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SIMPLE GRAPHS WHOSE 2-DISTANCE GRAPHS ARE PATHS OR CYCLES

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We study all finite simple graphs whose 2-distance graphs have maximum degree 2.

1. Introduction

Let (X,d) be a metric space with distance function $d : X \times X \to [0,\infty)$. For each set D of distances, the distance graph $G(X,D)$ is defined as a graph whose vertex set is X and two vertices $x,y \in X$ are adjacent if $d(x,y) \in D$. Even though distance graphs are extensively studied when X is a set of integers (see [1] for instance), but up to our knowledge there is no result when X is the set of vertices of a graph and d is the ordinary distance function between two vertices. Let Γ be a simple graph and n be a natural number. Clearly, $G(V(\Gamma),\{1,\dots,n\})$ is the natural nth power of Γ.

Definition 1.1. Let Γ be a graph. The n-distance graph Γ_n of Γ is a graph with vertex set $V(\Gamma)$ in which two vertices x and y are adjacent in Γ_n if and only if $d_\Gamma(x,y) = n$. In other words, $\Gamma_n := G(V(\Gamma),\{n\})$.

Concerning n-distance graphs the natural question is

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Question 1.2. Which graphs are the n-distance graph of some other graphs? In other words, when the equation $X_n \cong \Gamma$ has a solution X for a given graph Γ. If yes, find all the possible solutions of the equation.

Example 1.3. Let Γ be a graph with diameter 2. Then $\Gamma_2 = \Gamma^c$, the complement of Γ. Therefore, $(\Gamma^c)_2 = \Gamma$, whenever $\mathrm{diam}(\Gamma^c) = 2$. In particular, if T is a tree that is neither a star nor a double star, then $(T^c)_2 = T$. As well $(C^c_{2n+1})_2 = C_{2n+1}$, where C_{2n+1} is an odd cycle of length $2n+1 \geq 7$. However, C^c_{2n+1} is not the only solution to the equation $X_2 \cong C_{2n+1}$ since $(C_{2n+1})_2 \cong C_{2n+1}$.

The aim of this paper is to study the structure of those graphs Γ whose 2-distance graphs have special properties. Indeed, we shall classify all finite simple graphs whose 2-distance graphs are either paths or cycles. In what follows, all graphs will be finite undirected simple graphs without loops (for some notions on graph theory which does not appear in the paper the reader is referred to [2]).

2. Main results

Let us start by recalling the following definitions for graphs. If Γ is a graph, $\Delta(\Gamma)$ denotes the maximum degree of its vertices, for every vertex x in Γ, $N_\Gamma(x)$ is the set of all the neighbors of x and for any subset X of $V(\Gamma)$, $N_\Gamma(X)$ denotes the set of all vertices of $V(\Gamma) \setminus X$ adjacent to some vertex of X. Clearly, $N_\Gamma(\{x\}) = N_\Gamma(x)$ for all x in $V(\Gamma)$. Finally, $E_\Gamma(X,Y)$ denotes the set of all the edges $\{x,y\}$ of Γ such that $x \in X$ and $y \in Y$.

We are interested in graphs Γ such that Γ_2 is connected. So, either $V(\Gamma)$ consists of a single vertex or $|V(\Gamma)| \geq 4$.

For such graphs there is the following upper bound for the maximum vertex-degree of Γ_2.

Theorem 2.1. *Let Γ be a graph such that Γ_2 is connected. If Γ is not a single vertex, then $\Delta(\Gamma_2) \leq |V(\Gamma)| - 3$.*

Proof. If $\Delta(\Gamma_2) = |V(\Gamma)| - 1$, then $\deg_{\Gamma_2}(x) = |V(\Gamma)| - 1$ for some vertex x in Γ. Thus $\deg_\Gamma(x) = 0$ and this is not possible since in Γ_2 there is at least one vertex y adjacent with x and so in Γ a path of length 2 from x to y. Now, suppose that $\deg_{\Gamma_2}(x) = |V(\Gamma)| - 2$ for some $x \in V(\Gamma)$. Then $V(\Gamma) \setminus (\{x\} \cup N_{\Gamma_2}(x)) = \{y\}$ for some y, which implies that $N_\Gamma(y) = V(\Gamma) \setminus \{y\}$. Thus $\deg_{\Gamma_2}(y) = 0$, which is a contradiction since Γ_2 is connected. □

Although there is no graph with n vertices whose 2-distance graph has maximum degree $\geq n-2$, there are infinitely many graphs whose 2-distance graphs are connected and have maximum degree 2.

Theorem 2.2. *Let Γ be a graph such that Γ_2 is a path. Then either Γ is the complement of a path of length ≥ 4 or it is isomorphic to one of the following graphs:*

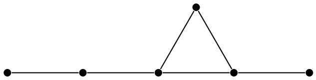

Figure 1

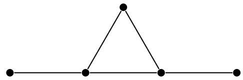

Figure 2

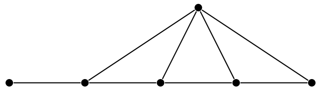

Figure 3

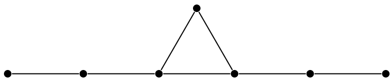

Figure 4

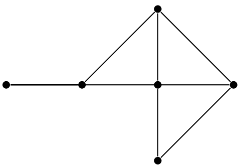

Figure 5

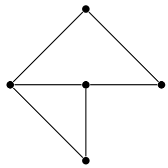

Figure 6

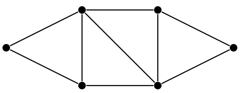

Figure 7

Proof. Let $x \in V(\Gamma)$ and $\deg_{\Gamma_2}(x) = 1$. Then there exists a unique vertex $y \in V(\Gamma)$ such that $d_\Gamma(x,y) = 2$. Hence $N_\Gamma(x) \cap N_\Gamma(y) \neq \emptyset$. We proceed in some steps.

Case 1. $N_\Gamma(x) \subseteq N_\Gamma(y)$. If $N_\Gamma(x) = N_\Gamma(y)$, then $\deg_{\Gamma_2}(y) = 1$ and the edge $\{x,y\}$ is a connected component of Γ_2 and hence Γ_2 is not connected, a contradiction. Since $\Delta(\Gamma_2) = 2$, we should have $|N_\Gamma(y) \setminus N_\Gamma(x)| \leq 2$. If $|N_\Gamma(x)| \geq 2$ and $|N_\Gamma(y) \setminus N_\Gamma(x)| = 2$, then Γ_2 contains a 4-cycle, which is a contradiction. Thus $(|N_\Gamma(x)|, |N_\Gamma(y) \setminus N_\Gamma(x)|) = (2,1)$, $(1,1)$, or $(1,2)$. If $(|N_\Gamma(x)|, |N_\Gamma(y) \setminus N_\Gamma(x)|) = (2,1)$, then x and its neighbors a and b have degree 1 in Γ_2 when a,b are adjacent and $N_\Gamma(y)$ is the set of vertices of a triangle in Γ_2 when a,b are not adjacent, which are both impossible. Now, suppose that $(|N_\Gamma(x)|, |N_\Gamma(y) \setminus N_\Gamma(x)|) = (1,1)$. Let $v_0 = x$, v_1 be the vertex adjacent to x, $v_2 = y$ and v_3 be the single vertex of $N_\Gamma(y) \setminus N_\Gamma(x)$. Since Γ_2 is connected, $\deg_{\Gamma_2}(v_2) = 2$ and hence $N_\Gamma(v_3) \setminus N_\Gamma(v_2) = \{v_4\}$ for some $v_4 \in V(\Gamma)$. Continuing this way, we obtain an infinite sequence of vertices $v_0, v_1, \ldots$ which determines an infinite path. But then Γ_2 is the union of two disjoint infinite paths, which is a contradiction. Thus $(|N_\Gamma(x)|, |N_\Gamma(y) \setminus N_\Gamma(x)|) = (1,2)$. Then a simple verification shows that Γ is isomorphic to one of the graphs in Figures 1 or 4 and $\Gamma_2 \cong P_6$ or P_7, respectively.

Case 2. $N_\Gamma(x) \not\subseteq N_\Gamma(y)$. Let $A = N_\Gamma(x) \setminus N_\Gamma(y)$, $B = N_\Gamma(x) \cap N_\Gamma(y)$ and $C = N_\Gamma(y) \setminus B$. First assume that $C \neq \emptyset$. Since $C \subseteq N_{\Gamma_2}(b)$ and $B \subseteq N_{\Gamma_2}(c)$ for all $b \in B$ and $c \in C$, it follows that $|B|, |C| \leq 2$. Hence, $2 \leq |B| + |C| \leq 3$ for otherwise $|B| = |C| = 2$ and $B \cup C$ contains a 4-cycle in Γ_2. Let $b \in B$. If $A \not\subseteq N_\Gamma(b)$, then $(A \setminus N_\Gamma(b)) \cup C \subseteq N_{\Gamma_2}(b)$ and so $|A \setminus N_\Gamma(b)| = |C| = 1$. Thus, $|C| = 2$ implies $|B| = 1$ and $A \subseteq N_\Gamma(b)$, from which it follows that $A \cup \{x\} \subseteq N_{\Gamma_2}(y)$. Thus $A = \{a\}$ is a singleton and $\{a, x, y\}$ is a connected component of Γ_2, which is a contradiction. Therefore $C = \{c\}$ is a singleton.

If $B = \{b_1, b_2\}$, then b_1, b_2 are adjacent and

$$(A \cap N_\Gamma(B)) \cup \{x\} \subseteq N_{\Gamma_2}(y),$$

which implies that $|A \cap N_\Gamma(B)| \leq 1$. In particular, if $|A| > 1$ then there exists $a \in A$ such that $a \notin N_\Gamma(B)$, hence $B \cup \{a, c\}$ induces a 4-cycle in Γ_2, which is a contradiction. Thus $A = \{a\}$ is a singleton. If $a \notin N_\Gamma(B)$, then again $B \cup \{a, c\}$ is a 4-cycle in Γ_2, a contradiction. Thus $a \in N_\Gamma(B)$. If $a \in N_\Gamma(b_1) \cap N_\Gamma(b_2)$, then Γ_2 is the union of two disjoint paths of length 2, which is impossible. Thus $a \in N_\Gamma(B) \setminus N_\Gamma(b_1) \cap N_\Gamma(b_2)$ and Γ is isomorphic to Figure 5 and $\Gamma_2 \cong P_6$.

If $B = \{b\}$ is a singleton, then since $A \subseteq N_{\Gamma_2}(b) \cup N_{\Gamma_2}(y)$, it follows that $|A| \leq 2$. If $A = \{a_1, a_2\}$, then $|A \cap N_{\Gamma_2}(b)| = |A \cap N_{\Gamma_2}(y)| = 1$, and a_1, a_2 are non-adjacent. Hence Γ is isomorphic to Figure 1 and $\Gamma_2 \cong P_6$.

If $A = \{a\}$ is a singleton, then we have two possibilities. If a and b are adjacent, then $\{a, x, y\}$ induces a connected component of Γ_2, which is a contradiction. Thus a and b are not adjacent. Since Γ_2 is connected, by putting $v_0 = a$, $v_1 = x$, $v_2 = b$, $v_3 = y$ and $v_4 = c$, and applying the same argument as in Case 1, one can easily see that Γ is an infinite path, which is again a contradiction.

Now suppose that $C = \emptyset$. First suppose that $|B| \leq 2$. If $|B| = 1$ then $|A| \leq 3$ and if $|B| = 2$ then $|A| \leq 2$ for Γ_2 is acyclic of the maximum degree two. If $|A| = |B| = 1$, then Γ_2 is disconnected, which is a contradiction. Hence either $|A| > 1$ or $|B| > 1$. Sicne $\deg_{\Gamma_2}(y) = 2$ we observe that $E_\Gamma(A, B) \subseteq \{\{a, b\} : b \in B\}$ for some $a \in A$. But then $|E_\Gamma(A, B)| = 1$ for otherwise $B = \{b_1, b_2\}$ and a is adjacent to b_1 and b_2, form which it follows that Γ_2 is disconnected. Therefore, $E_\Gamma(A, B) = \{\{a, b\}\}$ for some $b \in B$, which implies that $E_{\Gamma_2}(A, B) = \{\{x, y\} : x \in A, y \in Y\} \setminus \{\{a, b\}\}$. Now, it is easy to see that Γ is isomorphic to the graphs drawn in Figures 2, 3, 6 and 7 whenever $(|A|, |B|) = (2,1)$, $(3,1)$, $(1,2)$ or $(2,2)$, respectively. As a result, $\Gamma_2 \cong P_5$ or P_6.

If $|B| \geq 3$, then by the same argument as before $|A| = 1$. Let $B = \{b_1, \ldots, b_m\}$ and assume that $A = \{b_0\}$. Then by a relabeling of the b_i if necessary, b_i is adjacent to $b_{i+2}, \ldots, b_m$. Therefore, Γ is the complement of a path of length $m+2$. The complement of Γ is drawn in Figure 8. □

Theorem 2.3. *Let Γ be a graph such that Γ_2 is a cycle. Then Γ is an odd cycle of length ≥ 5, the complement of a cycle of length ≥ 6 or it is isomorphic to one of the following graphs:*

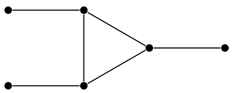

Figure 8

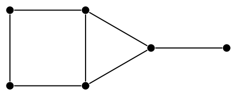

Figure 9

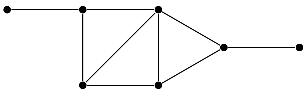

Figure 10

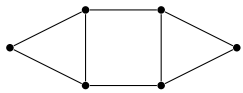

Figure 11

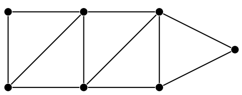

Figure 12

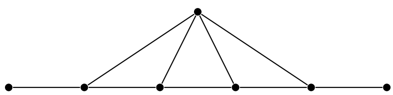

Figure 13

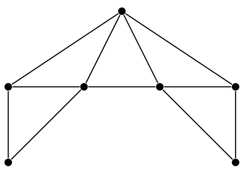

Figure 14

Proof. Let $x \in V(\Gamma)$. Since $\deg_{\Gamma_2}(x) = 2$, there exist exactly two distinct vertices y and z in $V(\Gamma)$ such that $d_\Gamma(x,y) = d_\Gamma(x,z) = 2$. Let

$$A = (N_\Gamma(x) \cap N_\Gamma(y)) \setminus C,$$
$$B = (N_\Gamma(x) \cap N_\Gamma(z)) \setminus C,$$

$$\begin{aligned}
C &= N_\Gamma(x) \cap N_\Gamma(y) \cap N_\Gamma(z), \\
D &= N_\Gamma(x) \setminus (A \cup B \cup C), \\
E &= N_\Gamma(y) \setminus (A \cup B \cup G), \\
F &= N_\Gamma(z) \setminus (A \cup B \cup G), \\
G &= (N_\Gamma(y) \cap N_\Gamma(z)) \setminus N_\Gamma(x).
\end{aligned}$$

We consider two cases:

Case 1. y and z are not adjacent in Γ.

If $C \neq \emptyset$ or $G \neq \emptyset$, then $\{x,y,z\}$ induces a cycle in Γ_2, which is a contradiction. Therefore $C = G = \emptyset$. First suppose that $E_\Gamma(A,B) \neq \emptyset$. Let $\{a,b\} \in E_\Gamma(A,B)$, where $a \in A$ and $b \in B$. If $D \neq \emptyset$, then clearly $E_\Gamma(A,D) = E_\Gamma(B,D) = \emptyset$ for otherwise either $d_{\Gamma_2}(y) \geq 3$ or $d_{\Gamma_2}(z) \geq 3$, which is a contradiction. Let $d \in D$. Then $\{x,y,b,d,a,z,x\}$ is a cycle in Γ_2, which implies that $A = \{a\}$, $B = \{b\}$, $D = \{d\}$ and $E = F = \emptyset$. Hence Γ is isomorphic to the graph in Figure 8 and $\Gamma_2 \cong C_6$. Now suppose that $D = \emptyset$. If $E = F = \emptyset$, then $(|A|,|B|) = (1,2)$, $(2,1)$ or $(2,2)$. If $(|A|,|B|) = (1,2)$ or $(2,1)$, then A,B are independent sets and Γ should be isomorphic to the graph in Figure 9 and $\Gamma \cong C_6$. If $(|A|,|B|) = (2,2)$, then clearly $E_\Gamma(A,B) = \{\{a,b\}\}$ and $A \cong B \cong K_2$, which implies that Γ is isomorphic to the graph in Figure 14 and $\Gamma_2 \cong C_7$. If $E \neq \emptyset$ and $F = \emptyset$, then a is adjacent to some vertex $e \in E$, from which it follows that $E_\Gamma(A,B) = \{\{a,b\}\}$, $A \cong K_2$, $B = \{b\}$ and $E = \{e\}$. Hence Γ is isomorphic to the graph in Figure 10 and $\Gamma_2 \cong C_7$. If $E = \emptyset$ and $F \neq \emptyset$, then similarly Γ is isomorphic to the graph in Figure 10. If $E,F \neq \emptyset$, then $|A| = |B| = |E| = |F| = 1$, which leads to a contradiction.

Second assume that $E_\Gamma(A,B) = \emptyset$. If $D \neq \emptyset$, then we may assume that $E_\Gamma(B,D) \neq \emptyset$ for otherwise, by symmetry, $E_\Gamma(A,D) = E_\Gamma(B,D) = \emptyset$ whereas $\{a,b,d\}$ induces a cyclic in Γ_2 for all $a \in A$, $b \in B$ and $d \in D$, a contradiction. Let $\{b,d\} \in E_\Gamma(B,D)$. If d is adjacent to some vertex a of A, then $\{x,y,d,z\}$ induces a cyclic in Γ_2, which is a contradiction. Thus $E_\Gamma(A,\{d\}) = \emptyset$, which implies that $A \cup \{z\} \subseteq N_{\Gamma_2}(d)$, hence $A = \{a\}$ is a singleton. Then $B \cup \{d\} \subseteq N_{\Gamma_2}(a)$, from which it follows that $B = \{b\}$ and $E = \emptyset$. Hence there exists a vertex $d' \in D$ adjacent to a. If $d'' \in D \setminus \{d,d'\}$, then either $\{x,d',d''\} \subseteq N_{\Gamma_2}(y)$ or $\{b,d,d''\} \subseteq N_{\Gamma_2}(a)$, which is impossible. Thus $D = \{d,d'\}$. A simple verification shows that d' is adjacent to d but not to b and consequently $F = \emptyset$. So Γ is isomorphic to the graph in Figure 13 and $\Gamma_2 \cong C_7$.

Now suppose that $D = \emptyset$. Since $E_{\Gamma_2}(A,B) = \{\{a,b\} : a \in A, b \in B\}$, it follows that $|A|,|B| \leq 2$. If $|A| = 2$, then since $A \cup F \subseteq N_{\Gamma_2}(B)$, it follows that $F = \emptyset$ and consequently $N_{\Gamma_2}(z) = \{x\}$, which is a contradiction. Thus $A = \{a\}$ and similarly $B = \{b\}$ are singletons. Hence $E = \{e\}$ and $F = \{f\}$ are singleton too. If e and f are adjacent, then $\Gamma_2 \cong C_7$. Otherwise, it is easy to see that $N_\Gamma(e) = \{y, e'\}$ and $N_\Gamma(f) = \{z, f'\}$. Again, if e' and f' are adjacent, then $\Gamma_2 \cong C_9$. Otherwise, we may continue the process in the same way and reach to any odd cyclic of length ≥ 7. Therefore, $\Gamma \cong \Gamma_2 \cong C_{2n+1}$ for some $n \geq 3$.

Case 2. y and z are adjacent in Γ.

First suppose that $C = \emptyset$. Then $A, B \neq \emptyset$. Since $A \cup E \cup \{x\} \subseteq N_{\Gamma_2}(z)$ and $B \cup F \cup \{x\} \subseteq N_{\Gamma_2}(y)$, it follows that $A = \{a\}$ and $B = \{b\}$ are singletons and $E = F = \emptyset$. If $D \neq \emptyset$, then $E_\Gamma(A, D) = E_\Gamma(B, D) = \emptyset$ for otherwise either $\deg_{\Gamma_2}(y) > 2$ or $\deg_{\Gamma_2}(z) > 2$, which is impossible. Hence $D = \{d\}$ is a singleton and $G = \emptyset$. Thus a and b are adjacent and Γ is isomorphic to the graph in Figure 9 and $\Gamma_2 \cong C_6$. If $D = \emptyset$, then we have two cases. If $E_\Gamma(A, B) = \emptyset$, then $G = \emptyset$ and $\Gamma \cong \Gamma_2 \cong C_5$. If $E(A, B) \neq \emptyset$, then $G = \{g\}$ is a singleton. Thus Γ is isomorphic to the graph in Figure 11 and $\Gamma_2 \cong C_6$.

Now, assume that $C \neq \emptyset$ and $c \in C$. If $E \neq \emptyset$ then since $E \cup \{x\} \subseteq N_{\Gamma_2}(z)$, it follows that $E = \{e\}$ is a singleton and $A = \emptyset$. Similarly, if $F \neq \emptyset$ then $F = \{f\}$ is a singleton and $B = \emptyset$. We have three cases:

If $E, F \neq \emptyset$ then since $\{e, f\} \subseteq N_{\Gamma_2}(c)$, it follows that $G = \emptyset$ and consequently $D = \emptyset$. Hence Γ is isomorphic to the graphs in Figures 8 and 9 according to e and f are adjacent or not, respectively. In both cases $\Gamma_2 \cong C_6$.

If $E \neq \emptyset$ and $F = \emptyset$, then since $C \cup \{z\} \subseteq N_{\Gamma_2}(e)$, it follows that $C = \{c\}$ is a singleton. If $G \neq \emptyset$ then since $G \cup \{e\} \subseteq N_{\Gamma_2}(c)$ we should have $G = \{g\}$ is a singleton and consequently $D = \emptyset$, and e, g are adjacent. If $B \neq \emptyset$, then $B \subseteq N_{\Gamma_2}(y) \cap N_{\Gamma_2}(g)$, which implies that $B = \{b\}$ is a singleton. Clearly b, c are adjacent. Hence, Γ is isomorphic to the graph in Figure 12 and $\Gamma_2 \cong C_7$. If $B = \emptyset$ then e is adjacent to a vertex h different from g, which implies that Γ is isomorphic to the graph in Figure 10 and $\Gamma \cong C_7$. Now, assume that $G = \emptyset$. If $B \neq \emptyset$ then since $B \cup \{x\} \subseteq N_{\Gamma_2}(y)$, we should have $B = \{b\}$ is a singleton. If b and c are not adjacent, then $D = \emptyset$, which implies that Γ is isomorphic to the graph in Figure 9 and $\Gamma_2 \cong C_6$. If b and c are adjacent, then since $D \subseteq N_{\Gamma_2}(b) \cap N_{\Gamma_2}(c)$, it follows that $D = \{d\}$ is a singleton and $E_\Gamma(B, D) = E_\Gamma(C, D) = \emptyset$. Hence Γ is isomorphic to the graph in Figure 10 and $\Gamma_2 \cong C_7$. If $B = \emptyset$ then $D \neq \emptyset$ and there exists $d \in D$, which is not adjacent to c. If $D \neq \{d\}$ then there exists $d' \in D \setminus \{d\}$, which implies that $d' \in N_{\Gamma_2}(c) \cup N_{\Gamma_2}(z)$, which is impossible. Thus $D = \{d\}$ is a singleton. But then $N_{\Gamma_2}(d) = \{c\}$, which is a contradiction. Similarly, if $E = \emptyset$ and $F \neq \emptyset$, then the result follows.

Finally, assume that $E, F = \emptyset$. If $G \neq \emptyset$ then since $G \subseteq N_{\Gamma_2}(C)$, it follows that $|G| \leq 2$ and $|C| + |G| \leq 3$. If $|G| = 2$ then $C = \{c\}$ is a singleton and $A \cup B \cup D \subseteq N_\Gamma(c)$. Since $G \subseteq N_{\Gamma_2}(A) \cap N_{\Gamma_2}(B)$, we should have $A = B = \emptyset$. If $D \neq \emptyset$ then $\{x, y, d, z\}$ induces a cycle in Γ_2 for all $d \in D$, which is a contradiction. Thus $D = \emptyset$. Clearly, the vertices of G are adjacent and $E(G, V(\Gamma) \setminus (\{x, y, z, c\} \cup G)) \neq \emptyset$. Let $\{g, h\}$ be an edge, where $g \in G$ and $h \in V(\Gamma) \setminus (\{x, y, z, c\} \cup G)$. But then

$\{x,y,h,z\}$ induces a cycle in Γ_2, which is a contradiction. Thus $G=\{g\}$ is a singleton. As $A\cup B\cup C\subseteq N_{\Gamma_2}(g)$, either $A=\emptyset$ or $B=\emptyset$. Without loss of generality, assume that $B=\emptyset$. If $A\neq\emptyset$, then $A=\{a\}$ and $C=\{c\}$ are singletons, and a,c are adjacent. Since $N_{\Gamma_2}(c)\subseteq\{g\}\cup D$, it follows that $D\neq\emptyset$. Let $d\in D$, which is not adjacent to c. Then a and d are adjacent, from which it follows that $D=\{d\}$ is a singleton. Thus Γ is isomorphic to the graph in Figure 12 and $\Gamma_2\cong C_7$. If $A=\emptyset$ then $|C|\leq 2$. If $C=\{c_1,c_2\}$ then $D\neq\emptyset$ for $C\subsetneq N_{\Gamma_2}(g)$. Hence there exists $d\in D$, which is not adjacent to c_1, say, but it is adjacent to c_2, which implies that $\{c_1,y,z\}\subseteq N_{\Gamma_2}(d)$, a contradiction. Thus $|C|=1$, which implies that $N_{\Gamma_2}(g)=C$, again a contradiction.

Now, suppose that $G=\emptyset$. Since $A\cup\{x\}\subseteq N_{\Gamma_2}(z)$ and $B\cup\{x\}\subseteq N_{\Gamma_2}(y)$, we have $|A|,|B|\leq 1$. If $A=\{a\}$ and $B=\{b\}$ are not empty sets, then a and b are adjacent. If $D\neq\emptyset$ and $d\in D$, then d is not adjacent to a and b, which implies that $\{x,z,a,d,b,y\}$ form a cycle in Γ_2, a contradiction. Thus $D=\emptyset$. A simple verification shows that $N_\Gamma(x)$ is a complement of a path starting and ending at a and b, respectively. Hence, Γ is isomorphic to the complement of a cycle of length $n+6$ for some $n\geq 0$. If $A=\{a\}$ is a singleton and $B=\emptyset$, then $E_\Gamma(A,D)\neq\emptyset$. Let $d\in D$ be a vertex adjacent to a. Clearly, $E_\Gamma(A,D\setminus\{d\})=\emptyset$ for otherwise there exists $d'\in D$ adjacent to a, and hence $\{x,d,d'\}\subseteq N_{\Gamma_2}(y)$, which is a contradiction. Thus $(D\setminus\{d\})\cup\{z\}\subseteq N_{\Gamma_2}(a)$, which implies that $|D|\leq 2$. If $|D|=2$, then $D=\{d,d'\}$ for some d' and this implies that $E_\Gamma(C,D)=\emptyset$. Thus $C\cup\{y\}\subseteq N_{\Gamma_2}(d)$, from which it follows that $C=\{c\}$ is a singleton. Clearly, a,c and d,d' are adjacent, respectively, and consequently Γ is isomorphic to the graph in Figure 12 and $\Gamma_2\cong C_7$. If $D=\{d\}$ then again $E_\Gamma(C,D)=\emptyset$, which implies that $C=\{c\}$ is a singleton. Clearly, a and c are not adjacent. Hence, Γ is isomorphic to the graph in Figure 11 and $\Gamma_2\cong C_6$. If $A=B=\emptyset$, then $E_\Gamma(C,D)\neq\emptyset$. Without loss of generality, we may assume that $\{c,d\}\in E_\Gamma(C,D)$ for some $d\in D$. But then $\{x,y,d,z\}$ induces a cycle in Γ_2, which is a contradiction. Thus, the assertion is completely proved. □

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