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# SIMPLE $G$-GRADED ALGEBRAS AND THEIR POLYNOMIAL IDENTITIES

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Abstract. Let $G$ be any group and $F$ an algebraically closed field of characteristic zero. We show that any $G$-graded finite dimensional associative $G$-simple algebra over $F$ is determined up to a $G$-graded isomorphism by its $G$-graded polynomial identities. This result was proved by Koshlukov and Zaicev in case $G$ is abelian.

## Introduction

The purpose of this article is to prove that finite dimensional (associative) simple $G$-graded algebras over an algebraically closed field $F$ of characteristic zero are determined up to $G$-graded isomorphism by their $G$-graded identities. Here $G$ is any group. In case $G$ is abelian, the result was established by Koshlukov and Zaicev [9]. Analogous results were obtained for Lie algebras by Kushkulei and Razmyslov [8] and for Jordan algebras by Drensky and Racine [6] and recently for nonassociative algebras by Shestakov and Zaicev [10].

The structure theory of finite dimensional $G$-graded algebras and in particular of simple $G$-graded algebras plays a crucial role in the proof of the representability theorem for $G$-graded PI algebras and in the solution of the Specht problem (that is, that the $T$-ideal of $G$-graded identities is finitely based) for such algebras (see Aljadeff and Kanel-Belov [2]).

Recall that the representability theorem for $G$-graded algebras says in particular that if $W$ is an affine $G$-graded algebra which is PI as an ordinary algebra, then there exists a finite dimensional algebra $A$ which satisfies precisely the same $G$-graded identities as $W$.

A fundamental part of the proof of the representability theorem is the construction of special finite dimensional $G$-graded algebras which are called basic. It turns out that if $B$ is a basic algebra, then $B$ admits $G$-graded polynomials which are called Kemer. These are multilinear polynomials, nonidentities, which admit alternating sets of homogeneous elements of degree $g \in G$ whose cardinalities are maximal possible [2]. In the special case where the basic algebra has no radical, it is in fact $G$-simple, and in that case, the cardinalities of the alternating sets in a Kemer polynomial coincide with the dimensions of the homogeneous components. Clearly, no nonidentity polynomial of a $G$-simple algebra (or in fact of any $G$-graded algebra) can have larger alternating sets.

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The key point in the proof of representability is that the finite dimensional algebra $A$ which satisfies the same $G$-graded identities as $W$ can be expressed as the direct sum of basic algebras and hence the $T$-ideal of $G$-graded identities of $A$ is the intersection of the corresponding ideals of identities of the basic algebras which appear in the decomposition. However, the basic algebras that appear in the decomposition of $A$ are not known to be unique. In fact, even the basic algebras themselves are not determined in general by their identities (as a result of the interaction of the simple components via the radical). But if the basic algebra is $G$-semisimple (and hence $G$-simple), the main result of the paper says that in that case the answer is positive.

To state the result precisely we recall some basic definitions. Let $k$ be an arbitrary field and let $G$ be a group. A $k$-algebra $A$ is said to be $G$-graded if for each $g \in G$ there is a $k$-subspace $A_g$ of $A$ (possibly zero) such that for all $g, h \in G$, we have $A_g A_h \subseteq A_{gh}$. Such a $G$-graded algebra is said to be a simple $G$-graded algebra (or a $G$-simple algebra) if there are no nontrivial homogeneous ideas, or equivalently if the ideal generated by each nonzero homogeneous element is the whole algebra.

A $G$-graded polynomial is a polynomial in the free algebra $k\langle X_G \rangle$ where $X_G$ is the union of sets $X_g$, $g \in G$ and $X_g = \{x_{1,g}, x_{2,g}, ...\}$. In other words, $X_G$ consists of countably many variables of degree $g$ for every $g \in G$. We say that a polynomial $p(x_{1,g_{i_1}}, ..., x_{n,g_{i_n}})$ in $k\langle X_G \rangle$ is a $G$-graded identity of a $G$-graded algebra $A$ if $p$ vanishes for every graded evaluation on $A$. The set of $G$-graded identities of $A$ is an ideal of $k\langle X_G \rangle$ which we denote by $Id_G(A)$. Moreover it is a $T$-ideal, that is, it is closed under $G$-graded endomorphisms of $k\langle X_G \rangle$.

It is known that if $k$ has characteristic zero, the $T$-ideal of identities is generated as a $T$-ideal by multilinear polynomials, that is, graded polynomials whose monomials are a permutation of each other (up to a scalar from the field). Moreover we may assume in addition that all of the monomials have the same homogeneous degree. We can now state the main result of the paper.

**Theorem.** *Let $A$ and $B$ be two finite dimensional simple $G$-graded algebras over $F$ where $F$ is an algebraically closed field of characteristic zero. Then $A$ and $B$ are $G$-graded isomorphic if and only if $Id_G(A) = Id_G(B)$.*

A key ingredient in the proof is the result of Bahturin, Sehgal and Zaicev ([5], Theorem 1.1) that determines the structure of a simple $G$-graded algebra as a combination of a fine graded algebra and an elementary graded algebra. In section 1 we state this result and use it to define the notion of a presentation of the given $G$-simple algebra. It is a consequence of our main theorem that any two graded isomorphic $G$-simple algebras have equivalent presentations. However one can prove this uniqueness result directly, without the use of identities, and we present such a proof in the last section of this paper.

Another motivation for studying $G$-graded polynomial identities of finite dimensional $G$-simple algebras is the possible existence of a "versal" object. It is well known that if $A$ is the algebra of $n \times n$-matrices, the corresponding algebra of generic elements has a central localization which is an Azumaya algebra and is versal with respect to all $k$-forms (in the sense of Galois descent) of $A$ where $k$ is any field of characteristic zero. Furthermore, extending the center to the field of fractions, one obtains a division algebra, the so-called generic division algebra, which is a form of $A$. The algebra of generic elements can be constructed in a different way.

It is well known that it is isomorphic to the relatively free algebra of $A$, namely, the free algebra on a countable set of variables modulo the $T$-ideal of identities.

Given a $G$-graded finite dimensional algebra one can construct the corresponding $G$-graded relatively free algebra, and it is of interest to know whether there exists a versal object in this case as well. It turns out that this is so for some specific cases as in [1], [3] and [4].

Clearly, if two nonisomorphic finite dimensional $G$-simple algebras $A$ and $B$ had the same $T$-ideal of identities, there could not exist a versal object for $A$ (or $B$). So in view of our main theorem, it is natural to ask whether for an arbitrary finite dimensional $G$-simple algebra there exists a corresponding versal object.

## 1. Preliminaries

We start by recalling some terminology. Let $G$ be any group and $A$ a finite dimensional simple $G$-graded algebra. As mentioned in the introduction, our proof is based on a result of Bahturin, Sehgal and Zaicev [5] in which they present any finite dimensional $G$-graded simple algebra by means of two types of $G$-gradings, fine and elementary. Before stating their theorem let us give two examples, one of each kind.

Given a finite subgroup $H$ of $G$ we can consider the group algebra $FH$ with the natural $H$-grading. This algebra is $H$-simple, in fact an $H$-division algebra in the sense that every nonzero homogeneous element is invertible. Moreover we can view the algebra $FH$ as a $G$-graded algebra where the $g$-homogeneous component is set to be 0 if $g$ is not in $H$. More generally we may consider any twisted group algebra $F^{\alpha}H$, where $\alpha$ is a 2-cocycle of $H$ with invertible values in $F$, again as a $G$-graded algebra. As in the case where the cocycle is trivial, the algebra $F^{\alpha}H$ is a finite dimensional $G$-division algebra. We refer to such a grading as a fine grading. The second type of grading is called elementary. Let $M_r(F)$ be the algebra of $r \times r$ matrices over the field $F$. Fix an $r$-tuple $(p_1, \ldots, p_r) \in G^{(r)}$, and assign the elementary matrix $e_{i,j}$, $1 \leq i, j \leq n$, the homogeneous degree $p_i^{-1}p_j$. Note that the product of the elementary matrices is compatible with their homogeneous degrees and so we obtain a $G$-grading on $M_r(F)$. Furthermore, since $M_r(F)$ is a simple algebra it is also $G$-simple.

The result of Bahturin, Sehgal and Zaicev [5] is the graded version of Wedderburn's structure theorem for finite dimensional simple algebras. Their result says that every finite dimensional $G$-simple algebra is isomorphic to a $G$-graded algebra which is the tensor product of two $G$-simple algebras, one with fine grading (hence a graded division algebra) and the other a full matrix algebra with an elementary grading. Here is the precise statement.

**Theorem 1.1** ([5]). *Let $A$ be a finite dimensional $G$-simple algebra over an algebraically closed field $F$ of characteristic zero. Then there exists a finite subgroup $H$ of $G$, a 2-cocycle $\alpha : H \times H \to F^*$ where the action of $H$ on $F$ is trivial, an integer $r$ and an $r$-tuple $(p_1, p_2, \ldots, p_r) \in G^{(r)}$ such that $A$ is $G$-graded isomorphic to $C = F^{\alpha}H \otimes M_r(F)$ where $C_g = span_F\{u_h \otimes e_{i,j} : g = p_i^{-1}hp_j\}$. Here $u_h \in F^{\alpha}H$ is a representative of $h \in H$ and $e_{i,j} \in M_r(F)$ is the $(i,j)$ elementary matrix.*

*In particular the idempotents $1 \otimes e_{i,i}$ as well as the identity element of $A$ are homogeneous of degree $e \in G$.*

**Definition 1.2.** Given a finite dimensional $G$-simple algebra $A$, let $H$, $\alpha \in Z^2(H, F^*)$ and $(p_1, \ldots, p_r) \in G^{(r)}$ be as in the theorem above. We denote the triple $(H, \alpha, (p_1, \ldots, p_r))$ by $P_A$ and refer to it as a presentation of the $G$-graded algebra $A$. We will refer to $r$ as the matrix size of $P_A$.

Clearly, a presentation determines the $G$-graded structure of $A$ up to a $G$-graded isomorphism. On the other hand, a $G$-graded algebra may admit more than one presentation and so we need to introduce a suitable equivalence relation on presentations.

We start by establishing some conditions on presentations which yield $G$-graded isomorphic algebras.

**Lemma 1.3.** *Let $A$ be a finite dimensional $G$-simple algebra with presentation $P_A = (H, \alpha, (p_1, \ldots, p_r))$. The following "moves" (and their composites) on the presentation determine $G$-graded algebras $G$-graded isomorphic to $A$.*

(1) *Permuting the $r$-th tuple, that is $A' \cong F^{\alpha_A} H_A \otimes_F M_r(F)$, and the elementary grading is given by $(p_{\pi(1)}, \ldots, p_{\pi(r)})$ where $\pi \in Sym(r)$.*
(2) *Replacing any entry $p_i$ of $(p_1, \ldots, p_r)$ by any element $h_0 p_i \in H p_i$ (changing right $H$-coset representatives).*
(3) *For an arbitrary $g \in G$,*
  (a) *replacing $H$ with the conjugate $H^g = gHg^{-1}$,*
  (b) *replacing the cocycle $\alpha$ by $\alpha^g$ where*
$$\alpha^g(gh_1g^{-1}, gh_2g^{-1}) = \alpha(h_1, h_2)$$
  *and*
  (c) *shifting the tuple $(p_1, \ldots, p_r)$ by $g$, that is, replacing the tuple $(p_1, \ldots, p_r)$ by $(gp_1, \ldots, gp_r)$.*

*Proof.* We describe the isomorphism maps.

(1)
$$u_h \otimes e_{k,l} \longmapsto u_h \otimes e_{\pi(k),\pi(l)}.$$
(2)
$$u_h \otimes e_{k,l} \longmapsto u_h \otimes e_{k,l}$$
if $k \neq i$ and $l \neq i$.
$$u_h \otimes e_{i,l} \longmapsto u_{h_0} u_h \otimes e_{i,l}$$
if $l \neq i$.
$$u_h \otimes e_{k,i} \longmapsto u_h u_{h_0}^{-1} \otimes e_{k,i}$$
if $k \neq i$.
$$u_h \otimes e_{i,i} \longmapsto u_{h_0} u_h u_{h_0}^{-1} \otimes e_{i,i},$$
(3)
$$u_h \otimes e_{k,l} \longmapsto u_{ghg^{-1}} \otimes e_{k,l}.$$
We leave the reader the task of showing that these maps are indeed isomorphisms. ☐

We will call these isomorphisms basic moves of type (1), (2), or (3). We will call presentations $P_A$ of the $G$-simple algebra $A$ and $P_B$ of the $G$-simple algebra $B$ equivalent if one is obtained from the other by a (finite) sequence of basic moves. This is clearly an equivalence relation on presentations. It follows from the lemma that algebras with equivalent presentations are $G$-graded isomorphic.

Let $A$ be $G$-simple with presentation $P_A$. Our proof requires, in terms of the given presentation $P_A$, a rather precise understanding of the structure of the subalgebra $A_N = \sum_{g\in N} A_g$ (of $A$) where $N$ is an arbitrary subgroup of $G$. To this end we introduce an equivalence relation on the elements of the $r$-tuple $(p_1, \dots, p_r)$: We will say $i, j \in \{1, \dots, r\}$ are $N$-*related* in $P_A$ if there exists $h \in H_A$ such that $p_i^{-1}hp_j \in N$. It is easy to see that this is indeed an equivalence relation. We may assume (after permuting the elements of the tuple $(p_1, \dots, p_r)$ if needed) that the tuple is decomposed into subtuples whose elements are the corresponding equivalence classes. We denote the classes by $(p_{i_1}, p_{i_1+1}, \dots, p_{i_1+k_1-1})$, $(p_{i_2}, p_{i_2+1}, \dots, p_{i_2+k_2-1})$,$\dots$, $(p_{i_d}, p_{i_d+1}, \dots, p_{i_d+k_d-1})$.

In order to get a better understanding of the $N$-elements in the presentation $P_A$, we focus our attention on one equivalence class, say $(p_{i_1}, p_{i_1+1}, \dots, p_{i_1+k_1-1})$, and so, for convenience we change the notation by letting $k = k_1$ and setting $(g_1, \dots, g_k) = (p_{i_1}, p_{i_1+1}, \dots, p_{i_1+k_1-1})$. We let $A_{N,1}$ denote the $F$-space spanned by the elements $u_h \otimes e_{i,j}$ where $i, j \in \{1, \dots, k\}$ and $g_i^{-1}hg_j$ is in $N$.

For $i = 1, \dots, k$ we consider the following subgroup of $N$:

$$\Omega_{g_i} = g_i^{-1}Hg_i \cap N$$

and let $d_i$ be its order.

**Proposition 1.4.** *With the notation as above, the following hold.*

(1) *For $1 \le i, j \le k$ the subgroups $\Omega_{g_i}$ and $\Omega_{g_j}$ are conjugate to each other by an element of $N$. In particular $d_i = d_j$.*

(2) *For $i, j \in \{1, \dots, k\}$ the set*

$$g_i^{-1}Hg_j \cap N$$

*is a left $\Omega_{g_i}$-coset and a right $\Omega_{g_j}$-coset. In particular the order of*

$$g_i^{-1}Hg_j \cap N$$

*is $d_i (= d_j)$.*

(3) *The subalgebra $A_{N,1}$ is $G$-simple with presentation*

$$P_{A_{N,1}} = (N \cap g_1^{-1}Hg_1, g_1(\alpha), (n_1, \dots, n_k))$$

*for some elements $n_1, \dots, n_k$, where $n_j \in N \cap g_1^{-1}Hg_j$.*

*Proof.* This is straightforward. We will prove only the first statement. By the equivalence condition, there are elements $h \in H$ and $n \in N$ such that $g_i^{-1}hg_j = n$. Hence

$$\begin{aligned}\Omega_{g_i} &= g_i^{-1}Hg_i \cap N\\ &= ng_j^{-1}h^{-1}Hhg_jn^{-1} \cap N\\ &= n(g_j^{-1}Hg_j \cap N)n^{-1}\\ &= n(\Omega_{g_i})n^{-1}\end{aligned}$$

as desired. □

*Remark* 1.5. Based on the presentation of the $N$-simple algebra above, we see that the appearance of an $N$-simple component constitutes of a diagonal block of the $r \times r$-matrix algebra. We will refer to the number $d_i$ as the number of *pages* in that component. So each $N$-simple component sits on the diagonal with a certain matrix size and a certain number of pages.

## 2. Proofs

Our aim is to show that algebras $A$ and $B$ (finite dimensional and $G$-simple) with nonequivalent presentations $P_A$ and $P_B$ have different $T$-ideals of $G$-graded identities and hence are $G$-graded nonisomorphic. This will imply

(1) $G$-graded (finite dimensional) $G$-simple algebras $A$ and $B$ are $G$-graded isomorphic if and only if any two presentations $P_A$ and $P_B$ are equivalent.
(2) $G$-graded (finite dimensional) $G$-simple algebras are characterized (up to $G$-graded isomorphism) by their T-ideal of $G$-graded identities.

*Remark* 2.1. In section 3 we will give a proof of statement (1) that does not depend on identities.

Generally speaking, we proceed step by step where in each step we show that if $A$ and $B$ satisfy the same $G$-graded identities, then the presentations $P_A$ and $P_B$ must coincide on certain "invariants/parameters" up to applications of basic moves.

Let us start by exhibiting a list of such invariants of a presentation

$$P_A = (H_A, \alpha, (p_1, \dots, p_r))$$

of an algebra $A$.

(1) The dimensions of the homogeneous components $A_g$, for all $g \in G$ (and so, in particular, the dimension of $A$).
(2) The multiplicities of right $H$-coset representatives in the $r$-tuple $(p_1, \dots, p_r)$.
(3) The order of $H$.
(4) The group $H$ up to conjugation.
(5) The group $H$.

Based on (5), for the rest of the invariants we will assume the subgroup $H$ is determined. The next sequence of invariants is determined by the $r$-tuple $T = (p_1, \dots, p_r)$. We decompose $T$ into subtuples where each subtuple consists of all elements in $(p_1, \dots, p_r)$ lying in the same right coset $N(H)g$ of the normalizer of $H$ in $G$.

Let us denote the full tuple by $T$ and the subtuples by

$$T_1 e, T_2 g_2, \dots, T_k g_k.$$

Each $T_i$ consists of representatives $\sigma_{i,j}$ of $H$ in $N(H)$ with multiplicity $d_{i,j}$.

(6) The vector of multiplicities of representatives in each $T_i$.
(7) The coset representatives $\{g_1 = e, g_2, \dots, g_k\}$ of $N(H)$ in $G$ that appear in the tuples, with multiplicities.
(8) The elements of $T$ up to left multiplication by an element of $N(H)$. Note that by the basic moves this determines the presentation up to the 2-cocycle on $H$.

For the rest of the invariants we will assume the subgroup $H$ and the tuple $T$ are determined.

(9) The 2-cocycle on $H$ up to conjugation by an element of $N(H)$.

For each element $t_{i,j} \in T_i$ we consider the cocycle on $H$ obtained by conjugation of $\alpha$ by $t_{i,j}^{-1}$ (note that conjugating with $t_{i,j}g_i$ gives a cocycle on $H^{g_i^{-1}}$). Then each $T_i$ determines a set of cocycles (on $H$).

(10) The set of cocycles (with multiplicities!) as determined by the elements of $T_i$.

Then finally

(11) The presentation $P_A$ of $A$.

We will refer to this list of steps as the outline of the proof.

Let $A$ and $B$ be $G$-graded algebras, finite dimensional $G$-simple with presentations

$$P_A = (H_A, \alpha, (p_1, \dots, p_r))$$

and

$$P_B = (H_B, \beta, (q_1, \dots, q_s)).$$

Suppose $A$ and $B$ satisfy the same $G$-graded identities. Our task will be to add (in each step) an invariant from the list above on which the presentations $P_A$ and $P_B$ must coincide (up to basic moves). The basic strategy is to establish a suitable connection between the invariants described above and the structure of some extremal $G$-graded nonidentities of $A$. But more than that, the polynomials we construct will establish a strong connection between the invariants and the structure of *any* nonzero evaluation of them (with a suitable basis).

*Remark* 2.2. Given a presentation $P_A$ of an algebra $A$, it is well known that in order to test whether a $G$-graded multilinear polynomial is an identity of $A$ it is sufficient to consider evaluations on any $G$-graded basis of $A$ and so, from now on, *we always choose the basis consisting of the elements* $u_h \otimes e_{i,j}$, for all $h \in H$ and all $i, j$. We will refer to this basis as the standard basis for $A$ (of course it really depends on the presentation of $A$). This will play a key role in the proof since the *connection* we make via nonzero evaluations between the structure of $A$ and the structure of the polynomials will be based precisely on that particular $G$-graded basis of $A$. In particular, all subspaces we consider will be linear spans of subsets of that basis.

We want to be more precise about what we mean by polynomials that establish a strong connection between their nonzero evaluations and the $G$-graded structure of $A$. Let $V = \oplus_g V_g \subseteq \oplus_g A_g$ be a $G$-graded subspace of $A$. Let $d_g = \dim_F(A_g)$ and $\delta_g = \dim_F(V_g)$, $g \in G$. We say that a multilinear $G$-graded polynomial $p$ *allocates* the $G$-graded subspace $V$ of $A$ if the following hold:

(1) $p = p(Z_G)$ is obtained from a single multilinear monomial $Z_G$ by homogeneous multialternation. This means that we choose disjoint sets of homogeneous variables in $Z_G$ (each set constitutes elements of the *same* homogeneous degree in $G$) and we alternate the elements of each set successively.
(2) For every $g \in G$ with $V_g \neq 0$, we have a subset $T_g$ of $g$-variables in $Z_G$ of cardinality $d_g$, and a subset $S_g$ of $T_g$ of cardinality $\delta_g$.
(3) The set $T_g$ is alternating on $p(Z_G)$, for every $g$ with $V_g \neq 0$.
(4) $p(Z_G)$ is a $G$-graded nonidentity of $A$.
(5) If $\phi$ is *any* nonzero evaluation of $p(Z_G)$ on $A$ (with elements of the form $u_h \otimes e_{i,j}$!), then all monomials but one vanish, and for the unique monomial of $p(Z_G)$ which does not vanish, say $Z_G$, the elements of the set $S_g$ assume precisely all basis elements of $V_g$.

Roughly speaking we construct alternating polynomials which are not only nonidentities of $A$, but also have the property that by means of *any* nonvanishing evaluation we are able to allocate the elements of $V_g$, $g \in G$, to the variables in

$S_g$ (in the sense of (5) above). In this case we will also say that the polynomial $p$ allocates the elements of $V_g$. The upshot of this is that since $A$ and $B$ satisfy the same $G$-graded identities, we will be able to allocate homogeneous basis elements of $B$ determined by the presentation $P_B$.

In what follows we will show how to construct such polynomials for certain $G$-graded subspaces $V$ of $A$ which correspond to the invariants mentioned above. In order to construct the polynomials (roughly speaking) we proceed as follows. We identify in the algebra $A$ (say) the spaces $(V_g)$ as well as the full $g$-component of $A$ for any $g$ which appears as a homogeneous degree in the $V_g$'s (no damage is done if we add a homogeneous degree $g$ for which $V_g = 0$). We write a *nonzero* monomial with the basis elements $u_h \otimes e_{i,j}$ where we pay special attention to the spaces in the $V_g$'s.

For each basis element $u_h \otimes e_{i,j}$ which is to be part of an alternating set we insert on its left the idempotent $1 \otimes e_{i,i}$ and on its right the idempotent $1 \otimes e_{j,j}$. We refer to these idempotents as <u>frames</u>. Next we consider the homogeneous degrees of the basis elements and we construct a (long!) multilinear monomial, denoted by $Z_G$, with homogeneous variables whose homogeneous degrees are as prescribed by the just constructed monomial in $A$. Finally we alternate the homogeneous sets of cardinality equal to the full dimension of the $g$-homogeneous component in $A$.

*Remark* 2.3. We use the adjective "long" for the multilinear monomial above, since the monomial to be constructed will consist of several bridged segments (i.e. sequences of variables) which correspond to certain subspaces of the algebra $A$ by any nonzero evaluation.

We start with **step 1**, the dimensions of the homogeneous components. It is well known that there is a nonzero product of the form

$$e_{1,1} \times e_{1,2} \times \cdots \times e_{i,1} = e_{1,1},$$

of all elementary matrices $e_{i,j}$, $1 \leq i, j \leq r$. Clearly, for every $h \in H_A$, the product of the monomial $\Sigma_h = u_h \otimes e_{1,1} \times u_h \otimes e_{1,2} \times \cdots \times u_h \otimes e_{i,1}$ is nonzero and is of the form $\lambda_h u_{h^{r^2}} \otimes e_{1,1}$, where the scalar $\lambda_h \in F^*$ depends on the 2-cocycle $\alpha$ on $H$. Clearly, the product $\Pi_h \Sigma_h$ of the monomials $\Sigma_h$ yields a nonzero product of the form $\lambda_H u_{h_0} \otimes e_{1,1}$ for some $\lambda_H \in F^*$ and some $h_0 \in H$. Let us denote the entire product by $\Sigma_H$. We refer to its elements as designated elements. Now we insert frame elements of the form $1 \otimes e_{i,i}$ on the left and and on the right of any basis element in $\Sigma_H$ so that the entire product $\tilde{\Sigma}_H$ is nonzero. The key property that we need here is that the pairs of indices $(i, j)$ and $(k, s)$, of any two different basis elements $u_h \otimes e_{i,j}$ and $u_{h'} \otimes e_{k,s}$ having the same homogeneous degree, must be different. Consequently, if we permute designated elements of $\tilde{\Sigma}_H$, of the same homogeneous degree, we obtain zero.

Consider the homogeneous degree of all basis elements which appear in $\tilde{\Sigma}_H$ and produce a (long) multilinear monomial $Z_G$ whose elements are homogeneous of degrees as prescribed by the elements of $\tilde{\Sigma}_H$. We denote variables which correspond to designated basis elements by $z_{i,g}$ and refer to them as designated variables. Variables which correspond to frame elements will be denoted by $y_{j,e}$. Note that by construction, the number of designated variables of degree $g$ coincides with the dimension of $A_g$, for every $g \in G$. Now, for every $g \in G$, we alternate the designated variables of degree $g$ in $Z_G$ and denote the polynomial obtained by $p$. By construction $p$ is a $G$-graded nonidentity of $A$ and so by assumption it is also a $G$-graded nonidentity of $B$. But by the alternation property of $p$ we have that $\dim(A_g) \leq \dim(B_g)$, for every $g \in G$, and so we are done by symmetry. This completes the proof of **step 1**.

We proceed to **step 2** and **step 3**. For **step 2** we will present two proofs which differ only in style. The first is more "algorithmic" while the second uses more precise notation. We find the second presentation more cumbersome and so we do it (as an illustration) only for that step.

Consider the $e$-component of $A$. By Proposition 1.4 and Remark 1.5, it is isomorphic to the direct sum of simple algebras which can be realized in blocks along the diagonal. By permuting the elements of the $r$-tuple (which provides the elementary grading) we can order the $e$-blocks in decreasing order. Construct a monomial $Z_G$ with segments which pass through each one of the $e$-blocks, bridged by an element (necessarily) outside the $e$-component. We insert frames of idempotents around the elements of the $e$-blocks. The prescribed sets $V$ here are determined as follows. For the maximal size (say $d_1$) of $e$-blocks, we have $r_1$ blocks, for the second size ($d_2$) we have $r_2$ blocks, and so on. So we have $r_1$ $e$-spaces of the largest dimension $(d_1)^2$, and so on. We produce the alternating polynomial as above.

**Proposition 2.4.** *The polynomial above allocates the $e$-blocks, where the $e$-blocks of the same dimension are determined up to permutation.*

*Proof.* First note that the polynomial $p$ is a nonidentity of $A$. To see this let us show that the evaluation (which determined the monomial $Z_G$) is indeed a nonzero evaluation. Clearly the monomial $Z_G$ does not vanish by construction. On the other hand in any nontrivial alternation, elements of the $e$-blocks will meet the wrong idempotent frames and so we get zero.

Next let us show that for any nonzero evaluation of the polynomial (on the standard basis) we have that all monomials but one vanish and for the one that does not vanish, the evaluation allocates the $e$-blocks as prescribed. Indeed, we note first that by the alternation property we are forced to evaluate the full $e$-alternating set by a full basis of the $e$-component (for otherwise we get zero), so taking a basis of $e$-elements of the form $u_h \otimes e_{i,j}$ we are forced to use all of them and each one exactly once.

Next we analyze the evaluation of any monomial whose value is nonzero. Elements of the $e$-component that are substituted for variables of the same segment must belong to the same block, for otherwise we obtain zero: Indeed, segments consist only of $e$-variables and basis elements of different blocks can be bridged only by (homogeneous) elements of degree $\neq e$. In other words variables of any segment must be evaluated only by elements of the same $e$-block. Consider a segment of largest size. Since it must be evaluated by elements of one single block, it must exhaust one of the blocks of size $d_1^2$. Proceeding to the next segment we see that we must substitute elements from the $e$-block of the next largest (perhaps the same) size. Continuing in this way we obtain the desired allocation property. □

Having constructed the polynomial $p$, we would like to see what can be deduced from the fact that $p$ is also a nonidentity of the $G$-graded algebra $B$. Without loss of generality let us assume that the configuration of the multiplicities (i.e. the sizes of the $e$-blocks) for $A$ is larger than for $B$ (with lexicographic order). It follows that in the largest $e$-segment we must put a full $e$-block and so we must have an $e$-block of the corresponding size in $B$. Continuing in this way we see that the multiplicities in $B$ must be the same as in $A$ and so we have **step 2**.

For **step 3**, note that because the size of the matrix part in $P_A$ (resp. $P_B$) is the sum of the sizes of the $e$-blocks of $P_A$ (resp. $P_B$), the size of the matrix part of $P_A$ and $P_B$ must be the same. But we have seen that $A$ and $B$ have the same dimension. It follows that the subgroups $H_A$ and $H_B$ have the same order.

Before we proceed to the next step, we now present our second proof for **step 2** using more precise notation.

Let $P_A = (H, \alpha, (p_1, \ldots, p_r))$ be the given presentation of the algebra $A$. Applying basic moves we know that elements $p_i$ may be replaced by right $H$-cosets representatives and so we write the $r$-tuple $(p_1, \ldots, p_r)$ as

$$(g_{(1,i_1)}, g_{(2,i_1)}, \cdots, g_{(d_1,i_1)}, g_{(1,i_2)}, g_{(2,i_2)}, \cdots, g_{(d_2,i_2)}, \cdots, g_{(1,i_m)}, g_{(2,i_m)}, \cdots, g_{(d_m,i_m)})$$

where $g_{(k,i_s)} = g_{(l,i_s)}$ for all $s$, $1 \leq s \leq m$ and all $k$ and $l$ in $\{1, 2, \ldots, d_s\}$, and $Hg_{(k,i_s)} \neq Hg_{(l,i_t)}$ for $s \neq t$. Clearly, $d_1 + d_2 + \ldots + d_m = r$.

With this notation, the $e$-component is spanned by the basis elements $u_e \otimes e_{i,j}$ where $d_1 + d_2 + \ldots + d_{k-1} + 1 \leq i, j \leq d_1 + d_2 + \ldots + d_k$, $1 \leq k \leq m$. Furthermore, the $e$-component is decomposed into the direct sum of $m$ simple algebras $A_i$, which are clearly isomorphic to the matrix algebras $M_{d_i}(F)$.

It is well known that for each one of the simple algebras $A_k$ (of degree $d_k$) there is a nonzero product $\overrightarrow{E}_k$ of precisely all basis elements, starting with $1 \otimes e_{t,t}$ and ending with $1 \otimes e_{s,t}$, where $t = d_1 + d_2 + \ldots + d_{k-1} + 1$ and $s = d_1 + d_2 + \ldots + d_k$. Next we expand each monomial $\overrightarrow{E}_k$ by bordering every basis element $1 \otimes e_{i,j}$ which appears in it by idempotents $1 \otimes e_{i,i}$ and $1 \otimes e_{j,j}$ from left and right respectively. We denote the monomial obtained by $\overrightarrow{E}_{k,fr}$ ("fr" stands for framed). We view the basis elements $1 \otimes e_{i,j}$ (of $\overrightarrow{E}_k$) as "designated" elements (which are about to alternate) and the idempotents $1 \otimes e_{j,j}$ as "frame" elements. The product of basis elements $\overrightarrow{E}_{k,fr}$ consists of designated elements as well as frames. Note that all basis elements in $\overrightarrow{E}_{k,fr}$ are homogeneous of degree $e$.

*Remark* 2.5. Note that in the nonzero product above of the basis elements we do not insist (although it is possible here) that each basis element appears precisely once but rather that it appears at least once. In case we have repetitions we may include the additional basis elements as part of the frame.

Now consider basis elements which bridge the different blocks. For instance the elements $a_{k,k+1} = 1 \otimes e_{i,j}$, $(i, j) = (d_1 + d_2 + \ldots + d_{k-1} + 1, d_1 + d_2 + \ldots + d_k + 1)$, $k = 1, \ldots, m-1$, bridge the $k$-th and $k$-th $+1$ block respectively. From the structure of the $r$-tuple we see that the homogeneous degree of $a_{k,k+1}$ is $\neq e$ (say $g_{\pi_k}$). We obtain that the product of basis elements $\overrightarrow{E}_{1,fr} a_{1,2} \overrightarrow{E}_{2,fr} a_{2,3} \cdots a_{m-1,m} \overrightarrow{E}_{m,fr}$ is nonzero (in fact the product is $1 \otimes e_{1,d_1+d_2+\ldots+d_{m-1}+1}$). We see that any nontrivial permutation on the designated basis elements (and leaving the other elements fixed) gives a zero product.

Now we create the multilinear monomial $Z_G$. The designated basis elements will be replaced by "designated variables" $z_{i,e}$, whereas the rest of the basis elements (frames and bridges) will be denoted by $y_{j,g}$, where $g$ is the corresponding homogeneous degree. To sum up, we have the following. From each product of basis elements $\overrightarrow{E}_{k,fr}$ we construct a multilinear monomial $\overrightarrow{T}_{k,fr}$ of $e$-variables (designated variables and frames). Then the monomial $Z_G$ is given by the product

$$\overrightarrow{T}_{1,fr}y_1\overrightarrow{T}_{2,fr}y_2\cdots y_{m-1}\overrightarrow{T}_{m,fr},$$

where $y_i$ has weight $g_{\pi_i}$. Finally, the polynomial $p(Z_G)$ is obtained by alternating the variables $z_{i,e}$ of $Z_G$. Note that the polynomial has precisely $dim_F(A_e)$ variables $z_{i,e}$.

Now we consider the degrees of the above $e$-blocks. Assume the $e$-blocks are ordered with degrees in decreasing order, so $d_1 \geq d_2 \geq \ldots \geq d_m$. Let $\chi_1$ be the number of blocks of degree $d_1$, $\chi_2$ the number of blocks of degree $d_{\chi_1+1}$, and finally $\chi_\nu$ the number of blocks of lowest degree.

In terms of the terminology above we have vector spaces $V_{1,1,e},\ldots,V_{1,\chi_1,e}$ which are the first $\chi_1$ diagonal blocks and are of dimension $d_1^2$, $V_{2,1,e},\ldots,V_{2,\chi_2,e}$ are the next $\chi_2$ diagonal blocks and are of dimension $d_2^2$, and so on.

We claim that the polynomial $p(Z_G)$ satisfies the allocation property for the spaces $V_{i,j,e}$. Clearly, by construction, $p(Z_G)$ is a multilinear $G$-graded nonidentity of $A$.

Next we need to see that any nonzero evaluation with basis elements allocates the vector spaces $V_{i,j,e}$ up to a permutation of the second index. By construction, the polynomial $p(Z_G)$ alternates on the set of designated variables (of degree $e$) whose cardinality equals the dimension of $A_e$ over $F$. Consequently, in any nonzero evaluation, the designated variables must precisely assume elements which form a basis of $A_e$ and so, choosing (as we may) a basis of $A_e$ of the form $1 \otimes e_{i,j}$, the designated variables must assume each of these elements exactly once. Next we show that (in a nonzero evaluation) each set of designated variables in $\overrightarrow{T}_{k,fr}$ must precisely assume all basis elements of a unique $e$-block. Indeed, basis elements of different $e$-blocks cannot be bridged by $e$-homogeneous elements and so designated variables $\overrightarrow{T}_{k,fr}$ must get values from a unique $e$-block. But the cardinalities of the sets of designated variables of $\overrightarrow{T}_{k,fr}$ coincide with the dimensions of the different $e$-blocks and so the result follows from the following obvious lemma.

**Lemma 2.6.** *Let $\Omega_A$ and $\Omega_{\bar{A}}$ be finite collections of finite sets $A_1,\ldots,A_n$ and $\bar{A}_1,\ldots,\bar{A}_n$ respectively. Assume the sets $A_i$, $i=1,\ldots,n$, are pairwise disjoints (likewise for the $\bar{A}_i$'s).*

*Suppose the cardinality of $A_i$ and $\bar{A}_i$ coincide for $i=1,\ldots,n$. Let $\mathcal{A}$ and $\bar{\mathcal{A}}$ be the union of the $A_i$'s and the $\bar{A}_i$'s respectively. Suppose*

$$\phi : \mathcal{A} \to \bar{\mathcal{A}}$$

*is a bijection such that any two elements of different $A_k$'s (say $a_f \in A_f$ and $a_h \in A_h$, $f \neq h$) are mapped to different $\bar{A}_i$'s.*

*Then there is a permutation $\pi \in S_n$ such that the map $\phi$ establishes a bijection of $A_i$ with $\bar{A}_{\pi i}$ for $i=1,\ldots,n$. Furthermore, if (to begin with) the sets $A$ and $\bar{A}$ are ordered in decreasing order, then the permutation $\pi$ permutes only sets of the same order.*

The rest of the argument is the same as for the first proof.

At this point we have that the multiplicities of the right $H_A$-coset representatives in the $r$-tuple $(p_1,\ldots,p_r)$ (of the presentation $P_A$) coincides with the multiplicities of right $H_B$-cosets in the corresponding tuple of $P_B$. In particular the matrix size of the presentations $P_A$ and $P_B$ coincide. Moreover the subgroups $H_A$ and $H_B$ have the same order.

The next step, **step 4** of the outline, is to show that the subgroups $H_A$ and $H_B$ are conjugate in $G$. For this and later steps we introduce a polynomial that generalizes the one above. We may arrange the tuples of coset representatives for $A$ and $B$ so that representatives of the same coset (of $H_A$ in $G$ for $A$ and of $H_B$ in $G$ for $B$) are grouped together and so that we use the same group elements for the same coset. We have proved that the number of coset representatives (with multiplicities) is the same for $A$ and $B$. Now let $T$ be an arbitrary subgroup of $G$. We have seen that $A_T$, the $T$-component of $A$, is a sum of $T$–simple algebras that appear in blocks in $A$. For each block we produce a nonzero product of the standard basis elements that lie in that block, each used exactly once, with the extra condition that the first part of the product uses those standard basis elements in that block with weight $e$. In other words the product begins with a nonzero product of the standard basis elements determined by that part of the $e$-component that lies in that block. This part of the $e$-component is a semisimple algebra. For each simple component we produce a nonzero product of the standard basis elements from that component. We then add frames of weight $e$ between every pair of these basis elements. We then add frames of weights in $T$ but necessarily not of weight $e$ between these simple components. We then complete the product for the rest of the standard basis elements in that block. Finally we put these block products in some order, and between each pair of successive blocks we put another standard basis element (necessarily of weight outside of $T$) so that the entire product is nonzero. We now form a monomial from this product. Denote it by $Z_{T,A}$. We then alternate the variables of the same weight (in $T$) that came from the standard basis elements in each block. Denote the resulting polynomial $f_{T,A}$. We claim that this polynomial is a $G$-graded nonidentity of $A$. Indeed, replacing the monomial $Z_{T,A}$ with the original basis elements we obtain a nonzero product. Let us now show that for any nontrivial alternation, some standard basis element will be bordered by elements which annihilate it. To see this note that two basis elements with the same $(i,j)$ position cannot have the same homogeneous degree. This shows that elements with equal homogeneous degrees are bordered by basis elements of the form $1 \otimes e_{i,i}, 1 \otimes e_{j,j}$ and $1 \otimes e_{i',i'}, 1 \otimes e_{j',j'}$ where the pairs $(i,j)$ and $(i',j')$ are different. It follows easily that any nontrivial alternation yields a zero value. This proves the claim. Of course we can do the same thing for $B$ and we denote the resulting polynomial by $f_{T,B}$.

**Proposition 2.7.** *Let $T$ be a subgroup of $G$. There is a one-to-one correspondence between the $T$–simple components of $A_T$ and the $T$–simple components of $B_T$ such that corresponding components have the same dimension and the same matrix size. Moreover for corresponding components the vector of multiplicities of the coset representatives (from the tuple for $A$ and the tuple for $B$) is the same.*

*Proof.* Because $f_{T,A}$ is a nonidentity for $A$, it must be a nonidentity for $B$, so some monomial $Z$ of $f_{T,A}$ must be nonzero on $B$. (In fact because each of the monomials of $f_{T,A}$ is an alternation of $Z$, if $Z$ has some nonzero evaluation so does every monomial, so we could assume $Z = Z_{T,A}$.) Under the evaluation of $Z$ no two blocks of $B_T$ can be substituted into the segment coming from a single block of $A_T$ because elements of different blocks annihilate each other. So consider the $T$–simple component of $B_T$ of smallest dimension. When we evaluate $Z$ on $B$ this block must completely fill some segment. In other words the dimension of this smallest component must be at least as large as the dimension of the smallest component of

$A_T$. Since we can use the same argument for $f_{T,B}$ we infer that the dimension of this smallest component is the same as the dimension of the smallest component of $A_T$. Continuing with the component of the next smallest dimension and so on, we see that we have a one-to-one correspondence between the $T$–simple components of $A_T$ and the $T$–simple components of $B_T$ such that corresponding components have the same dimension. Moreover we see that in any nonzero evaluation of $Z$ on $B$ we must substitute elements from a given $T$–simple component of $B_T$ into a segment coming from a $T$–simple component of $A_T$ of the same dimension.

Next we claim that under such a substitution the component of $B_T$ must involve the same number of elements of the tuple for $B$ with the same multiplicities as the component for $A_T$ in whose segment of $Z$ the component of $B_T$ is placed. To see this, label these corresponding components $U_A$ and $U_B$. Under the nonzero evaluation of $Z$ the elements of the $e$-component of $U_B$ must be substituted in the first part of the segment, the part formed from the $e$-component of $U_A$, and must fill that part of the segment. In particular the dimension of the $e$-component of $U_B$ must be greater than or equal to the dimension of the $e$-component of $U_A$. Because this is true for every component of $A_T$ and we know the dimensions of $A_e$ and $B_e$ are the same, we see that the dimension of the $e$-component of $U_A$ equals the dimension of the $e$-component of $U_B$. But in fact more is true. Each of these $e$-components is a semisimple (ungraded) algebra. Under the evaluation we cannot substitute two elements from different simple components of the $e$-component in $U_B$ into a segment coming from a single simple component of the $e$-component in $U_A$ because such elements annihilate each other. Therefore the number of simple components of the $e$-component in $U_B$ must be no larger than the number of simple components of the $e$-component in $U_A$. But the total number of simple components of $A_e$ is the same as the total number of simple components of $B_e$. Again because we have the inequality for all components of $A_T$ we see that the number of simple components of the $e$-component in $U_B$ must equal the number of simple components of the $e$-component in $U_A$. Finally since the dimension of each of the simple components of the $e$-component of $U_B$ must be greater than or equal to the dimension of the simple component of the $e$-component of $U_A$ in which it is evaluated, we see that the dimensions of the simple components of the $e$-component in $U_B$ must equal the dimensions of the simple components of the $e$-component in $U_A$. (In other words the $e$-component of $U_A$ is isomorphic as an $F$–algebra to the $e$-component of $U_B$.) But the sum of the matrix sizes of the simple components of the $e$-component of $U_A$ is the matrix size of $U_A$, so $U_A$ and $U_B$ have the same matrix size. Moreover the matrix sizes of the simple components of the $e$-component of $U_A$ are exactly the multiplicities of the elements of the tuple for $A$ that appear there, so these are the same for $U_B$. $\square$

We can now complete **step 4** and **step 5** of the outline. Let $H = H_A$. By applying a basic move to the presentation for $A$ we may assume that $e$ appears in the tuple and that it appears with the highest multiplicity. Call this multiplicity $d$. In the algebra $A_H$ there will then be an $H$-simple component of dimension $d^2|H|$ coming from the single coset representative $e$ in the tuple. By the proposition there must be a simple component of $B_H$ of the same dimension and matrix size coming from a single coset representative $g$ (say) of $H_B$ in $G$ that appears in the tuple for $B$. Because the matrix size is the same as the multiplicity, we see that $g$ has multiplicity $d$. Hence the dimension of the corresponding component is $d^2|H \cap g^{-1}H_B g|$. So we must have $d^2|H| = d^2|H \cap g^{-1}H_Bg|$. Hence $|H| = |H \cap g^{-1}H_Bg|$. Because $H$ and $H_B$ have the same cardinality it follows that $H = g^{-1}H_Bg$, so $H_A$ and $H_B$ are conjugate. By applying a basic move we may assume that $H_A = H_B$. We will denote this common subgroup by $H$. We also have that the multiplicities arising in each $H$–simple component are the same (up to permutation of the blocks) in $A$ and $B$.

We now proceed to **step 6** and **step 7**. We decompose the tuples for $A$ and $B$ as described before **step 5** of the outline. Let $g$ be a coset representative of $N(H)$ in $G$ that appears in the tuple for $A$. By Proposition 2.7 we know that there is a one-to-one correspondence between the $g^{-1}Hg$–simple components of $A_{g^{-1}Hg}$ and the $g^{-1}Hg$–simple components of $B_{g^{-1}Hg}$ such that corresponding components have the same dimension and matrix size. Moreover for corresponding blocks the vector of multiplicities of the coset representatives (from the tuple for $A$ and the tuple for $B$) is the same. In particular because an element of $N(H)g$ appears in the tuple we see that we have blocks coming from a single coset representative. As in the case where $g = e$ this implies that the same is true for $B_{g^{-1}Hg}$ and so an element of $N(H)g$ also appears in the tuple for $B$. In fact again as in the case where $g = e$ we see that the number of tuple elements for $A$ that lie in the coset $N(H)g$ is the same as for the tuple for $B$ including multiplicities. This proves **step 6**. It also proves **step 7**.

*Remark* 2.8. Note that if $H$ is $e$, then all we have so far is that the multiplicities in the $r$–tuples for $A$ and $B$ are the same. In particular $A$ and $B$ have the same matrix size.

Our next goal (**step 8**) is to show that the tuples of the elementary grading in $A$ and in $B$ are obtained from one another by multiplication on the left by a single element of $N(H)$. This will lead to the situation where the groups $H_A$ and $H_B$ are still the same and the tuples are the same. Then the final parameter we will need to deal with will be the 2-cocycle on the group $H$.

We consider a (very) special case of the statement above, namely where $H$ is $e$. We have the tuple for $A$, and based on it we can construct the polynomial $f_{\{e\},A}$. Let us recall the construction. We consider the $e$-blocks arising from the multiple representatives. We produce $e$-segments for each block bridged by non-$e$-elements. We know that the monomial is a nonidentity of $A$ and if we put frames we know that any nontrivial permutation of the designated $e$-elements gives a zero product of basis elements. We construct a monomial out of the product above which we denote by $Z_{\{e\},A}$ (see the notation in Proposition 2.7 and the paragraph preceding it) and alternate the designated variables. The polynomial obtained is denoted by $f_{\{e\},A}$.

We denote by $\sigma_1, \dots, \sigma_n$ the distinct $H$-coset representatives in the tuple for $A$ and by $\tau_1, \dots, \tau_n$ the distinct coset representatives in the tuple for $B$. Note that, because $H = \{e\}$, distinct coset representatives just mean distinct elements. Also, we remind the reader that by previous steps, the vector of multiplicities of $\sigma_1, \dots, \sigma_n$ and $\tau_1, \dots, \tau_n$ is the same. Let $d_1, \dots, d_n$ be the vector of multiplicities, which we may assume are in decreasing order. By Proposition 2.7 a nonzero evaluation on $B$ gives rise to a permutation $\pi$ on $\{\tau_1, \dots, \tau_n\}$ so that the segment for $\sigma_i$ is being evaluated by the elements in the $\tau_{\pi(i)}$ block. For every pair $i, j \in \{1, 2, \dots, n\}$ the elements that can bridge between the $i$-th block and the $j$-th block must have weight $\sigma_i^{-1}\sigma_j$. It follows that the bridging elements between the $\pi(i)$ block and the $\pi(j)$

block have the same weight and so we obtain the relations $\sigma_i^{-1}\sigma_j = \tau_{\pi(i)}^{-1}\tau_{\pi(j)}$ for all $i,j$. Rewriting these equations, we see that for all $i,j \in \{1,2,\ldots,n\}$, $\sigma_j\tau_{\pi(j)}^{-1} = \sigma_i\tau_{\pi(i)}^{-1}$, and so all the elements $\sigma_i\tau_{\pi(i)}^{-1}$ are the same. We see then that for all $i \geq 1$, $\sigma_i = (\sigma_1\tau_{\pi(1)}^{-1})\tau_{\pi(i)}$, and so we have found an element $g \in G$ such that $\sigma_i = g\tau_i$ for all $i$. That ends the case where $H = \{e\}$.

*Remark* 2.9. Note that not every permutation $\pi$ is allowed. For instance, a permutation that exchanges elements with different multiplicities would lead to a contradiction. In other words we cannot substitute an $e$-block of size $d_i$ with an $e$-block of size $d_j \neq d_i$. It is important to note (as mentioned above) that if a segment was determined by a block of size $d_i$, arising from an element $\sigma_i$ (say) (with multiplicity $d_i$), then in any nonzero evaluation on $B$ (or on $A$) the segment will assume values of precisely one block arising from $\tau_j$ where necessarily $d_j = d_i$. Nevertheless, for the proof, we only need to know the existence of a permutation $\pi$ as above.

In fact a similar argument will work when $H$ is normal in $G$. Because we will use it in the general case, when $H$ is not necessarily normal, we outline the normal case here:

Applying Proposition 2.7 to $A_H$ and $B_H$ we see (by constructing the polynomial $f_{H,A}$) that there is a permutation $\pi$ on $\{\tau_1,\ldots,\tau_n\}$ with the requirement that for every $i,j \in \{1,2,\ldots,n\}$, the corresponding bridging elements must have the same weight. The set of weights of possible bridging elements from the $i$-th to the $j$-th block in this case is the set $\sigma_i^{-1}H\sigma_j$. Therefore the necessity of having bridging elements of the same weight for passing from the $\pi(i)$ block to the $\pi(j)$ means that for every $i,j \in \{1,2,\ldots,n\}$ the intersection

$$\sigma_i^{-1}H\sigma_j \cap \tau_{\pi(i)}^{-1}H\tau_{\pi(j)}$$

is nonempty. But because the $\sigma$'s and $\tau$'s normalize $H$, these two sets are in fact cosets of $H$, and so must be equal. It follows that there are elements $h_j \in H$, $j \in \{1,2,\ldots,n\}$, such that for all $j$,

$$\sigma_j = (\sigma_1\tau_{\pi(1)}^{-1})h_j\tau_{\pi(j)}.$$

To complete the argument in this case recall that by basic move (2) we may replace any element of the tuple for $B$ by a different representative of the $H$-coset (that is, replace $\tau_{\pi(j)}$ by $h_j\tau_{\pi(j)}$). We therefore see that the tuple for $A$ is obtained from the tuple for $B$ by multiplying on the left by an element from $N(H)(=G)$.

We can now consider the general case where the group $H$ is not necessarily normal in $G$. We decompose the tuple for $A$ into subtuples coming from different $N(H)$-representatives in $G$. We will refer to these subtuples as “big blocks”. We know that the multiplicities in each subtuple coincide. We construct a monomial which corresponds to that configuration: We start with the monomials $Z_{g_i^{-1}Hg_i,A}$ constructed in the paragraph preceding Proposition 2.7, where $g_1 = e, g_2, \ldots, g_k$ are the distinct coset representatives of $N(H)$ in $G$ appearing in the tuple for $A$ which we have shown can also be taken to be the distinct coset representatives of $N(H)$ in $G$ appearing in the tuple for $B$. We then form the product of these monomials bridging successive monomials with variables whose weights allow a nonzero evaluation using the standard basis elements for $A$. Call this big monomial $Z_A$. We then perform successive alternations of the designated variables of a given weight appearing in each of the monomials $Z_{g_i^{-1}Hg_i,A}$. (Note: the alternations above are performed among designated variables (with the same homogeneous degree) which lie in the same monomial $Z_{g_i^{-1}Hg_i,A}$ for any $i$. No alternation is performed among variables coming from monomials $Z_{g_i^{-1}Hg_i,A}$, for different $i$'s.) Call this polynomial $f_A$. This is a nonidentity for $A$ and so must be a nonidentity for $B$. So one of the monomials of $f_A$ must be nonzero on the standard basis of $B$ and as we saw in the proof of Proposition 2.7 we may assume this monomial is $Z_A$. In particular each of the submonomials $Z_{g_i^{-1}Hg_i,A}$ must have a nonzero evaluation. We denote by $\sigma_{i,k}$ a typical representative of the cosets of $H$ in $N(H)$ such that $\sigma_{i,k}g_i$ appears in the tuple for $A$ and by $\tau_{i,m}$ a typical representative of the cosets of $H$ in $N(H)$ such that $\tau_{i,m}g_i$ appears in the tuple for $B$. The nonzero evaluation of $Z_A$ then produces a permutation $\pi$ on the tuple for $B$ that preserves the subtuples coming from each coset representative $g_i$ of $N(H)$ in $G$ that takes a block corresponding to the coset representative $\sigma_{i,k}g_i$ to the block coming from the coset representative $\tau_{i,\pi(k)}g_i$. As before the bridge between the $\sigma_{i,k}g_i$ block and the $\sigma_{j,m}g_j$ block must also serve as a bridge between the $\tau_{i,\pi(k)}g_i$ block and the $\tau_{j,\pi(m)}g_j$ block and so the set

$$g_i^{-1}\sigma_{i,k}^{-1}H\sigma_{j,m}g_j \cap g_i^{-1}\tau_{i,\pi(k)}^{-1}H\tau_{j,\pi(m)}g_j$$

must be nonempty. Canceling we obtain

$$\sigma_{i,k}^{-1}H\sigma_{j,m} \cap \tau_{i,\pi(k)}^{-1}H\tau_{j,\pi(m)}$$

is nonempty. It follows that there are elements $h_{j,m} \in H$, $j \in \{1, 2, \ldots, n\}$, such that for all $j, m$

$$\sigma_{j,m} = (\sigma_{1,1}\tau_{1,\pi(1)}^{-1})h_{j,m}\tau_{j,\pi(m)}.$$

To complete the argument in this case recall that by basic move (2) we may replace any element of the tuple for $B$ by a different representative of the $H$-coset (that is, replace $\tau_{j,\pi(m)}$ by $h_{j,m}\tau_{j,\pi(m)}$). We therefore see that the tuple for $A$ is obtained from the tuple for $B$ by multiplying on the left by an element from $N(H)$.

So by applying basic moves, we may now assume that the fine gradings of $A$ and $B$ are determined by the same group $H$ and the elementary grading is determined by the same $r$-tuple $(p_1, \ldots, p_r) \in G^{(r)}$. We now proceed to show that the cocycles $\alpha$ and $\beta$ may be assumed to be the same.

We start with the case where the grading is fine, that is, $A$ and $B$ are twisted group algebras. Before stating the proposition, recall (Aljadeff, Haile and Natapov [1]) that the $T$-ideal of $H$-graded identities of a twisted group algebra $F^{\alpha}H$ is generated as a $T$-ideal by the multilinear binomial identities of the form

$$B(\alpha) = x_{i_1,h_1}x_{i_2,h_2}\cdots x_{i_s,h_s} - \lambda_{((h_1,\ldots,h_s),\pi)}x_{i_{\pi(1)},h_{\pi(1)}}x_{i_{\pi(2)},h_{\pi(2)}}\cdots x_{i_{\pi(s)},h_{\pi(s)}},$$

where

(1) $h_i \in H$, $i = 1, \ldots, s$,
(2) $\pi \in Sym(s)$,
(3) the products $h_1h_2\cdots h_s$ and $h_{\pi(1)}h_{\pi(2)}\cdots h_{\pi(s)}$ coincide in $H$,
(4) $\lambda$ is a nonzero element (root of unity) $\in F$ determined by the $s$-tuple $h_1, h_2, \ldots, h_s$ and the permutation $\pi$.

*Remark* 2.10. In fact more is true. If we allow repetitions of the homogeneous variables, the binomial identities obtained span the $T$-ideal of $H$-graded identities (in case the grading is fine) as an $F$-vector space. However we will not need this fact here.

**Proposition 2.11.** *Given twisted group algebras $F^{\alpha}H$ and $F^{\beta}H$, then the cocycles are cohomologous if and only if the algebras satisfy the same graded identities.*

*Proof.* The idea of the proof appeared already in [1], where we considered the particular case where the group $H$ is of central type and the twisted group algebra $F^{\alpha}H$ is the algebra of $k \times k$-matrices over $F$ where $ord(H) = k^2$. However, the same construction holds in general. For the reader's convenience, let us recall the construction here.

It is well known, by the universal coefficient theorem, that the cohomology group $H^2(H, F^*)$ is naturally isomorphic to $Hom(M(H), F^*)$ where $M(H)$ denotes the Schur multiplier of $H$. It is also well known that $M(H)$ can be described by means of presentations of $H$, namely the Hopf formula. Indeed, let $\Gamma = \Gamma\langle x_{h_1}, \ldots, x_{h_m}\rangle$ be the free group on the variables $x_{h_i}$ where $H = \{h_1, \ldots, h_m\}$. Consider the presentation

$$\{1\} \to R \to \Gamma \to H \to \{1\}$$

where the epimorphism is given by $x_{h_i} \longrightarrow h_i$.

One knows that the Schur multiplier $M(H)$ is isomorphic to

$$R \cap [\Gamma, \Gamma]/[R, \Gamma].$$

Given a 2-cocycle $\alpha$ on $H$ (representing $[\alpha] \in H^2(H, F^*)$) it determines an element of $Hom(M(H), F^*)$ as follows: Let $[z]$ be an element in $M(H)$ where $z$ is a representing word in $R \cap [\Gamma, \Gamma]$. For each variable $x_h$ consider the element $u_h$ in the twisted group algebra $F^{\alpha}H$ representing $h$. Then the value of $\alpha$ on $z$ is the root of unity which is the product in $F^{\alpha}H$ of the elements $u_h$ (which correspond to the variables $x_h$ of $z$). One knows that the value $[\alpha]([z])$ depends on the classes $[\alpha] \in H^2(H, F^*)$ and $[z] \in M(H)$ and not on their representatives. Note that by the isomorphism of $H^2(H, F^*))$ with $Hom(M(H), F^*)$ we have that for two noncohomologous 2-cocycles $\alpha$ and $\beta$ there is $z \in R \cap [\Gamma, \Gamma]$ with $\alpha(z) \neq \beta(z)$. Let us now show how $H$-graded polynomial identities come into play.

Let

$$z = x_{h_{i_1}}^{\epsilon_1} x_{h_{i_2}}^{\epsilon_2} \cdots x_{h_{i_r}}^{\epsilon_r}$$

where $\epsilon_i = \{\pm 1\}$. Because $z$ is in $R$ we have that $h_{i_1}^{\epsilon_1} h_{i_2}^{\epsilon_2} \cdots h_{i_r}^{\epsilon_r} = e$, and because $z$ lies in $[\Gamma, \Gamma]$, we have that the sum of the exponents $\epsilon_i$ which decorate any variable $x_h$ which appears in $z$, is zero.

Our task is to construct out of $z$ and the value $\alpha(z) \in F^*$ an $H$-graded binomial identity of the twisted group algebra $F^{\alpha}H$. Pick any variable $x_h$ in $z$ and let $n$ be the order of $h$ (in $H$). Clearly the commutator $[x_h^n, y]$, $y \in \Gamma$, is in $[R, \Gamma]$ and so multiplying $z$ (say on the left) with elements $x_h^n$ and $x_h^{-n}$, and moving them (to the right) successively along $z$ by means of the relation $[x_h^n, y]$, we obtain a representative of $[z]$ in $M(H)$ of the form

$$z_1 z_2^{-1}$$

where the variables in $z_1$ and $z_2$ appear only with positive exponents.

The binomial identity which corresponds to $z$ and $\alpha(z)$ is given by

$$Z_1 - \alpha(z) Z_2$$

where $Z_i$ is the monomial in the free $H$-graded algebra whose variables are in one-to-one correspondence with the variables of $z_i$. We leave the reader with the task of showing that indeed $Z_1 - \alpha(z)Z_2$ is a $G$-graded identity. Clearly, from the construction it follows that twisted group algebras $F^\alpha H$ and $F^\beta H$ satisfy the same $G$-graded identities if and only if the cocycles $\alpha$ and $\beta$ are cohomologous. This completes the proof of the proposition. □

*Remark* 2.12. (1) The binomial identity obtained above, say for $\alpha$, may not be multilinear. In order to obtain a multilinear binomial identity assume $x_h$ appears $k$-times in each monomial $Z_i$, $i = 1, 2$. Then replacing the variables by $k$ different variables $x_{1,h}, \dots, x_{k,h}$ in each monomial (any order!) we obtain an $H$-graded (binomial) identity which is on variables whose homogeneous degree is $h$. Repeating this process for every $h \in H$ gives a multilinear (binomial) identity.

(2) It follows that any two noncohomologous cocycles can be separated by suitable binomial identities in the sense that for any ordered pair, $(\alpha, \beta)$ (where $\alpha \neq \beta$), there is a binomial $B(\widehat{\alpha}, \beta)$ which is an identity of $\beta$ (abuse of language) and not an identity of $\alpha$.

(3) Assume $\beta_1, \dots, \beta_k$ are cocycles on $H$ which are different from $\alpha$ (noncohomologous to $\alpha$). Then by (2) above there is a binomial identity $B(\widehat{\alpha}, \beta_i)$ of $\beta_i$ which is a nonidentity of $\alpha$. Then if we take the product of these binomials (with different variables), we see that the product is a multilinear identity of any of the $\beta_i$'s and not an identity of $\alpha$. This follows from the fact that in the twisted group algebra the product of two nonzero homogeneous elements is nonzero.

We now come to an important lemma which is due to Yaakov Karasik, in which we extend the preceding proposition to algebras with presentations in which the elementary grading is trivial.

**Lemma 2.13.** *Let $A$ and $B$ be finite dimensional $G$-simple algebras with presentations $P_A$ and $P_B$ respectively. Suppose $P_A$ and $P_B$ are given by $F^\alpha H \otimes M_r(F)$ and $F^\beta H \otimes M_r(F)$ respectively, both with trivial elementary grading on $M_r(F)$. If $\alpha$ and $\beta$ are noncohomologous, then there is an identity of $A$ which is a nonidentity of $B$.*

*Remark* 2.14. In case the group $G$ is abelian, this was proved by Koshlukov and Zaicev [9] using certain modifications of the standard polynomial. However this approach (at least in its straightforward generalization) seems to fail for nonabelian groups.

*Proof.* As above let $B(\widehat{\alpha}, \beta)$ denote a binomial identity of $F^\beta H$ which is a nonidentity of $F^\alpha H$. Then $B(\widehat{\alpha}, \beta)$ has the form

$$B(\widehat{\alpha}, \beta) = z_{h_1} z_{h_2} \cdots z_{h_s} - \lambda_{(\widehat{\alpha}, \beta, (h_1, \dots, h_s), \pi)} z_{h_{\pi(1)}} z_{h_{\pi(2)}} \cdots z_{h_{\pi(s)}}.$$

Next, consider the Regev polynomial $p(X, Y)$ on $2r^2$ variables (each of the sets $X$ and $Y$ consists of $r^2$ variables). It is multilinear (of degree $2r^2$) and central on $M_r(F)$. Any evaluation of $X$ or $Y$ on a proper subset of the $r^2$ elementary matrices $e_{i,j}$ yields zero, whereas in case $X$ and $Y$ assume the full set of elementary matrices the value is a central, nonzero (and hence invertible) matrix (see [7]).

Now, for each variable $z_h$ of $B(\widehat{\alpha}, \beta)$ we construct a Regev polynomial on $2r^2$ variables where we pick one variable from $X$ (no matter which) and we determine its homogeneous degree to be $h$. The rest of the $x$'s and all of the $y$'s in $Y$ are determined as variables of homogeneous degree $e$. We denote the corresponding

Regev polynomial by $p_h(X_{r^2}, Y_{r^2})$. Now, we consider a basis of the algebra $F^\alpha H \otimes M_k(F)$ consisting of elements of the form $u_h \otimes e_{i,j}$. Note that there are precisely $r^2$ basis elements of degree $e$ and $r^2$ basis elements of degree $h$. We see that if we evaluate the polynomial $p_h(X_{r^2}, Y_{r^2})$ with elements in $\{1 \otimes e_{i,j}, u_h \otimes e_{i,j}\}_{i,j}$, the result will be zero if the elementary matrix constituent of the basis elements is *not* the full set of $r^2$ matrices (either for $X$ or for $Y$) and $u_h \otimes \lambda \operatorname{Id}$ otherwise. It follows that if we replace every variable $z_h$ in $B(\widehat{\alpha}, \beta)$ by the Regev polynomial $p_h(X_{r^2}, Y_{r^2})$ we obtain a polynomial

$$R(\widehat{\alpha}, \beta, r)$$

which is an identity of $B$ and a nonidentity of $A$. □

Before we continue, recall that a big block of $M_r(F)$ is any block which is determined by elements of the tuple $\{p_1, \ldots, p_r\}$ which belong to the same right $N(H)$-coset of $G$. A subblock of a big block is called "basic" if it is determined by all elements of the tuple $\{p_1, \ldots, p_r\}$ which belong to the same right $N(H)$-coset of $G$ <u>and</u> have the same multiplicity.

Let $\sigma_1, \ldots, \sigma_t$ be coset representatives of the cosets of $H$ in $N(H)$ and $g_1 = e, g_2, \ldots, g_n$ be coset representatives of the right cosets of $N(H)$ in $G$. For any element $g \in \{g_1 = e, g_2, \ldots, g_n\}$, we consider the representatives (for the right cosets of $H$ in $G$) given by $\sigma_1 g, \ldots, \sigma_r g$. Clearly these representatives determine a big block. We refer to this set as the set of representatives of the big block determined by $g$. Note that in the elementary grading for $A$ or for $B$ these representatives may appear with different multiplicities. Next we fix a subblock of the big block determined by $g$ by fixing the coset representatives $\sigma_1 g, \ldots, \sigma_m g$ of $H$ in $G$.

Now, each one of these representatives, say $\sigma_k g \in N(H)g$, conjugates the cocycle $\alpha$ into a cocycle $\alpha^{(\sigma_k g)^{-1}}$ on the group $H^{g^{-1}} = g^{-1}Hg$ and we <u>claim</u> that the sets of cocycles on $H^{g^{-1}}$ which are obtained by conjugating the cocycles $\alpha$ and $\beta$ by representatives of one subblock are the same (with multiplicities).

Consider the set of cocycles $(\alpha^{(\sigma_1 g)^{-1}}, \ldots, \alpha^{(\sigma_m g)^{-1}})$ obtained in the algebra $A$. For every cocycle $\alpha^{(\sigma_k g)^{-1}}$ we choose a set of representatives for all cohomology classes on $H^{g^{-1}}$ which are different from the class represented by $\alpha^{(\sigma_k g)^{-1}}$ and denote this set by $S_{g,k}$.

For each cocycle $\gamma$ in $S_{g,k}$ we may construct, by Lemma 2.13, a polynomial which is an identity for $F^\gamma H^{g^{-1}} \otimes M_d(F)$ (where the elementary grading is trivial) but not for $F^{\alpha^{(\sigma_k g)^{-1}}} H^{g^{-1}} \otimes M_d(F)$. Moreover, if we take the product of these polynomials (using different sets of variables) we obtain a multilinear polynomial $R(g, \widehat{\alpha}^{(\sigma_k g)^{-1}}, d)$ which is an identity for the algebra $F^\gamma H^{g^{-1}} \otimes M_d(F)$ for all $\gamma \in S_{g,k}$, and is a nonidentity if $\gamma = \alpha^{(\sigma_k g)^{-1}}$.

We now construct a monomial for the $g = g_i$ big block. We begin with the monomial $Z_{g^{-1}Hg,A}$ we considered in **step 8**. This monomial was constructed by considering the graded simple components of $A_{g^{-1}Hg}$. Each such simple component has a group part that is a subgroup of $H^{g^{-1}}$. We will alter the segments that come from simple components in which the group part is all of $H^{g^{-1}}$. Such a component comes from a single coset representative $\sigma g$ in $N(H)g$ (with its multiplicity). For each such component we change the corresponding segment of $Z_{g^{-1}Hg,A}$ by inserting, between the end of the segment and the bridge to the next segment, the polynomial $R(g, \widehat{\alpha}^{(\sigma g)^{-1}}, d)$ (with new variables), where $d$ is the multiplicity of the representative $\sigma g$ (which is also the matrix size of the simple component). Denote this new monomial $\tilde{Z}_{g^{-1}Hg,A}$. We now use alternation to produce a multilinear polynomial which we will denote $\tilde{f}_{g^{-1}Hg,A}$.

By its construction $\tilde{f}_{g^{-1}Hg,A}$ is clearly a $G$-graded nonidentity of $A$ and hence, by assumption, it is also a $G$-graded nonidentity of $B$. We may therefore assume that there is a nonzero evaluation of the monomial $\tilde{Z}_{g^{-1}Hg,A}$ on $B$. By the proof of Proposition 2.7 there is a one-to-one correspondence between the simple components of the big block of $A$ coming from the coset $N(H)g$ and the simple components of the big block for $B$ corresponding to the coset $N(H)g$ such that in the evaluation of the monomial the segments of the monomials determined by the simple components of the big block of $A$ are evaluated at the corresponding simple components of that big block for $B$. But because of the inserted polynomials, say $R(g, \widehat{\alpha}^{(\sigma g)^{-1}}, d)$, we claim the evaluation can be nonzero only if the cocycle on $H^{g^{-1}}$ determined by the coset representative of $H$ which evaluates the segment, say $\tau g$, is cohomologous to $\alpha^{(\sigma g)^{-1}}$. This will say that the cocycles $\alpha^{(\sigma g)^{-1}}$ and $\beta^{(\tau g)^{-1}}$ are cohomologous in $Z^2(H^{g^{-1}}, F^*)$ as desired. To prove the last claim we note that the $Y$ variables in Regev's polynomials which appear in $R(g, \widehat{\alpha}^{(\sigma g)^{-1}}, d)$ are $e$ variables and their cardinality is $d^2$. Next, since the set is alternating we must evaluate the $Y$ variables by linearly independent elements. In the $X$ variables of a Regev polynomial which appears in $R(g, \widehat{\alpha}^{(\sigma g)^{-1}}, d)$, all but one degree $e$ and one variable is of homogeneous degree in $H^{g^{-1}}$. At any rate all variables are $H^{g^{-1}}$ variables and so they must come from one single $H^{g^{-1}}$-block. But if this $H^{g^{-1}}$-block is not the same as determined by the segment evaluation, we get zero. The upshot of this is that the $H^{g^{-1}}$-block of $B$ which evaluates the segment (in a nonzero evaluation) must determine a cocycle which does not annihilate $R(g, \widehat{\alpha}^{(\sigma g)^{-1}}, d)$. In other words the cocycle must be cohomologous to $\alpha^{(\sigma g)^{-1}}$. This completes the proof that the cocycles appearing in a subblock of $A$ and the corresponding subblock of $B$ must be the same up to permutation.

We are now reduced to the situation where the multiplicities of the cocycles appearing in each basic block for the algebras $A$ and $B$ coincide. In particular we know that the cocycles $\alpha$ and $\beta$ are conjugate by an element of $N(H)$ (**step 9** and **step 10** of the outline).

The final step will be to show that up to equivalence we may assume $\alpha$ and $\beta$ are actually cohomologous. To do this we will produce an element $b$ in $N(H)$ such that left multiplication of the $r$-tuple $(p_1, \dots, p_r)$ permutes the representatives in each big block in such a way that it preserves multiplicities and conjugates $\alpha$ to $\beta$. Then by our basic moves, the presentations $P_A$ and $P_B$ will be equivalent. The argument is similar to the proof of **step 8**.

We start with the monomials $\tilde{Z}_{g_i^{-1}Hg_i,A}$ constructed above, where $g_1 = e, g_2, \dots, g_s$ are the distinct coset representatives of $N(H)$ in $G$ appearing in the tuple for $A$ (and $B$). We then form the product of these monomials bridging successive monomials with variables whose weights allow a nonzero evaluation using the standard basis elements for $A$. Call this big monomial $\tilde{Z}_A$. We then perform successive alternations of the variables of a given weight appearing in each of the monomials $\tilde{Z}_{g_i^{-1}Hg_i,A}$. Call this polynomial $\tilde{f}_A$. This is a nonidentity for $A$ and so must be a nonidentity for $B$. So one of the monomials of $\tilde{f}_A$ must be nonzero on the standard basis of $B$, and as we saw in the proof of Proposition 2.7 we may assume this monomial is $\tilde{Z}_A$. In particular each of the submonomials $\tilde{Z}_{g_i^{-1}Hg_i,A}$ must have a nonzero evaluation. We denote by $\sigma_{i,k}$ a typical representative of the cosets of $H$ in $N(H)$ such that $\sigma_{i,k}g_i$ appears in the tuple for $A$ (and $B$). The nonzero evaluation of $\tilde{Z}_A$ then produces a permutation $\pi$ on the tuple for $B$ that preserves the subtuples coming from each coset representative $g_i$ of $N(H)$ in $G$ and so takes a block corresponding to the coset representative $\sigma_{i,k}g_i$ to the block coming from the coset representative $\sigma_{i,\pi(k)}g_i$. We also know that the cocycles $\alpha^{(\sigma_{i,k}g_i)^{-1}}$ and $\beta^{(\sigma_{i,\pi(k)}g_i)^{-1}}$ (and hence $\alpha^{\sigma_{i,k}^{-1}}$ and $\beta^{\sigma_{i,\pi(k)}^{-1}}$) are cohomologous.

As before, the bridge between the $\sigma_{i,k}g_i$ block and the $\sigma_{j,m}g_j$ block must also serve as a bridge between the $\sigma_{i,\pi(k)}g_i$ block and the $\sigma_{j,\pi(m)}g_j$ block, and so the set

$$g_i^{-1}\sigma_{i,k}^{-1}H\sigma_{j,m}g_j \cap g_i^{-1}\sigma_{i,\pi(k)}^{-1}H\sigma_{j,\pi(m)}g_j$$

must be nonempty. Canceling we obtain that

$$\sigma_{i,k}^{-1}H\sigma_{j,m} \cap \sigma_{i,\pi(k)}^{-1}H\sigma_{j,\pi(m)}$$

is nonempty. It follows that there are elements $h_{j,m} \in H$, $j \in \{1, 2, \ldots, n\}$, and $m$ running over the subscripts for the representatives in the $j$-th big block, such that

$$\sigma_{j,m} = (\sigma_{1,1}\sigma_{1,\pi(1)}^{-1})h_{j,m}\sigma_{j,\pi(m)}.$$

Recall that by basic move (2) we may replace any element of the tuple for $B$ by a different representative of the $H$-coset (that is, replace $\sigma_{j,\pi(m)}$ by $h_{j,m}\sigma_{j,\pi(m)}$). We therefore see that the tuple for $A$ is obtained from the tuple for $B$ by multiplying on the left by $\gamma = (\sigma_{1,1}\sigma_{1,\pi(1)}^{-1})$. This element is in the normalizer so we are left with showing $\beta^{\gamma}$ is cohomologous to $\alpha$. But we know that $\alpha^{\sigma_{1,1}^{-1}}$ and $\beta^{\sigma_{1,\pi(1)}^{-1}}$ are cohomologous, so we are done. This completes the proof of the main theorem.

## 3. Uniqueness

We end the paper with a proof, without the use of identities, of the uniqueness of the decomposition given in Theorem 1.1.

**Proposition 3.1.** *Let $A$ and $B$ be simple $G$–graded algebras with presentations $P_A = (H_A, \alpha, (p_1, \ldots, p_r))$ and $P_B = (H_B, \beta, (q_1, \ldots, q_s))$, respectively. If $A$ and $B$ are $G$–graded isomorphic, then $P_A$ and $P_B$ are equivalent.*

*Proof.* Let $\phi : A \to B$ be a $G$-graded isomorphism. Then $\phi$ must take the $e$–component of $A$ to the $e$–component of $B$, that is, $\phi(A_e) = B_e$. So $\phi|_{A_e} : A_e \to B_e$ is an $F$–algebra isomorphism between these two semisimple (ungraded) $F$–algebras. This isomorphism will take each simple component of $A_e$ onto a simple component of $B_e$. In particular the dimensions of these corresponding components will be the same. It follows that $r = s$, the multiplicities of the $p_i$'s is the same as that of the $q_i$'s and that by possibly rearranging the tuple for $B$ (a basic move), we may assume the $i$–th block of $A_e$ is sent to the $i$-th block of $B_e$. Also because $r = s$, the elementary parts of $P_A$ and $P_B$ have the same dimension. Because $A$ and $B$ have the same dimension we infer that $H_A$ and $H_B$ have the same cardinality.

The next step is to show that the subgroups $H_A$ and $H_B$ are conjugate in $G$. By shifting the tuple of the elementary grading of $A$ (basic move (3)) we may assume that in the tuple $(p_1, \dots, p_r)$ the elements $p_i$ are listed in order of nonincreasing multiplicity and that $p_1 = e$.

Let $H = H_A$. In view of Proposition 1.4 and Remark 1.5, the subalgebra $A_H$ decomposes into a direct sum of $H$-simple subalgebras where each $H$-simple is a diagonal block with a certain number of pages. Note that at least one of the blocks (for example the block coming from the representative $e$) is "full" in the sense that it has the maximal possible number of pages (namely, $|H|$). In fact we get this maximal number precisely when the coset representative is in the normalizer of $H$. Now we consider $B_H = \phi(A_H)$. If $H$ is not conjugate to $H_B$, then there will be no block in the decomposition of $B_H$ in which we get a full number of pages, because the number of pages in each block is the cardinality of $q_i^{-1} H_B q_i \cap H$ for some $i$. But applying $\phi$ to one of the full blocks of $A_H$ will have to give a full block in $B_H$, so we have a contradiction. Therefore $H_A$ and $H_B$ are conjugate. Applying a basic move we may assume $H_A = H_B$.

So at this point we have reduced to the case where $H_A = H_B$, $r = s$, the multiplicities in the tuples are the same, and under the isomorphism $\phi$ the $i$–th block of $A_e$ is sent to the $i$-th block of $B_e$. We next show that we can assume the tuples for $P_A$ and $P_B$ are the same.

Let $H = H_A (= H_B)$. Let $m$ denote the number of distinct $H$–coset representatives in each tuple and let $d_1 \geq d_2 \geq d_3 \geq \cdots \geq d_m$ denote the multiplicities. Now consider the matrix units $e_{i,j} (= 1 \otimes e_{i,j})$. We look at the $F$–span of the images $\phi(e_{11}), \phi(e_{2,2}), \dots, \phi(e_{d_1 d_1})$. This is the $F$–span of commuting semisimple elements in the first component of $B_e$ and so is conjugate in the first component to the space of diagonal matrices there. The conjugating element $b_1$ is an element (of weight $e$) in the first block of $B_e$. We do the same for all $m$ blocks of $A_e$, obtaining elements $b_1, b_2, \dots, b_m$. The sum $b = b_1 + b_2 + \cdots + b_m$ is an invertible element in $B_e$ and the composite of $\phi$ with conjugation by $b$ will take the space of diagonal elements of $A_e$ to the space of diagonal elements of $B_e$. Because $b$ is an invertible element of weight $e$, conjugation by $b$ does not change the presentation for $B$ at all. So by possibly reordering the elements in the tuple for $B$ we may assume $\phi(e_{ii}) = e_{ii}$ for all $i$, $1 \leq i \leq r$. It follows that for all $i, j$, $1 \leq i, j \leq r$, $\phi(e_{ij}) = \phi(e_{ii} e_{ij} e_{jj}) = e_{ii} \phi(e_{ij}) e_{jj}$. Moreover if $t \neq i$, then $e_{tt} \phi(e_{ij}) = 0$, and if $t \neq j$, then $\phi(e_{ij}) e_{tt} = 0$. Hence we must have $\phi(e_{ij}) = \gamma_{ij} u_{h_{ij}} \otimes e_{ij}$, for some $h_{ij} \in H$ and $\gamma_{ij} \in F^{\times}$. In particular, letting $i = 1$, we see that $\phi(e_{1j}) = \gamma_{1j} u_{h_{1j}} \otimes e_{1j}$, where $\gamma_{11} = 1$ and $h_{11} = e$. But the weight of $e_{1j}$ is $p_j$ (recall that $p_1 = e$) and the weight of $\gamma_{1j} u_{h_{1j}} \otimes e_{1j}$ is $q_1^{-1} h_{1j} q_j$. Hence for all $j$, $1 \leq j \leq r$, $p_j = q_1^{-1} h_{1j} q_j$. In other words $q_1 p_j = h_{1j} q_j$. Also, because the first block of $B_e$ has a full number of pages (it corresponds to the first block of $A_e$ which has a full number of pages), the element $a_1$ must normalize $H$. So by applying a basic move of type 3 and one of type 1 we can replace $(q_1, \dots, q_r)$ by $(p_1, \dots, p_r)$ without changing $H$.

Notice also at this point that $\phi$ takes the elementary part of $P_A$ to the elementary part of $P_B$ (in fact for all $i, j$, $\phi(e_{ij})$ is a nonzero constant multiple of $e_{ij}$), and so takes the centralizer of the elementary part of $P_A$ to the centralizer of the elementary part of $P_B$. In other words $\phi$ takes $F^{\alpha} H$ to $F^{\beta} H$ and is a graded isomorphism between these two $G$–graded algebras. It is easy to see that it follows that $\alpha$ and $\beta$ are cohomologous, so we are done. $\square$

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