Simple Proof of Equivalence between Adiabatic Quantum Computation and the Circuit Model

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We prove the equivalence between adiabatic quantum computation and quantum computation in the circuit model. An explicit adiabatic computation procedure is given that generates a ground state from which the answer can be extracted. The amount of time needed is evaluated by computing the gap. We show that the procedure is computationally efficient.

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Introduction.—In the effort to realize a quantum computer, adiabatic quantum computation (AQC) [1] offers a promising alternative to the standard ''circuit model'' [2,3]. In comparison to the circuit model, AQC alleviates the need to perform fast quantum logic operations and measurements, which is particularly troublesome in the context of fault-tolerant quantum computation [4]. In AQC, the answer to a calculation is contained in the ground state of a quantum Hamiltonian. By placing a system in the ground state of a simple Hamiltonian and then adiabatically changing until the desired Hamiltonian is reached, one carries the system into the computationally meaningful state. The AQC model was known from the outset to be efficiently simulatable by the standard model [1,5], but for some time researchers wondered whether AQC could efficiently simulate the standard model.

Recently, a number of relatively complex proofs of the equivalence between the circuit model and AQC were given. One proof showed that AQC using Hamiltonians with long-range five- or three-body interactions, or nearest-neighbor two-body interactions with six-state particles, can efficiently simulate the circuit model [6]. This result was soon modified to qubits with two-body interactions [7,8], and then it was shown that AQC using qubits with nearest-neighbor two-body interactions on a 2D lattice can efficiently simulate the standard circuit model [9]. The proofs in Refs. [6–9] all start from a five-body interaction Hamiltonian that arises in Kitaev's quantum NP-complete ''local Hamiltonians'' problem [10]. They then require a reduction to two-body Hamiltonians and a proof that the spectral gap of the AQC Hamiltonian thus constructed is properly lower bounded. (A simplified construction has appeared recently, but it leads to a three-body Hamiltonian [11].) Let the number of algorithm steps (number of single- and two-qubit gates) be N. The running time is $O(N^5)$ with five-body interactions and $O(N^{14})$ with three-body interactions [6], which was improved to $O(N^{12})$ with two-body interactions in Ref. [8] (throughout we use the O notation to mean ''of order''). An additional improvement by a factor of N was given in Ref. [12], where relatively simple methods were used to provide a lower bound on the minimal energy gap.

Here we provide an alternative, constructive proof of the equivalence between the standard circuit model and AQC that is physically and mathematically transparent, amenable to implementation and yields a running time T of order $(MN)^2$ or better, where M is the number of qubits. For example, in the case of Shor's algorithm for factoring an L-bit integer using a linear nearest-neighbor qubit array [13], this translates into $T \sim [(2L + 4)(8L^4)]^2 \sim 256L^{10}$ compared to $T \sim (8L^4)^{11} \sim 10^{10}L^{44}$ using the previous $O(N^{11})$ scaling. We do this by setting up an explicit Hamiltonian involving at most two-body, nearest-neighbor interactions between particles on a 2D lattice. Our construction uses the method of ground state quantum computation (GSQC), which was independently proposed in Refs. [14–16] around the same time as AQC and also studied in Ref. [17]. In contrast to the previous equivalence proofs [6–9,11], our proof does not rely on Feynman's ''global clock particle'' idea. Instead, we synchronize the particles locally via CNOT gates.

In GSQC, one executes an algorithm by producing a ground state that *spatially* encodes the entire temporal trajectory of the algorithm, from input to output. This requires N times as much hardware but provides some robustness against decoherence. GSQC was deliberately constructed to simulate the standard model [14]. However, little attention was devoted to the process of reaching the desired ground state. Here, we marry together AQC and GSQC. The result is a formalism supplying an explicit Hamiltonian $H(s)$ acting on qubits with at most two-body nearest-neighbor interactions for any algorithm formulated in the circuit model. The initial Hamiltonian $H(0)$ and its ground state are simple. The intermediate Hamiltonian $H(s)$ $(0 \leq s \leq 1)$ has a gap and a nondegenerate ground state for all s (the dimensionless time). The final Hamiltonian $H(1)$ has a ground state containing the solution to the algorithm. Using the adiabatic theorem, we provide an upper bound on the time needed to reach $H(1)$ while keeping the system in its ground state. This

bound scales polynomially in the algorithm steps and qubits; the calculation is efficient.

Single qubit.—The ground state that contains the result of a given standard algorithm is specified as follows [14]. First consider a particularly simple computation involving only a single qubit with basis states $|0\rangle$ and $|1\rangle$. In the circuit model the qubit evolves through $N+1$ time steps: its initial state and a state after each algorithm step. If the initial state is $|0\rangle$ and algorithm step i consists of application of a 2×2 unitary gate U_i, then the two amplitudes at time step i appear in the state $U_i\cdots U_1|0\rangle$, where $1\le i\le N$. Since there are two amplitudes at each of the $N+1$ steps, the whole trajectory can be described by giving $2(N+1)$ complex amplitudes. In GSQC, instead of a time-dependent state in a two-dimensional Hilbert space, the qubit has a time-independent state in a $2(N+1)$-dimensional Hilbert space, with basis states $c_{i,0}^\dagger|\text{vac}\rangle$ and $c_{i,1}^\dagger|\text{vac}\rangle$, $i=0,\ldots,N$. Here, $c_{i,x}^\dagger$ ($c_{i,x}$) is a fermionic creation (annihilation) operator for a particle in state $x\in\{0,1\}$ in mode $i\in\{0,1,\ldots,N\}$. The amplitude of the time-independent wave function in basis state $c_{i,0}^\dagger|\text{vac}\rangle$ ($c_{i,1}^\dagger|\text{vac}\rangle$) contains the amplitude of the time-dependent system in the basis state $|0\rangle$ ($|1\rangle$) after algorithm step i.

To illustrate this with a concrete physical system, imagine a two-dimensional array of quantum dots with $N+1$ columns and 2 rows. Column i contains one state localized on the dot in the left row ($c_{i,0}^\dagger|\text{vac}\rangle$) and one on the dot in the right row ($c_{i,1}^\dagger|\text{vac}\rangle$). One spin-polarized electron is placed in the array; its state is a superposition of the $2(N+1)$ localized states.

It is convenient to group creation operators into row vectors $C_i^\dagger\equiv[c_{i,0}^\dagger c_{i,1}^\dagger]$. Then, the (unnormalized) ground state containing the results of the algorithm is

$$|\Psi^N\rangle=\left(C_0^\dagger\begin{bmatrix}1\\0\end{bmatrix}+C_1^\dagger U_1\begin{bmatrix}1\\0\end{bmatrix}+\cdots+C_N^\dagger U_N\cdots U_1\begin{bmatrix}1\\0\end{bmatrix}\right)|\text{vac}\rangle.$$

The results, stored in the states $c_{N,0}^\dagger|\text{vac}\rangle$ and $c_{N,1}^\dagger|\text{vac}\rangle$, can be extracted reliably [16]. To execute the GSQC, one realizes a specific Hamiltonian $H(1)$ whose ground state is the time-independent state we have identified. When the computation involves a single qubit, the Hamiltonian takes the form $H(1)=\sum_{i=1}^N h^i(U_i)$, where

$$h^i(U_i)\equiv\mathcal{E}[C_i^\dagger-C_{i-1}^\dagger U_i^\dagger][C_i-U_iC_{i-1}],\qquad(1)$$

and where $\mathcal{E}$ sets the energy scale. Here we have used the quadratic form $C_i^\dagger VC_j=\sum_{x,y\in\{0,1\}}v_{xy}c_{i,x}^\dagger c_{j,y}$, where v_{xy} are the matrix elements of V. In the quantum dots illustration mentioned above, $H(1)$ controls the on-site energy of each dot and the tunneling coupling between each dot in one column and each dot in the next column. One can confirm $|\Psi^N\rangle$ is an eigenstate with eigenvalue 0 [$H(1)|\Psi^N\rangle=0$]; since every term (1) is clearly positive semidefinite, $|\Psi^N\rangle$ is the ground state [18].

We have a time-independent state that contains the result of any given standard algorithm for one qubit and is the ground state of a known Hamiltonian $H(1)$. Now we show that placing the system in this ground state can be done efficiently via the method of AQC, which constitutes a proof that AQC can simulate the circuit model (for a single qubit). To do so we introduce the Hamiltonian $H(s)=\sum_{i=1}^N h^i(\lambda(s)U_i)$, where λ: $s\in[0,1]\mapsto[0,1]$, such that $\lambda(0)=0$ and $\lambda(1)=1$. If $\lambda=0$, then $h^i(0)=\mathcal{E}C_i^\dagger C_i$ reduces to a simple on-site energy term. There is then no tunneling from algorithm step i to algorithm step $i+1$. If $\lambda=1$, we recover $H(1)=\sum_{i=1}^N h^i(U_i)$ with $h^i(U_i)$ being the full operator (1).

The (unnormalized) ground state of $H(s)$ is simply $|\Psi^N\rangle$ as written above, but with a factor of $\lambda(s)$ in front of each unitary operator U_i. Alternatively, the ground state is given by the recursion relation

$$|\Psi^j(s)\rangle=\{1+C_j^\dagger(\lambda(s)U_j)C_{j-1}\}|\Psi^{j-1}(s)\rangle.\qquad(2)$$

Intuitively, the state of a j step calculation, $|\Psi^j(s)\rangle$, is formed by adding to the state of a $j-1$ step calculation $|\Psi^{j-1}(s)\rangle$ a term which annihilates the particle at $j-1$ and creates a particle at j with $\lambda(s)U_j$ applied to its state. The initial input state is simply

$$|\Psi^0(s)\rangle=C_0^\dagger\begin{bmatrix}1\\0\end{bmatrix}|\text{vac}\rangle.$$

The wave function is localized on the input row when $\lambda=0$. As λ increases, the on-site energy rises on rows $0,\ldots,N-1$. The tunneling matrix elements in (1) also begin to turn on, and the ground state wave function starts to spill into all states $c_{i,0}^\dagger|\text{vac}\rangle$ and $c_{i,1}^\dagger|\text{vac}\rangle$. When λ reaches 1, the wave function reaches its final form.

The traditional statement of the adiabatic theorem [19] is that the increase in $\lambda(s)$ must be sufficiently gradual that the system does not transition to an excited state as s goes from 0 to 1. If the time to take s from 0 to 1 is T, transitions are suppressed if $T\gg\hbar\max_s|\langle\chi^N(s)|\frac{dH(s)}{ds}|\Psi^N(s)\rangle|/(E_\chi(s)-E_\Psi(s))^2$, where $|\chi^N(s)\rangle$ is any excited eigenstate of $H(s)$, and $E_\chi(s)$ is its energy. Recent work has emphasized that this is not necessarily the right condition, since what really matters is not suppression of transitions throughout the entire adiabatic quantum algorithm, but rather that the overlap between the ground state of $H(1)$ and the final adiabatic wave function be large [20,21]. An adiabatic condition arises of the form $T=\hbar O(\Delta E_{\min}^{-1})$, where $\Delta E_{\min}$ is the minimum energy gap between the ground and first excited state, as s goes from 0 to 1 [20,21]. Here we use this latter condition to prove that AQC can simulate the circuit model efficiently; we omit the (similar but more complicated) proof that uses the traditional adiabatic condition. It is, however, necessary to use knowledge of the gap structure (e.g., position of

$\Delta E_{\min}$ as a function of s) in order to achieve the running time $T = \hbar O(\Delta E_{\min}^{-1})$. For this reason the position of the minimum gap $\Delta E_{\min}$ is given below.

To compute the minimum gap for an $N+1$ step calculation, we look for solutions of $D_{N+1}^2(\bar{E}) = 0$, where $D_{N+1}^2 \equiv \det(H(s)/\mathcal{E} - \bar{E})$, $H(s)$ is the $2(N+1) \times 2(N+1)$ Hamiltonian matrix, and $\bar{E} \equiv E/\mathcal{E}$. We first make a unitary transformation to new operators $\tilde{C}_i \equiv (U_i^\dagger \cdots U_1^\dagger) C_i$, which transforms $H(s) = \sum_{i=1}^N h^i(\lambda(s) U_i)$ to $\sum_{i=1}^N h^i(\lambda I)$, where I is the 2×2 identity matrix. Writing out the matrix $H(s)$, we find the iterative relation $D_{N+1} = (1 + \lambda^2 - \bar{E}) D_N - \lambda^2 D_{N-1}$. The solutions to $D_{N+1} = 0$ identify the exact single-qubit eigenenergies, which we find to be $E_{0,s} = 0$ and $E_{n,s} = \mathcal{E}[(1 - \lambda(s))^2 + 2\lambda(s)(1 - \cos\frac{\pi n}{N+1})]$ for $n = 1, \ldots, N$. By minimizing the first excited state energy $E_{1,s}$ with respect to λ, one sees that $E_{1,s} \geq \mathcal{E}\sin^2\frac{\pi}{(N+1)} = \mathcal{E}O(1/N^2)$. The single-qubit minimum occurs when $\lambda(s) = \cos\frac{\pi}{N+1}$. (This can also be used to estimate the position of the minimum in the multiple-qubit case below.) Thus $\Delta E_{\min} = E_{1,s} - E_{0,s} = \mathcal{E}O(1/N^2)$, and the simulation time is $T = \hbar O(\Delta E_{\min}^{-1}) = (\hbar/\mathcal{E})O(N^2)$. This is polynomial, so we see AQC can efficiently simulate the circuit model.

Multiple qubits.—The recursion relation (2) generalizes immediately to algorithms involving M noninteracting qubits: $|\Psi^j(s)\rangle = \Pi_{A=1}^M (1 + C_{A,j}^\dagger(\lambda U_{A,j}) C_{A,j-1}) |\Psi^{j-1}(s)\rangle$, where

$$|\Psi^0(s)\rangle = \Pi_{A=1}^M C_{A,0}^\dagger \begin{bmatrix} 1 \\ 0 \end{bmatrix} |\text{vac}\rangle.$$

The multiple-qubit Hamiltonian is just the sum of the single-qubit Hamiltonians $H(s) = \sum_{A=1}^M \sum_{i=1}^N h_A^i(\lambda(s) U_{A,i})$; one can verify that $H(s)|\Psi^N(s)\rangle = 0$ for arbitrary λ. The AQC procedure for noninteracting qubits simply involves the single-qubit procedure applied independently to each.

Now, we allow the qubits to interact via two-qubit gates such as a controlled-NOT (CNOT). Suppose the algorithm specifies a CNOT gate between qubits A and B at step j. Then, instead of applying the factors $(I + C_{A,j}^\dagger(\lambda U_{A,j}) C_{A,j-1})$ and $(I + C_{B,j}^\dagger(\lambda U_{B,j}) C_{B,j-1})$ to $|\Psi^{j-1}(s)\rangle$, we write

$$|\Psi^j(s)\rangle = (I + c_{A,j,0}^\dagger \lambda c_{A,j-1,0} C_{B,j}^\dagger(\lambda I) C_{B,j-1} + c_{A,j,1}^\dagger \lambda c_{A,j-1,1} C_{B,j}^\dagger(\lambda \sigma_x) C_{B,j-1}) |\Psi^{j-1}(s)\rangle.$$

If qubit A is in state 0, this operator applies $[I + C_{B,j}^\dagger(\lambda I) C_{B,j-1}]$ to qubit B. This is just the usual recursion relation factor that subjects qubit B to an IDENTITY gate. The factor for a NOT gate, $(I + C_{B,j}^\dagger(\lambda \sigma_x) C_{B,j-1})$, is applied to B if A is in state 1.

When a CNOT gate is present, $H(s)$ needs to be changed so that we still have $H(s)|\Psi^N(s)\rangle = 0$. One replaces terms $h_A^j(\lambda U_{A,j})$ and $h_B^j(\lambda U_{B,j})$ in $H(s)$, with two-body interactions

$$h_{A,B}^j(\lambda, \text{CNOT}) = h_{A,B}^j(\text{ID}) + h_{A,B}^j(\text{N}) + h_{A,B}^j(\text{P}). \quad (3)$$

Here $h_{A,B}^j(\text{ID}) = \mathcal{E}(C_{B,j} c_{A,j,0} - \lambda^2 C_{B,j-1} c_{A,j-1,0})^\dagger \times (C_{B,j} c_{A,j,0} - \lambda^2 C_{B,j-1} c_{A,j-1,0})$, and $h_{A,B}^j(\text{N}) = \mathcal{E}(C_{B,j}\, c_{A,j,1} - \lambda^2\, \sigma_x\, C_{B,j-1}\, c_{A,j-1,1})^\dagger\, (C_{B,j}\, c_{A,j,1} - \lambda^2 \sigma_x C_{B,j-1} c_{A,j-1,1})$ are two-particle analogues of the one-particle IDENTITY gate $h_A^j(\lambda I)$ and NOT gate $h_A^j(\lambda \sigma_x)$ defined by (1). The third term $h_{A,B}^j(\text{P}) = \mathcal{E}\sum_{i<j,k\geq j} C_{A,i}^\dagger C_{A,i} C_{B,k}^\dagger C_{B,k} + C_{A,k}^\dagger C_{A,k} C_{B,i}^\dagger C_{B,i}$ imposes an energy penalty on states in which one qubit has gone through the CNOT gate without the other. Since $h_{A,B}^j(\lambda, \text{CNOT})$ is positive semidefinite with all other terms in $H(s)$, verifying $H(s)|\Psi^N(s)\rangle = 0$, one sees $|\Psi^N(s)\rangle$ is the ground state.

To determine the effect of a CNOT gate on the gap, consider first a simple calculation with $M = 2$ qubits and a single CNOT at row j. Divide the Hamiltonian into $H = H_0 + H_1$, where H_1 is just the CNOT gate (3) and H_0 contains all of the single-qubit terms. We know all of the exact eigenstates and eigenvalues of H_0 from the single-qubit analysis of D_{N+1} above. If the interaction H_1 were absent, the qubits would simply occupy these eigenstates of H_0 independently. Let $|Z\rangle$ be a two-qubit state satisfying $H_0|Z\rangle = 0$, where both qubits are in zero-energy ground states. Let $|\bar{Z}\rangle$ be a state in which at least one qubit is excited. Our single-qubit analysis of D_{N+1} yields the exact result $\langle\bar{Z}|H_0|\bar{Z}\rangle \geq \mathcal{E}\sin^2\frac{\pi}{(N+1)} = \mathcal{E}O(1/N^2)$.

The CNOT Hamiltonian H_1 couples these eigenstates of H_0. It can be diagonalized analytically [22] in the small basis of ground states $|Z\rangle$; for all states $|Z\rangle$ orthogonal to the computationally meaningful $|\Psi^N(s)\rangle$, we find $\langle Z|H|Z\rangle = \langle Z|H_1|Z\rangle = \mathcal{E}O(1/N^2)$. This exact result will

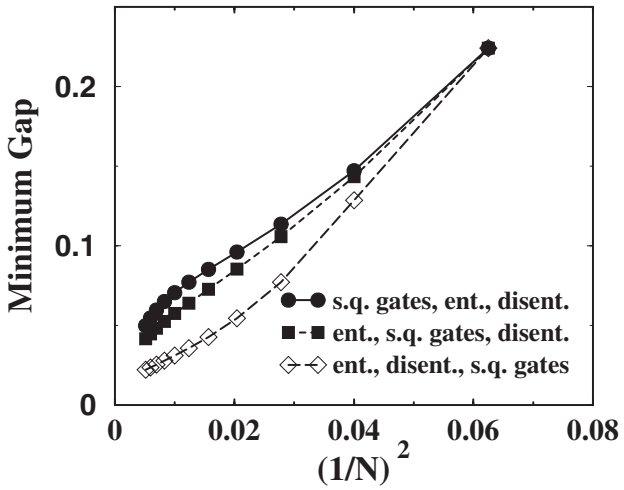

FIG. 1. Minimum gap in units of $\mathcal{E}$ for a two-qubit system of N steps, $4 \leq N \leq 14$. Results are shown for computations that entangle the two qubits into a Bell state and then disentangle them. A string of single-qubit gates is also included at one of three stages of the computation. The minimum gap is roughly a linear function of $1/N^2$.

be used momentarily to derive a lower bound of the true gap. But first, it supplies a rigorous variational *upper bound* of the true gap (because the small number of $|Z\rangle$ states are not a basis for the full $[2(N+1)]^2$-dimensional Hilbert space), and it is also a good estimate of the *exact* energy gap, as supported by the two-qubit numerical calculation shown in Fig. 1. As a result, AQC requires a time $T = (\hbar/\mathcal{E})O(N^2)$. While this estimate is intuitively correct and numerically verified, what we need for our equivalence proof is a lower bound, which we now show is $\mathcal{E}O(1/N^4)$. To obtain this bound, note that an arbitrary excited state of $H = H_0 + H_1$ can be written as $\alpha|Z\rangle + \beta|\bar{Z}\rangle$ for some unique normalized $|Z\rangle$ and $|\bar{Z}\rangle$. Since H_1 is positive semidefinite, we find $\langle Z|H_1|Z\rangle\langle\bar{Z}|H_1|\bar{Z}\rangle \geq |\langle Z|H_1|\bar{Z}\rangle|^2$. Given this inequality, we minimize $\langle H\rangle$ with respect to α and β and find

$$\langle H\rangle \geq \langle Z|H|Z\rangle\langle\bar{Z}|H_0|\bar{Z}\rangle/(\langle Z|H|Z\rangle + \langle\bar{Z}|H|\bar{Z}\rangle). \quad (4)$$

(This is most easily derived by computing the lowest eigenvalue of the two-by-two matrix H in the $|Z\rangle$, $|\bar{Z}\rangle$ basis.) We can estimate the numerator using the $\mathcal{E}O(1/N^2)$ bounds on $\langle\bar{Z}|H_0|\bar{Z}\rangle$ and $\langle Z|H|Z\rangle$ stated above. Since the denominator of (4) certainly does not increase with N, the energy $\langle H\rangle$ is at least $\mathcal{E}O(1/N^4)$.

A similar argument works even when the system has many qubits and many CNOT gates. We write the many-qubit Hamiltonian as $H = H_0 + H_1$, where H_1 includes all of the CNOT gates (3) and H_0 contains all of the single-qubit terms. The exact eigenstates and eigenenergies of H_0 are immediately known from the single-qubit analysis. An arbitrary state can be written $\alpha|Z\rangle + \beta|\bar{Z}\rangle$, where in $|Z\rangle$ all qubits are in ground states of H_0, while in $|\bar{Z}\rangle$ there is at least one excited qubit. We still have $\langle\bar{Z}|H_0|\bar{Z}\rangle \geq \mathcal{E}\sin^2\frac{\pi}{(N+1)} = \mathcal{E}O(1/N^2)$. We can also show that $\langle Z|H|Z\rangle \geq \mathcal{E}O(1/N^2)$, since for each term in $|Z\rangle$ there is always at least one CNOT gate that contributes to its energy. Using (4), we find the same $\mathcal{E}O(1/N^4)$ bound on $\langle H\rangle$. To facilitate extraction of the results of the computation, it is important that when the system is measured every qubit has a large amplitude on the final row N. As Ref. [15] shows, it is straightforward to rescale tunneling to the final row of the computation to ensure this happens. However, the reduction of qubit amplitude at earlier stages of the calculation leads to a reduction of the gap. The estimate/upper bound becomes $\mathcal{E}O(1/N^2M)$ and the lower bound is $\mathcal{E}O(1/N^4M^2)$.

We have presented an explicit adiabatic procedure that will carry a system adiabatically into a ground state containing the result of an arbitrary standard quantum computation. Ref. [16] shows how to use quantum teleportation to trade an arbitrary GSQC with N steps and M qubits for a different GSQC with 7 steps, $(2N-1)M$ qubits. Once the Hamiltonian is adjusted to facilitate extraction of the results [15], the running time of the new calculation is $T = (\hbar/\mathcal{E})O(N^2M^2)$. The upper bound/estimate of the gap yields $T = (\hbar/\mathcal{E})O(NM)$, which is proportional to the "volume" of the algorithm.

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[18] Note that it is always necessary to introduce a perturbation to select $|\Psi^N\rangle$ as the unique ground state over a state that looks just like $|\Psi^N\rangle$ but starts in $|1\rangle$ rather than $|0\rangle$. This corresponds to selecting the input to the calculation to be 0 rather than 1. The perturbation we add is $-\delta\mathcal{E}\sum_{A=1}^{M} c^\dagger_{A,0,0}c_{A,0,0}$, where $\delta\mathcal{E} \ll \mathcal{E}$. This ensures a gap of $O(1/N)$ between $|\Psi^N\rangle$ and states with undesired input values, pushing them far above the low-lying excited states of energy $\leq O(1/N^2)$. So, they play no role in determining the gap, except in the teleportation scheme discussed at the end of the paper. See [16].
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