Simple singularities and $N = 2$ supersymmetric Yang-Mills theory

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Abstract

We present a first step towards generalizing the work of Seiberg and Witten on $N = 2$ supersymmetric Yang-Mills theory to arbitrary gauge groups. Specifically, we propose a particular sequence of hyperelliptic genus $n-1$ Riemann surfaces to underlie the quantum moduli space of $SU(n)$ $N = 2$ supersymmetric gauge theory. These curves have an obvious generalization to arbitrary simply laced gauge groups, which involves the A-D-E type simple singularities. To support our proposal, we argue that the monodromy in the semiclassical regime is correctly reproduced. We also give some remarks on a possible relation to string theory.

1. Introduction

In two beautiful papers [1,2], Seiberg and Witten have investigated $N = 2$ supersymmetric $SU(2)$ gauge theories and solved for their exact nonperturbative low energy effective action. For an arbitrary gauge group G, such supersymmetric theories are characterized by having flat directions for the Higgs vacuum expectation values, along which the gauge group is generically broken to the Cartan subalgebra. Thus, the effective theories contain $r = \text{rank}(G)$ abelian $N = 2$ vector supermultiplets, which can be decomposed into r $N = 1$ chiral multiplets A^i plus r $N = 1$ vector multiplets W^i_α. The $N = 2$ supersymmetry implies that the effective theory depends only on a single holomorphic prepo-

tential $\mathcal{F}(A)$. More precisely, the effective lagrangian in $N = 1$ superspace is

$$
\mathcal{L} = \frac{1}{4\pi}\text{Im}\left[\int d^4\theta\left(\sum \frac{\partial\mathcal{F}(A)}{\partial A^i}\bar{A}^i\right) + \int d^2\theta\,\frac{1}{2}\left(\sum \frac{\partial^2\mathcal{F}(A)}{\partial A^i\partial A^j}W^i_\alpha W^j_\alpha\right)\right].
$$
(1)

The holomorphic function $\mathcal{F}$ determines the quantum moduli space and, in particular, its metric. This space has singularities at points or surfaces where additional fields become massless, that is, where the effective action description breaks down. A crucial insight is that the electric and magnetic quantum numbers of the fields that become massless at a given singularity are determined by the eigenvalue-$(+1)$ eigenvectors of the monodromy matrix associated with the singularity.

For $G = SU(2)$ considered in [1,2], besides the point at $u = \infty$ there are singularities at $u = \pm\Lambda^2$, where Λ is the dynamically generated scale of the theory, and $u = \frac{1}{2}\langle a^2\rangle$, where $a \equiv A|_{\theta=0}$. (On the other hand, $u = 0$ is not singular in the exact quantum the-

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[2] Work supported in part by the US-Israel Binational Science Foundation and GIF - the German-Israeli Foundation for Scientific Research.

ory, which means that, in contrast to the classical theory, no massless non-abelian gauge bosons arise here). One of these points corresponds to a massless, purely magnetically charged monopole, and the other to a massless dyon. The parameter region near $u = \infty$ describes the semiclassical, perturbative regime, which is governed by the one-loop beta function [3,4]. It gives rise to a non-trivial monodromy as well (arising from the logarithm in the one-loop beta function), but there are no massless states associated with it.

The singularity structure and knowledge of the monodromies allow to completely determine the holomorphic prepotential $\mathcal{F}$. The monodromy group is $\Gamma(2) \subset SL(2, \mathbb{Z})$ consisting of all matrices congruent to $\mathbb{1}$ modulo 2. The matrices act on the vector $(a_D; a)^t$, where a_D is the magnetic dual of a, that is, $a_D \equiv \frac{\partial F(a)}{\partial a}$. The quantum moduli space, namely the u-plane punctured at $\pm \Lambda^2$ and ∞, can thus be thought of $\mathbb{H}/\Gamma(2)$, where $\mathbb{H}$ is the upper half-plane.

The basic idea [1] in solving for the effective theory is to consider the following family of holomorphic curves parametrized by $\mathbb{H}/\Gamma(2)$:

$$y^2 = (x - \Lambda^2)(x + \Lambda^2)(x - u) . \quad (2)$$

These curves represent a double cover of the x-plane with branch points at $0, \pm\Lambda^2$ and ∞, and describe a genus one Riemann surface. That is, the quantum moduli space of the $SU(2)$ super Yang-Mills theory coincides with the moduli space of a particular torus; this torus becomes singular when two branch points in (2) coincide. The derivatives of the electric and magnetic coordinates $(a_D; a)^t$ with respect of u are just given by the periods. Computing the period integrals (related to the two homology cycles) thus yields, upon integration, the dependence of a, a_D in terms of u, and integrating a_D finally determines the prepotential $\mathcal{F}(a)$.

In the present paper, we make a first step towards generalizing the work of Seiberg and Witten to (pure) super Yang-Mills theory with $SU(n)$ gauge group (we also will hint at how it might work for arbitrary simply laced groups). More precisely, we will propose what we think the appropriate curves are, and give circumstantial evidence to the fact that our choice is correct. We will present a more detailed analysis of the monodromies and period integrals in a follow-up paper [5].

2. Semiclassical regime

To be specific, we will consider mainly the gauge group $G = SU(3)$, but from our setup it will be clear that all of our arguments immediately generalize to arbitrary $SU(n)$. We will denote the gauge invariant order parameters (Casimirs) by

$$u = \tfrac{1}{2}\mathrm{Tr}\left\langle \phi^2 \right\rangle , \quad v = \tfrac{1}{3}\mathrm{Tr}\left\langle \phi^3 \right\rangle , \quad (3)$$

where we can always take the scalar superfield component to be $\phi = \mathrm{diag}(\alpha, \beta - \alpha, -\beta)$, such that, classically, $u = \alpha^2 + \beta^2 - \alpha\beta$, $v = \alpha\beta(\alpha - \beta)$. The residual global $\mathbb{Z}_6$ symmetry act as $u \to e^{2\pi i/3}u$, $v \to -v$. For generic eigenvalues of ϕ, the $SU(3)$ gauge symmetry is broken to $U(1) \times U(1)$, whereas if any two eigenvalues are equal, the unbroken symmetry is $SU(2) \times U(1)$. These classical symmetry properties are encoded in the following, gauge and globally $\mathbb{Z}_6$ invariant discriminant:

$$\Delta_0 = 4u^3 - 27v^2 = (\alpha + \beta)^2(2\alpha - \beta)^2(\alpha - 2\beta)^2 . \quad (4)$$

The lines $\Delta_0 = 0$ in (u, v) space correspond to unbroken $SU(2) \times U(1)$, and have a cusp singularity at the origin, where the $SU(3)$ symmetry is restored. As we will see, in the full quantum theory the cusp is smoothed out, $\Delta_0 \to \Delta_\Lambda = 4u^3 - 27v^2 + O(\Lambda^3)$, which, in particular, prohibits a phase with massless non-abelian gluons.

It is straightforward to compute the prepotential $\mathcal{F}$ in the perturbative regime, with the result

$$\mathcal{F}_{\mathrm{class}} = \frac{i}{4\pi} \sum_{i<j}^{3} (e_i - e_j)^2 \log[(e_i - e_j)^2/\Lambda^2] . \quad (5)$$

Here, e_i denote the roots of the equation

$$W_{A_2}(x, u, v) \equiv x^3 - u x - v = 0 , \quad (6)$$

whose bifurcation set is given by Δ_0 in (4). Whenever two roots coincide, the discriminant vanishes. In terms of the variables α, β we have

$$e_1 - e_2 = (\alpha + \beta)$$
$$e_1 - e_3 = (2\alpha - \beta)$$
$$e_2 - e_3 = (\alpha - 2\beta) \quad (7)$$

and thus, accordingly

$$W_{A_2}(x,\alpha,\beta) = (x-\alpha)(x-(-\beta))(x-(\beta-\alpha)) . \tag{8}$$

The Casimirs u,v are gauge invariant and, in particular, invariant under the Weyl group $\mathcal{W}$ of $SU(3)$. This group is generated by any two of the reflections

$$\begin{aligned} r_1 : \quad (\alpha,\beta) &\rightarrow (\beta-\alpha,\beta) \\ r_2 : \quad (\alpha,\beta) &\rightarrow (\alpha,\alpha-\beta) \\ r_3 : \quad (\alpha,\beta) &\rightarrow (-\beta,-\alpha) . \end{aligned} \tag{9}$$

Due to the multi-valuedness of the inverse map $(u,v) \rightarrow (\alpha,\beta)$, paths in (u,v) space will in general not close in (α,β) space, but will in general close only up to Weyl transformations. Such a monodromy will be non-trivial if a given path encircles a singularity in (u,v) space — in our case, the singularities will be at "infinity" and along the lines where the discriminant vanishes.

It is indeed well-known [6] that the monodromy group of the simple singularity of type A_2 (6) is given by the Weyl group of $SU(3)$, and acts as Galois group on the e_i (and analogously for $W_{A_{n-1}}$ related to $SU(n)$). This will be the starting point for our generalization.

What we are interested in, of course, is not just the monodromy acting on (α,β), but the monodromy acting on $(a_{D;1},a_{D;2};\alpha,\beta)^t$, where

$$a_{D,i} \equiv \mathcal{F}_i = \frac{\partial}{\partial a_i} \mathcal{F}(\alpha,\beta) . \tag{10}$$

Performing the Weyl reflection r_1 on $(\alpha,\beta)^t$, we easily find

$$\begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{pmatrix} + N \begin{pmatrix} 2 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} . \tag{11}$$

The winding number N that arises from the logarithms is not determined by the finite, "classical" Weyl transformation acting on the a_i, but depends on the chosen path in (u,v) space. We note that for large $Z \equiv 4u^3 - 27v^2$, which corresponds to the semiclassical limit, the prepotential behaves like

$$\mathcal{F}_{class} \sim u \log\left[\frac{Z}{\Lambda^6}\right] , \tag{12}$$

from which we can read off the winding number for any given path in the semiclassical regime. We find that by choosing appropriate paths, one can have N jump by even integers, and that the minimal winding number is $N = 1$. (An example for such a closed loop is given by $(u(a_i(t)),v(a_i(t)))$ for $t = 0,\ldots,1$, where $a_1(t) = e^{i\pi t}a_1 + \frac{1}{2}(1 - e^{i\pi t})a_2$, $a_2(t) = a_2$.) Therefore, the matrix representation of r_1 acting on $(a_{D;1},a_{D;2};\alpha,\beta)^t$ is

$$r_1 = \begin{pmatrix} -1 & 0 & 2 & -1 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv r_1^{class} T^{-1} , \tag{13}$$

where r_1^{class} is the "classical" Weyl reflection (given by the block diagonal part of r_1), and T the "quantum monodromy"

$$T = \begin{pmatrix} \mathbb{1} & C \\ 0 & \mathbb{1} \end{pmatrix} , \quad \text{where} \quad C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{14}$$

is the Cartan matrix of $SU(3)$. The other Weyl reflections are given analogously by $r_i = r_i^{class} T^{-1}$. The r_i are related to each other by conjugation, and, in particular, rotate into each other via the Coxeter element, $r_{cox}^{class} = r_1^{class} r_2^{class}$.

3. The curves for $SU(n)$

Our aim is now to find a sequence of curves $\mathcal{C}$ that reproduces the quantum moduli space of supersymmetric $SU(n)$ Yang-Mills theories. Since we do not know how to derive these curves from first principles, we will make a proposal for the curves that is consistent with various requirements, and subsequently verify that at least the monodromies at "infinity" reproduce the above matrices r_i. This will be our only nontrivial consistency check for the time being. To really show that the choice of curves is correct physicswise, requires in addition to check the various other monodromies, which correspond to the condensation of monopoles and dyons. A detailed discussion of these matters will be presented elsewhere [5].

Let us now list the requirements that we impose on the curves $\mathcal{C}$. First, we seek surfaces with $2n$ periods (corresponding to $(a_{D;i}; a_i)$), whose period matrices are positive definite. It will be pointed out below that this condition can be satisfied by choosing genus

$n - 1$ Riemann surfaces, in direct generalization of [1]. Secondly, we require that for $\Lambda \to 0$ the classical situation is recovered. That is, the discriminant of C should have, for $\Lambda = 0$, a factor of Δ_0 given in (4). This means that for $\Lambda = 0$ the curves should have the form $y^m = C(x) \equiv W_{A_{n-1}} \times (\ldots)$ for some m. This then also implies that the monodromy groups will have something to do with the Weyl groups of $SU(n)$, and this is what we want as well. Thirdly, the curves must behave properly under the cyclic global transformations acting on the Casimirs $\{c_2, c_3, \ldots\} \equiv \{u, v, \ldots\}$; in other words, there should be a natural dependence on Casimirs, for all groups. Finally, from [2] we know that Λ should appear in $C(x)$ with a power that corresponds to the charge violation of the one-instanton process.

Taking these requirements together suggests the surfaces for $SU(n)$ Yang-Mills theory to be the following, genus $g = n - 1$ hyperelliptic curves:

$$y^2 = C_n(x) \equiv \left(W_{A_{n-1}}(x, c_i)\right)^2 - \Lambda^{2n}, \qquad (15)$$

where

$$W_{A_{n-1}}(x, c_i) = x^n - \sum_{i=2}^{n} c_i x^{n-i} \qquad (16)$$

are the A-type simple singularities related to $SU(n)$. Morally speaking, the square of W reflects having both electric and magnetic degrees of freedom. Since there is a general relationship [6] between Arnold's simple A-D-E singularities, perturbations by Casimirs, monodromy and Weyl groups, we conjecture that (15) describes surfaces for the other simply laced gauge groups G as well, by simply replacing $W_{A_{n-1}}(x, c_i)$ by the corresponding D- or E-type singularity, and Λ^{2n} by Λ^{2h}, where h is the corresponding Coxeter number.

Note in passing that even though $C_2(x)$ does not have the form as one of the curves given in [1,2], it is equivalent to the $\Gamma_0(4)$ modular curve given in [2], since the modular invariants [3] coincide: $j(u) = \frac{1}{27\Lambda^8} \frac{(3\Lambda^4 - 4u^2)^3}{\Lambda^4 - u^2}$. Note also that the points $u^2 = \Lambda^4$ and $u = \infty$ are exchanged for the two curves in [1,2], i.e., the parameters of the $\Gamma(2)$ and $\Gamma_0(4)$ modular curves are related as follows:

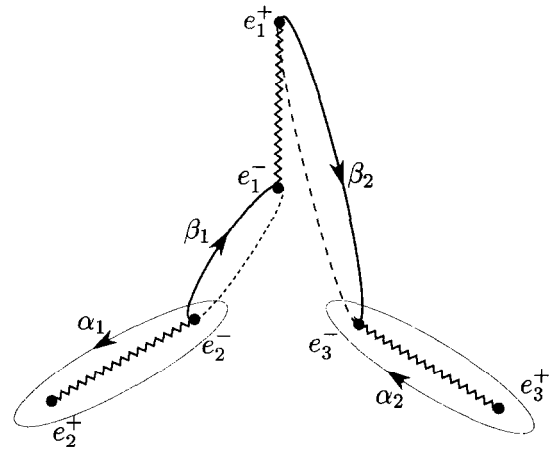---

[3] Obtained by transforming to the Weierstrass normal form.

Fig. 1. Branched x-plane with cuts linking pairs of roots of $C_3 = 0$. We depicted our choice of basis for the homology cycles. Condensation of monopoles or dyons occurs when two branch points approach each other. The monodromy of the corresponding vanishing cycle then determines the electric and magnetic quantum numbers.

$$u_{(4)} = \frac{u_{(2)}}{\sqrt{u_{(2)}^2 - \Lambda^4}} \Lambda^2 . \qquad (17)$$

Returning to $SU(n)$, it is useful to write

$$C_n(x) = \left(W_{A_{n-1}}(x, c_i) + \Lambda^n\right)\left(W_{A_{n-1}}(x, c_i) - \Lambda^n\right)$$

$$= \prod_{i=1}^{n}(x - e_i^+)(x - e_i^-) . \qquad (18)$$

Critical surfaces occur whenever two roots of $C(x)$ coincide, that is, whenever the discriminant $\Delta_\Lambda = \prod_{i<j}(e_i^\pm - e_j^\pm)^2$ vanishes. Physicswise we expect when this happens, monopoles or dyons condense whose quantum numbers are determined by the corresponding monodromy matrices. For example, for $G = SU(3)$ the quantum discriminant is

$$\Delta_\Lambda = \Lambda^{18} \Delta_\Lambda^+ \Delta_\Lambda^-, \quad \Delta_\Lambda^\pm = 4u^3 - 27(v \pm \Lambda^3)^2 . \quad (19)$$

By construction, the hyperelliptic curves (15) are represented by branched covers over the x-plane. More precisely, we have n $\mathbb{Z}_2$ cuts, each linking a pair of roots e_i^+ and e_i^-, $i = 1, \ldots, n$. As an example, we present the picture for $G = SU(3)$ in Fig. 1. In the classical theory, where $\Lambda \to 0$, the branch lines shrink to n doubly degenerate points: $e_i^- \to e_i^+ \equiv e_i$. These points, given for $SU(3)$ in Eq. (7), correspond to the

weights of the n-dimensional fundamental represen-
tation (the picture represents a deformed projection
of the weights onto the unique Coxeter eigenspace
with $\mathbb{Z}_n$ action). This means that the branched x-plane
transforms naturally under the finite "classical" Weyl
group that permutes the points. This finite Weyl group
is all there is in the classical theory, and is just the
usual monodromy group of the A_{n-1} singularity al-
luded to earlier. In the quantum theory, where the de-
generate dots are resolved into branch lines, there are
in addition possibilities for "quantum monodromy",
which involves braiding of the cuts.

4. Periods and monodromies for $G = SU(3)$

What we are interested in is the monodromy at in-
finity, which happens to be an "unstable" situation in
that more than two points collide simultaneously. This
would in principle require to find an appropriate com-
pactification of the moduli space to make the degen-
eration stable. [4] However, instead of trying to resolve
this subtle problem, we will rather employ a trick to
get at the monodromy at infinity in a more direct way.

Specifically, we will use the fact the the monodromy
factors into a classical and a quantum part, just as in
(13) for $SU(3)$. From the above it is quite clear that
the classical part of the monodromy is obtained by
simply permuting the branches in the x-plane. This is
easily implemented by choosing appropriate paths just
like the one above Eq. (13).

The quantum part is associated with the logarithm
in (12). The crucial observation is that we can mimic
the effect of looping around in the Z-plane (where
$Z \equiv 4u^3 - 27v^2$), by formally rotating

$$\mathcal{T} : \quad \Lambda^6 \to e^{2\pi i t}\Lambda^6, \quad t = 0, \ldots, 1, \quad (20)$$

along a small cycle around the origin. Such a rota-
tion of a singularity $\tilde{W} = W_0 + \epsilon$, $\epsilon = e^{2\pi i t}$, is indeed
well-known in the mathematical literature [6], where
it is, ironically, called "classical monodromy". For the

A-D-E simple singularities it corresponds to the Cox-
eter element of the Weyl group, whereas in the present
context, it gives the quantum monodromy in the semi-
classical regime.

Now what $\mathcal{T}$ does on the x-plane is to transform
the e_i^+ and e_i^- into each other – this is obvious from
(18). Therefore, the quantum monodromy is given by
the product of all the monodromy matrices associated
with the vanishing cycles around the branch cuts in
the x-plane. So what needs to be done is to determine
the precise form of these matrices.

Clearly, the monodromy matrices must reflect the
action of braiding and permuting the cuts on the vector
$(a_{D;i}; a_i)^t$. This action is expressed in terms of the
action on the homology cycles via

$$a_{D;i} = \oint_{\beta_i} \lambda, \quad a_i = \oint_{\alpha_i} \lambda, \quad (21)$$

where α_i, β_j is some symplectic homology basis with
$\langle \alpha_i, \beta_j \rangle = -\langle \beta_j, \alpha_i \rangle = \delta_{ij}$, $\langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle = 0$,
$i, j = 1, \ldots, g$. From the theory of Riemann surfaces it
is clear that the monodromy group must be contained
in $\mathrm{Sp}(2g, \mathbb{Z}) = \mathrm{Sp}(2n - 2, \mathbb{Z})$. For $G = SU(3)$, we
have depicted our choice of homology basis in Fig. 1.

In (21), λ denotes a suitably chosen meromorphic
differential. The holomorphic differentials on a genus
$g = n - 1$ hyperelliptic curve are given by $\omega_i = x^{i-1}\frac{dx}{y}$,
$i = 1, \ldots, g$. They give rise to the period matrices as
$A_{ij} = \int_{\alpha_j}\omega_i$ and $B_{ij} = \int_{\beta_j}\omega_i$, which are related to
$(a_{D;i}, a_i)$ as follows:

$$A_{ij} = \frac{\partial a^i(u)}{\partial u_j}, \quad B_{ij} = \frac{\partial a_D^i(u)}{\partial u_j}. \quad (22)$$

This represents a non-trivial integrability condition,
and this is what determines λ in (21). Specifically,
we can have that $\omega_i = \frac{\partial \lambda}{\partial u_i}$ if we (formally) choose,
for example, the differential as follows:

$$\lambda = -dx \log[-W_{A_{n-1}} - \sqrt{(W_{A_{n-1}})^2 - \Lambda^{2n}}]. \quad (23)$$

Note that due to the identification (22), the second
Riemann bilinear relation, $\mathrm{Im}(A^{-1}B) > 0$, ensures
the positivity of the metric:

$$(ds)^2 = \mathrm{Im}\frac{\partial^2\mathcal{F}}{\partial a^i\partial a^j}da^id\bar{a}^j = \mathrm{Im}\sum_{i=1}^{g}da_{D;i}d\bar{a}^i, \quad (24)$$

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[4] For $G = SU(2)$, one can do this by just fixing three points on
the x-plane. Doing this has the effect that $u = \pm\Lambda^2$ and $u = \infty$
get exchanged, precisely according to the reparametrization (17).
Thus, what one might call monodromy at infinity gets exchanged
with what one might call monodromy at $u = \Lambda^2$. The monodromy
matrices, however, are essentially the same (up to conjugation and
inversion), so the difference does not seem to matter.

similar as for genus one [1].

The action of braiding the branch points on the homology can be obtained elementarily by tracing the deformations of the cycles induced by the movement of the branch points. Somewhat easier and more elegant is however to use the Picard-Lefshetz formula [6]. Denoting by ν_{ij} the vanishing cycle that vanishes as one moves the i'th branch point along a specified path γ to the j'th point, the action on any cycle of the counter-clockwise braiding of the two points along that path is given by

$$S_{\nu_{ij}}\gamma = \gamma + \langle \gamma, \nu_{ij} \rangle \nu_{ij} , \qquad (25)$$

where $\langle \gamma, \nu_{ij} \rangle$ denotes the intersection of the two cycles. (For the A-D-E simple singularities, this formula coincides with the well-known formula for Weyl reflections.)

Specifically, the effect of braiding the cut between e_1^+ and e_1^- on our homology basis, defined in Fig. 1, comes out to be as follows:

$$B_{e_1^+,e_1^-} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

There is no additional sign since the forms ω_i are invariant for this particular braid. Similarly,

$$B_{e_2^+,e_2^-} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B_{e_3^+,e_3^-} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

According to what we said above, the quantum monodromy is then given by the product of these matrices, and it indeed coincides with (14):

$$B_{e_1^+,e_1^-} B_{e_2^+,e_2^-} B_{e_3^+,e_3^-} \equiv T^{-1} . \qquad (26)$$

The classical monodromy depends on the particular path, and is trivially given by the corresponding classical Weyl group element. For example, for the path given above Eq. (13), the branches 1 and 2 in Fig. 1 are exchanged, and the action of this braid on the homology is given by r_1^{class}. Hence, the full monodromy

is $r_1 = r_1^{class}T^{-1}$, in accordance with Eq. (13). We believe that this is a non-trivial verification of our proposal for the hyperelliptic curves.

Obtaining the monodromies around the other vanishing cycles is harder, and is deferred to the future [5]. However, we can make an easy guess of what the monodromies corresponding to the purely magnetically charged monopoles might be (possibly up to conjugation), namely by considering

$$r_i^{(mag)} = \Omega \, r_i^{class}T \, \Omega^{-1} = (r_i^{-1})^t , \\ i = 1, 2, 3 , \qquad (27)$$

where

$$\Omega = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

is the symplectic metric. These matrices would look like

$$r_1^{(mag)} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & -1 & -1 & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix} , \quad etc.$$

According to [1,2], the magnetic charges of the monopoles that become massless at the corresponding singularities are given by the eigenvalue-$(+1)$ eigenvectors of the monodromy matrices. Specifically, for the matrices (27) we find the following (magnetic;electric) charges for the massless monopoles,

$$r_1^{(mag)} \quad : \quad (0,1;0,0)$$
$$r_2^{(mag)} \quad : \quad (1,0;0,0)$$
$$r_3^{(mag)} \quad : \quad (-1,1;0,0) , \qquad (28)$$

which is something one would expect in accordance with semiclassical stability [7].

5. Remarks about a relation to string theory

The A-D-E singularities play also a well-known rôle in string theory [8]. They give rise to exactly solvable $d = 2$ $N = 2$ supersymmetric Landau-Ginzburg models, whose tensor products can be used to represent Calabi-Yau string compactifications. We can in-

deed relate our curves as well to LG models, by simply going to homogenous coordinates. The LG superpotentials are of the form $W_{LG} = y^2 + x^{2n} + (\Lambda z)^{2n} + c_2 x^{2n-2} z^2 + \ldots$, where c_i are now dimensionless moduli, and Λ is an irrelevant, non-zero number. These potentials describe tensor products of two $N = 2$ minimal models of type A_{2n-1}.

As is well-known [9], such $N = 2$ theories, when viewed as topological field theories, are characterized by prepotentials $\mathcal{F}_{LG}(a_i)$, where a_i are the flat coordinates corresponding to the LG moduli c_i. The point is that the computation of $\mathcal{F}_{LG}$ is, essentially, the same as the computation that leads to $\mathcal{F}$ in (1), and therefore these two prepotentials are very closely related. Therefore, if we consider type IIB string compactification with $N = 2$ space-time supersymmetry in $d = 4$, on a superconformal background that contains one of the above LG models as a tensor product piece of it, the string effective action [10,11] contains a piece that is very similar to the the $N = 2$ Yang-Mills effective action (1). It might be possible that, upon decoupling the gravitational sector ("rigid special geometry" [1,12]) and appropriately freezing the various other fields, the effective actions do coincide. It is indeed well-known that the abelian gauge group in the RR-sector of a type II string compactification never enlarges to a non-abelian group, and this may be thought as a reflection of what happens for quantum $N = 2$ Yang-Mills theory.

Does this potentially mean that a low-energy observer cannot distinguish between this subsector of the type IIB string compactification and the effective Yang-Mills theory? The answer could be related to the conjecture [13] about the equivalence of string theory with its effective field theory, when all solitons of the effective theory are taken into account. The above would also imply that the complex functional dependence of a given $\mathcal{F}$ on the moduli a could be attributed either to world-sheet instanton effects, or, equally well, to space-time non-perturbative effects, and this relation would seem to be quite non-trivial.

We believe that these matters urgently deserve further study.

References

[1] N. Seiberg and E. Witten, Nucl. Phys. B 426 (1994) 19, hep-th/9407087.
[2] N. Seiberg and E. Witten, Monopoles, Duality, and Chiral Symmetry Breaking in $N = 2$ Supersymmetric QCD, preprint RU-94-60, IASSNS-94-55, hep-th/9408099.
[3] V. Novikov, M. Schifman, A. Vainstein, M Voloshin and V. Zakharov, Nucl. Phys. B 229 (1983) 394;
V. Novikov, M. Schifman, A. Vainstein and V. Zakharov, Nucl. Phys. B 229 (1983) 381, 407; Phys. Lett. B 166 (1986) 329.
[4] N. Seiberg, Phys. Lett. B 206 (1988) 75.
[5] A. Klemm, W. Lerche, S. Theisen and S. Yankielowicz, in preparation.
[6] See e.g., V. Arnold, A. Gusein-Zade and A. Varchenko, Singularities of Differentiable Maps I, II, Birkhäuser 1985.
[7] R. Brandt and F. Neri, Nucl. Phys. B 161 (1979) 253.
[8] E. Martinec, Phys. Lett. B 217 (1989) 431;
C. Vafa and N.P. Warner, Phys. Lett. B 218 (1989) 51.
[9] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B 352 (1991) 59.
[10] N. Seiberg, Nucl. Phys. B 303 (1988) 286.
[11] S. Ferrara and A. Strominger, in "Strings 89", eds. R. Arnowitt et al. (World Scientific, Singapore, 1989) p. 245.
[12] A. Ceresole, R. D'Auria and S. Ferrara, Phys. Lett. 339 (1994) 71, hep-th/9408036.
[13] P. Townsend, Phys. Lett. B 202 (1988) 53;
C. Hull and P. Townsend, Unity of superstring dualities, preprint QMW-94-30, hep-th/9410167.