Simple proofs of classical results on zeros of $\mathbf{J}_\nu(\mathbf{x})$ and $\mathbf{J}'_\nu(\mathbf{x})$

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Abstract

The Bessel functions $J_\nu(x)$ and their derivatives $J'_\nu(x)$ can be represented by infinite series and infinite products. Using these representations we give very simple proofs for known results concerning the zeros of the above functions.

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1 Introduction

It is well known [4, 5] that the Bessel function $J_\nu(x)$ and its derivative $J'_\nu(x)$ can be represented by the infinite series:

$$J_\nu(x) = \big(\frac{x}{2}\big)^\nu \sum_{n=0}^{\infty} \frac{\big(-\frac{x^2}{4}\big)^n}{n!\Gamma(\nu+n+1)}, \quad \nu > -1 \tag{1.1}$$

$$J'_\nu(x) = \frac{1}{2}\big(\frac{x}{2}\big)^{\nu-1} \sum_{n=0}^{\infty} \frac{\big(-\frac{x^2}{4}\big)^n (2n+\nu)}{n!\Gamma(\nu+n+1)}, \quad \nu > 0 \tag{1.2}$$

as well as by infinite products:

$$J_\nu(x) = (\frac{x}{2})^\nu \frac{1}{\Gamma(\nu+1)} \Pi_{n=1}^{\infty}(1 - \frac{x^2}{j_{\nu,n}^2}), \quad \nu > -1 \tag{1.3}$$

and

$$J'_\nu(x) = \frac{1}{2}(\frac{x}{2})^{\nu-1} \frac{1}{\Gamma(\nu)} \Pi_{n=1}^{\infty}(1 - \frac{x^2}{(j'_{\nu,n})^2}), \quad \nu > 0 \tag{1.4}$$

respectively. By $j_{\nu,n}$ and $j'_{\nu,n}$, $n = 1, 2, ...$ we indicate the n-th positive zeros of $J_\nu(x)$ and $J'_\nu(x)$ respctively. Using only these representations for $J_\nu(x)$ and $J'_\nu(x)$ we obtain very easily well known [1, 2, 3, 5] results concerning the zeros of these functions.

2 Results on the zeros of $J_\nu(x)$

By equating the right hand side of (1.1) and (1.3) we obtain

$$\sum_{n=0}^{\infty} \frac{(-\frac{x^2}{4})^n}{n!\Gamma(\nu+n+1)} = \frac{1}{\Gamma(\nu+1)} \Pi_{n=1}^{\infty}(1 - \frac{x^2}{j_{\nu,n}^2}). \tag{2.1}$$

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Let us consider the first terms of the series on the left and the first terms of the products on the right, so:

$$\frac{1}{\Gamma(\nu+1)} - \frac{1}{4}x^2\frac{1}{\Gamma(\nu+2)} + \frac{1}{4^2}x^4\frac{1}{2!\Gamma(\nu+3)} - \frac{1}{4^3}x^6\frac{1}{3!\Gamma(\nu+4)} + ... \tag{2.2}$$

$$= \frac{1}{\Gamma(\nu+1)}(1-\frac{x^2}{j^2_{\nu,1}})(1-\frac{x^2}{j^2_{\nu,2}})(1-\frac{x^2}{j^2_{\nu,3}})... \tag{2.3}$$

Using the equality $\Gamma(x+1) = x\Gamma(x)$, it becomes:

$$1 - \frac{1}{4}x^2\frac{1}{\nu+1} + \frac{1}{4^2}x^4\frac{1}{2!(\nu+1)(\nu+2)} - \frac{1}{4^3}x^6\frac{1}{3!(\nu+1)(\nu+2)(\nu+3)} + ... \tag{2.4}$$

$$= (1-\frac{x^2}{j^2_{\nu,1}})(1-\frac{x^2}{j^2_{\nu,2}})(1-\frac{x^2}{j^2_{\nu,3}})... \tag{2.5}$$

1)By equating the coefficients of $x^0, x^2, x^4, ...$ of (2.5) we obtain respectively

$$1 = 1, \tag{2.6}$$

$$\frac{1}{4(\nu+1)} = \sum_{n=1}^{\infty}\frac{1}{j^2_{\nu,n}}, \tag{2.7}$$

$$\frac{1}{4^2 2!(\nu+1)(\nu+2)} = \sum_{n=1}^{\infty}\frac{1}{j^2_{\nu,n}}\sum_{k=1,k\neq n}^{\infty}\frac{1}{j^2_{\nu,k}} \tag{2.8}$$

Taking in account (2.7) the sums of the right hand side of (2.8) can be written

$$\sum_{n=1}^{\infty}\frac{1}{j^2_{\nu,n}}\sum_{k=1,k\neq n}^{\infty}\frac{1}{j^2_{\nu,k}} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{j^2_{\nu,n}}(\sum_{k=1}^{\infty}\frac{1}{j^2_{\nu,k}} - \frac{1}{j^2_{\nu,n}}) \tag{2.9}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{j^2_{\nu,n}}(\frac{1}{4(\nu+1)} - \frac{1}{j^2_{\nu,n}}) = \frac{1}{2}[(\frac{1}{4(\nu+1)})^2 - \sum_{n=1}^{\infty}\frac{1}{j^4_{\nu,n}}] \tag{2.10}$$

so, the equation (2.8) takes the form

$$\frac{1}{4^2 2!(\nu+1)(\nu+2)} = \frac{1}{2}[(\frac{1}{4(\nu+1)})^2 - \sum_{n=1}^{\infty}\frac{1}{j^4_{\nu,n}}] \tag{2.11}$$

or

$$\sum_{n=1}^{\infty}\frac{1}{j^4_{\nu,n}} = \frac{1}{2^4(\nu+1)^2(\nu+2)}. \tag{2.12}$$

Remark 2.1. If we continue using the analogous procedure by equating the coefficients of x^6,..., we'll obtain the sums $\sum_{n=1}^{\infty}\frac{1}{j^{2k}_{\nu,n}}$, $k = 3, ...$

Remark 2.2. We mention that the sums $\sum_{n=1}^{\infty}\frac{1}{j^{2k}_{\nu,n}}$, $k = 1, 2, 3, ...$ are well known [1, 2, 3, 5] but their proof is much more complicated.

Remark 2.3. It is obvious that using the sums $\sum_{n=1}^{\infty}\frac{1}{j_{\nu,n}^{2k}}$, $k = 1,2,3,...$ we obtain [5] known inequalities for the first zero of $J_\nu(x)$. For example using (2.12) we obtain the lower bound $j_{\nu,1}^2 > 4(\nu+1)(\nu+2)^{1/2}$, for $\nu > -1$.

2) Putting $\nu = 1/2$ in (2.5)and since $j_{1/2,n} = n\pi$, it becomes:

$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+... = (1-\frac{x^2}{\pi^2})(1-\frac{x^2}{2^2\pi^2})(1-\frac{x^2}{3^2\pi^2})... \tag{2.13}$$

or

$$sinx = x\Pi_{n=1}^{\infty}(1-\frac{x^2}{n^2\pi^2}) \tag{2.14}$$

which is the known [4] infinite product expansion for *sinx*.

3) Similarly, by putting $\nu = -1/2$ in (2.5)and since $j_{-1/2,n} = (2n-1)\frac{\pi}{2}$, it becomes:

$$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+... = (1-\frac{4x^2}{\pi^2})(1-\frac{4x^2}{3^2\pi^2})(1-\frac{4x^2}{5^2\pi^2})... \tag{2.15}$$

or

$$cosx = \Pi_{n=1}^{\infty}(1-\frac{4x^2}{(2n-1)^2\pi^2}) \tag{2.16}$$

which is the known [4] infinite product expansion for *cosx*.

4) We put iy instead of x in (2.5), so it becomes:

$$1+\frac{1}{4}y^2\frac{1}{\nu+1}+\frac{1}{4^2}y^4\frac{1}{2!(\nu+1)(\nu+2)}+\frac{1}{4^3}y^6\frac{1}{3!(\nu+1)(\nu+2)(\nu+3)}+... \tag{2.17}$$

$$=(1+\frac{y^2}{j_{\nu,1}^2})(1+\frac{y^2}{j_{\nu,2}^2})(1+\frac{y^2}{j_{\nu,3}^2})... \tag{2.18}$$

and y are the zeros of the modified Bessel function $I_\nu(y)$. By putting $\nu = 1/2$ in (2.18) we have

$$1+\frac{y^2}{3!}+\frac{y^4}{5!}+\frac{y^6}{7!}+... = (1+\frac{y^2}{\pi^2})(1+\frac{y^2}{2^2\pi^2})(1+\frac{y^2}{3^2\pi^2})... \tag{2.19}$$

or

$$sinhy = y\Pi_{n=1}^{\infty}(1+\frac{y^2}{n^2\pi^2}) \tag{2.20}$$

which is the known [4] infinite product expansion for *sinhy*.

5) Similarly, by putting $\nu = -1/2$ in (2.18)we have:

$$1+\frac{y^2}{2!}+\frac{y^4}{4!}+\frac{y^6}{6!}+... = (1+\frac{4y^2}{\pi^2})(1+\frac{4y^2}{3^2\pi^2})(1+\frac{4y^2}{5^2\pi^2})... \tag{2.21}$$

or

$$coshy = \Pi_{n=1}^{\infty}(1+\frac{4y^2}{(2n-1)^2\pi^2}) \tag{2.22}$$

which is the known [4] infinite product expansion for *coshy*.

Remark 2.4. From (2.14) we also obtain the well known [4] result that $\lim_{x\to 0}\frac{sinx}{x} = 1$.

Remark 2.5. The equations (2.7) and (2.12) for $\nu = 1/2$ and $\nu = -1/2$ give the known summable series $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$, $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$ and $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$, $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$ respectively.

3 Results on the zeros of $J'_\nu(x)$

By equating the right hand side of (1.2) and (1.4) we obtain

$$\sum_{n=0}^{\infty} \frac{(-\frac{x^2}{4})^n(2n+\nu)}{n!\Gamma(\nu+n+1)} = \frac{1}{\Gamma(\nu)}\Pi_{n=1}^{\infty}(1-\frac{x^2}{(j'_{\nu,n})^2}). \tag{3.1}$$

We are working similarly as in section 2, so, we consider the first terms of the series on the left and the first terms of the products on the right, so:

$$\frac{\nu}{\Gamma(\nu+1)} - \frac{x^2}{4}\frac{(2+\nu)}{\Gamma(\nu+2)} + \frac{x^4}{4^2}\frac{(4+\nu)}{2!\Gamma(\nu+3)} - \frac{x^6}{4^3}\frac{(6+\nu)}{3!\Gamma(\nu+4)} + ... \tag{3.2}$$

$$= \frac{1}{\Gamma(\nu)}(1-\frac{x^2}{(j'_{\nu,1})^2})(1-\frac{x^2}{(j'_{\nu,2})^2})(1-\frac{x^2}{(j'_{\nu,3})^2})... \tag{3.3}$$

and using the equality $\Gamma(x+1) = x\Gamma(x)$, it becomes:

$$1 - \frac{1}{4}x^2\frac{(2+\nu)}{\nu(\nu+1)} + \frac{1}{4^2}x^4\frac{(4+\nu)}{2!\nu(\nu+1)(\nu+2)} - \frac{1}{4^3}x^6\frac{(6+\nu)}{3!\nu(\nu+1)(\nu+2)(\nu+3)} + ... \tag{3.4}$$

$$= (1-\frac{x^2}{(j'_{\nu,1})^2})(1-\frac{x^2}{(j'_{\nu,2})^2})(1-\frac{x^2}{(j'_{\nu,3})^2})... \tag{3.5}$$

By equating the coefficients of $x^0, x^2, x^4, ...$ we obtain respectively

$$1 = 1, \tag{3.6}$$

$$\frac{(2+\nu)}{4\nu(\nu+1)} = \sum_{n=1}^{\infty}\frac{1}{(j'_{\nu,n})^2}, \tag{3.7}$$

$$\frac{(4+\nu)}{4^2 2!\nu(\nu+1)(\nu+2)} = \sum_{n=1}^{\infty}\frac{1}{(j'_{\nu,n})^2}\sum_{k=1,k\neq n}^{\infty}\frac{1}{(j'_{\nu,k})^2} \tag{3.8}$$

As in the previous section, the sum in right hand side of (3.8) can be written

$$\sum_{n=1}^{\infty}\frac{1}{(j'_{\nu,n})^2}\sum_{k=1,k\neq n}^{\infty}\frac{1}{(j'_{\nu,k})^2} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(j'_{\nu,n})^2}(\sum_{k=1}^{\infty}\frac{1}{(j'_{\nu,k})^2} - \frac{1}{(j'_{\nu,n})^2}) \tag{3.9}$$

so we obtain

$$\sum_{n=1}^{\infty}\frac{1}{(j'_{\nu,n})^4} = \frac{(\nu^2+8\nu+8)}{4^2\nu^2(\nu+1)(\nu+2)}. \tag{3.10}$$

Remark 3.1. If we continue using the analogous procedure by equating the coefficients of x^6,..., we'll obtain the sums $\sum_{n=1}^{\infty}\frac{1}{(j'_{\nu,n})^{2k}}$, $k = 3, ...$

Remark 3.2. We mention that the sums $\sum_{n=1}^{\infty}\frac{1}{(j'_{\nu,n})^{2k}}$, $k = 1, 2, 3, ...$ are well known [1, 3] but their proof is much more complicated.

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