simple currents than the ones just mentioned, but it would of course be very useful to prove this.

The exceptional simple current that occurs for E_8 level 2 has spin 3/2 and satisfies level matching, but it does not yield a new modular invariant. A half-integer spin current yields a non-diagonal modular invariant by pairing the two members of the Ramond orbits with each other. The orbits of the E_8 level 2 simple current are formed by the representations (1),(3875) in the Neveu-Schwarz sector, and the single field (248) in the Ramond sector. The fusion rules are identical to those of the Ising model. Since the only Ramond orbit is a fixed point, there is no new invariant. However, there are more interesting possibilities in tensor products of E_8 level 2 with itself or other conformal field theories. For example the tensor product of two copies of E_8 level 2 has 9 primary fields, including a spin-3 simple current (3875,3875). This current can be used to build the modular invariant

$$|(1,1)+(3875,3875)|^2+|(1,3875)+(3875,1)|^2+2|(248,248)|^2 \,.$$

The other simple currents generate a center of the Kac-Moody algebra which is isomorphic to the center of the Lie-algebra. It is then not hard to guess what the charges of the primary fields with respect to these simple currents must be. They are indeed nothing but the charges corresponding to the conjugacy classes of the ground state representations.

Writing down modular invariant partition functions that can be obtained with a simple current is now very simple. We will illustrate this with some examples. Consider first $SU(3)$, level 5. There are 21 primary fields which are organized into 7 orbits. The extended Dynkin labels (a_1, a_2, a_x) of these orbits are

$(0,0,5)_0$	$(0,5,0)_1$	$(5,0,0)_2$
$(2,2,1)_0$	$(2,1,2)_1$	$(1,2,2)_2$
$(1,4,0)_0$	$(4,0,1)_1$	$(0,1,4)_2$
$(4,1,0)_0$	$(1,0,4)_1$	$(0,4,1)_2$
$(3,0,2)_0$	$(0,2,3)_1$	$(2,3,0)_2$
$(0,3,2)_0$	$(3,2,0)_1$	$(2,0,3)_2$
$(1,1,3)_0$	$(1,3,1)_1$	$(3,1,1)_2$

To correspond to a unitary highest weight representation these labels must satisfy the condition $a_1 + a_2 + a_x = 5$. The subscript indicates the triality of the representation, which is equal to $a_1 + 2a_2$ mod 3. The charge Q is $\frac{1}{3}$ of this value. The automorphism generating the orbits is simply a cyclic permutation of the 3 labels. The current $(0,5,0)$ generates a modular invariant partition function in which the sum of the left and right-moving charge is an integer. This partition function has thus the form

$$\begin{aligned}&|\mathcal{X}_{00}|^2 + \mathcal{X}_{05}\mathcal{X}^*_{50} + \mathcal{X}_{50}\mathcal{X}^*_{05} + |\mathcal{X}_{22}|^2 + \mathcal{X}_{12}\mathcal{X}^*_{21} + \mathcal{X}_{21}\mathcal{X}^*_{12}\\ &+|\mathcal{X}_{14}|^2 + \mathcal{X}_{40}\mathcal{X}^*_{01} + \mathcal{X}_{01}\mathcal{X}^*_{40} + |\mathcal{X}_{41}|^2 + \mathcal{X}_{10}\mathcal{X}^*_{04} + \mathcal{X}_{04}\mathcal{X}^*_{10}\\ &+|\mathcal{X}_{30}|^2 + \mathcal{X}_{02}\mathcal{X}^*_{23} + \mathcal{X}_{23}\mathcal{X}^*_{02} + |\mathcal{X}_{03}|^2 + \mathcal{X}_{32}\mathcal{X}^*_{20} + \mathcal{X}_{20}\mathcal{X}^*_{32}\\ &+|\mathcal{X}_{11}|^2 + \mathcal{X}_{13}\mathcal{X}^*_{31} + \mathcal{X}_{31}\mathcal{X}^*_{13}\end{aligned} \tag{5.4}$$

where the subscripts indicate a_1 and a_2 (which are sufficient to identify a representation).

An entirely different structure is obtained if the level is a multiple of 3. For example $SU(3)$ level 3 has the following set of orbits

$(0,0,3)_0$	$(0,3,0)_0$	$(3,0,0)_0$
$(1,0,2)_1$	$(0,2,1)_1$	$(2,1,0)_1$
$(0,1,2)_2$	$(1,2,0)_2$	$(2,0,1)_2$
$(1,1,1)_0$		

In this case the orbits with non-zero triality are projected out completely, and one gets the following modular invariant partition function

$$|\mathcal{X}_{00} + \mathcal{X}_{03} + \mathcal{X}_{30}|^2 + 3|\mathcal{X}_{11}|^2 . \tag{5.5}$$

Partition functions (5.4) and (5.5) are both obtained from the same general formula (4.7), including the important factor of 3 multiplying the contribution of the short orbit in the last example.

To demonstrate some other features, consider $SO(7)$. At level 2 the 7 primary fields are organized into the following orbits

$(0,0,0,2)_0$	$(2,0,0,0)_0$
$(0,0,1,1)_1$	$(1,0,1,0)_1$
$(0,0,2,0)_0$	
$(0,1,0,0)_0$	
$(1,0,0,1)_0$	

Here the automorphism is interchange of the labels a_1 and a_x. Unitary highest weight representations must satisfy the condition $a_1 + 2a_2 + a_3 + a_x = 2$. The charge is equal to $\frac{1}{2}$ times the subscript, which is equal to a_3 mod 2. Thus spinors have half-integral charge, and are projected out in the modular invariant partition function generated by the integer spin current $(2,0,0,0)$. This partition function is

$$|\mathcal{X}_{000} + \mathcal{X}_{200}|^2 + 2|\mathcal{X}_{002}|^2 + 2|\mathcal{X}_{010}|^2 + 2|\mathcal{X}_{100}|^2 .$$

As a final example, consider $SO(7)$ level 3. Now the set of orbits is

$(0,0,0,3)_0$	$(3,0,0,0)_0$
$(0,0,1,2)_1$	$(2,0,1,0)_1$
$(0,0,2,1)_0$	$(1,0,2,0)_0$
$(0,1,0,1)_0$	$(1,1,0,0)_0$
$(1,0,0,2)_0$	$(2,0,0,1)_0$
$(1,0,1,1)_1$	
$(0,1,1,0)_1$	
$(0,0,3,0)_1$	

Although superficially this looks very similar to the previous case there is an important difference, since the current $(3,0,0,0)_0$ now has half-integer spin. Thus the parameter r, appearing for example in (4.7), is equal to 2 (mod 4) whereas previously it was zero. The only way in which this is revealed by the orbit charges is that the orbits of length 1, which previously had charge 0, now have charge $\frac{1}{2}$. As a result, the modular invariant partition function is totally different:

$$|\mathcal{X}_{000}|^2+|\mathcal{X}_{300}|^2+|\mathcal{X}_{002}|^2+|\mathcal{X}_{102}|^2+|\mathcal{X}_{010}|^2+|\mathcal{X}_{110}|^2+|\mathcal{X}_{100}|^2+|\mathcal{X}_{200}|^2$$
$$+\mathcal{X}_{001}\mathcal{X}^*_{201}+\mathcal{X}_{201}\mathcal{X}^*_{001}+|\mathcal{X}_{003}|^2+|\mathcal{X}_{011}|^2+|\mathcal{X}_{101}|^2.$$

Here the first line contains the Neveu-Schwarz fields and the second line the Ramond fields, with charge 0 and $\frac{1}{2}$ respectively. The first two terms on the second line show an off-diagonal pairing that is typical for Ramond fields. Because in this example all but one of the Ramond orbits have length one, the result is nearly diagonal. The analogous partition function at level 1 is in fact completely diagonal, just as the E_8 level 2 example discussed above.

We could obviously fill many more pages with examples, but the general idea should now be clear.

6. Fixed points

Very often the orbits of a simple current do not all have the same length. As explained in section 3, the shorter orbits should have a length d that is a divisor of the length N of the identity orbit. Furthermore if an orbit has length d the spectrum of h values on such an orbit should have periodicity d rather than N.

It is simplest to discuss this problem for the special case N prime, so that d can only be N or 1 (other values of N are discussed at the end of this section). If there is an orbit of length 1 then this means that there exists a field ϕ so that

$$J\phi = J^c\phi = \phi\ .$$

We call such a ϕ a fixed point of the current. A condition on the spin of the current J can be derived as follows. From (3.2) we get

$$Q(\phi) = h(\phi) + h(J) - h(J\phi) = h(J) \bmod 1$$
$$Q^c(\phi) = -Q(\phi) = h(\phi) + h(J^c) - h(J^c\phi) = h(J) \bmod 1\ .$$

Hence $2h(J) = 0 \bmod 1$, so that $h(J)$ must be integer or half-integer.

The effect of fixed points on the form of modular invariant partition functions can be read off from (4.7), and is somewhat different in these two cases. If the current has half-integer spin the field ϕ (which belongs to the Ramond sector) is simply paired with itself, and all entries of M_{ij} are 1. If the current has integer spin a fixed point appears in the partition function with multiplicity N.

The general form of a partition function with fixed points is

$$P(\tau,\bar\tau) = \sum_i |\sum_p \mathcal{X}_{J^p i}|^2 + N\sum_f |\mathcal{X}_f|^2\ , \qquad (6.1)$$

where "i" labels the normal orbits and "f" the fixed point orbits. A partition function of this form can be viewed as a diagonal partition function of a theory with a larger algebra, with characters equal to the ones appearing inside the squares of (6.1) (see section 2). Since this theory has fewer characters than the one we started with, it obviously has a different modular transformation matrix S. However, since we do know how the original characters transform, we know at least something about that new matrix.

A first guess for the matrix $\hat{S}$ that acts on the new characters is obtained by going to an orthonormal basis

$$\begin{aligned}\hat{\mathcal{X}}_i &= \sqrt{\frac{1}{N}}(\mathcal{X}_i + \mathcal{X}_{Ji} + \ldots \mathcal{X}_{J^{N-1}i}) \\ \hat{\mathcal{X}}_f &= \mathcal{X}_f \ .\end{aligned} \tag{6.2}$$

On these characters (6.2) the modular transformation $\tau \to -1/\tau$ is represented by the matrix

$$\hat{S}_{ab} = \sqrt{N_a N_b} S_{ab} \ , \tag{6.3}$$

where "a" stands for i or f, and $N_i = N$, $N_f = 1$. Since the basis transformation on the space of characters is orthonormal, the matrix $\hat{S}$ is obviously unitary.

Nevertheless the matrix $\hat{S}$ constructed above is inadequate if there are fixed points. One problem is that it acts on characters (6.2) whose leading coefficient is not an integer. Furthermore one cannot make an overall change in the normalization so that simultaneously the normal characters and the fixed point ones have integer coefficients (remember that we assumed N to be prime). A related problem is that the fusion rules derived from $\hat{S}$ using (4.1) do not have integral coefficients.

A way out of this problem has been found by several authors (see *e.g.* [20], [26],[6]). They observed that in many cases the appearance of coefficients $N \neq 1$ as in (6.1) indicates that this contribution to the partition function is

due to N primary fields rather than just one. After splitting the corresponding rows and columns of $\hat{S}$ into N different ones, one should obtain a new matrix $\tilde{S}$ which satisfies all relevant consistency conditions, including that of having correct fusion rules. We will refer to this procedure as "resolving the fixed points".

Thus we are led to define a new matrix, $\tilde{S}$, which acts on characters

$$\mathcal{X}_{a_\alpha} = \sqrt{N_a}\hat{\mathcal{X}}_a \ , \alpha = 1, \ldots, N/N_a \ , \tag{6.4}$$

with $\hat{\mathcal{X}}_a$ defined as in (6.2). Note that all these new characters have integral coefficients. The matrix elements of $\tilde{S}$ are related to those of $\hat{S}$ by the known transformation of the characters $\hat{\mathcal{X}}_a$. However, there is some freedom in defining $\tilde{S}$ since $\hat{S}$ does not tell us how $\tilde{S}$ acts on the labels α. For the transformation of the new characters we find

$$\mathcal{X}_{a_\alpha}(-1/\tau) = \sqrt{\frac{N_a}{N}} \sum_b \hat{S}_{ab} \mathcal{X}_b(\tau) \ . \tag{6.5}$$

On the other hand the right-hand side should be equal to

$$\sum_{b,\beta} \tilde{S}_{a_\alpha b_\beta} \mathcal{X}_{b_\beta}(\tau) = \sum_{b,\beta} \tilde{S}_{a_\alpha b_\beta} \sqrt{\frac{N_b}{N}} \mathcal{X}_b(\tau) \ , \tag{6.6}$$

so that

$$\sum_\beta \tilde{S}_{a_\alpha b_\beta} = \sqrt{\frac{N_a}{N_b}} \hat{S}_{ab} \tag{6.7}$$

The most general parametrization of $\tilde{S}$ that is consistent with (6.7) is

$$\begin{aligned}\tilde{S}_{a_\alpha b_\beta} &= \sqrt{\frac{N_a N_b}{N^2}} \hat{S}_{ab} E_{\alpha\beta} + \Gamma^{ab}_{\alpha\beta} \\ &= \frac{N_a N_b}{N} S_{ab} E_{\alpha\beta} + \Gamma^{ab}_{\alpha\beta} \ ,\end{aligned} \tag{6.8}$$

where $E_{\alpha\beta} = 1$ (independent of α and β) and Γ must satisfy

$$\sum_\alpha \Gamma^{ab}_{\alpha\beta} = \sum_\beta \Gamma^{ab}_{\alpha\beta} = 0 \ . \tag{6.9}$$

and

$$\Gamma^{ab}_{\alpha\beta} = \Gamma^{ba}_{\beta\alpha} \ .$$

The matrices Γ are restricted by the consistency conditions for $\tilde{S}$, namely $\tilde{S}\tilde{S}^\dagger = \mathbf{1}$, $\tilde{S}^2 = C$ (with $C^2 = \mathbf{1}$), $(\tilde{S}T)^3 = C$ plus the condition that the fusion coefficients should be integers. The first two conditions lead to

$$\begin{aligned} &\sum_{b,\beta} \Gamma^{ab}_{\alpha\beta}\Gamma^{bc*}_{\beta\gamma} = \delta_{ac} P^a_{\alpha\gamma} \\ &\sum_{b,\beta} \Gamma^{ab}_{\alpha\beta}\Gamma^{bc}_{\beta\gamma} = C_{ac} \sum_\delta C^a_{\alpha\delta} P^a_{\delta\gamma} \ , \end{aligned} \tag{6.10}$$

where $P^a_{\alpha\beta}$ is a projection operator

$$P^a_{\alpha\beta} = \delta_{\alpha\beta} - \frac{N_a}{N} E_{\alpha\beta} \ .$$

The appearance of this operator is related to (6.9), since it projects on the non-zero modes of Γ. The matrices C appearing in (6.10) are the charge conjugation matrix of the original theory and a new matrix $C^a_{\alpha\beta}$, which defines the charge conjugation matrix of the new theory.

In exactly the same way one can derive a relation from $(ST)^3 = C$. It is convenient to write this as $STS = T^{-1}ST^{-1}$. For Γ we find then essentially the same relation

$$\Gamma^{ab}_{\alpha\beta} T_{bb} \Gamma^{bc}_{\beta\gamma} = T^{-1}_{aa} \Gamma^{ac}_{\alpha\gamma} T^{-1}_{cc} \ . \tag{6.11}$$

The "correction matrix" Γ (which of course exists only on the subspace of the fixed points) is thus seen to satisfy conditions that look a lot like those of

the modular group. We can make this more precise by making the following *ansatz*

$$\Gamma^{ab}_{\alpha\beta} = \gamma^{ab} P_{\alpha\beta} \ . \tag{6.12}$$

This ansatz is certainly correct if $N = 2$ (where, as before, N is the order of the simple current) as a consequence of (6.9), but for larger values of N more general expressions could be considered. This ansatz leads to the following conditions for γ

$$\begin{aligned} &\gamma\gamma^\dagger = \mathbf{1} \\ &\gamma\rho\gamma = \rho^{-1}\gamma\rho^{-1} \ , \end{aligned} \tag{6.13}$$

where ρ is equal to T, restricted to the fixed points. For the third condition we find

$$(\gamma^2)_{ab} P_{\alpha\beta} = C_{ab} \sum_\delta C^b_{\alpha\delta} P_{\delta\beta} \ .$$

This can only be satisfied if

$$\sum_\delta C^b_{\alpha\delta} P_{\delta\beta} = \eta_b P_{\alpha\beta} \tag{6.14}$$

with

$$(\gamma^2)_{ab} = \eta_b C_{ab} \ . \tag{6.15}$$

The first equation, combined with the knowledge that C must be a permutation of order two, can be satisfied only if $C^b_{\alpha\beta} = \delta_{\alpha\beta}$, or if $N = 2$ and $C^b = \sigma_1$. The value of η_b is $+1$ and -1 respectively in these two cases. Note that it is not necessary to consider the choice $C^b = \sigma_1$ unless ϕ_b is self-conjugate, since otherwise one can always rearrange the basis to make C^b diagonal.

Thus to resolve the fixed points we have to find a representation of the modular group, consisting of matrices γ and ρ, where ρ is the restriction of

T to the fixed points. The only way in which this representation differs from those one normally encounters in conformal field theory is that the entries of the charge conjugation matrix may be -1. The general form, (6.15), suggests that the entries of C could have different signs, but we have only encountered cases where all signs are the same.

It is a non-trivial problem to find, given a matrix T, a related matrix S so that the two matrices form a representation of the modular group. For arbitrary T such a matrix S does in fact not exist. If one finds such a matrix, it is natural to try to examine the fusion rules obtained from it by using (4.1). Of course this requires a choice of the identity. It is certainly not true that every representation of the modular group yield sensible fusion rules for any choice of the identity. However, it is a remarkable empirical fact that for matrices ρ obtained from fixed points of a simple current there does not only exist a matrix γ (this should be true in view of the foregoing discussion), but furthermore this matrix does lead to positive integer fusion coefficients.

More precisely what we find is that the matrix ρ obtained from the fixed points of an integer spin current is related by an overall phase to the matrix T of some conformal field theory. If this phase were arbitrary, we would still not be able to find a matrix γ belonging to ρ. However, for every solution to the conditions for modular group representations

$$(ST)^3 = S^2 \quad , \quad S^4 = \mathbf{1} \tag{6.16}$$

there is a set of rescalings leading to other solutions. If we set

$$T \rightarrow \lambda T \; ; \qquad S \rightarrow \mu S$$

then we see from (6.16) that λ and μ must satisfy $(\lambda)^3 = \mu^{-1}$ and $\mu^4 = 1$. In

other words, λ must be a 12^{th} root of unity. Indeed, we find that

$$\rho = e^{2\pi i \frac{m}{12}} T \; ; \qquad \gamma = e^{-2\pi i \frac{m}{4}} S \; , \tag{6.17}$$

where S and T are modular transformation matrices satisfying the 4 conditions listed in the beginning of section 4. Note that the phase in S does not affect the fusion coefficients, but if m is odd it does change the sign of C and of the three-point coupling coefficients N_{ijk}.

Note that (6.17) only fixes the values of $h - \frac{c}{24}$ modulo integers. However, in most cases we can identify T with a unitary or non-unitary conformal field theory. In all those cases the values of $h - \frac{c}{24}$ of the fixed points and the corresponding conformal field theory differ exactly by $\frac{m}{12}$, for some integer m.

Fixed points of half-integer spin currents do not require any such treatment, but if one tensors two theories with half-integer spin currents one may combine the two half-integer spin currents to one with integer spin. The fixed points of this integer spin current are simply all products of the fixed points of the two half-integer spin currents. One would therefore expect that our conjecture has an extension to fixed points of half-integer spin currents, an indeed it does. The only difference is that the shift in $h - \frac{c}{24}$ is an *odd* multiple of $\frac{1}{24}$, whereas previously it was an *even* multiple. Therefore there is no obvious solution for γ, but this does not matter, since we need γ only if the current has integer spin.

Up to now we have discussed the resolution of fixed points for prime N, and for modular invariants obtained by simple currents. If one can identify the fixed point conformal field theory (more about this in the next section) this problem is probably solved, although one has to verify that the candidate for the matrix $\tilde{S}$ as defined in (6.8) does indeed yield positive integer fusion coefficients. If N is not prime one may proceed in steps, extending in each step the chiral algebra by a prime factor of N. In each step one resolves the fixed points as explained above. Exceptional invariants can also have features similar to fixed points of simple currents (*i.e.* squares with fewer terms than the identity, and a multiplicity larger than one) and in all examples we know the multiplicities can be resolved into separate fields in exactly the same way as for simple currents.

7. Fixed point conformal field theories

According to the conjecture formulated in the previous section the values of $h - \frac{c}{24}$ for the fixed points of simple currents differ from those of some conformal field theory by an even multiple of $\frac{1}{24}$ if the current has integer conformal spin, and by an odd multiple if it has half-integer conformal spin. In [5] some examples of fixed point conformal field theories were given, and the complete classification was given for $SU(N)$. In this section we will extend this result to all other Kac-Moody algebras.

For the single fixed point of the exceptional simple current occurring for E_8 level 2 we find $h - \frac{c}{24} = \frac{7}{24}$, which agrees with the conjecture, since this current has half-integral spin. The orbits of all other known simple currents can be described by means of automorphisms of the extended Dynkin diagram, which makes it very simple to list the fixed points. The results are summarized in table I.

In most cases the fixed point CFT of a Kac-Moody algebra $\mathcal{K}$ turns out to be a Kac-Moody algebra itself. We will denote the latter as $\hat{\mathcal{K}}$. The precise relation between the Dynkin labels of the fixed points and the primary fields of Kac-Moody algebras associated with them is shown in figures 1-7. To prove the correspondence one can proceed as follows. The conformal weights h of Kac-Moody primary fields with highest weight Λ are given by (5.1). The Casimir eigenvalues can be computed conveniently in Dynkin basis. Denoting the Dynkin labels as Λ_i and the inverse Cartan matrix as G_{ij}, we may write (5.1) as

$$h = \frac{1}{2}\frac{1}{k+g}\left[(\Lambda_i+\delta_i)G_{ij}(\Lambda_j+\delta_j) - (\delta_i G_{ij}\delta_j)\right] ,$$

KM-algebra				Fixed Point CFT		
Lie-algebra	level	current	order	Lie-algebra	level	shift
A_n	$\frac{p(n+1)}{l}$	J_l	$\frac{n+1}{l}$	A_{l-1}	p	$\frac{p[(n+1)^2-l^2]}{24l}$
B_n	$2p$	J_v	2	$\mathcal{B}_{n-1,p}$		$\delta_h \equiv 0$
B_n	$2p+1$	J_v	2	C_{n-1}	p	$\frac{2n+6p+1}{24}$
C_{2n+1}	$2p$	J	2	C_n	p	$\frac{(2n+1)p}{8}$
C_{2n}	$2p$	J	2	$\mathcal{B}_{n,p}$		$\delta_h = \frac{p(n-1)}{4}$
C_{2n}	$2p+1$	J	2	C_n	p	$\frac{np}{4} + \frac{5n}{24}$
D_n	$2p$	J_v	2	C_{n-2}	p	$\frac{p}{4}$
D_n	$2p+1$	J_v	2	No fixed points		
D_{2n}	$2p$	J_s	2	B_n	p	$\frac{pn}{8}$
D_{2n}	$2p+1$	J_s	2	No fixed points		
D_{2n+1}	$4p$	J_s	4	C_{n-1}	p	$\frac{(2n+3)p}{8}$
E_6	$3p$	J	3	G_2	p	$\frac{2}{3}p$
E_7	$2p$	J	2	F_4	p	$\frac{3}{8}p$

Table I. *Fixed point conformal field theories of Kac-Moody algebras. For levels that are not listed in column 2 the current does not have integral or half-integral spin. Column 3 indicates the current, as defined in section 5. The A_n current J_l is assumed to be chosen so that l is a divisor of $n+1$. The meaning of $\mathcal{B}_{m,p}$ in column 5 is explained in the text. The last column gives the common shift δ_h in the values of $h - \frac{c}{24}$.*

where $\delta_i = 1$. If Λ represents a fixed point, some of its entries are equal to each other as a consequence of the automorphism of the extended Dynkin diagram. Hence we can "fold up" the matrix G by adding together those rows and columns that are mapped into each other by the automorphism. The new, folded, matrix $\tilde{G}$ will have one row and one column that corresponds to the extended root, since the extended root is combined with one or more simple roots by the extended Dynkin diagram automorphism. The argument continues in a somewhat different way if the Kac-Moody algebra is simply laced (of type A, D or E) or non-simply laced (of type B, C, F or G). We start with the former class.

7.1 Simply laced algebras

For simply-laced algebras one finds, after some computations, that the matrix $\tilde{G}$ takes the following form

$$\tilde{G} = \begin{pmatrix} \alpha & \alpha v_a \\ \alpha v_b & N\hat{G}_{ab} + \alpha v_a v_b \end{pmatrix} . \tag{7.1}$$

Here N is the order of the automorphism, and the basis has been arranged so that the first entry corresponds to the extended root. It turns out that the matrix $\hat{G}_{ab}$ is always equal to the Cartan matrix of some Lie-algebra of rank m. The fixed point conformal field theory $\hat{\mathcal{K}}$ will turn out to be a Kac-Moody algebra corresponding to this Lie-algebra, with level $\frac{k}{N}$. The vector v_a is equal to $\hat{G}_{ab}\psi_b$, where ψ_b is the highest root (in Dynkin basis). Finally, α is some rational number that depends on the extended Dynkin diagram and the automorphism one considers.

The Casimir eigenvalue entering in the computation of h now takes the form

$$(\Lambda_i + \delta_i)G_{ij}(\Lambda_j + \delta_j) = \vec{\mu}\tilde{G}\vec{\mu} ,$$

where $\vec{\mu}$ is the vector

$$\vec{\mu} = (\hat{k} - \sum_{a=1}^{m} v_a\lambda_a + 1, \lambda_1 + 1, \ldots, \lambda_m + 1) ,$$

and $\hat{k} = \frac{k}{N}$. It is then a matter of straightforward calculation to obtain the

following expression for h

$$h = \frac{N}{2}\frac{1}{k+g}\lambda_a\hat{G}_{ab}(\lambda_b + 2\delta_b) + \frac{1}{2(k+g)}\left[\alpha(\hat{k}+\hat{g})^2 - \frac{1}{12}gd + \frac{N}{12}\hat{g}\hat{d}\right] . \tag{7.2}$$

Here $\hat{g}$ is the dual Coxeter number of the fixed point CFT $\hat{\mathcal{K}}$, whose inverse Cartan matrix is $\hat{G}_{ab}$. We have used the relation $\hat{g} = \sum_a v_a\psi_a + 1$. Furthermore we have used the Freudenthal-de Vries strange formula, which equates the norm of the vector δ with $\frac{1}{12}gd$, where d is the dimension of the Lie algebra ($\hat{d}$ denotes the dimension of the Lie algebra of $\hat{\mathcal{K}}$).

We can identify the conformal weights of the fixed points, (7.2), with those of $\hat{\mathcal{K}}$ (up to a constant) provided that $(k+g) = N(\hat{k}+\hat{g})$ (*i.e.* $\hat{g} = \frac{g}{N}$). One may check that this relation does indeed hold for simply laced algebras. If we now consider instead of h the quantity $t = h - \frac{c}{24}$ we find

$$t = \hat{t} + \frac{\alpha}{2N}(\hat{k}+\hat{g}) - \frac{d}{24} + \frac{\hat{d}}{24} . \tag{7.3}$$

Finally, one may verify that for all simply laced algebras

$$d - \hat{d} = \frac{12\alpha\hat{g}}{N} , \tag{7.4}$$

so that we may rewrite the result as

$$t = \hat{t} + \frac{\alpha\hat{k}}{2N} = \hat{t} + \frac{\alpha k}{2N^2} . \tag{7.5}$$

By computing α for all simply-laced Lie-algebras one may verify that the shift in t is indeed an even or odd multiple of $\frac{1}{24}$, as required. The formulas for this shift are given in table I.

7.2 Non-simply laced algebras

The only non-simply laced Kac-Moody algebras with a non-trivial center are B_n and C_n. If n is odd, the arguments for C_n are identical to the ones given above for simply-laced algebras. Equation (7.4) holds also in this case, so that shift in t is given by (7.3) or equivalently (7.5). The conformal spin of the current is $\frac{nk}{4}$, and therefore there can be fixed points only for even k.

For B_n and C_n, n even the calculation is different. It will be convenient to discuss simultaneously B_{m+1} and C_{2m}. Note first of all that the relation $g = N\hat{g}$ (here with $N = 2$) cannot hold, since g is odd. It turns out that the fixed point conformal field theory can still be identified with a Kac-Moody algebra if the level, k, is *odd*. This can be shown by a small modification of the calculation presented above. First of all one has to multiply the last row and column of $\tilde{G}$ by 2, and divide the last entry of the vector $\vec{\mu}$ by 2 (the last entry corresponds to the short root of B_{m+1}, and to the point in the middle of the extended Dynkin diagram of C_{2m}). Now the vector μ has the form

$$\vec{\mu} = (\frac{k}{2} - \sum_{a=1}^{p} u_a\lambda_a + 1, \lambda_1 + 1, \ldots, \frac{1}{2}(\lambda_m + 1)) ,$$

where $\vec{u} = (1,1,1,\ldots,1,1,\frac{1}{2})$. After changing the normalization of the last column, the matrix $\tilde{G}$ has the form (7.1), with $\hat{G}$ equal to the inverse Cartan matrix of C_m.

The last entry of $\vec{\mu}$ can be written as $\frac{1}{2}(\lambda_m - 1) + 1$. Furthermore the first entry can be written as

$$\frac{k-1}{2} - \sum_a v_a\lambda'_a + 1 ,$$

where

$$\vec{\lambda}' = (\lambda_1, \ldots, \lambda_{m-1}, \frac{1}{2}(\lambda_m - 1)) ,$$

and $\vec{v} = (1,1,1,\ldots,1)$, so that $v_a = \hat{G}_{ab}\psi_b$ for C_m. We have now succeeded in writing the expression for h exactly as before, but with $\vec{\lambda}'$ instead of $\vec{\lambda}$, and with $\hat{k} = \frac{1}{2}(k-1)$. Furthermore $(k+g) = 2(\hat{k}+\hat{g})$ is satisfied, so that (7.3) holds. On the other (7.4) does not hold, so that the formula for the shift in t is a little bit more complicated. It is given in table I.

The result derived above can only be interpreted as a spectrum of a Kac-Moody algebra if λ' consists of integers, and if $\hat{k}$ is an integer. It is easy to see that both statements are true if and only if k is odd (this follows from the fact that the first entry of $\vec{\mu}$ must be an integer.) Hence for k odd we again get a fixed point CFT that is a Kac-Moody algebra, namely C_m, level $\frac{k-1}{2}$.

7.3 A new class of conformal field theories?

If k is even this interpretation cannot be made, and if one investigates the simplest examples one discovers that the fixed point h-values do not correspond to the h-values of any Kac-Moody algebra. The values of h that one gets can be computed using the procedure described above, but there is no need anymore to change the normalization of the last row and column. If instead one just folds up the B_{m+1} or C_{2m} Cartan matrices one arrives at the following characterization of the fixed point h-values. Consider a set of Dynkin-labels $(\lambda_1, \ldots, \lambda_m)$ that satisfy the condition

$$\frac{k}{2} - \sum_{a=1}^{m} \lambda_a \leq 0 . \qquad (7.6)$$

The fixed points of B_{m+1} or C_{2m} level k are in one-to-one correspondence with these labels. The precise correspondence is exactly as in figs. 2 and 3, with $2\lambda_m$ instead of $2\lambda_m - 1$ attached to the last node of B_{m+1} or the middle one of C_{2m}. To write down the conformal weights we define

$$\tilde{\lambda} = (\lambda_1, \ldots, \lambda_{m-1}, 2\lambda_m) ,$$

which is to be regarded as a set of B_m Dynkin labels. In terms of $\tilde{\lambda}$ the h-values of the fixed points are given by

$$h = \frac{1}{2(k+g)} C_B(\tilde{\lambda}) + \begin{cases} \frac{k(k+2g)}{8(k+g)} & \text{for } B_{m+1} \\ \frac{mk(k+2g)}{8(k+g)} & \text{for } C_{2m} \end{cases} , \tag{7.7}$$

where $g = 2m+1$. Here C_B is the quadratic Casimir operator of B_m, defined as in (5.2).

Note that the conformal weights resulting from (7.7) are not those of B_m, not even up to a constant, for two reasons: the numerator $k+g$ is not the correct one (g is the dual Coxeter number of B_{m+1} or C_{2m}, but not of B_m) and the labels $\tilde{\lambda}$ do not correspond to a complete set of primary fields of B_m. The labels $\tilde{\lambda}$ are in fact precisely those of the *tensor* representations of B_m level p (where $p \equiv \frac{k}{2}$). This allows us to write (7.7) in a different way. All tensor representations of B_m level p can be described by means of Young tableaux, which may have at most m rows and p columns. In this notation, the Casimir eigenvalues are given by

$$C_B(f_i) = 2mr + \sum_i f_i^2 - \sum_j g_j^2 ,$$

where f_i is the length of the i^{th} row and g_j the length of the j^{th} column, and $r = \sum f_i = \sum g_j$ (*i.e.* the number of boxes of the Young tableaux). The values of g_j depend of course on those of f_i.

This notation can be used to demonstrate a remarkable symmetry under interchange of m and p. Note first of all that the number of fixed points is equal to $\frac{(m+p)!}{m!p!}$, which is symmetric in m and p. The formula for h reads now

$$h = \frac{2mr + \sum_i f_i^2 - \sum_j g_j^2}{2(2p+2m+1)} + \text{constant} , \tag{7.8}$$

where the constant is as in (7.7). We can map the tensor representations of B_m level p to those of B_p level m by defining

$$\begin{aligned} f'_j &= m - g_j \\ g'_i &= p - f_i \end{aligned} , \tag{7.9}$$

so that $r' = mp - r$. The Casimir eigenvalue for this B_p representation is

$$\begin{aligned} C_B(f'_j) &= 2pr' + \sum_j (f'_j)^2 - \sum_i (g'_i)^2 \\ &= mp^2 + pm^2 - C_B(f_i) \end{aligned} . \tag{7.10}$$

Thus the set of h-values changes sign (up to a constant) upon interchange of p and m.

Do these h-values correspond to conformal field theories? If our conjecture is correct they should, but for the moment we only have some suggestive facts, but no general proof. First of all, the fact that the conformal weights of the fixed points of C_{2m} and B_{m+1} differ by a constant suggests that they are related to the same CFT. If this is true, the values of $t \equiv h - \frac{c}{24}$ of each of these two theories differ by a multiple of $\frac{1}{12}$ from those of some CFT, and hence they must differ by a multiple of $\frac{1}{12}$ from each other (note that the currents have integer spin). For definiteness we will choose the fixed points of B_{m+1} to fix the values of t of our putative conformal field theory $\mathcal{B}_{m,p}$. Those values are

$$t^B_{m,p}(\tilde{\lambda}) = \frac{C_B(\tilde{\lambda})}{2(2p+2m+1)} + \frac{p}{12(2p+2m+1)}(6p+7m+3-2m^2) . \tag{7.11}$$

In the following we will refer to this set of values of t as "$\mathcal{B}_{m,p}$", a notation already used in table I. For the fixed points of C_{2m} level $2p$ we find then

$$t^C_{m,p}(\tilde\lambda) = \frac{C(\tilde\lambda)}{2(2p+2m+1)} + \frac{p}{12(2p+2m+1)}(6mp+4m+4m^2)$$
$$= t^B_{m,p}(\tilde\lambda) + \frac{(m-1)p}{4} ,$$

a difference which is indeed a multiple of $\frac{1}{12}$.

Because of the exchange symmetry in p and m described above there is a second set of values of t that differ by a constant, and by the same logic we might expect this constant to be a multiple of $\frac{1}{12}$ as well. These two sets are $\mathcal{B}_{m,p}$ and the complement of $\mathcal{B}_{p,m}$, where the t-values of the complement of a theory are defined as minus those of the theory itself. Using (7.10) we can write down the following formula for the t-values of the complement of $\mathcal{B}_{p,m}$

$$-t^B_{p,m}(\tilde\lambda') = \frac{C(\tilde\lambda')}{2(2p+2m+1)}$$
$$- \frac{1}{12(2p+2m+1)}(4p^2m+6m^2p+6m^2+7mp+3m)$$
$$= t^B_{m,p}(\tilde\lambda) - \frac{(2pm+3p+3m)}{12} ,$$

where the relation between $\tilde\lambda$ and $\tilde\lambda'$ is defined via Young tableaux, as in (7.9). Note that the shift is indeed a multiple of $\frac{1}{12}$, and that this requires a non-trivial cancellation of the denominator factor $(2p+2m+1)$.

These facts provide additional support for our conjecture, but do not prove it. In some special cases we can identify the set of h-values with a conformal field theory. The fixed points of $SO(5)$ (*i.e.* for $B_2 = C_2$) at level $2p$ are associated with $\mathcal{B}_{1,p}$. They are parametrized by a single integer $r \le p$, corresponding to f_1 in (7.8). As was shown in [5] the resulting t-values can be identified with members of the minimal series of non-unitary Virasoro

representations

$$h = \frac{(sq'-s'q)^2-(q-q')^2}{4qq'} ,$$
$$s=1,\dots,q-1, \quad s'=1,\dots,q'-1, \quad s'q \le q's , \tag{7.12}$$

with central charge

$$c = \frac{1-6(q-q')^2}{qq'} .$$

We will denote these theories as $[q,q']$. The $h-\frac{c}{24}$ values of $\mathcal{B}_{1,p}$, given by (7.11), are equal to those of $[2,2p+3]$ up to an overall constant $\frac{p}{4}$ (the precise relation with (7.8) and (7.11) follows by substituting $s'=p-r+1$). It was also observed in [5] that the fixed points of $SO(2N+1)$ level 2 yield the complements of the same non-unitary series. This was a first hint of the $p \to m$ exchange symmetry discussed above.

The foregoing observations are summarized in table II. Here the relations between various theories are shown up to overall constant contributions to h which are multiples of $\frac{1}{12}$. The precise definition of the notation $X \approx Y^c$ used in this table is that the $h-\frac{c}{24}$ values of the theory X can be obtained by taking those of Y, changing their sign and adding an appropriate multiple of $\frac{1}{12}$. Note that this relation (which is a reflection symmetry with respect to the diagonal) holds for the exact values of $h-\frac{c}{24}$, and not just their fractional part. The same is true for the identification with elements of the minimal series. The theories on the diagonal are "self-complementary". This means that their spectra, again up to an overall constant, are symmetric under a sign change of h.

We expect that the set of T-matrices furnished by $\mathcal{B}_{m,p}$ correspond to conformal field theories, perhaps in a slightly more general sense than usual, not only for $m=1,p$ arbitrary and vice-versa, but also for arbitrary values of m and p. More concretely, we would expect that for all values of m and p a

group → level	$SO(5) \equiv Sp(4)$	$SO(7)$ or $Sp(8)$	$SO(9)$ or $Sp(12)$	...
2	$\mathcal{B}_{1,1} \approx [2,5]^c$	$\mathcal{B}_{2,1} \approx [2,7]^c$	$\mathcal{B}_{3,1} \approx [2,9]^c$	...
4	$\mathcal{B}_{1,2} \approx [2,7]$	$\mathcal{B}_{2,2} \approx \mathcal{B}^c_{2,2}$	$\mathcal{B}_{3,2} \approx \mathcal{B}^c_{2,3}$	...
6	$\mathcal{B}_{1,3} \approx [2,9]$	$\mathcal{B}_{2,3} \approx \mathcal{B}^c_{3,2}$	$\mathcal{B}_{3,3} \approx \mathcal{B}^c_{3,3}$	...
⋮	⋮	⋮	⋮	⋱

Table II. *The fixed point conformal field theories of $SO(2m+1)$ and $Sp(4m)$ at even levels, most of which are unidentified. The known identifications, as well as self-identifications within the diagram are shown; $\approx$ indicates that the identifications are up to an overall constant in h, which can be found in the text. The minimal series defined by* (7.12) *is denoted as* $[q,q']$.

matrix S exists so that S and T form a representation of the modular group. Furthermore there should be a set of characters, not necessarily with positive coefficients, that transforms with S and T. Finally previous experience suggests that the matrix S should define sensible fusion rules.

Without a positive identification of the spectra $\mathcal{B}_{m,p}$ with a known conformal field theory this is hard to investigate. However, it is possible to perform some additional explicit calculations for the simplest unidentified example, $\mathcal{B}_{2,2}$. The values of $h - \frac{c}{24}$, obtained from the fixed points of $SO(7)$ level 4 are $\frac{7}{18}, \frac{11}{18}, \frac{13}{18}, \frac{17}{18}, \frac{19}{18}$ and $\frac{23}{18}$. One can now attempt to determine S by simply solving the equations that it has to satisfy, namely $SS^\dagger = 1$, $S = S^T$, $S^2 = 1$ and $ST^3 = 1$. (Note that the charge conjugation matrix must be diagonal since there are no degenerate eigenvalues of T. For matrices S used for fixed point resolution we could also consider $C = -1$, but this yields no solution.) In total we have found 128 different solutions to these equations, but they belong to just two equivalence classes. Obviously from any solution one can construct another one by multiplying row i and column i by -1, which leads

to a total of 64 possible sign choices.

From S one can compute the fusion rules using Verlinde's formula provided that one knows which field corresponds to the identity. Since we suspect that this theory (assuming it exists) is non-unitary, we cannot simply assume that the identity is the representation with smallest h-value. Instead we simply consider all possibilities for using one of the six fields as the identity. We find that all 64 matrices from the first class yield integer fusion coefficients for all six possible choices of the identity. However, none of the matrices in the second class yields integer fusion rules. The fusion coefficients are not in general *positive* integers. However, for each choice of the identity one of the 64 possible sign choices leads to positive integer fusion coefficients. One of these matrices with positive integer fusion coefficients is

$$S = \begin{pmatrix} -\frac{1}{3} & a & -\frac{1}{3} & -c & \frac{1}{3} & -b \\ a & -\frac{1}{3} & -b & -\frac{1}{3} & c & -\frac{1}{3} \\ -\frac{1}{3} & -b & -\frac{1}{3} & a & \frac{1}{3} & -c \\ -c & -\frac{1}{3} & a & -\frac{1}{3} & b & -\frac{1}{3} \\ \frac{1}{3} & c & \frac{1}{3} & b & -\frac{1}{3} & -a \\ -b & -\frac{1}{3} & -c & -\frac{1}{3} & -a & -\frac{1}{3} \end{pmatrix}$$

Here $a = .62646174191, b = .115765451778$ and $c = .510696295413$. The identity is assumed to be the first entry. The fusion rules are

$$(2) \times (2) = (1) + (3) + (4)$$
$$(2) \times (3) = (2) + (3) + (5)$$
$$(2) \times (4) = (2) + (5)$$
$$(2) \times (5) = (3) + (4) + (5) + (6)$$
$$(2) \times (6) = (5) + (6)$$
$$(3) \times (3) = (1) + (2) + (3) + (4) + (5) + (6)$$
$$(3) \times (4) = (3) + (4) + (5)$$
$$(3) \times (5) = (2) + (3) + (4) + 2(5) + (6)$$
$$(3) \times (6) = (3) + (5) + (6)$$
$$(4) \times (4) = (1) + (3) + (6)$$
$$(4) \times (5) = (2) + (3) + (6)$$
$$(4) \times (6) = (4) + (5)$$
$$(5) \times (5) = (1) + (2) + 2(3) + (4) + 2(5) + (6)$$
$$(5) \times (6) = (2) + (3) + (4) + (5)$$
$$(6) \times (6) = (1) + (2) + (3)$$

It would be interesting to know whether there exists a character representation for these matrices S and T. Assuming that there exists a set of characters with positive integer coefficients one makes the following consistency check, using an argument of [20]. Suppose that the field with the smallest value of h is (m). Then

$$\frac{S_{km}}{S_{mm}} = \lim_{q\to 0} \frac{\sum_l S_{kl}\mathcal{X}_l(q)}{\sum_l S_{ml}\mathcal{X}_l(q)} = \lim_{q\to 1} \frac{\mathcal{X}_k(q)}{\mathcal{X}_m(q)} > 0 .$$

This implies that all entries in the m^{th} row and column must have the same sign. The matrix given above satisfies this with $m = 6$. The other 6 matrices that yield positive integer fusion coefficients for other choices of the identity satisfy the consistency check for 5 other values of m. The other 58 sign

choices that do not yield positive signs in the fusion rules do not satisfy the consistency check either.

The fact that the identity and the field with the lowest h-value do not coincide shows that this theory cannot correspond to a unitary CFT. This is not surprising, since we know already that this is true for $m = 1$ and for $p = 1$.

An important remaining question is now whether this matrix S really resolves the fixed points of $SO(7)$ level 4. We find that it does: the conditions on $(ST)^3$ and S^2 of the theory obtained after resolving the six fixed points are satisfied, and the fusion coefficients are positive integers. This is true for all 64 sign choices of the solutions in the first class discussed above. Those in the second class (for which the fixed point CFT itself does not have sensible fusion rules) do not give integral fusion coefficients. The 64 sign choices have a rather trivial interpretation. They simply correspond to an interchange of the two fields that belong to a given fixed point, *i.e.* a mere relabelling of the basis.

The fixed point conformal field theory of $Sp(8)$ level 4 is equivalent to that of $SO(7)$ level 4, and indeed we find that the same matrix S resolves its fixed points, yielding correct fusion rules.

We regard these facts as strong evidence that in general the fixed point conformal field theory should itself have sensible fusion rules, although there is no obvious direct connection between the fusion rules of the fixed point CFT and those of the modular invariant theory whose fixed points it helps resolving.

8. Applications

8.1 Free, untwisted bosons

One does not really need simple currents to write down all modular invariant partition functions of free, untwisted bosons, since it is already known they can be obtained from even self-dual Lorentzian lattices [27]. (It is easy to prove that every such lattice yields a modular invariant partition function; the converse is less obvious, but can also be proved.) Nevertheless it may be instructive to consider this rather trivial example first. Rational conformal field theories of this type are always characterized by a root lattice Λ_R with vectors of even norm (for more details about lattices see *e.g.* [28]). The dual of the root lattice can be decomposed in terms of a finite number of cosets of the form $\vec{w}_i + \Lambda_R$. The primary fields are in one-to-one correspondence with the coset representatives w_i. The fusion rules are nothing but addition of the vectors w_i modulo the root lattice. It follows that all fusion rules have just one term on the left-hand side, so that all primary fields are simple currents. One can now use any of these currents to build a new modular invariant partition function, provided that the level matching condition is satisfied. This condition requires N times the conformal weight of a current of order N to be an integer. In terms of lattice vectors $\vec{w}$ this implies $\vec{w}^2 = \frac{2k}{N}, k \in \mathbf{Z}$. This is precisely the level matching condition for shift vectors of order N, and indeed the simple current procedure is nothing but the construction of a new self-dual lattice from a given one by means of shift vectors. Note that we require the shift-vector to be a "weight", *i.e.* to lie on the dual of Λ_R. Such a restriction is not necessary if one works directly with lattices. However, if one does not use a weight one leaves the original conformal field theory, and passes to a new one with more primary fields, and a root lattice with fewer vectors. This new conformal field theory is then the one to use to describe the construction in terms of simple currents.

8.2 Orbifolds

Although our discussion in section 3 uses orbifold-like arguments, it is not true that the usual orbifold construction is a trivial application of simple currents. The inverse of the orbifold construction on the other hand can be described in our language.

We are using the word "orbifold" here strictly in the sense of twisted torus compactifications. In such a construction new fields are introduced, the twist fields. They are not present among the set of fields ∂X and e^{ikX} built from free bosons. On the other hand, the twist field $J(z)J^c(\bar{z})$ used in section 3 is entirely built out of fields that were present already.

It follows that in order to describe the relation between the torus and orbifold partition functions one should start with a conformal field theory that already contains all primary fields that can appear, *i.e.* one should start with the orbifold conformal field theory.

We will illustrate this by means of $c = 1$ orbifolds. The operator algebra of these orbifolds is described in [29]. One has the following fields:

$\mathbf{1}$	$h = 0$
J	$h = 1$
ψ^i	$h = \frac{M}{4}$ $(i = 1,2)$
ϕ_q	$h = \frac{q^2}{4M}$ $(q = 1, \ldots, M-1)$
σ_i	$h = \frac{1}{16}$ $(i = 1,2)$
τ_i	$h = \frac{9}{16}$ $(i = 1,2)$

These fields form the conformal field theory of the $\mathbf{Z}_2$-orbifold of a rational torus of radius $R^2 = 2M, M \in \mathbf{Z}$. The diagonal partition function containing all these fields is the orbifold partition function. The modular transformation matrix S for the characters of this theory was computed in [29]. It follows from this computation that the spin-1 field J as well as the two fields ψ^i are simple currents. These four fields yield a center $\mathbf{Z}_2 \times \mathbf{Z}_2$ if M is even and $\mathbf{Z}_4$ if M is odd. The torus partition function is now obtained by using the current J, which has order 2, to build an integer spin invariant.

The orbit structure under J is as follows: The orbits of length 2 are $(\mathbf{1}, J)$, (ψ^1, ψ^2) and $(\sigma^i, \tau^i), i = 1, 2$. The fields $\phi_q, q = 1, M-1$ are all fixed points of J, and the twist fields σ^i and τ^i are projected out. The fixed point conformal field theory is $SU(2)$ level $M-2$ [5] (note that there are no fixed points for $M = 1$, and that $SU(2)$ level 0 must be interpreted as the identity). The operator J that extends the algebra plays the rôle of the free boson ∂X. Upon resolving the fixed points each field ϕ_q produces two fields, corresponding to the two opposite charges that the operators $e^{\pm ikX}$ have with respect to ∂X. In the orbifold these opposite charges are of course identified. Although the fields ϕ_q are not simple currents in the orbifold model, the fixed point resolution splits up their fusion rules, in such a way that they do become simple currents of the torus-compactified theory. Thus the construction of the torus partition out of orbifold characters is indeed nothing but an application of the simple current formalism, combined with the resolution of fixed points. Obviously one can apply our formalism also to the currents ψ^i.

8.3 Tensor products

The simple currents of a tensor product of two or more conformal field theories are obviously precisely all possible products of simple currents (including the identity) of the factors.* In general, the theory generated by the product $J_1 J_2$ of two currents from two different factors is *not* the tensor product of the theories generated by J_1 and J_2 separately.

* Additional simple currents can sometimes appear if one resolves the fixed points of an integer spin modular invariant. Examples are given in sections 8.2 and 8.4.

The center of the tensor product is the product of the centers of the separate factors. Thus in general it is a product of several $\mathbf{Z}_N$ factors. The number of different modular invariant partition functions increases very rapidly with the number of such factors. This is true not only because there is a large number of currents to choose from (*i.e.* a large number of subgroups of the center), but also because one can consider products of the matrices $M(J)$ obtained by means of a current J.

It is obvious that a product $M(J_1)M(J_2)$ is a matrix of positive integers which commutes with S and T. A slightly more delicate point is the normalization. In general in such a product the identity appears more than once. However, it can be shown that one can divide the product matrix by the multiplicity of the identity without encountering fractional multiplicities for other fields [30]. In general $M(J_1)M(J_2) \neq M(J_1 J_2)$, even after normalizing the identity. The product of two matrices M is of course in general not a symmetric matrix. In particular this makes it possible to obtain different extensions of the chiral algebra for left- and right-movers. All these additional possibilities appear only if the center has more than one $\mathbf{Z}_N$ factor. If the center consists of a single $\mathbf{Z}_N$ factor multiplication of two matrices M yields nothing new.

At present we do not know a simple characterization of all the modular invariants for a given center that are different; we only know this for a simple center, where they are in one-to-one correspondence with the subgroups of the center.

8.4 Free fermions

A simple application of the foregoing is the construction of modular invariant partition functions for free fermions. The basic building block is the Ising model, whose primary fields are $\mathbf{1}$, ψ and σ, with conformal weights 0, $\frac{1}{2}$ and $\frac{1}{16}$ respectively. The field ψ is a simple current, and σ is a fixed point of this current. If one tensors many Ising models, there is a huge number of modular invariant partition functions that can be written down as explained above.

However, this is still not all. In the tensor product of two Ising models one can extend the chiral algebra with the spin-1 current $\psi\psi$. This operator has a fixed point $\sigma\sigma$, which can easily be resolved into two fields. It turns out that these two fields are simple currents themselves, and that the resulting $c = 1$ theory is simply the D_1 lattice. In tensor products one may now combine these extra simple currents with each other or with other fields ψ, to get a much larger set of modular invariants. Fixed point resolution is needed in general if one tensors $2n$ Ising models and uses a current ψ^{2n}, but only for $n = 1$ the fixed point yields new simple currents.

The resulting set of partition functions will probably coincide with those obtained by different methods in [31] and [32], for fermions with periodic and anti-periodic boundary conditions. Of course one cannot discuss within this framework more general boundary conditions; this requires an extension of the conformal field theory with extra fields.

8.5 Selection rules

An integer spin invariant always has the property that certain fields of the original theory do not appear. If the chiral algebra of this invariant is built out of simple currents, the fields that do not appear all have a non-zero charge with respect to one or more of these currents. The converse is also true: Suppose one has a modular invariant partition function that satisfies a certain charge selection rule, so that all fields that appear have zero charge with respect to some set of simple currents. Then those simple currents must appear in the spectrum (for a proof see [33]).

Note that in formulating the converse we did not require that the simple currents have integral conformal spin. This is not necessary for invariance under S, but fractional spin currents would violate invariance under T. Thus in attempting to build a modular invariant that satisfies a certain selection

rule one may encounter a contradiction, showing that such a modular invariant cannot exist.

For example, suppose one wishes to build an invariant in which only the zero-triality representations of $SU(3)$, level 1 appear. Then the simple current of $SU(3)$ must appear as a chiral current in the spectrum. However, this current has conformal spin $\frac{1}{3}$, and cannot appear. Hence there is no such partition function. Note that this argument is completely independent of any other factors that might appear in the theory besides $SU(3)$.

A more interesting example is provided by the standard model of the strong and electro-weak interactions. The presently known spectrum belongs to representations of $SU(3)\times SU(2)\times U(1)$, with a certain correlation between $SU(3)$-triality, the integrality of the $SU(2)$ spin, and the $U(1)$ charge. This selection rule corresponds to the observed quantization of electric charge. Using the foregoing argument one can now investigate under which conditions this selection rule can be realized in string theory [33].

8.6 Projections in string theory

Consistent four-dimensional superstring models – or at least those that are presently known – are based on modular invariant partition functions of some conformal field theory, which must satisfy some additional requirements. The additional requirements include world-sheet and space-time supersymmetry. The task of building partition functions satisfying simultaneously all these requirements is made straightforward if one uses simple currents.

The main problem one usually has with world-sheet supersymmetry is that one has to tensor two (or more) conformal field theories with $N = 1$ supersymmetry in such a way that the product also has that property. The two theories may be a Neveu-Schwarz-Ramond model and an $N = 1$ internal sector, or two components of the internal sector. The product has a supercurrent which is the sum of the supercurrents of the components, and all fields should have a well-defined monodromy of either Neveu-Schwarz (NS) or Ramond (R) type with respect to the total current. This means that only combinations of the two NS-fields or two R-fields of the two original theories may appear, whereas mixed combinations should be removed.

This is precisely a selection rule of the type discussed in the previous subsection. An $N = 1$ superconformal field theory is nothing but a conformal field theories which has a simple current of spin-$\frac{3}{2}$ and order 2. All fields have charges 0 or $\frac{1}{2}$ with respect to this current, which divides them into Neveu-Schwarz and Ramond states respectively. These states are organized into orbits by the spin-$\frac{3}{2}$ currents, and these orbits are the supermultiplets. The selection rule on tensor products is that the sum of the charges of the two factors should be an integer for all combinations of fields that appear. To impose this one can use the product of the simple currents of the two factors, which is a spin-3 current. If one puts this current in the chiral algebra, the selection rule is automatically satisfied. Furthermore this is the only way to satisfy it, since the selection rule requires the product of the supercurrents to be present in the chiral algebra. If one has a tensor product consisting of more than two factors, one should do this for all pairs.

A necessary and sufficient condition for space-time supersymmetry (assuming all other consistency conditions are satisfied) is simply that there should be a gravitino in the spectrum. This is equivalent to requiring that the chiral algebra should contain a spin-1 operator which transforms as a spinor representation of the NSR-model. The combination of the requirements of world-sheet and space-time supersymmetry has the interesting consequence that the $N = 1$ world-sheet supersymmetry extends to $N = 2$ [34]. An easy way of seeing this is to use the bosonic string map [35], which replaces the fermionic sector(s) of four-dimensional heterotic (type-II) strings by a bosonic ones, by mapping the NSR model to $SO(10) \times E_8$, while preserving modular invariance (for more details see [28]).

In this language a heterotic string is described by a conformal field theory $(C_{22})_L \times (SO(10) \times E_8 \times C_9)_R$, where C_n is a conformal field theory with central

charge n. As a first step, we require the combination to be modular invariant, and furthermore C_9 should have a simple current T of spin $\frac{3}{2}$, so that it has $N = 1$ supersymmetry. Now, as explained above, we combine this simple current with the vector v of $SO(10)$ (which has the same monodromies as the space-time supercurrent) and put this combination vT in the chiral algebra. The spin-1 operator needed to get space-time supersymmetry is necessarily built out of the spin field of $SO(10)$ and a Ramond ground state (with $h = \frac{3}{8}$) of C_9. Although the $SO(10)$ spin field s is a simple current, a Ramond ground state R_0 in general is not, and hence it is not *a priori* obvious how to achieve our goal. However, suppose there exists a modular invariant partition function which has such a spin-1 operator sR_0 in the chiral algebra. Then, since sR_0 is an $SO(10)$ spinor, its presence implies that $SO(10)$ is embedded in E_6 (or E_7, E_8). This implies in its turn that the chiral algebra contains a spin-1 field H which is a singlet of $SO(10)$ (and hence lives entirely within C_9) namely the extra Cartan subalgebra generator. It is not difficult to show using the E_6 vertex operator algebra that H extends the $N = 1$ algebra to $N = 2$. This possibility to extend the algebra is a genuine property of C_9 by itself. Indeed, one can use the simple current s of $SO(10)$ (in combination with an operator from the left-moving sector C_{22} with the same conformal weight, up to integers*) to project out the combinations vT and sR_0 and decouple $SO(10)$ completely from C_9. The resulting C_9 partition function still has the operator H in its chiral algebra, and hence has $N = 2$ supersymmetry.

In a theory with $N = 2$ world-sheet supersymmetry at least one of the Ramond ground states is a simple current, namely the one connected to the identity by "spectral flow". If one combines it with the NSR spin field to build a spin-1 simple current, it is trivial to add this current to the chiral algebra without destroying modular invariance.

Thus if one starts with an internal sector that already has $N = 2$ world-sheet supersymmetry, then the entire procedure for constructing supersym-

⋆ Such an operator certainly exists if we take the left-moving sector identical to the right-moving one.

metric string theories is a straightforward application of simple currents, and the proof given in section 4 guarantees modular invariance no matter which $N = 2$ theory one uses. On the other hand, one may start with an $N = 1$ internal sector, and build first an off-diagonal partition function of that theory which has $N = 2$ supersymmetry. This is only possible if the $N = 1$ theory has a spin-1 operator H that can be put in the chiral algebra. A sufficient condition is that H is a simple current (for example the $N = 1, c = 1$ theory has such a field, which extends it to the $N = 2, c = 1$ theory). Of course there might also exist "exceptional" invariants, in which H appears in the chiral algebra without being a simple current, but we don't know any examples. Having constructed an $N = 2$ theory from the $N = 1$ theory, one proceeds as before.

This description applies to all heterotic and type-II theories constructed so far. Historically this construction has been carried out first for theories built out of free bosons or fermions, where the projections are fairly straightforward. Models built with more general superconformal field theories were first discussed in [36], where it was shown how to make the space-time and world-sheet supersymmetry projections for tensor products of $N = 2$ minimal models. Although making the projections appears more difficult in this case, it is in fact as easy as for "free" theories if one uses simple currents.

The supersymmetry projections can be applied to any modular invariant of the theory. Tensor products of $N = 2$ minimal models for example have a huge number of modular invariant partition functions, most of which can be obtained by means of simple currents [30]. Although in the simplest cases the projections on the right-moving sector automatically act on the left-moving one as well, this is not necessary. By multiplying several matrices M (see section 8.3) one obtains easily examples of $(2,1)$ and $(2,0)$ models, which are far more numerous than the $(2,2)$ models. A list of most of the $(2,2)$ and $(2,1)$ models that can by obtained from tensor products of minimal $N = 2$ theories by means of simple currents is available [37] (the total number of such models is about 10^4).

8.7 Coset theories

The set of primary fields of a coset conformal field theory G/H is labelled by pairs (Λ, λ), where Λ represents a primary field of G and λ one of H [17]. The conformal weight of such a field is $h_\Lambda - h_\lambda$ mod 1. If all such pairs of G and H representations do indeed correspond to primary fields of the coset theory the discussion is very simple. For all purposes related to modular invariance and fusion rules one can then view the coset theory as a tensor product of the Kac-Moody algebra G with the complement of the Kac-Moody algebra H. In particular the modular transformation matrix S is then given by $S^G_{\Lambda\Lambda'}(S^H_{\lambda\lambda'})^*$. The fusion rules are obtained by trivially combining the G and H fusion rules.

However, this simple description is correct in only very few cases. In most cases not all pairs of G and H representations yield a primary field in the coset theory. The reason is that there may be Lie-algebra selection rules forbidding the appearance of the H-representation λ in the decomposition of the G-representation Λ [38], [39], [40].*

The results of the previous subsection tell us exactly how to deal with this. The selection rules of Lie-algebra decompositions are always related to maps between the centers of G and H. If one thinks of the coset theory in terms of $G \times H^c$ this implies that only fields with zero charge with respect to a set of simple currents can exist. To write down modular invariant partition functions that only involve fields that exist, one has to make sure that all these simple currents appear in the chiral algebra. We call these currents the "identification currents" of the theory, for reasons that will become clear below.

There is an important difference between a coset theory G/H and the tensor product $G \times H^c$. Although both theories have the same matrices S and T, and hence the same fusion rules, they do not have the same characters. The

* We will assume that this is the only reason why a pair of G and H representations can be absent. Possible caveats are discussed in [5]. Most probably the only exceptions to this assumption are trivial coset theories with $c = 0$, *i.e.* conformal embeddings.

characters of the coset theory are (at least to first approximation) the branching functions, which can be computed by decomposing the G-representations level-by-level into H-representations:

$$\mathcal{X}_\Lambda(\tau) = \sum_\lambda b_\lambda^\Lambda(\tau)\mathcal{X}_\lambda(\tau) .$$

Obviously the branching function b_λ^Λ vanishes if λ never appears in the decomposition of Λ. Of course the characters of $G \times H^c$ are simply the product of those of G and H^c, and exist for any pair of labels.

The fact that branching functions that do not satisfy the selection rule vanish can be used to show that all characters of fields that *do* satisfy the selection rule are identical along the orbits of the identification currents. This is known as ***field identification***, and is discussed in [39], [38], [40] and [5]. Note that in particular the currents themselves have characters equal to the identity character, and hence they have conformal spin 0. The identification currents are thus identified with the identity.

Therefore, unlike the "isomorphic" modular invariant of $G \times H^c$, we cannot simply regard the invariant generated by the identification currents as an ordinary integer spin invariant. Since all the identification currents are to be regarded as copies of the identity, the result is that the identity appears more than once. Since the same identification takes place on all other orbits as well, this is still no problem if all orbits have the same length. Then one can simply divide the entire partition function by the multiplicity of the identity without generation fractional coefficients. The S and T matrices and the fusion rules are not exactly those of $G \times H^c$, but they are trivially related to them, by projecting on the orbits of non-vanishing branching functions.

The latter situation exists in several coset theories, such as $SU(2)_k \times SU(2)_1/SU(2)_{k+1}$, but in general we know that the simple currents of Kac-Moody algebras have fixed points, so that very often the identification currents

have fixed points as well. This leads to two complications. The first is that the matrix S of the coset theory cannot be straightforwardly derived from those of G and H. One has to resolve the fixed points, thus introducing an entirely new matrix into the problem. This has already been discussed in section 6, and the discussion there applies directly to coset theories (see also [38]).

We want to emphasize strongly that in general ***the fusion rules of a coset theory are not simply the "tensor product" of those of G and H***, despite claims in the literature. The proper derivation requires knowledge of the fixed point conformal field theories. In most cases these are Kac-Moody algebras, and hence their modular matrices S are known explicitly. Then one has to go through the construction of section 6 to obtain the correct matrix S of the coset theory, and then one can finally compute the correct fusion rules using Verlinde's formula. Perhaps a simpler calculation method can be found, but in any case the fusion rules that involve fixed points are not simply those of G and H combined.

The second complication introduced by the presence of fixed points is that it makes the normalization of the identity term impossible, since the fixed point fields would appear with fractional coefficients. Unlike the previous problem, this one is typical for coset theories and does not arise in ordinary integer spin modular invariants. This problem was pointed out in [40], and a solution was proposed in [5]. It turns out that one has the freedom to modify the fixed point branching functions by characters of the fixed point conformal field theory. In non-trivial examples we have found that it was possible to choose these modifications in such a way that the partition function can be normalized properly. These modified characters are the proper characters of the coset theory, and the fixed point branching functions are not. This still has to be proved in general, but it is likely to be true since it is the only way out of the dilemma.

All this can be summarized as follows. For the purpose of discussing modular invariant partition functions and fusion rules, one can mimic a coset theory by the tensor product $G \times H^c$. The selection rules imply that only a special subset of the modular invariant partition functions of this tensor product should be considered, namely those in which the identification currents are in the extended algebra. Fixed points are to be resolved in the usual way. However, for the description of the characters this is not adequate. The description of characters in terms of branching functions is closer to the truth, but still not correct if there are fixed points. The true characters are obtained by modifying the appropriate branching functions by the characters of the fixed point CFT.

Finally a remark regarding the simple currents of a coset theory. To find them all, one should as a first step list all possible combinations of simple currents of G and H^c. Because of the selection rule, some of these currents are eliminated. Because of the identification, some others are identified with each other or with the identity. The ones that remain (*i.e.* one representative of each identification orbit) are simple currents of the coset theory. There may be more if there are fixed points: after the resolution of fixed points, some of the fixed point fields may have become simple currents, as we have seen in examples earlier in this section.

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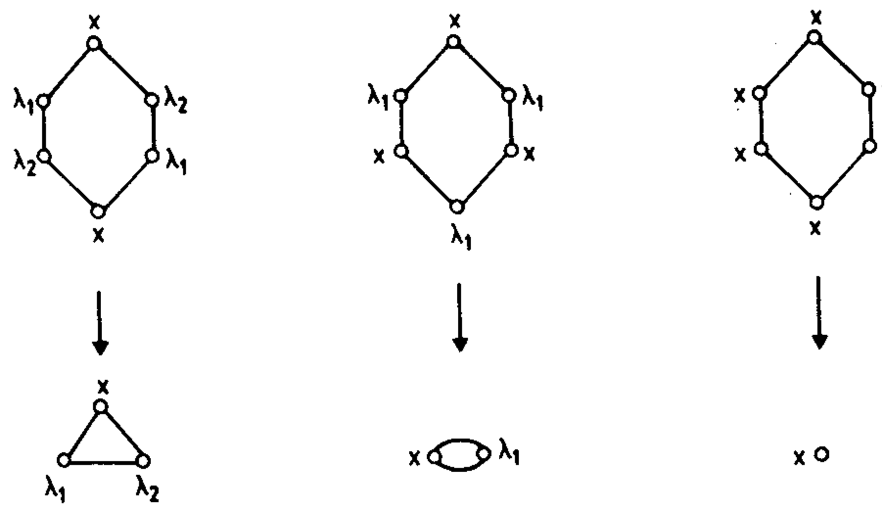
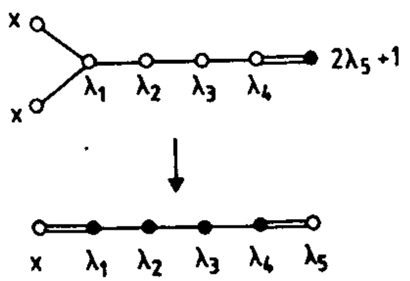
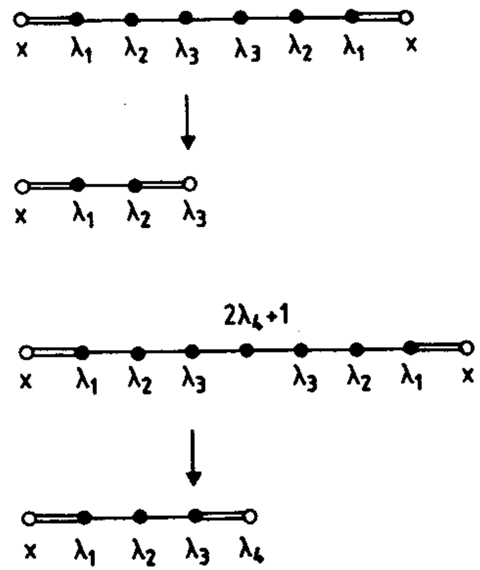


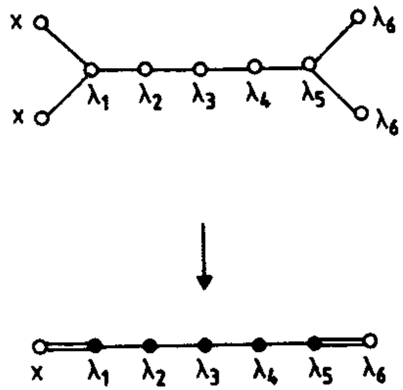
1. Fixed points of $SU(N)$ and their relation to the unitary highest weight representations of the fixed point conformal field theory. The example shows $SU(6)$ level k with currents J_3, J_2 and J respectively. The fixed point conformal field theories are $SU(3)$ level $\frac{k}{2}$, $SU(2)$ level $\frac{k}{3}$ and the identity, if k is divisible by 2,3 and 6 respectively.



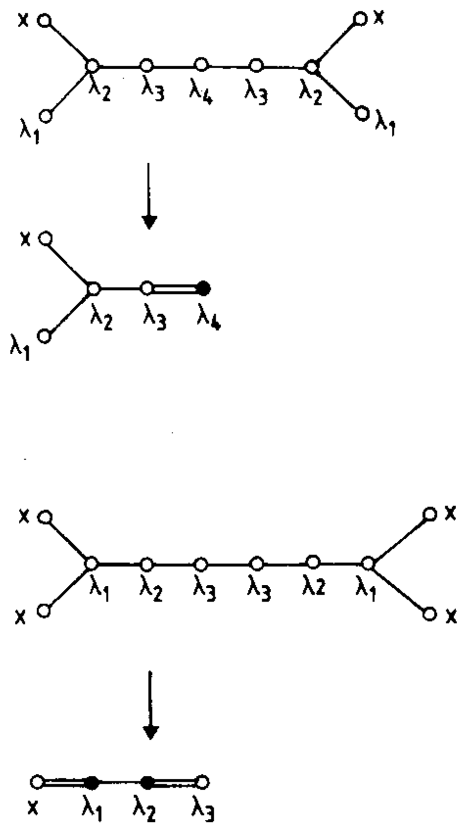
2. Same as Fig. 1 for B_n, odd levels. See Table I for more details.



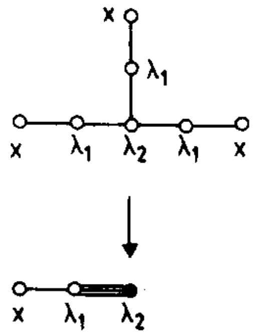
3. Same as Fig. 1 for C_{2n+1}, even levels (top) and C_{2n}, odd levels (bottom)



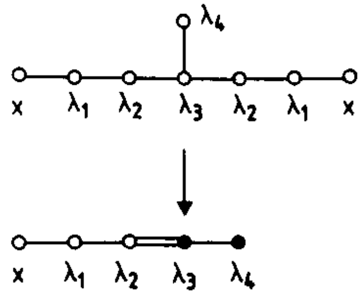
4. Same as Fig. 1 for D_n, even levels, with the current J_v.



5. Same as Fig. 1 for D_{2n}, even levels, with the current J_s (top) and D_{2n+1} with level $4p$ and current J_s (bottom).



6. Same as Fig. 1 for E_6, level $3p$.



7. Same as Fig. 1 for E_7, level $2p$.