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FOREWORD

One of the main tasks of IIASA's Regional Issues Project is to develop a theoretical modeling apparatus suitable for characterizing the cycles, oscillations, and discontinuities observed in the dynamics of urban housing, transportation, and industrial development. Furthermore, in order to maintain analytic and computational tractability, a great premium is placed upon the "simplicity" of the models.

This report addresses many of these issues from a theoretical modeling standpoint, showing by precept, as well as by example, the mathematical methods associated with questions of model simplification, catastrophes, and cycles. In the report a specific regional development model is discussed as a fundamentally dynamic problem. It is shown that oscillatory rather than steady-state behavior of metropolitan populations and income levels is to be expected, and that such behavior has actually been observed in the United States for the period 1940–77.

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SIMPLE MODELS, CATASTROPHES AND CYCLES*

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It is often observed in practice that the essential behavior of mathematical models involving many variables can be captured by a much smaller model involving only a few variables. Further, the simpler model very often displays oscillatory behavior of some sort, especially when critical problem parameters are varied in certain ranges. This paper attempts to supply arguments from the theory of dynamical systems for why oscillatory behavior is so frequently observed and to show how such behavior emerges as a natural consequence of focusing attention upon so-called "essential" variables in the process of model simplification. The relationship of model simplification and oscillatory behavior is shown to be inextricably intertwined with the problems of bifurcation and catastrophe in that the oscillations emerge when critical system parameters, i.e. those retained in the simple model, pass through critical regions. The importance of the simplification, oscillation and bifurcation pattern is demonstrated here by consideration of several examples from the environmental, economic and urban areas.

1 ARTIFACTS, ATTRACTORS AND MEDIUM-SCALE PHENOMENA

One of the most obvious features of human and natural resource systems is that they oscillate. Whether the system involves fluctuations in a macroeconomic indicator, change in population of a forest insect pest or the regular beat of the human heart, the most easily observed aspect of its behavior is that it is oscillatory, and often periodic. Terms such as the respiratory "cycle", the Kondratieff "wave" and the circadian "rhythm" have been introduced to dignify and acknowledge this most basic aspect of the dynamical behavior of living systems. But what is it that accounts for this ubiquitous oscillatory behavior? Is there a common mechanism at work here that forces human processes into a periodic mode or does each process have its own eccentric, individualistic, vibration-generating scheme with no common thread linking it to other superficially similar processes? Part of our story in this paper is to provide a systematic explanation for why oscillatory behavior is the *expected* way for systems to behave and to show why long-term behavior such as point equilibria is, in the absence of special problem constraints, an extremely rare occurrence in real systems.

A second commonly observed behavioral feature in natural systems is that the amplitudes and/or phases of the oscillations often exhibit rapid jumps, or discontinuities. We are all depressingly familiar with stock market crashes, plagues of locusts and outbreaks of warfare, but there are many other less dramatic but equally interesting "bifurcations" arising from oscillatory process. Such bifurcations are *prima facie* evidence of non-linear interactions underlying the observed system behavior, and a description of the linkage between the oscillation-generating mechanism and the bifurcation-generating mechanism is a second goal of this paper.

Finally, we come to the interface between what we can actually observe at a macroscopic level and the microlevel interactions giving rise to the macro-patterns. It has been empirically observed in many modelling exercises that the essential behavioral properties of a system which involves interactions of many variables can be captured by centering attention upon a small number of macrolevel variables formed, generally, as some (usually non-linear) combination of microvariables. Usually, the observed macrovariables exhibit the characteristic oscillations, bifurcations, etc., and what is needed is some sort of *meso-level* theory enabling us to translate back-and-forth between the micro-variables, which we cannot see or know, and the macro-patterns. This type of *model simplification* question forms the final piece of the system modeling mosaic addressed in this report.

Leaving aside for the moment the mathematical formulation of oscillations and bifurcations, let us consider what the connection could be between elementary oscillators such as a pendulum or a vibrating string, and a complex natural system like an economy, a human nervous system or an indus-

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trial organization. For sake of definiteness, we consider a national economy. We cannot possibly measure all the important events taking place in an economy; nevertheless, it is possible to imagine a dynamical system of sufficient complexity to model these events. On general grounds, such dynamical systems have attractors which can bifurcate. Further, even though we cannot measure those attractors, we can sometimes observe their bifurcations by means of *artifacts*. For example, the rate of national growth is an artifact whose behavior could not possibly give any significant information about the internal dynamics of the economy. Nonetheless, when the attractor inside the economy makes a catastrophic jump, the artifact outside may well also exhibit a sudden qualitative change in behavior. When such a change occurs at the same time as a significant change in some internal microvariable like coal production, then it strongly suggests that the artifact is reflecting important bifurcations and jumps within the economy. Thus, although the artifact may be a pale shadow of the internal dynamics, yet its catastrophes may furnish an important indicator of significant events. In this sense, the artifact may provide a nontrivial qualitative model for the underlying economic events.

What kind of machinery can we invoke in order to put the foregoing idea onto a more concrete mathematical footing? In this paper, we shall employ results from the theory of dynanimical systems, as well as concepts from singularity and catastrophe theory in order to provide a basis for a unified view of model simplification, oscillatory behavior and sudden, catastrophic change in important system variables. Among the questions to be addressed are:

—how can we "split" the artifacts into "essential" and "inessential" variables so that the oscillatory and/or bifurcating behavior occurs only in the essential variables?

—can we "explain" the appearance of oscillatory behavior in almost every human system?

—how can we predict qualitative changes in the amplitudes and/or phases of a system's observed oscillatory behavior?

—can we regulate or control oscillatory behavior and/or system bifurcations?

While we cannot pretend to a complete answer to any of these questions, the results of the paper shed considerable light on these and related issues and provide a basis for a more detailed study of specific processes. As illustration of how the methods work in practice, the paper concludes with a discussion of the oscillatory/bifurcation/simplification question for economic cycles, the emergence of social and historical trends and the cyclic behavior of large ecosystems, as well as a discussion of how control theory may be used to influence a system's natural oscillatory motions.

2 IS NATURE OSCILLATORY?

Suppose that we model a natural system N by a mathematical dynamic process Σ, where Σ consists of a multidimensional manifold of states M, together with a vector field $X: M \to M$. In the terms used above, M represents the set of internal microvariables, while X is the rule specifying the state transitions. In the case of a national economy, we are quite prepared for the dimension of M to be as great as 10^9 or more. Regardless of the dimension of M, the C°-Density Theorem[1,2] asserts that if the vector field X is structurally stable (which can be guaranteed by making an arbitrarily small continuous perturbation of X), then the only attractors of X are fixed points (= stable equilibria) and closed orbits (= limit cycles).

Point attractors are easy to understand and if Σ contains parameters which represent its interactions with an external driving system, then the equilibria will bifurcate only according to the so-called "elementary" catastrophes. Such attractors imply that X is a gradient dynamic, i.e. $X = -\text{grad}\ \phi$, where ϕ: $M \to R$ is some system energy or potential function.

However, there are two compelling reasons why the closed orbit attractors are more interesting than the point attractors. First, the empirical evidence in nature strongly suggests that periodicity is the rule, and static equilibrium the exception. Second, on evolutionary grounds a system N that can respond to the environment more swiftly than its neighbors has a competitive advantage. If N had only point attractors, it would remain stable when weakly coupled to any other stable system (e.g., the environment) and, hence, could not respond to external disturbances. On the other hand, a system N with closed orbits can resonate with, and lock-on to, the attractors of any system it is even weakly coupled with, thus N can respond quickly. Consequently, we expect N to evolve non-gradient dynamics and limit cycles. By contrast a *developing* system does *not* want to be too perturbed by the environment during its crucial stage of development; hence, we would expect it to evolve gradient dynamics and equilibrium states in its *embryonic* phase, which is exactly what one observes in the early phases of most natural systems.

In passing, let us note that the foregoing abstract generalities can be brought down to the level of elementary differential equations by noting the following

Periodicity Lemma. If y (t) *is a measured closed orbit of an arbitrary dynamical system, then there exists a 2nd-order differential equation having* y (t) *as its unique attractor.*

Proof. Without loss of generality, assume that the closed orbit has period 2π (this can always be arranged by reparameterizing the time-scale).

Consider the 2nd-order system

$$\ddot{x} = 2[\dot{y}(t) - \dot{x}] + 2[y(t) - x] + \ddot{y}(t).$$

It can be verified that the general solution of this equation is

$$x(t) = y(t) + A\,e^{-t}\cos(t - \alpha),$$

Where A and α are constants. Thus, all solutions decay to $y(t)$ as claimed.

The Periodicity Lemma shows that we need not look beyond a simple 2nd-order differential equation if we desire to model a periodic phenomena characterized by the scalar quantity $y(t)$. In other words, oscillators described by 2nd-order systems form the building blocks for all systems exhibiting periodic behavior. We return to this point later.

Before proceeding, it is worthwhile to note that in this report a distinction will be made between use of the word "oscillatory" and the term "periodic". A system can exhibit oscillatory behavior without being periodic, but not conversely. For instance, the system $\ddot{x} + a\dot{x} = 0$, $a > 0$, is oscillatory, having trajectories which are spirals in the $(x, \dot{x})$-plane, but the trajectories are periodic only if $a = 0$. It is most likely the case that real systems exhibit behavior which is oscillatory of this nature, rather than truly periodic since perturbations of one sort or another continually push the system off one orbit and onto another. Most of the discussions which follow will be seen to apply equally well to either oscillatory or periodic motion, so we shall make a clear distinction only in those situations where confusion may arise.

3 A 'ZOO' OF OSCILLATORS

As natural building blocks for oscillatory behavior, let us consider some of the classical oscillators and a few of their main features.

(a) *The Simple Harmonic Oscillator*—here the dynamics are

$$\ddot{x} + x = 0.$$

In the $(x, \dot{x})$-phase plane, the trajectories of this system are concentric circles, whose radii depend upon the initial position and velocity. This flow is *not* structurally stable as the introduction of arbitrarily small damping changes the topological type of the orbits.

(b) *The Van der Pol Oscillator*—This is one of the simplest structurally stable nonlinear perturbations of the harmonic oscillator. The dynamical equation is

$$\ddot{x} + \varepsilon(x^2 - 1)\,\dot{x} + x = 0, \varepsilon > 0, \varepsilon \ll 1.$$

In the phase plane, the orbits of this system are shown in Figure 1. The system has a repellor at the origin and an attracting limit cycle of radius near 2.

(c) *Duffing's Equation*—another structurally stable perturbation of the harmonic oscillator is

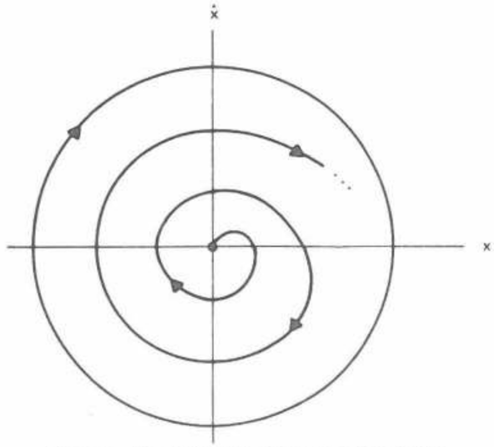

Figure 1. The orbits of the Van der Pol Oscillator.

Duffing's oscillator, which is described by the equation

$$\ddot{x} + \varepsilon k\dot{x} + \varepsilon\alpha x^3 = \varepsilon F \cos \Omega t,$$

where $\varepsilon > 0$, $\varepsilon \ll 1$, $k, F > 0$, $\Omega = 1 + \varepsilon\omega$, α, ω are real parameters. If the perturbation away from the harmonic oscillator is small (α, ω small), the attractors of the flow are either one attracting limit cycle or two point equilbria and one saddle-type limit cycle. The amplitude A and phase ϕ of the limit cycles are given (to order ε) by the equations

$$A^2(3/4\alpha A^2 - 2\omega)^2 = F^2 - k^2A^2 \quad (1)$$

$$\tan\phi = \frac{4k}{3\alpha A^2 - 8\omega} \quad (2)$$

The graph of A as a function of the parameters α, ω as given by equation (1) has two cusp catastrophes. After eliminating A from equation (1), we obtain the following equation for the cusp points.

$$(\alpha, \omega) = \pm\left(\frac{k\sqrt{3}}{2}, \frac{32k^3}{9F^2\sqrt{3}}\right).$$

Geometrically, the picture is as shown in Figure 2. At each cusp, the upper and lower sheets represent attractors, while the middle sheet is saddles. When $\alpha = 0$ the equation is linear and there is always a unique attractor, whose amplitude reaches a maxi-

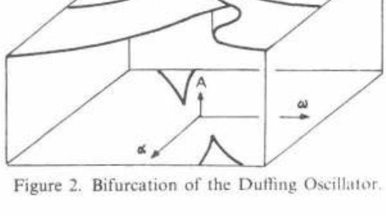

Figure 2. Bifurcation of the Duffing Oscillator.

mum $A = F/k$ when $\omega = 0$, i.e., when the forcing frequency equals that of the original oscillator causing resonance.

When $\alpha > k\sqrt{3}/2$ then the cusp catastrophe can occur. If ω is then slowly increased from negative to positive values, A smoothly increases to the maximum $A = F/k$ at a point inside the cusp $\omega = 3\alpha F^2/8k^2$. If ω is further increased then the larger attractor will coalesce with the saddle and disappear, causing a catastrophic jump to the lower (smaller) attractor. Conversely, a decrease of ω will cause a catastrophic increase in amplitude and phase-shift.

By adding further nonlinear terms to the Duffing oscillator, for example, by replacing αx^3 by $\alpha_1 x^3 + \alpha_2 x^5 + \ldots$, then the graph of A over the enlarged parameter space will exhibit higher catastrophes such as the butterfly. The important conclusion from this analysis is to observe that smooth changes in the frequency of the forcing term can cause both smooth and catastrophic changes in amplitude and phase of the oscillator. In other words, by even weakly coupling the original oscillating system to another oscillating "environment", the original system can exhibit catastrophic changes in amplitude and frequency brought on by its interaction with the environment.

(d) *The Hopf Bifurcation*—The simplest and most important example of a stable bifurcation of an oscillator which is *not* governed by elementary catastrophe theory is the so-called Hopf bifurcation, in which a stable point equilibrium loses its stability and turns into a repellor together with the appearance of a stable limit cycle. This case is not governed by elementary catastrophe theory since it can be shown that there does not exist a stably bifurcating Lyapunov function governing the appearance of the limit cycle.

To illustrate the Hopf bifurcation more concretely consider again the Van der Pol oscillator with parameter γ:

$$\ddot{x} + \varepsilon(x^2 - \gamma)\dot{x} + x = 0.$$

when $\gamma < 0$ the flow in the phase plane has only an attracting point equilibrium at the origin; when $\gamma > 0$ the origin turns into a repellor and an attracting limit cycle appears of radius $\cong 2\sqrt{\gamma}$. If we are in the situation where $\gamma > 0$ and ε *large*, then the limit cycle has the shape indicated in Figure 3 and the form of x itself resembles a square wave.

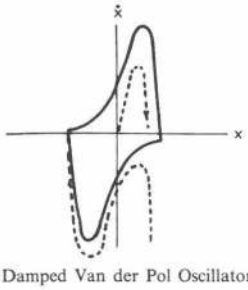

Figure 3. The Damped Van der Pol Oscillator with large ε.

(e) *Van der Pol Oscillator with large Damping*—If ε is large, Figure 3 shows that $\dot{x}$ is no longer a suitable variable with which to characterize the flow since it can become very large. So, it is better to use $z = \int x$ in the following way.

Let the initial values of x and $\dot{x}$ be x_0 and $\dot{x}_0$, respectively and let

$$z(t) = z_0 - \frac{1}{K}\int_0^t x(\tau)d\tau,$$

where $z_0 = 1/3x_0^3 - \gamma x_0 - \dot{x}_0/K$. We now have

$$\dot{z} = -\frac{x}{K}.$$

Substituting into the original equations we obtain

$$\ddot{x} + K(x^2\dot{x} - \gamma\dot{x} - \dot{z}) = 0,$$

which we can write as the 2nd-order system

$$\dot{x} = -K\left(\frac{x^3}{3} - \gamma x - z\right), \qquad (\text{"fast"})$$

$$\dot{z} = -\frac{x}{K}. \qquad (\text{"slow"})$$

These equations are termed "fast" and "slow" because with K large, the rate of change of x is much greater than that of z. Thus, z may be regarded as a *parameter* for the behavior of x. The equilibria of x are given by the equation

$$\frac{x^3}{3} - \gamma x - z = 0,$$

which leads to a cusp catastrophe since we are treating z as a parameter.

We geometrically interpret this situation in Figure 4 as follows. Off the surface M, the fast equation ensures that the trajectories are very nearly parallel to the x-axis. The system will then quickly move to the surface M. This makes $\dot{x} = 0$, so the system is then governed entirely by the slow equation. If $\gamma > 0$, the system moves in an orbit (like that shown in Figure 4), exhibiting sudden jumps and hysteresis. We shall consider this system in greater detail in Section 9.

(f) *The Lorenz Attractor and Chaos*—At first glance it might appear that the regularity implicit in oscillatory and periodic motion would be antipodal to the idea of chaotic and totally unpredictable behavior. Yet the two concepts have much to do with each other as the following prototypical example, due to Lorenz[3] shows. The dynamical equations for the Lorenz attractor are

$$\dot{x} = \sigma(x + y),$$
$$\dot{y} = rx - y - xz,$$
$$\dot{z} = xy - bz,$$

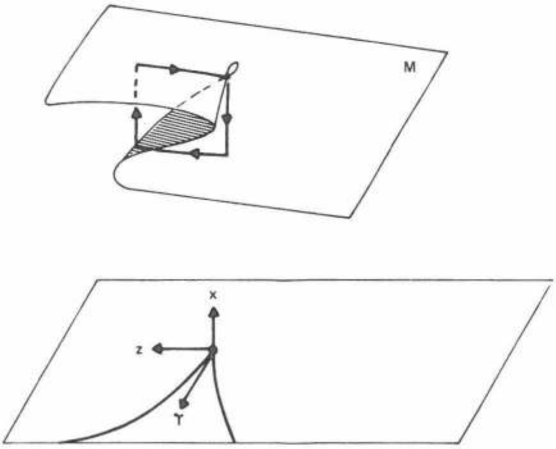

Figure 4. The Van der Pol equation as a cusp catastrophe.

where σ, r and b are constants. It can be shown that when

$$\sigma > b + 1$$

and

$$r > \sigma(\sigma + b + 3)/(\sigma - 1 - b)$$

the motion of the above system is *chaotic*, i.e. there are accountably infinite number of periodic orbits of infinitely long period, as well as an uncountable number of initial conditions for which trajectories, although bounded, do not settle into any cycle. The important point to note here is that the initial conditions lying on aperiodic orbits (i.e. those which lead to chaotic behavior) form a dense set in R^3. Thus, under the conditions on σ, b and r stated above, the *expected* behavior of the system is chaos, even though an infinite number of initial conditions lead to periodic motion.

All of the standard models of oscillatory behavior sketched here have involved simple low-dimensional systems of equations. It is natural to wonder whether or not such elementary systems can actually provide adequate building blocks for the rich variety of oscillatory behavior seen in natural and human phenomena. The Periodicity Lemma given earlier provides some of the motivation for such a claim, showing that 2nd-order systems are rich enough to mimic *any* scalar oscillatory process. But now we wish to turn to the issue of model simplifications and show how it is possible to look at a high-dimensional, complex process in such a way that we can systematically "factor out" a lower-dimensional piece, called the "center manifold" for study of the system cyclic character. In other words, for study of the essentially nonlinear phenomena of bifurcation and oscillation we can study a simplified version of the original system consisting of that part of the system "living" on the center manifold. The problem is how to *find* such a center manifold.

4 THE CENTER MANIFOLD THEOREM

The basic idea underlying the Center Manifold Theorem is an abstraction of the idea of uncoupled equations. Here we follow the development in ref. 4. Consider the system

$$\begin{aligned} \dot{x} &= Ax + f(x, y) \\ \dot{y} &= By + g(x, y) \end{aligned} \qquad (\dagger)$$

where $x \in R^n$, $y \in R^m$ and A and B are constant matrices such that the characteristic values of A are all purely imaginary, i.e., $Re\lambda_i(A) = 0, i = 1, 2, \ldots n$. Further, assume B is a stability matrix and that the functions f and g are smooth with $f(0, 0) = g(0, 0) = f'(0, 0) = g'(0,0) = 0$ (where f' denotes the Jacobian matrix of f).

If f and g are identically zero then the system has two invariant manifolds, namely $x = 0$ and $y = 0$. The manifold $y = 0$ is called the *stable* manifold since if we restrict initial conditions to $y = 0$, all solutions of the system tend to zero. The manifold $x = 0$ is called the *center* manifold. In general, if $y = \mathrm{h(x)}$ is an invariant manifold for (†) and h is smooth, then it is called a *center manifold* if $h(0) = h'(0) = 0$.

Note that if $f = g = 0$, then all solutions of (†) tend exponentially fast as $t \to \infty$ to solutions of

$$\dot{x} = Ax$$

That is, the equation on the center manifold determines the asymptotic behavior of the entire system up to exponentially decaying terms. The Center Manifold Theorem justifies extending this conclusion to the case when f and g are non-zero.

The three theorems that follow taken together comprise the content of the Center Manifold Theorem.

Theorem 1. There exists a center manifold $y = h(x)$ *for the system* (†) *for* $|x|$ *sufficiently small.*

The behavior of (†) on the center manifold is governed by the n-dimensional system

$$\dot{u} = Au + f[u, h(u)] \qquad (*)$$

Notice here that the existence of the center manifold $h(x)$ means that there exists a transformation of the x coordinates such that $y = h(x)$. In other words, we can replace the y variables in the first equation in (†) by a suitable combination of x variables and thereby decouple the x and y equations. The next theorem tells us that all the information needed to determine the asymptotic behavior of solutions of equation (†) near the origin is contained in the equation (*).

Theorem 2. (a) Suppose that the zero solution of equation (*) is stable (asymptotically stable) (unstable). Then the zero solution of equation (†) is also stable (asymptotically stable) (unstable).
(b) Suppose that the zero solution of equation (*) is stable. Let $x(t)$, $y(t)$ *be a solution of equation* (†) *with* $|x(0)|$ $|y(0)|$ *sufficiently small. Then there exists a solution* $u(t)$ *of equation* (†) *such that as* $t \to \infty$

$$x(t) = u(t) + O(e^{-\gamma t})$$
$$y(t) = h[u(t)] + O(e^{-\gamma t})$$
$$\gamma > 0.$$

Now the question arises as to how to actually calculate the center manifold $y = h(x)$. If we substitute $y = h[x(t)]$ into the second equation in (†) we obtain

$$h'(x)\{Ax + f[x, h(x)]\} = Bh(x) + g[x, h(x)].$$

This equation, together with the conditions $h(0) = h'(0) = 0$ is the equation which must be solved for the center manifold. In general, this is impossible since it is equivalent to solving the original problem (†). However, Theorem 3 below shows that, in principle, the center manifold can be approximated to any desired degree of accuracy.

For purposes of notation, let $\phi: R^n \to R^m$ be a continuously differentiable function and define the operation $[M\phi](x)$ as follows

$$[M\phi](x) \doteq \phi'(x)\{Ax + f[x, \phi(x)]\} - B\phi(x) - g[x, \phi(x)]$$

Note that if $\phi = h$, then $[Mh](x)] = 0$.

Theorem 3. Let ϕ *be as above with* $\phi(0) = \phi'(0) = 0$. *Suppose that as* $x \to 0$, $[M\phi](x) = O(|x|^q)$ *for* $q > 1$. *Then as* $x \to 0$, $|h(x) - \phi(x)| = O(|x|^q)$.

Thus, we can employ the function $\phi(x)$ to approximate $h(x)$ up to terms $O(|x|^q)$.

In order to fix the ideas inherent in the Center Manifold Theorem, it is useful to consider examples of the applications of Theorems 1–3.

Example 1. Consider the system

$$\dot{x} = xy + ax^3 + by^2x,$$
$$\dot{y} = -y + cx^2 + dx^2y,$$

with x and y scalar variables, a, b, c, d constant.

By Theorem 1, this system has a center manifold $y = h(x)$. To approximate $h(x)$, we set

$$[M\phi](x) = \phi'(x)[x\phi(x) + ax^3 + bx\phi^2(x)] + \phi(x) - cx^2 - dx^2\phi(x).$$

For any function $\phi(x)$ such that $\phi(x) = O(|x|^2)$, $[M\phi](x) = \phi(x) - cx^2 + O(|x|^4)$. Hence, if we take $\phi(x) = cx^2$, then $[M\phi](x) = O(|x|^4)$, so by Theorem 3, $h(x) = cx^2 + O(|x|^4)$.

By Theorem 2, the equation which determines the stability of the original system is

$$\dot{u} = uh(u) + au^3 + buh^2(u) = (a + c)u^3 + O(|u|^5).$$

Thus, the origin is stable if $(a + c) < 0$ and unstable if $(a + c) > 0$. If $a + c = 0$ then we have to obtain a better approximation to h.

Suppose $a + c = 0$. Let $\phi(x) = cx^2 + \psi(x)$, where $\psi(x) = O(|x|^4)$. Then

$$[M\phi](x) = \psi(x) - cdx^4 + O(|x|^6).$$

Thus, if $\phi(x) = cx^2 + cdx^4$, $[M\phi](x) = O(|x|^6)$ and, by Theorem 3, $h(x) = cx^2 + cdx^4 + O(|x|^6)$. The equation governing stability of the original system is now

$$\dot{u} = uh(u) + au^3 + buh^2(u) = (cd + bc^2)u^5 + O(|u|^7).$$

Hence, if $a + c = 0$, the origin of the original system is stable if $cd + bc^2 < 0$ and unstable if $cd + bc^2 > 0$. Again, if $cd + bc^2 = 0$, we have to obtain a better approximation to h.

It is important to emphasize here again what the Center Manifold Theorem has accomplished. By defining the new variable $y = h(x) \cong \phi(x)$, the asymptotic behavior of the original 2-dimensional system in the $x - y$ variables has been *reduced* to the study of the asymptotic behavior of the 1-dimensional system in the u-variable. Thus, by the nonlinear "coordinate change", $x \to h(x)$, the original system has been decoupled in such a fashion that the asymptotic behavior is determined only by the behavior of the original system on the center manifold $y = h(x)$. In a rather precise way, the function $h(x)$ tells us the "right" way to combine the x-variables in order to decouple the problem, and to reduce its study to a lower-dimensional "simpler" problem.

Example 2. Bifurcations. Let us consider the system

$$\dot{z} = F(z, \lambda), \quad F(0, \lambda) = 0,$$

where $z \in R^{n+m}$ and λ is a p-dimensional parameter.

Suppose that the linearization of the system about $z = 0$ is

$$\dot{z} = F(\lambda)z.$$

If the characteristic values of $F(0)$ all have non-zero real parts then, for $|\lambda|$ small, small solutions of the original system behave like solutions of the linearized systems so that $\lambda = 0$ is not a bifurcation point. Thus, the only interesting situation is when $F(0)$ has characteristic values on the imaginary axis.

Suppose $F(0)$ has n purely imaginary roots and m roots in the left half-plane (we assume there are no unstable roots since we are interested only in the bifurcation of stable phenomena). We can now rewrite the original system as

$$\begin{aligned} \dot{x} &= Ax + f(x, y, \lambda), \\ \dot{y} &= Bx + g(x, y, \lambda), \quad (\Sigma) \\ \dot{z} &= 0 \end{aligned}$$

where f and g vanish together with their derivatives at $(x, y, \lambda) = (0, 0, 0)$.

By Theorem 1, the system (Σ) has a center manifold $y = h(x, \lambda)$, for $|x|$, $|\lambda|$ small. By Theorem 2 the behavior of small solutions of (Σ) is governed by the equation

$$\begin{aligned} \dot{u} &= Au + f[u, h(u, \lambda), \lambda], \\ \dot{\lambda} &= 0. \end{aligned}$$

In applications n is usually 1 or 2 so the reduction from the original system in generally very significant.

Before leaving the Center Manifold Theorem, it is useful to mention some of the ways in which it may be extended, since the results given here are only the simplest result of this type.

(i) Under rather weak assumptions, we can replace the equilibrium point at the origin by invariant sets. This enables us to consider the behavior of a system in a neighborhood of a periodic orbit rather than just a point equilibrium;

(ii) the assumptions that the characteristic values of the linearized problem all have non-positive real parts can be dropped;

(iii) similar results can be obtained for certain classes of infinite-dimensional equations involving time-delays and/or partial differential equations;

(iv) the results given here for the continuous-time case (flows) can be extended to discrete-time case (maps)

5 A BIOMEDICAL EXAMPLE—LIMIT CYCLES IN IMMUNE RESPONSE

A problem which illustrates application of most of the ideas presented above arises in the study of the immune response to an antigen. The mathematical model of this process is given by the system[4]

$$\begin{aligned} \varepsilon\dot{x} &= -[x^3 + (a - 1/2)x + (b - 1/2)], \\ \dot{a} &= 1/2\delta(1 - x) - a - \gamma_1 ab, \qquad \text{(IR)} \\ \dot{b} &= -\gamma_1 ab + \gamma_2 b, \end{aligned}$$

where $\varepsilon, \delta, \gamma_1, \gamma_2$ are positive parameters. Here a and b represent concentrations of the antigens, while x measures the stimulation of the immune system in response to the antigens. The stimulation is assumed to take place on a much faster time-scale than the antigen dynamics, so we take $\varepsilon \ll 1$. This situation is of exactly the same "fast-slow" type discussed earlier in connection with the damped Van der Pol oscillator with large damping. Here we will employ center manifold theory to show that the system (IR) has a periodic solution bifurcating from a fixed point for certain values of the parameters, i.e., there is a Hopf bifurcation.

Let (x^*, a^*, b^*) be an equilibrium point for the stystem (IR). If $b^* \neq 0$, then $a^* = \gamma_2/\gamma_1 - \gamma_1$ and x^* and b^* satisfy the equations

$$\begin{aligned} x^{*3} + (\gamma_2/\gamma_1 - 1/2)x^* + b^* - 1/2 &= 0, \\ 1/2\delta(1 - x^*) - \gamma_2/\gamma_1 b^* &= 0. \end{aligned}$$

For the remainder of our discussion, assume that $a^* = \gamma_2/\gamma_1$.

If we let

$$y = a - a^*, z = b - b^*, \quad w = -\psi(x - x^*) - x^*y - z$$

with

$$\psi = 3x^{*2} + a^* - 1/2,$$

then if $\psi \neq 0$,

$$\begin{aligned} \varepsilon\dot{w} &= g(w, y, z, \varepsilon), \\ \dot{y} &= f_2(w, y, z, \varepsilon), \\ \dot{z} &= f_3(w, y, z, \varepsilon), \end{aligned}$$

where

$$\begin{aligned} g(w, y, z, \varepsilon) &= f_1(w, y, z, \varepsilon) - \varepsilon x^* f_2(w, y, z, \varepsilon) - \varepsilon f_3(w, y, z, \varepsilon), \\ f_1(w, y, z, \varepsilon) &= -\psi w + N(w + x^*y + z, y), \\ f_2(w, y, z, \varepsilon) &= (\delta/2\psi^{-1}x^* - 1 - \gamma_1 b^*)y + (\delta/2\psi^{-1} - \gamma_2)z + \delta/2\psi^{-1}w - \gamma_1 yz, \\ f_3(w, y, z, \varepsilon) &= -\gamma_1 b^* y - \gamma_1 yz, \\ N(\alpha, y) &= -\psi^{-2}\alpha^3 + 3\psi^{-1}x^*\alpha^2 - y\alpha \end{aligned}$$

To put all equations of the above system on the same time-scale, let $s = t/\varepsilon$ and now denoting differentiation with respect to s by $'$, we can rewrite the w-y-z system as

$$w' = g(w, y, z, \varepsilon),$$
$$y' = \varepsilon f_2(w, y, z, \varepsilon),$$
$$z' = \varepsilon f_3(w, y, z, \varepsilon),$$
$$\varepsilon' = 0.$$

Suppose $\psi > 0$. Then the linearized version of the above system has one negative characteristic value and 3 zero roots. Hence, by Theorem 1, there exists a center manifold

$$w = h(y, z, \varepsilon).$$

By Theorem 2, the local behavior of the solutions to the system is determined by the equations

$$y' = \varepsilon f_2[h(y, z, \varepsilon), y, z, \varepsilon],$$
$$z' = \varepsilon f_3[h(y, z, \varepsilon), y, z, \varepsilon]$$

or, in terms of the original time-scale,

$$\dot{y} = f_2[h(y, z, \varepsilon), y, z, \varepsilon], \qquad (\#)$$
$$\dot{z} = f_3[h(y, z, \varepsilon), y, z, \varepsilon].$$

We must study the system (#) to see about the possibility of a Hopf bifurcation.

The linear part of the (#) system near $y = z = 0$ is given by

$$J(\varepsilon) = \begin{bmatrix} \delta/2\,\psi^{-1}x^* - 1 - \gamma_1 b^* & \delta/2\,\psi^{-1} - \gamma_2 \\ -\gamma_1 b^* & 0 \end{bmatrix}$$

If (#) is to have a Hopf bifurcation, then we must have

$$\text{trace } J(\varepsilon) = 0$$

and

$$\delta/2\,\psi^{-1} - \gamma_2 > 0.$$

From the earlier analysis, we also know that x^* and b^* are solutions of the equilibrium equations and for the problem to make physical sense, we must also have $|x^*| < 1, b^* > 0$ and $\psi > 0$. The satisfaction of these requirements is assured by the following result.

Lemma. Let $\gamma_1/\gamma_2 < 2$. Then for each $\varepsilon > 0$, there exists $a\delta(\varepsilon), x^*(\varepsilon)$ and $b^*(\varepsilon)$ such that $0 < x^*(\varepsilon) < 1/2$, $b^*(e) > 0$, $\delta(\varepsilon)\psi^{-1} - 2\gamma_2 > 0$, $\psi > 0$, trace $J(\varepsilon) = 0$ and the equilibrium equations for $x^*(\varepsilon)$ and $b^*(\varepsilon)$ are satisfied.

In other words, no matter what "fast-slow" time-scale ε is employed, there always exists a value of δ which will send the immune response bifurcating into oscillatory behavior from an equilibrium.

The preceding example shows very clearly the power of center manifold theory to reduce the study of bifurcation phenomena from the original 3-dimensional system to the associated 2-dimenstional center manifold system (#).

6 OSCILLATIONS AND BIFURCATIONS IN ECONOMICS, URBAN GROWTH AND ECOLOGY

The Center Manifold Theorem makes it evident that any "bad" behavior of a dynamical process will arise from the system's local behavior on the center manifold. Here, of course, "bad" is interpreted in the sense of unstable oscillations and/or bifurcations emerging from stable processes due to changes in system parameters and/or the operating environment. In this section we review the appearance of such behavior in some models in the economic, energy and ecological areas. Each of these examples has been chosen to illuminate an important aspect of the use of the material discussed earlier on oscillations, chaos and bifurcation and, taken together, these examples act as a strong testament to the employment of dynamical systems-theoretic concepts in applied modeling analyses.

6.1. *Economic Chaos*[5]

The neoclassical theory of capital accumulation provides an explanation of investment cycles that lies exclusively in the interaction of the propensity to save and the productivity of capital when sufficient nonlinearities and a production lag are present. This theory can be used to establish the existence of irregular economic oscillations which need not converge to a cycle of any regular periodicity. Moreover, because they are unstable, errors of parameter estimation or errors in initial conditions, however minute, will accumulate rapidly into substantial forecasting errors. Such irregular fluctuations can emerge after a period of apparently balanced growth so that the "future" behavior of a model solution cannot be anticipated from its "past".

While it certainly cannot be proved that real economies are chaotic in the above sense, the example below shows that irregular fluctuations of a highly unstable nature constitute one characteristic mode of behavior in dynamic economic models and that they may emerge in standard economic theories.

It is also of interest to note here that the past behavior of a nonlinear system may be a poor guide for inferring even *qualitative* let alone *quantitative* patterns of change in the future since the type of model discussed here may evolve through apparently different regimes even though no *structural* change has occurred.

Assuming homogeneity of the production function and an exponentially growing population, the difference equation describing capital accumulation is

$$k_{t+1} = s(k_t)/(1+\lambda),$$

where

k = capital-labor ratio,
λ = population growth rate
$s(\cdot)$ = the per capital savings relation.

We consider conditions on $s(\cdot)$ in which growth occurs but in which the steady-state is unstable and the oscillations which emerge fail to converge to a cycle of any order even though cycles of every order exist.

The occurrence of a sufficient reduction in capital to cause unstable oscillations can arise from two distinct forces or their combination. Let $f(k)$ be the production function and let $h(k)$ be the consumption wealth function. Per capita consumption depends upon wealth, interest rates and income, but we use the production function to eliminate income and equate marginal productivity of capital $f'(k)$ with interest to arrive at $h(k)$. Since, by definition, we have $s(k) = f(k) - h(k)$, the capital accumulation dynamics are

$$k_{t+1} = [f(k_t) - h(k_t)]/(1+\lambda).$$

In order to demonstrate the occurrence/non-occurrence of investment cycles or chaos, let us first consider the standard neoclassical case in which the steady-state is globally stable and oscillations cannot arise. Here we take the production function to be

$$f(k) = Bk^{\beta}, \quad B, \beta \text{ positive constants.}$$

The consumption wealth function is

$$h(k) = (1-\sigma)f(k),$$

where σ is the marginal propensity to save. Thus,

$$k_{t+1} = \sigma Bk^{\beta}/(1+\lambda).$$

For $\beta > 0$ investment cycles cannot occur and instead growth converges to a steady-state with an equilibrium capital–labor ratio $(\sigma\beta/(1+\lambda))]^{1/(1-\beta)}$.

To illustrate the appearance of oscillations and chaos, suppose that we introduce a productivity inhibiting effect into the model. Thus, we multiply the production function by the inhibiting factor $(m-k)^{\gamma}$ and obtain the production function

$$f(k) = Bk^{\beta}(m-k)^{\gamma}, \quad \gamma > 0.$$

As $k \to m$, the inhibiting factor becomes important and output rapidly falls. This factor represents, for instance, the harmful effect, for whatever reason, of excessive concentrations of capital on output per worker. Keeping the constant savings factor σ, we obtain

$$k_{t+1} = Ak_t^{\beta}(m-k_t)^{\gamma},$$

where $A = \sigma B/(1+\lambda)$.

For small values of A and for sufficiently small k_o, growth will be monotonically increasing converging to a stable steady-state. As A is increased, a bifurcation point is reached after which further increases in A result in bounded oscillatory behavior as long as

$$\frac{\beta m}{\beta+\gamma} < A\left(\frac{\beta}{\beta+\gamma}\right)^{\beta}\left(\frac{\gamma}{\beta+\gamma}\right)^{\gamma} m^{\beta+\gamma} \leq m$$

Now choose A so that we have equality on the right-hand inequality above and let A'' be the value of A which yields equality, i.e.,

$$A'' = \left(\frac{\beta+\gamma}{\beta}\right)^{\beta}\left(\frac{\beta+\gamma}{\gamma}\right)^{\gamma} m^{1-\beta-\gamma}.$$

Now irregular investment and growth cycles occur and chaotic behavior ensues. Actually, it can be shown that there exists an $A' \leq A''$ such that for all $A' \leq A \leq A''$ chaos occurs. But, since $A = \sigma B/(1+\lambda)$, for fixed σ and λ, there exists an interval $[B', B'']$ such that for all productivity multipliers $(1+\lambda)A'/\sigma \leq B \leq (1+\lambda)A''/\sigma$ we have chaotic trajectories. Similar results have been obtained in the case of a variable savings ratio.[5]

The depressing aspect of the above results is that they provide a basis for skepticism of any modeling effort which relies upon parameter identification unless it can be demonstrated *in advance* that the parameters do not lie in the chaotic regime. If the parameters are in the chaotic region, then there is little hope that observations on the past behavior of the system will provide a basis for indentifying their values and such a model will certainly be a poor tool to use for discerning the system's future performance. Unfortunately, the results obtained here are symptomatic of a much broader class of models and there is evidence to indicate that chaotically unstable trajectories of this type are more likely to occur with weaker nonlinearities in higher dimensional models.[6] The implications for deterministic description of complex phenomena are obvious.

6.2. Oscillation of Urban Populations

Much of classical and even "new" urban economics emphasizes long-run, static equilibrium models as paradigms for the description of changes in urban population sizes. In a recent work,[7] issue was taken with this view and a comprehensive study was done of the population changes in the 90 largest metropolitan areas in the USA over the period 1940–1977. The dynamic patterns showed that 64 areas exhibited oscillatory behavior of some sort, while only 3 areas showed a steady-state type of behavior. The remaining 23 areas displayed behavior which were of a "perturbed" nature, indicating either a discontinuous shift of population levels or a transition from one mode of oscillatory behavior to another. Thus, the overwhelming empirical evidence supported the contention that urban population dynamics also fluctuate in a cyclical manner, as one might have expected from our earlier C^0-Density Lemma.

In order to mathematically account for the population cycles, a generalized Lotka-Volterra model was proposed in Ref. 7 consisting of the dynamics

$$\dot{x} = x(-a_1 - a_{11}x + a_{12}y),$$
$$\dot{y} = y(a_2 - a_{21}x),$$

where x = city population and y = per capital income of the city's inhabitants. The positive parameters $a_1, a_2, a_{21}, a_{11}, a_{12}$ represent factors influencing growth rates of population and income. Of special note is the parameter a_{11} which is an indicator of the presence of urban "friction" limiting the city's exponential growth tendencies. Presumably, this reflects crowding effects, i.e. the density-dependent nature of urban growth.

When a city changes from a form in which urban friction operates into one where it does not, and where we can characterize such transitions as a consequence of smooth changes in some parameter, then it is certainly reasonable to suppose that cities could undergo a Hopf-type bifurcation. In the above setting, regularly oscillating behavior should occur only when $a_{11} = 0$, i.e., the frictionless city must be very rarely observed. Furthermore, the transition from a mode of orbital oscillations to a convergent mode is also rare, as it would require a city poised right at the brink of a critical value of a key parameter undergoing an appropriate change of circumstances.

For all meaningful values of the parameters representing friction, limit cycle behavior *cannot* occur in the above model. What the model does predict is a stable focus behavior, i.e. a spiraling down of population to a stable equilibrium level, with movements of high amplitude followed by ones of low amplitude.

In order to characterize urban dynamics in a more concrete manner, the parameters of the above model were calibrated for the city of Tacoma, Washington. By expressing median family income as a percentage of national family income, the earlier model can be rewritten as

$$\dot{x} = x[\alpha(y-1) - \beta x],$$
$$\dot{y} = \gamma y(\bar{x} - x),$$

where $\bar{x}$ is the carrying capacity of the region, α and γ are parameters reflecting the speed of adjustment of the two variables, and β represents urban friction.

Using some standard parameter identification procedures,[6] it was found that with the values $\alpha = 1.12, \beta = 0.033, \gamma = 0.003, \bar{x} = 1.96$, the dynamical behavior of the model exhibited the sink-spiral pattern with population and income levels as depicted in Table 1. Extrapolation of this model suggests a steady-state level for Tacoma of $x^* = 1.96$, $y^* = 1.0578$, a median income somewhat above the national level.

TABLE I.[a]
Comparison of data vs. simulation for Tacoma, Washington

	Actual		Simulated	
Year	Pop.	Income	Pop.	Income
1940	0.00137	—	0.00137	1.000
1950	0.00180	1.1243	0.00191	1.0794
1960	0.00178	1.0518	0.00197	1.0524
1970	0.00201	1.0280	0.00195	1.0592
1975	0.00195	—	0.00198	1.0582

[a] Other results involving oscillatory behavior of economic and urban systems are presented in refs. 8–10.

6.3. Population Models with Time-Lags

It is commonly held in circles dealing with human affairs that oscillatory behavior is due to the inevitable presence of significant time-delays between the taking of a decision and its actual implementation in the system. While this bit of modeling "folklore" is certainly far from being either a necessary or sufficient condition for oscillatory behavior to emerge, as the preceding examples amply illustrate, it does contain enough of a germ of truth to warrant serious consideration as a *possible* oscillation-producing mechanism. Time-lags *can* generate oscillations, but they can also prevent them, or they can have no qualitative effect at all! The question in any specific case is "which is which?" As a particular illustration of how time-delays may result in a limit cycle emerging from a stable equilibrium, we consider the following generalized Volterra system studied in ref. 11.

The predator-prey dynamics are given by the integro-differential system

$$\frac{dN_1}{dt} = b_1 N_1 \left[1 - c_{11}N_1 - c_{12}\int_0^\infty N_2(t-u)k_1(u)du\right],$$
$$\frac{dN_2}{dt} = b_2 N_2 \left[-1 + c_{21}\int_0^\infty N_1(t-u)k_2(u)du\right],$$
$$c_{ij} > 0,\ b_i > 0,\ k_i(u) \geq 0,\ \int_0^\infty k_i(u)du = 1$$

Here N_1 and N_2 represent the levels of prey and predator, respectively, while the integrals represent interaction terms and account for delay effects. The coefficient c_{11} measures density effects within the prey population, with $1/c_{11}$ being the "carrying capacity" of the prey. The coefficients b_1 and b_2 are natural birth and death rates of prey and predator, in the absence of all constraints.

The only nontrivial equilibrium for the above system is at $E = (N_1^*, N_2^*)$ where

$$N_1^* = 1/c_{21}, N_2^* = (c_{21} - c_{11})/c_{12}c_{21}.$$

The asymptotic behavior of the predator and prey depends upon the relative values of the b_i and c_{ij}. There are two cases to consider:

(i) if $c_{11} - c_{21} < 0$, but $|c_{11} - c_{21}| < \varepsilon$ for ε sufficiently small, then it can be shown that E is *globally* asymptotically stable, i.e., all initial population levels eventually lead to E. Thus, the time-lags do not generate oscillatory behavior.

(ii) With c_{21} fixed, if c_{11} becomes sufficiently small then E becomes unstable. The loss of stability of E as c_{11} decreases suggests the existence of a bifurcation to a limit cycle. This conjecture has been investigated in ref. 11 with the following conclusion. Define

$$S_{ij}(n) = c_{ij} \int_0^\infty k_i(u) \sin(2n\pi/p) du,$$

$$C_{ij}(n) = c_{ij} \int^\infty k_i(u) \cos(2n\pi/p) du,$$

$$\Sigma_1(n) = S_{12}C_{21} + S_{21}C_{12},$$

$$\Sigma_2(n) = S_{12}S_{21} - C_{12}C_{21}.$$

Consider the hypotheses

(H_1) $\Sigma_1(n) > 0$, $\Sigma_2(n) < 0$ for some integer $n \geq 1$ and period $p > 0$;

(H_2) $C_{21}(n) \neq 0$ for n in (H_1);

(H_3) Either $n\,\Sigma_1(m) \neq m\,\Sigma_1(n)$ or $n^2\,\Sigma_1(n)\,\Sigma_2(m) \neq m^2\,\Sigma_1(m)\,\Sigma_2(n)$ for all $m \neq n$ (n as in H_1), m an integer ≥ 1.

Then the condition (H_1) for some n and p is necessary and the conditions (H_1) – (H_3) are sufficient for a periodic solution of period p to bifurcate from the equilibrium E as the birth and death rates b_1 and b_2 pass through the critical values

$$b_1^* = \frac{-(2\pi n/p)\,\Sigma_1(n)}{c_{11}\,\Sigma_2(n)}, \quad b_2^* = \frac{(2\pi n/p)c_{11}}{(c_{21} - c_{11})\,\Sigma_1(n)}$$

with $0 < c_{11} < c_{21}$.

Thus, the conclusion is that periodic solutions of *any* period p may emerge from the stable equilibrium E as the birth and death rates go through bifurcation points. Furthermore, oscillations need not occur, even in a system with an *uncountable* number of time-delays. We can conclude that the appearance or non-appearance of oscillatory behavior depends upon much more than the mere presence of time-lags. While such simple-minded arguments as "time-lags imply oscillations" may appear plausible in some settings, the issue is usually far more complicated and lies at a much deeper level than just finite speeds of information transfer: there is no general causal relationship between lags and oscillations!

7 CUSPOIDS AND LOGISTICS

Catastrophes, elementary and otherwise, have already been seen to be intimately related to oscillations of diverse sorts. In this section connections between the bifurcation geometry of the elementary cusp catastrophe and the ubiquitous logistic curve are explored with the perhaps surprising conclusion that every logistic-type function has "cusp-like" behavior as a *necessary* part of its dynamical motion. As a consequence of this fact, one may conclude that in *every* situation in which a logistic curve is used to represent the development of some problem variable, the cusp geometry *must* be present, i.e. the type of behavior which can be exhibited is exactly as complicated as that allowed by the cusp and no more so. The simple mathematics given below implies that logistic curves and cusp geometry are inextricably intertwined: they are two sides of the same coin and, as a result, there is no mystery or surprise in discovering a cusp in any model based upon logistic-like assumptions. The cusp must occur. What *is* surprising is that the cusp is the most complicated behavior that can follow from the logistic. The mathematics underlying this result involves deep results from singularity and transversality theory and can be seen, for instance, in ref. 12. Here we shall be content to only indicate the basic results and why they are plausibly true.

To fix ideas, consider the logistic curve

$$L(x) = 1/(1 + e^{-x})$$

and its intersection with the straight line

$$y = ax + b, \quad a > 0.$$

The corresponding 2-parameter equation of state

$$F(x, a, b) = L(x) - ax - b$$

has either 3 solutions or 1, unless $ax + b$ is tangent to the graph of L, in which case there are 3 solutions counting multiplicities. We study the variations of the solution of $F(x, a, b) = 0$ with variations in a and b. (Remark: this may be regarded as the study of how the equilibria of $\dot{x} = ax + b - L(x)$ vary with the parameters a and b.)

The bifurcation set B consists of points (a, b) for which

$$F(x, a, b) = 0,$$

$$\frac{\partial F}{\partial x}(x, a, b) = 0.$$

A small amount of algebra shows that this set consists of those points

$$a = e^{-x}/(1 + e^{-x})^2$$

$$b = 1/(1 + e^{-x}) - xe^{-x}/(1 + e^{-x})^2$$

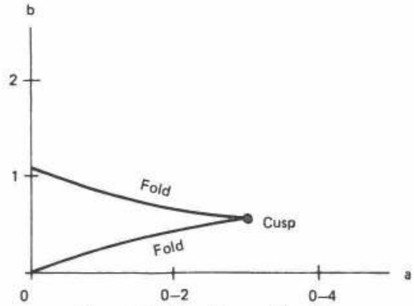

Figure 5. Bifurcations set for F.

Here x is regarded as a parameter (which gives the coordinate of the point of tangency of $y = ax + b$ with the graph of L). If we plot (a, b) as x runs through the real numbers, we obtain Figure 5. To justify use of the terms "fold" and "cusp" in Figure 5 and to conclude that the zeros of F are governed by the cusp geometry, some simple Taylor series arguments coupled with checking transversality conditions is required[12]. The conclusion is that for each point of the set

$$M = \{(x, a, b): F(x, a, b) = 0\},$$

the *local* geometry is of canonical fold or cusp type, up to a smooth change of coordinates [in (x, a, b)-space].

The above local result can be extended to a *global* result following the arguments in ref. 12. This implies that we have global cusp geometry for M, i.e. the picture seen in Figure 6 holds globally. Thus, despite the supposedly "local" nature of catastrophe theory, in this case it is possible to deduce global bifurcation geometry.

To show that the result is not a consequence of the special form chosen for the logistic curve $L(x)$, we quote the following theorem from ref. 12.

Logistic Bifurcation Theorem. Let U, V, W *be open intervals in* R *with* $U \subseteq \{x : x > 0\}$. *Then an equation of state*

$$ax + b = \Phi(x), \quad a \in U, \quad b \in V,$$

where $\Phi = W \rightarrow R$ *is smooth, has global bifurcation geometry diffeomorphic to the canonical cusp catastrophe if*

(*i*) Φ *has a unique inflection point at* ξ, *i.e.*,

$$\frac{d^2\Phi}{dx^2}(\xi) = 0;$$

(*ii*)

$$\frac{d^3\Phi}{dx^3}(\xi) > 0;$$

(*iii*) *Solutions to* $F(x, a, b) = 0$ *do not tend to infinity as* (a, b) *tends to any point inside* $U \times V$.

Thus, we conclude that global cusp geometry is to be expected for any process in which a "sigmoidal"

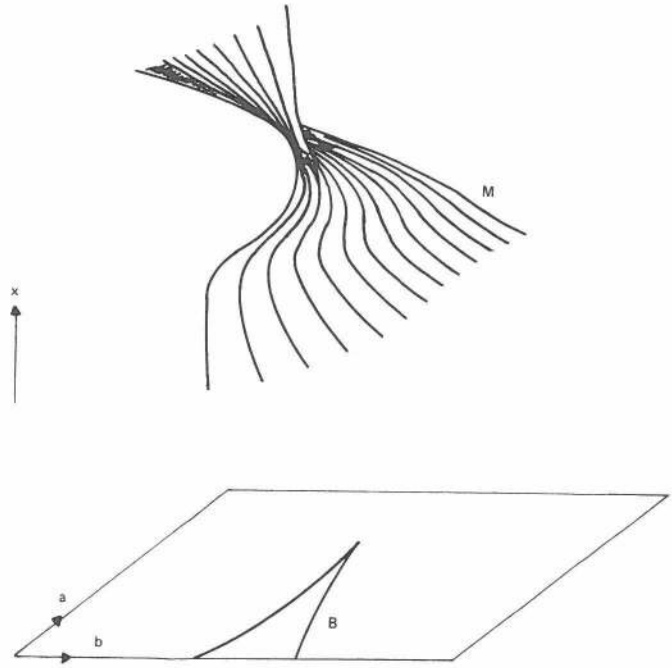

Figure 6. The global geometry of M.

function Φ is used to described logistic growth. Referring now to our earlier treatment linking the cusp catastrophe and relaxation oscillations, it is natural to conclude that processes exhibiting logistic growth can, and often will, display the same type of cyclical behavior as, for example, that displayed in the processes described in Section 3. In fact the results of ref. 12 show that when steady-states are described by a cusp, the dynamic behavior usually involves Hopf bifurcation to a limit cycle.

The characteristic "S-shape" of the logistic function is shared by a large number of others often seen in the literature. When two processes each involving such a function are interacting, the description often involves a weighted sum of sigmoid functions $\Phi(x) + c\psi(x - d)$, giving an overall response which increases in two observable "steps". It is tempting to conclude that in this case the relevant bifurcation geometry is the "butterfly" catastrophe and, indeed, *locally* this result is also established in ref. 12. A global result similar to the case of a single sigmoidal function appears likely, at least in large regions of parameter space.

For multiple interactions of other kinds, we would expect to find other higher cuspoid catastrophes of type A_{2k+1} (in Arnold's notation) for k coupled sigmoidal processes.

(*Technical Remark*: the foregoing results, when intepreted in dynamical terms as statements about the behavior of equilibria of a scalar dynamical process $dx/dt = F(x, a, b, \dots)$, are valid only for the equilibria shifts in *continuous-time* processes. As seen from the chaotic economics example, *discrete-time* dynamics may have much more complicated equilibria behavior, even for scalar processes. Such behavior is ruled-out for continuous time systems of order ≤ 2 by the Poincare'–Bendixson theory).

8 BIFURCATIONS, OSCILLATIONS AND FEEDBACK CONTROL

Oscillations and bifurcations are the expected behavior of most dynamical processes, as the foregoing discussion amply illustrates. But what if we do not want the particular types of cycles and discontinuities implied by a given dynamic? Is there any means of interfering with the process in order to change the oscillation and/or bifurcation into less costly or more advantageous behavior? At this point we enter the realm of control theory and the consideration of how external inputs may affect the dynamical behavior of a system.

In generals terms, we are concerned with the control system

$$\dot{x} = f(x, u, \lambda),$$

where u is a control function and λ is a set of parameters. When no control is exerted ($u \equiv 0$), the free dynamics of the system exhibits a bifurcation at some point $\lambda = \lambda^*$. The question is whether or not it is possible to "neutralize" the bifurcation to/from oscillatory motion at λ^* by application of a suitable feedback control law $u = u(x)$. In other words, the controlled system

$$x = f[x, u(x), \lambda]$$

should not exhibit undesirable behavior as λ passes through the point λ^*.

In general, this is, of course, an extremely difficult question to answer as it depends upon the structure of f, the way the control influences the state, constraints on the control and many other factors. But, to give an indication of the type of result obtainable in a specific case, consider the 3-dimensional system[13]

$$\begin{aligned}
\dot{x}_1 &= \lambda x_1 + \omega c x_2 + \omega s x_3 - x_1[x_1^2 + (cx_2 + sx_3)^2] \\
&\quad + g_{11}u_1 + g_{12}u_2, \\
\dot{x}_2 &= -\omega c x_1 + (\lambda c^2 - s^2)x_2 + (\lambda + 1)csx_3 \\
&\quad - c(cx_2 + sx_3)[x_1^2 + (cx_2 + sx_3)^2] \\
&\quad + g_{21}u_1 + g_{22}u_2, \\
\dot{x}_3 &= -\omega s x_1 + (\lambda + 1)csx_2 + (\lambda s^2 - c^2)x_3 \\
&\quad + s(cx_2 + sx_3)[x_1^2 + (cx_2 + sx_3)^2] \\
&\quad + g_{31}u_1 + g_{32}u_2,
\end{aligned}$$

where $G = [g_{ij}(x)]$ is assumed to be of full rank in a neighborhood of the origin, ω is a fixed, real parameter, $c \doteq \cos\alpha$, $s \doteq \sin\alpha$ with α and λ being parameters. If the controls $u_1 = u_2 = 0$, then standard results show that as λ passes from negative to positive values, a limit cycle emerges at $\lambda = 0$. In fact, if α is fixed, this system represents a normal form for third-order families of systems admitting a Hopf bifurcation, i.e., this is the canonical model for all third-order systems displaying a Hopf bifurcation. The limit cycle is confined to a 2-dimensional manifold W^c-which, for $\lambda = 0$, is the center manifold. The other invariant manifold W^s is stable for all λ, and its intersection with W^c is transverse. The basic picture is as shown in Figure 7.

Let us define the matrix

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and let $F(x, \lambda, \alpha)$ be the vector denoting the right side of the uncontrolled system. Then it can be shown that the feedback law

$$u(x) = -(HG)^{-1}HF(x, \lambda, \alpha) + (HG)^{-1}Uv$$

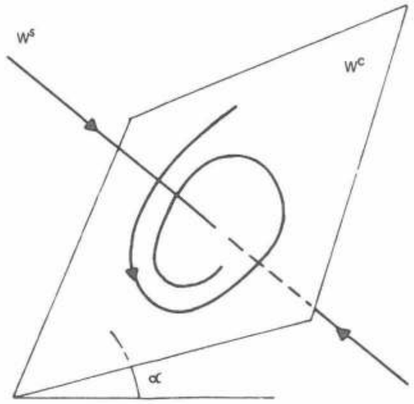

Figure 7. The center manifold W^c and its intersection with the stable manifold W^s.

decouples the original system into the new dynamics

$$\dot{x}_1 = \mu_1 v_1,$$
$$\dot{x}_2 = \mu_2 v_2,$$
$$\dot{x}_3 = Ax_1 + Bx_2 + Cx_3 + N(x_1, x_2, x_3) + \beta_1\mu_1 v_1 + \beta_2\mu_2 v_2$$

where $U = \text{diag}\ (\mu_1, \mu_2)$ is arbitrary,

$$G(HG)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \beta_1 & \beta_2 \end{bmatrix}$$

v_1 and v_2 are arbitrary parameters and

$$A = -\omega s - \beta_1\lambda + \beta_2\omega c,$$
$$B = (\lambda H)cs - \beta_1\omega c - \beta_2(\lambda c^2 - s^2),$$
$$C = (\lambda s^2 - c^2) - \beta_1\omega s - \beta_2(\lambda + 1)cs,$$

with $N(\cdot)$ involving cubic terms.

To study the stability and bifurcation properties of this decoupled system, we keep $x_1 = x_2 = 0$ by setting $v_1 = v_2 = 0$ and making the initial values of x_1 and x_2 vanish. The resulting system for x_3 is

$$\dot{x}_3 = C(\lambda, \alpha)x_3 + N(0, 0, x_3),$$

which is stable if and only if $C(\lambda, \alpha) < 0$. Since we are intereseted in stabilizing the Hopf bifurcation at $\lambda = 0$, we obtain the following result: *the Hopf bifurcation at $\lambda = 0$ is stabilized by the above feedback law $u(x)$ if and only if $C(0, \alpha) < 0$*, i.e. if x_1 and x_2 remain small, so does x_3.

Explicitly, the criterion on C is

$$-c^2 - \beta_1\omega s - \beta_2\omega cs < 0.$$

Thus, we see that either case can occur: the feedback may either be stabilizing or not, depending upon the parameters β_1, β_2, ω and α. For instance, if $\beta_1 = \beta_2 = 1 = \omega$, the criterion becomes

$$\cos^2\alpha + \sin\alpha + \cos\alpha \sin\alpha > 0.$$

For some angles α this inequality holds (e.g., $\alpha = 0$), while for others it does not ($\alpha = -\pi/4$). In each case it would be necessary to calculate sgn C.

The moral of this example is to show that it often is possible to stabilize undesired oscillations by feedback control, but that choice of the right control law may involve some delicacy.

The preceding example shows how feedback control can neutralize an inherent bifurcation in the uncontrolled dynamics. But, the opposite can also happen, i.e. a well-behaved system can be sent into oscillatory, or even chaotic behavior by introduction of control laws of unsuitable structure. This possibility is particularly insidious in situations where the control law is selected to optimize some performance criterion without proper attention being paid to its possibily bifurcation generating side effects. The possibility of chaos rather than order emerging from control actions is illustrated by the following result.

Consider the discrete-time *linear* system

$$x_{t+1} = \alpha x_t + u_t, \quad \alpha \text{ real}$$

Then it is easy to see that if $u_t \equiv 0$ (no control) the solutions do not oscillate. Now introduce the simple feedback control

$$u_t = \beta|x_t| - 1, \quad \beta \text{ real}.$$

It is established in ref. 14 that if the parameters α and β satisfy the conditions

$$\beta > 0, \quad \beta^2 - \alpha^2 \geq \alpha + \beta + 1$$

the controlled system has at least N_{n-2} distinct periodic trajectories of period n, where N_k = kth Fibonacci number. Furthermore, if strict inequality holds in the second relation above, then there are an uncountable number of bounded aperiodic trajectories with the property that if x^1 and x^2 are distinct members of this family then

$$|x_t^1 - x_t^2| \geq 2(\beta^2 - \alpha^2 - \alpha - \beta - 1)/[(\alpha - \beta)^2(\alpha + \beta)^2] > 0,$$

for arbitrarily large t. In other words, the controlled system displays *chaos*.

Other examples could be given to illustrate how feedback control can move, or even eliminate, undesirable bifurcations in the equilibrium behavior of various processes in ecology and economics. However, the above illustrations are sufficient to convey the message that external control can help in stabilizing system behavior, but it can also generate unstable oscillations if not applied with care.

9 THE PERIOD OF A LIMIT CYCLE

Once it has been established that a limit cycle ζ exists for a given dynamical process, its shape can then be determined by integration of the system equations in forward time using arbitrary initial conditions. Once the contour of the limit cycle has been determined, the period of the motion can be determined from the contour integral

$$T = \oint_C dt.$$

To be more specific, let us assume that we are studying the following structurally stable perturbation of the classical linear oscillator

$$\ddot{x} + \gamma f(x)\dot{x} + x = 0, \quad \gamma \geq 0.$$

The constant γ determines the strength of the damping term. The Van der Pol oscillator occurs as the special case of this equation when $f(x) = 3x^2 + a$. To ease the exposition, we write the dynamics as

$$\dot{x} = y - \gamma F(x),$$
$$\dot{y} = -x$$

where $d/dt\, F(x) = f(x)\dot{x}$. We wish to study the case of the Van der Pol oscillator when γ is very small and again when γ is very large. In particular, we shall examine how the period T of the limit cycle depends upon the parameter a.

First consider the case $|\gamma f(0; a)| \ll 1$. The focus at the origin is stable if $\gamma f(0, a) > 0$, unstable if $\gamma f(0; a) < 0$. Since the foci wind in or out slowly when $\gamma f(0; a)$ is small, the values $x(t), y(t)$ during the course of one revolution are given by

$$x \cong R \sin t, \quad y \cong R \cos t.$$

In order to find the radius of the approximately circular limit cycle (when it exists), we make use of the fact that, since $xdy + ydy - \gamma F(x)dy = 0$ and $xdx + ydy = 1/2\, d(x^2 + y^2)$, we have

$$\oint F(x)\, dy = 0.$$

Thus, when $F(x; a) = x^3 + ax$, we have

$$\oint [R^2 \sin^3 \theta + aR \sin \theta](-R \sin \theta) d\theta = 0$$

or

$$\pi R^2(3/4R^2 + a) = 0.$$

For $a \neq 0$ there is an isolated critical point at $R = 0$, which is stable if $a > 0$, unstable if $a < 0$. If $a > 0$, there are no other nearby solutions, while if $a < 0$ there is a stable limit cycle of radius

$$R = \sqrt{-4a/3} = 2\sqrt{-a/3}$$

The period of this limit cycle is

$$T = \oint dt = -\oint \frac{dt}{x} = -\oint \frac{d(R \cos\theta)}{R(\sin\theta)} = 2\pi.$$

Now let us examine the situation when γ is very large. Since γ large implies y is large, we introduce the new variable z

$$y = \gamma z, \quad 1/\gamma \ll 1.$$

The new system equations are

$$\dot{x} = \gamma[z - F(x)],$$
$$\dot{z} = -x/\gamma,$$

or

$$[z - F(x)]\frac{dz}{dx} = -x/\gamma^2 \cong 0.$$

For large γ, either

$$\frac{dz}{dx} \cong 0$$

or

$$z - F(x) \cong 0.$$

In the x–z plane, when $F(x) = x^3 + ax$ and $a < 0$, the system motion is shown in Figure 8. Here we see the familar type of relaxation oscillation described in earlier sections. If the system starts off the surface $z = F(x)$, as for instance at points A and A', the fast dynamic in the z-direction immediately pulls the system to the equilibrium manifold. Once the system reaches this curve, it cycles around the closed loop BCDEB in an alternating fast-slow-fast-slow periodic orbit. The time scales involved are of the order $1/\gamma$ for the fast jumps and order γ for the slow sections. Thus, the period is given by

$$T \cong 2\int_E^B -\frac{\gamma dz}{x} = -2\gamma \int_E^B \left(3x + \frac{a}{x}\right) dx$$
$$= \gamma \left[3\,(x_E^2 - x_B^2) + 2a \log \frac{x_E}{x_B}\right].$$

In the Van der Pol case, relaxation oscillations occur only when $a < 0$. In this event,

$$x_B = \sqrt{-a/3}, \quad x_E = 2\sqrt{-a/3}$$
$$T \cong |\gamma a|\,(3 - 2 \log 2)$$
$$\cong 1.6\, |\gamma a|$$

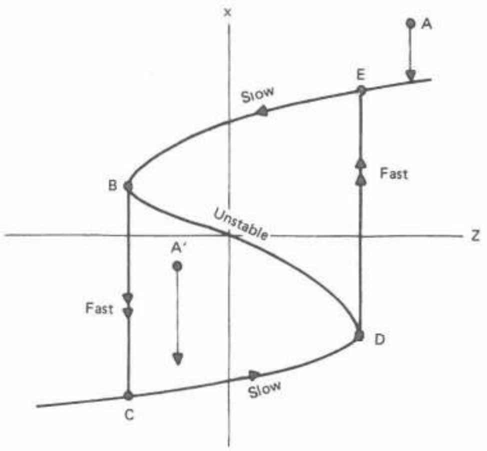

Figure 8. Relaxation oscillation for γ large.

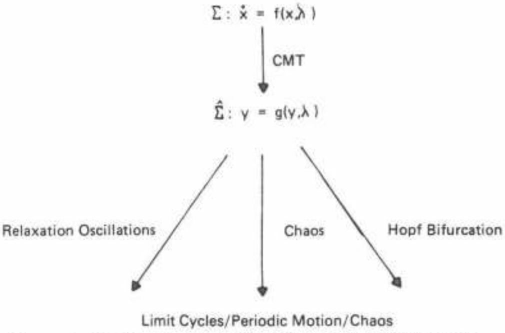

Figure 9. Model reduction, bifurcations and cyclic behavior.

Remark: In the above treatment, we have identified the Hopf bifurcation and the relaxation oscillations of the Van der Pol oscillator with the symmetry-restricted cusp catastrophe, i.e. we have restricted the general cusp $\phi(x) = x^3 + ax + b$ to the case $b = 0$ to obtain the symmetry relation $F(x) = -F(-x)$. Functions $F(x)$ associated with higher symmetry restricted catastrophes can be used to construct dynamical systems with even more interesting behavior.

The major message of the example considered here is that complicated dynamical systems become tractable when they involve multiple distinct and widely separated time scales.

10 SUMMARY AND DISCUSSION

System model simplifications, bifurcations from one type of behavior and oscillatory/cyclic motion have been seen to be related according to the diagram in Figure 9. The diagram should be interpreted in the following sense: we begin with the system Σ parameterized by λ. By means of the Center Manifold Theorem Σ is reduced to $\hat{\Sigma}$, a system of much smaller dimension and whose behavior at least *locally* characterizes the essential nonlinear aspects of Σ. It is only on the center manifold that local bifurcations can occur; away from the center manifold the system equations can be linearized. On the center manifold the sytem trajectory will bifurcate at certain values of λ into oscillatory behavior. The type of behavior will depend upon whether or not the intrinsic damping is weak or strong; if strong, then relaxation oscillations can be expected, if weak, then bifurcation to a limit cycle in the tradition Hopf sense is the usual pattern. In either case, the *expected* nonlinear aspects of the system display themselves through the loss of stability of a fixed point (equilibrium) as some parameter(s) pass through critical values.

When viewed within the above framework, the ubiquitous nature of oscillatory behavior in natural systems seems mostly to be due to the simple fact that, as λ varies, the characteristic roots of the linearized dynamics move across the imaginary axis. It is exactly the roots on the imaginary axis which give rise to the center manifold and, since the generic case is for a complex conjugate pair to cross the imaginary axis with non-zero speed, almost all bifurcations are of the Hopf-type and, furthermore, the center manifold is of dimension two, generically.

It has also been seen that bifurcation may be of chaotic, rather than cyclic behavior. Strictly speaking, this is also a kind of oscillatory but aperiodic motion. For continuous time systems, such behavior cannot occur unless the system is of order three or higher; in discrete-time chaos can emerge even for scalar processes.

The possibility of changing the system's behavior through introduction of feedback control has also been examined. In general, the question of what can and cannot be done about altering the system dynamics through feedback is a delicate one and requires the full machinery of nonlinear reachability theory[15] for its resolution. However, by example, it has been seen that feedback control can be both stabilizing and destabilizing, depending upon how it is employed. Thus, feedback control is one way out of undesirable oscillations—but only if adroitly applied.

Space considerations have required that a number of additional topics involving oscillatory behavior be eliminated. First on this list are issues surrounding spatial extension and oscillations not just in time, but also in space. Various infinite-dimensional extensions of the Center Manifold Theorem exist to deal with such processes, which are governed by non-linear partial differential equations of evolution-type. In another direction, the presence of time-lags in a process is another well-known oscillation-producing mechanism. We have touched briefly upon this matter here, but much more remains to be said. In particular, such processes also involve infinite-dimensional state spaces for which an appropriate extension of the Center Manifold Theorem is required. Such results exist[15], but must be deferred to another time. Finally, we have omitted entirely a consideration of how stochastic disturbances may induce (un)stable oscillations in a dynamic process. Such considerations lead to the theory of dissipative structures elaborated by Prigogine and others.[16] Note also the deep connections between stochastic processes and the deterministic chaos results discussed here and in refs. 16 and 17.

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