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SIMPLE GAMES: AN OUTLINE OF THE DESCRIPTIVE THEORY

L. S. Shapley

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SUMMARY

The elementary properties of "simple" games—n–person games in which each coalition is either able to win outright or completely impotent—are set forth.

A table of all simple games with four (or fewer) players is included.

SIMPLE GAMES: AN OUTLINE OF THE DESCRIPTIVE THEORY

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Introduction

The term "simple" was introduced by von Neumann and Morgenstern ([11], Chapter X) to distinguish those n-person games in which each coalition that might form is either completely victorious or completely impotent. These games are especially appropriate in the study of committees and legislatures, or more generally of any political structure in which power, rather than utility, is the fundamental objective.

Since the publication of Theory of Games and Economic Behavior, a good deal of work has been done on simple games, as a glance at the appended bibliography will confirm. The descriptive theory has been widened and deepened, and a number of new solution concepts have been developed to compete with (and supplement) the "stable-set" solutions of [11], which, nevertheless, continue to receive the lion's share of attention.

The present note is an attempt to set forth in a concise way the basic definitions and terminology of the theory, and to develop the significant structural properties of simple games. A complete listing of the n-person simple games for $n \leq 4$ has been included as an appendix. The theory as given here is based on lectures by the author at Princeton in 1953 [9], and has not yet been published elsewhere.

1. Some particular games

To ease into the subject as painlessly as possible, we shall suspend formalities for a few paragraphs while we go through some elementary examples. The main technical terms are underlined when they first appear in context.

Perhaps the simplest of simple games is the straight majority game M_n, with an odd number, n, of players. An example is the House of Representatives (when voting on ordinary legislation) = M_{435}. In M_n there are 2^n possible coalitions, if we count the empty set, and they are of two types: winning (more than n/2 members) and losing (less than n/2 members). Moreover, the complement of every losing coalition is winning.

The last remark is no longer true if n is even. Coalitions of exactly n/2 players are now possible; they neither win nor permit their complements to win. Thus, deadlocks can occur. Games without any such blocking coalitions are called strong.* (Other games—if a term be needed—must be called nonstrong, since we are reserving "weak" for a special purpose.) We shall still consider all non-winning coalitions to be losing, and thus subsume the blocking coalitions under the losing coalitions.

One might be tempted to describe the U.S. Senate on

*"Strong" has much the same force as "zero-sum" in [11].

ordinary majority votes as the nonstrong game M_{96}. However, a little thought shows that the tie-breaking vote of the presiding officer (the Vice President) makes M_{97} the more accurate model since any coalition of 49 or more individuals can win, and no others. Of course, Senate rules allow for a great deal of blocking before issues come to a vote. If we go to the extreme and imagine for a moment that each individual senator is a potential blocking coalition, we would have an example of the pure bargaining game B_{96}, in which all decisions require unanimous consent. B_n has just one winning coalition (the whole set) and $2^n - 1$ losing coalitions, all but one of them (the empty set) blocking. This is an extreme example of a weak game—that is, a game in which at least one player has veto power.

An even more extreme case, useful for formal purposes, is the null game on n players, O_n, in which there are no winning coalitions at all.

All the games so far mentioned are special cases of $M_{n,k}$, the game where there are n players in all and it takes k or more to win.* Thus:

$$M_n = M_{n,(n+1)/2} \qquad \text{(n odd)}$$
$$M_n = M_{n,(n+2)/2} \qquad \text{(n even)}$$
$$B_n = M_{n,n}$$
$$O_n = M_{n,n+1}$$

*These games are the subject of papers by Bott [1] and Gillies [2].

The $M_{n,k}$ games $(k > n/2)$ actually exhaust all simple games which are completely symmetric in the players. Of course, these symmetric games barely scratch the surface.

An example of a nonsymmetric game built up from symmetric games is

$$\underline{\text{Congress}} = M_{97} \times M_{435}$$

(majority in both houses needed to win); another is

$$\underline{\text{Congress overriding a veto}} = (M_{96,64} + O_1) \times M_{435,290}$$

(two-thirds in both houses required), the "$+O_1$" indicating that the 97th player has no vote. A somewhat different sort of example is

$$\underline{\text{UN Security Council}} = B_5 \times M_{6,2}$$

(seven out of eleven to win, but five veto powers). Observe here that $M_{6,2}$, by itself, is not a game in the usual sense. Indeed, completely separate groups of players can form "winning" coalitions simultaneously. We shall call such objects <u>pseudogames</u>. They will be quite useful in the theory of decomposable games, and are of some interest in their own right.*

2. Definitions

We now begin the formal exposition. Let N denote the set of players and $\mathcal{N}$ the set of subsets of N, or "coalitions." Let O denote the empty set of players, $\mathcal{O}$ the empty set of coalitions. (Script capitals will always stand for subsets of $\mathcal{N}$.) Let $\mathcal{S}^+$ denote the set of supersets, $\mathcal{S}^-$ the set of

* Put in terms of characteristic functions, the pseudogames are those whose functions are not superadditive. Strong games are those whose functions are constant-sum, and simple games in general are those whose functions assume just the values 0 and 1, or strategically equivalent games.

subsets, and $\mathcal{S}^*$ the set of complements, of elements of $\mathcal{S}$. Let $\bigcap\mathcal{S}$ denote the intersection, and $\bigcup\mathcal{S}$ the union, of the elements of $\mathcal{S}$.

A <u>game</u> G is an ordered pair:

$$G = (N, \mathcal{W})$$

where $\mathcal{W}$, the set of "winning coalitions," satisfies

(1) $\mathcal{W} = \mathcal{W}^+$

(2) $\mathcal{W} \cap \mathcal{W}^* = \emptyset$.

Define

(3) $\mathcal{L} = \mathcal{N} - \mathcal{W}$ (set of "losing coalitions")

(4) $\mathcal{B} = \mathcal{L} \cap \mathcal{L}^*$ (set of "blocking coalitions")

(5) $\mathcal{W}^m = \{\bigcap$ of all $\mathcal{S}$ for which $\mathcal{S}^+ = \mathcal{W}\}$ (set of "minimal winning coalitions)

We note that

$$\mathcal{L} = \mathcal{L}^- , \qquad \mathcal{B} = \mathcal{B}^* , \text{ and } \mathcal{W}^{m+} = \mathcal{W}.$$

A <u>pseudogame</u> P is an ordered pair

$$P = (N, \mathcal{W})$$

where $\mathcal{W}$ satisfies (1), but not (2). A game or pseudogame is

<u>strong</u> if $\mathcal{B} = \emptyset$

<u>weak</u> if $\bigcap\mathcal{W} \neq 0$

<u>null</u> if $\mathcal{W} = \emptyset$

<u>inessential</u> if $\mathcal{W} = \emptyset$ or $\{\{p\}\}^+$, some $p \in N$.*

*The null game and the dictator games on a fixed set of players are strategically equivalent in the sense of [11], since they all have additive characteristic functions. However, we must distinguish among them here.

The last three definitions might just as well have been given in terms of $\mathcal{W}^m$. We note that pseudogames may be strong or nonstrong, but are never weak, null, or inessential.

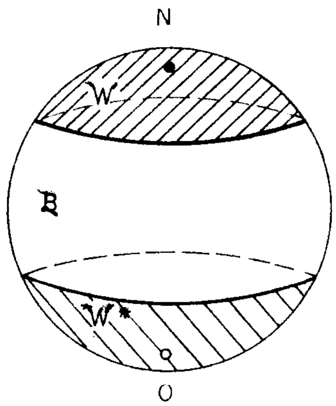
THEOREM. No essential game is both weak and strong.

Proof. Suppose G is weak. Take $p \in \bigcap \mathcal{W}$. Suppose G is also strong. Then one of $\{p\}$, $N - \{p\}$ is in $\mathcal{W}$. Since the latter is not possible, we have $\{p\} \in \mathcal{W}$. Hence $\{\{p\}\}^+ \subseteq \mathcal{W}$, and $N-\{p\} \in \mathcal{L}$. Hence $\{N-\{p\}\}^- \subseteq \mathcal{L}$. We conclude that $\mathcal{W} = \{\{p\}\}^+$, making G inessential. QED.

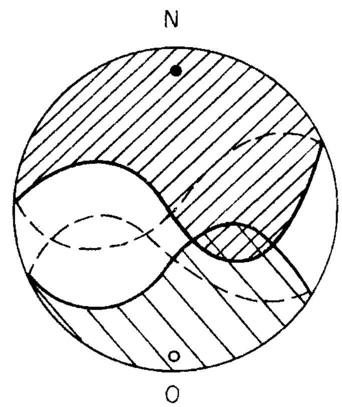
If $\mathcal{W}$ is enlarged (preserving property (1)), we shall say that the game or pseudogame in question has been strengthened; if $\mathcal{W}$ is diminished, we shall say it has been weakened. It can be verified that repeated strengthening of a game or pseudogame will eventually make it strong, moreover once it becomes strong it remains so. A similar statement holds in the case of repeated weakening.

3. A graphical representation

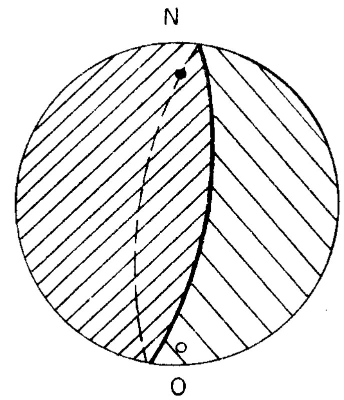
The foregoing may conveniently be visualized by taking the subsets of N to be points on a sphere, with complementary sets diametrically opposed. Let subsets of equal size be located on the same latitude, the size increasing from 0 at the south pole to n at the north pole. Set-inclusion can be represented by lines joining each set to its immediate subsets and supersets. (For $n > 3$ these lines cannot be drawn without crossing.) Sets of sets can be represented by regions on the sphere.



An $M_{n,k}$ game



A pseudogame
(non-strong)



A dictator game
(weak and strong)

For the game $M_{n,k}$, the set $\mathcal{W}$ is now a circular region centered on the north pole, and the set $\mathcal{W}^*$ is the antipodal cap around the south pole. $\mathcal{B}$ is the equatorial zone (possibly empty) between them. More generally, $\mathcal{W}$ is some neighborhood of N; $\mathcal{W}^*$ the corresponding neighborhood of O. In a pseudogame, $\mathcal{W}$ and $\mathcal{W}^*$ must overlap somewhere. In a strong game, $\mathcal{W}$ and $\mathcal{W}^*$ can be made to cover the sphere exactly without overlapping. Strengthening a game extends the polar caps; weakening it pulls them back. In a weak game, the $\mathcal{B}$ region extends right down to the south pole, since there is a one-person blocking coalition, and hence right up to the north pole too, since $\mathcal{B} = \mathcal{B}^*$.

4. Duality

The dual of a game or pseudogame $G = (N, \mathcal{W})$ is defined by

$$G^* = (N, \mathcal{L}^*).$$

For example, we have $M^*_{n,k} = M_{n,n-k+1}$. The following properties of duals are fairly obvious, if one bears the graphical representation in mind.

THEOREM: (a) $G = G^{**}$

(b) G^* is a game if and only if G is strong;

(c) $G = G^*$ if and only if G is a strong game.

The usefulness of the duality concept depends heavily on the inclusion of pseudogames in our theory, since, by (b) and (c), the dual of a game is either the same game over again, or a pseudogame.

5. Composition of simple games

Two games or pseudogames $G_1 = (N_1, \mathcal{W}_1)$ and $G_2 = (N_2, \mathcal{W}_2)$ will be said to be <u>isomorphic</u>, $G_1 \simeq G_2$, if there is a one-one function from N_1 onto N_2 which maps $\mathcal{W}_1$ onto $\mathcal{W}_2$. Hitherto we have been able to speak of isomorphic games as if they were identical; we must now be more careful. Let N' and N'' be fixed, disjoint sets of players, and let $N = N' \cup N''$.

The <u>product</u> $(N', \mathcal{W}') \times (N'', \mathcal{W}'')$ is defined to be $(N, \mathcal{W})$, where $\mathcal{W}$ comprises those sets S for which $S \cap N' \in \mathcal{W}'$ <u>and</u> $S \cap N'' \in \mathcal{W}''$.

The <u>sum</u> $(N', \mathcal{W}') + (N'', \mathcal{W}'')$ is defined to be $(N, \mathcal{W})$, where $\mathcal{W}$ comprises those sets S for which $S \cap N' \in \mathcal{W}'$ <u>or</u> $S \cap N'' \in \mathcal{W}''$.

If we want these operations to behave properly with respect to isomorphism (for instance: $G_1 \simeq G_2 \Rightarrow G_1 \times H \simeq G_2 \times H$), then we cannot define a unique product (or sum) for games whose sets of players overlap. Instead, we end up with a whole equivalence class of isomorphic products (sums) by first replacing the original games by non-overlapping isomorphic images, and then forming the product or sum. We shall avoid the necessity for this construction by consistently assuming all games and pseudogames appearing in composition to be disjoint.

Multiplication and addition satisfy the usual associativity and communtativity laws, but are not distributive. Instead we have the useful <u>duality principle</u>:

$$(G+H)^* = G^* x H^*$$

$$(GxH)^* = G^* + H^*$$

The product of two games is generally much weaker than their sum. We give below some of the elementary properties of the two operations.

<u>Products</u>. The product of two games, or of a game and a pseudogame, is a game. The product of two pseudogames is a pseudogame. A product is never strong unless one factor consists entirely of dummies (see below). A product is weak if and only if one of the factors is weak; null if and only if one of the factors is null.

<u>Sums</u>. A sum is always a pseudogame (unless one term consists entirely of dummies—see below), and hence is never weak. A sum is strong if and only if at least one term is strong.

<u>Weak games</u>. Any weak game can be factored

$$(N, \mathcal{W}) = B_W \times (N-W, \mathcal{W}')$$

where W is $\bigcap \mathcal{W}$, or any nonvoid subset of $\bigcap \mathcal{W}$, and $\mathcal{W}'$ is the set of intersections of elements of $\mathcal{W}$ with N-W. In particular, B_n is isomorphic to $B_1 \times B_1 \times \ldots \times B_1$ (n times).

Dummies. A game with dummies can be decomposed in two ways, in view of the identity

$$G + O_N = G \times O_N^* .$$

A dummy in a component remains a dummy in the product or sum.

Decomposition. A game or pseudogame which is neither a sum nor a product is said to be prime. Games or pseudogames, except for those with dummies, can be resolved in only one way into prime components (except for the order of terms). All strong games without dummies are prime.

Examples. A multicameral legislature provides an excellent example of a product (see §1 above). Sums, being pseudogames, are harder to find in nature, in the pure state. However, situations of the type

$$(G + H) \times K$$

are quite common—two bodies, each with power to act, but subject to the veto of a third. Sometimes the "x K" is not actually present, but is replaced by some kind of inertial or dynamical process, which mitigates the formal ambiguity of the pseudogame G + H. For example, the U.S. Constitution can be amended by the state legislatures, even if Congress is opposed; alternatively, it can be amended by Congress and special conventions in the states, over the opposition of the state legislatures. Here we have the sum of two games—formally a pseudogame. However, each term is so weak and the inertia so great that it is unlikely that disjoint winning coalitions will ever form on opposite sides of a crucial issue.

6. Weighted majority games

Often it is possible to specify the winning coalitions of a simple game by attributing "voting strengths" $w_1, w_2, \ldots, w_n$ to the players and finding a number m with the property that

$$S \in \mathcal{W} \text{ if and only if } \sum_{i \in S} w_i \geq m.$$

A game for which this can be done is known as a weighted majority game and is denoted by the symbol

$$[m; w_1, \ldots, w_n] .$$

The symbol for a particular game is, of course, never unique; in fact, one can always be found in which all the numbers are integers.

The dual of a weighted majority game is obtained simply by replacing m by w—m+1, w being the sum of the w_i (expressed as integers). It is easy to see that all weighted majority pseudogames are strong.

The ownership of a corporation by stockholders is a good example of a weighted majority game—w_i being the number of shares held by the i^{th} player. Of course, our ordinary majority games $M_{n,k}$ are trivially weighted majority games:

$$M_{n,k} = [k; 1, 1, \ldots, 1] .$$

A somewhat more unusual example, since the voting strengths are not explicit in the rules, is the U.N. Security Council, mentioned earlier, which can be represented

$$B_5 \times M_{6,2} = [27; 5, 5, 5, 5, 5, 1, 1, 1, 1, 1, 1].$$

There are many games which do not have such representations. In fact, it is not possible in general to impose a simple order on the players which ranks their relative effectivness in coalitions. For example, $M_3 \times M_3$ has no weighted majority symbol. (If it did, the weights could be made equal, by symmetry. But some four-person coalitions win and some lose, so there is no value of m which works.) However, almost all games and pseudogames with fewer than five players do possess symbols, and this fact makes them much easier to describe and catalogue (see below).

7. Solution theory

We confine ourselves here to a few remarks on the principle methods of solving simple games which are now available. Since the existence of optimal strategies for each coalition is already assumed in the formulation of the games, the questions to be "solved" have more to do with such things as the formation and stability of coalitions, and the distribution of power or payoff among the players that can be expected to occur.

Values. The relative power inherent in the different player positions can be defined and measured, the definition following uniquely from postulates of symmetry and additivity. (See [7], [9]. An informal account is given in [10].)

Coalition structures. The stability of a coalition structure under a given distribution of spoils can be defined, to provide insight into the question of what coalition structures and what distribution of spoils are most likely to occur. (See [5], [9].)

Stable sets. These are the "solutions" of von Neumann and Morgenstern, describing social structures more subtle and intricate than just coalitions, or sets of coalitions. They provide, in the simplest cases, the "asking price" for each player's participation in a victorious coalition; in other cases "bargaining curves" along which such prices are allowed to vary simultaneously, "discriminatory solutions" in which one or more players are excluded from bargaining, although not necessarily from the payoff, and many other phenomena. (See [1]. [2], [4], [6], [9], [11].)

Appendix

The table on the next page is a complete list (up to isomorphisms) of the 30 games and pseudogames with 4 or fewer players. The weighted majority symbol is given where available. In the other cases, the $(N, \mathcal{W})$ notation is used, in terms of players p, q, r, s. The factors given are all prime, with the exception of B_k and B_k^* for $k > 1$, which decompose into $B_1 \times \ldots \times B_1$ and $B_1 + \ldots + B_1$, respectively. Duals are listed side by side; the strong games are self-dual. The pseudogame "q" has the curious property of being isomorphic to its dual, the pseudogame $(\overline{pqrs}, \{\overline{pr}, \overline{qr}, \overline{qs}\}^+)$, but not equal to it.

FOUR-PERSON SIMPLE GAMES AND PSEUDOGAMES

(GAMES)		(PSEUDOGAMES)
a. $[1;0,0,0,0] = O_4$		a*. $[0;0,0,0,0] = O_4^*$
b. $[1;1,0,0,0] = B_1 + O_3$	weak; strong	
c. $[2;1,1,0,0] = B_2 + O_2$	weak	c*. $[1;1,1,0,0] = B_2^* + O_2$
d. $[3;1,1,1,0] = B_3 + O_1$	weak	d*. $[1;1,1,1,0] = B_3^* + O_1$
e. $[3;2,1,1,0] = (B_1 x B_2^*) + O_1$	weak	e*. $[2;2,1,1,0] = B_1 + B_2 + O_1$
f. $[2;1,1,1,0] = M_3 + O_1$	strong	
g. $[4;1,1,1,1] = B_4$	weak	g*. $[1;1,1,1,1] = B_4^*$
h. $[5;2,2,1,1] = B_2 x B_2^*$	weak	h*. $[2;2,2,1,1] = B_2^* + B_2$
i. $[4;2,1,1,1] = B_1 x M_3$	weak	i*. $[2;2,1,1,1] = B_1 + M_3$
j. $[3;1,1,1,1] = M_4$ (prime)		j*. $[2;1,1,1,1] = M_{4,2}$ (prime)
k. $[5;3,2,1,1] = B_1 x (B_1 + B_2)$	weak	k*. $[3;3,2,1,1] = B_1 + (B_1 x B_2^*)$
l. $[4;2,2,1,1]$ (prime)		l*. $[3;2,2,1,1]$ (prime)
m. $[5;3,2,2,1]$ (prime)		m*. $[4;3,2,2,1]$ (prime)
n. $[4;3,1,1,1] = B_1 x B_3^*$	weak	n*. $[3;3,1,1,1] = B_1 + B_3^*$
o. $[3;2,1,1,1]$ (prime)	strong	
p. $(\overline{pqrs}, \{\overline{pq}, \overline{rs}\}^+) = B_2 + B_2$		p*. $(\overline{pqrs}, \{\overline{pr}, \overline{ps}, \overline{qr}, \overline{qs}\}^+)$
q. $(\overline{pqrs}, \{\overline{pq}, \overline{qr}, \overline{rs}\}^+)$ (prime)		

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