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04-22

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Simple Utility Functions with Giffen Demand*

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October 27, 2004

Abstract

We present some simple utility functions whose Marshallian demand functions possess the Giffen property: at some price-wealth pairs, the demand for a good marginally increases in response to an increase in its own price. The utility functions satisfy standard preference properties throughout the usual consumption set of non-negative bundles: continuity, monotonicity, and convexity.

Keywords: Preferences, Giffen Good.

JEL Classification: D11 (Consumer Economics: Theory)

*The inspiration for this manuscript came from 10 classes of students in Microeconomic Theory. I am particularly grateful to a class of 2003 that gave feedback on one of the examples in their written exam.

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1. Introduction

Most microeconomics textbooks mention the theoretical possibility of Giffen goods. Many books explain this possibility using a graphical example showing two relevant indifference curves. It is then left to the (student) reader to imagine the family of indifference curves that fills the gap between the two portrayed ones. Those graphical examples may fail to fully convince some students. Commonly, most other stated properties of demand curves are illustrated using explicit utility functions — this is often considered more convincing. The main motivation for this paper is to showcase some simple, standard utility functions with the Giffen property.

In our main examples, the consumer has utility function $u(x) = \min\{u_1(x), u_2(x)\}$ where u_1 and u_2 are themselves standard utility functions. Such a utility function u represents preferences for perfect complements in the intermediate utility indices u_1 and u_2. Several interpretations are possible. Leaning on the familiar interpretation of perfect complements (e.g. Varian (1999), Chapter 4), imagine that the goods are types of nutrition. Taking in the bundle x, the consumer is enabled to perform one activity (walking) in amount u_1 and simultaneously another activity (thinking) in amount u_2. The consumer views these activities as complements,[1] and therefore walking u_1 and thinking u_2 provides utility $u = \min\{u_1, u_2\}$. In another interpretation, the bundle x must be bought before the consumer learns his actual preferences. His utility function is either u_1 or u_2, and the consumer is infinitely risk-averse so that his ex ante utility function is $u = \min\{u_1, u_2\}$. In a similar interpretation, the consumer has multiple selves, one with utility u_1 and another with u_2. The consumer's decision making process involves maximizing a Rawlsian welfare aggregate for the two selves, namely the welfare function $u = \min\{u_1, u_2\}$.

Example 1 below is perhaps the simplest. With two goods, we let $u_1(x_1, x_2) = x_1 + B$ and $u_2(x_1, x_2) = A(x_1 + x_2)$. It is easy to draw the indifference map for u, and simple to depict how good 2 is a Giffen good over a wide range of prices and incomes. Figure 2.1 provides an illustration.

In order to further develop our understanding of Giffen goods, Section 2 continues to present a number of closely related two-goods examples. One example leads to a family of demand functions that could be helpful in empirical work, other examples test the limits of the construction behind Example 1. Some examples are introduced since they have nicer properties than Example 1. For instance, Example 6 lets $\min\{u_1, u_2\}$ be approximated by a CES expression to obtain kink-free indifference curves.

[1] Aristotle enjoyed walking around to discuss ideas with colleagues and students, and thus co-founded the Peripatetic school of philosophy.

Section 3 provides an example with L goods, demonstrating the theoretical observation that any number $k < L$ of the goods can have the Giffen property simultaneously. The Section also establishes general properties of utility functions of the form $\min\{u_1, \ldots, u_K\}$.

Moffatt (2002) has offered the first example of a family of direct utility functions giving rise to demand with the Giffen property, while also obeying standard properties of preferences: continuity, monotonicity and convexity.[2] In the construction, he proceeds in a reverse direction from us. He first constructs a backward-bending expansion path, to which he attaches a string of nearly-kinked hyperbolic indifference curves.[3] We start from a family of utility functions $u = \min\{u_1, u_2\}$ with kinked indifference curves, and notice that a standard trick of changing the cardinality of u_1 or u_2 allows us to place a backward-bending expansion path.[4]

2. Two Goods

2.1. Example 1

Consider an individual consumer with standard consumption set $\mathbb{R}^2_+$ and utility function $u(x_1, x_2) = \min\{x_1 + B, A(x_1 + x_2)\}$ where $A > 1$ and $B > 0$. An interpretation is that the intake (x_1, x_2) of coffee and Danish pastry will allow $x_1 + B > 0$ of thinking and $A(x_1 + x_2) \geq 0$ of walking, and that these two activities are perfect complements for the consumer.[5]

It is simple to construct a map of indifference curves for this consumer. In the area where $x_1 + B < A(x_1 + x_2)$, i.e. where $x_2 > (B - (A-1)x_1)/A$, the indifference curves follow the vertical ones of utility function $x_1 + B$, while in the opposite area they follow the straight indifference curves for $A(x_1 + x_2)$. Indifference curves are glued together on the straight line $x_2 = (B - (A-1)x_1)/A$ which slopes down from bundle $(0, B/A)$ to bundle $(B/(A-1), 0)$. The left panel of Figure 2.1 illustrates some indifference curves.

It is a straightforward exercise to find the Marshallian demand function. Given income $m > 0$ and price vector $p = (p_1, p_2) \gg 0$, the consumer chooses (x_1, x_2) to maximize the

[2]Earlier attempts at constructing utility functions yielding the Giffen property have failed to satisfy the standard properties. See the discussion in Moffatt (2002).

[3]In the end, he offers two equations, (18) and (20), to be solved in order to arrive at a direct expression for his utility function, but he does not give this expression analytically.

[4]Figure 17.E.1 of Mas-Colell et al. (1995) illustrates preferences of a very similar nature. They have selected expansion curves that are parallel to the axes, which could be used to give examples of a Giffen neutrality effect, when the demand for a good does not respond to a marginal change in its own price.

[5]Since $B > 0$, this may stretch the interpretation that x is an intake of food while $x_1 + B$ is an activity (thinking). It means that the brain thinks a significant amount even at full starvation. Since the Giffen property is local, the shape of the utility function near 0 is inessential for the point of this example, but examples 3 and onwards improve on this feature.

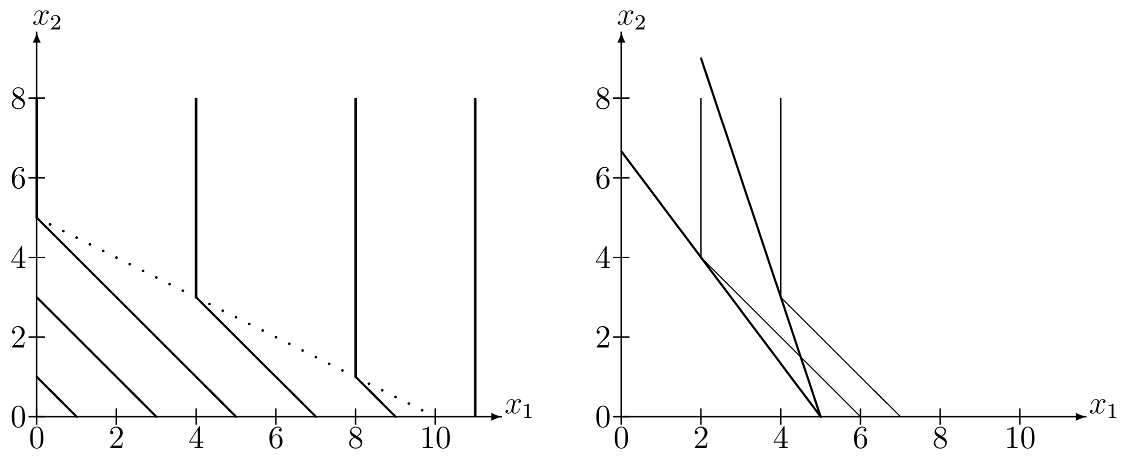

Figure 2.1: The left panel depicts some indifference curves for the consumer of Example 1, where $A = 2$ and $B = 10$. The straight dotted line indicates where the curves are kinked. The right panel illustrates the Giffen good effect, as the consumer's income is $m = 5$. The price of good 1 is fixed at $p_1 = 12$. When initially $p_2 = 4$, the consumer demands $x = (4, 3)$, but a price increase to $p_2 = 9$ changes the demand to $x = (2, 4)$. The Giffen effect is that the demand of good 2 rises from 3 to 4 as its price rises.

utility subject to the budget constraint $p_1x_1 + p_2x_2 \leq m$. We are particularly interested in the situation where the optimal choice lies on the kink line $x_2 = (B - (A-1)\, x_1)\,/A$. This happens when $p_1 > p_2$ and $Bp_2/A < m < Bp_1/\,(A-1)$. We can then find the demand as the solution to the kink line and budget line equations. This yields

$$x_2\,(p_1, p_2, m) = \frac{Bp_1 - (A-1)\,m}{Ap_1 - (A-1)\,p_2}.$$

Notice that the restrictions on (p_1, p_2, m) imply that both numerator and denominator are positive in this expression. It is then simple to see that x_2 is an increasing function of p_2 in this range. This means that x_2 is a Giffen good. For a graphical illustration of this Giffen effect, see the left panel of Figure 2.1.

2.2. Preparation for Further Examples

Generalizing the idea of Example 1, let us consider a consumer whose indifference curves are stitched together from two sets of straight indifference curves, with a general form of the kink locus. Consider thus a consumer with consumption set $\mathbb{R}^2_+$ and utility function of the form $u(x_1, x_2) = \min\{u_1(x_1, x_2), u_2(x_1, x_2)\}$, where $u_1(x_1, x_2) = x_1 + c_1x_2$ and $u_2(x_1, x_2) = f\,(x_1 + c_2x_2)$. Here $c_2 > c_1 > 0$ are parameters, and f is an arbitrary strictly increasing function.

The utility function u_1 represents preferences for perfect substitutes, under which the consumer requires only a units of good 1 to compensate for the loss of 1 unit of good 2 (see e.g. Varian (1999), Chapter 4). The indifference curves are parallel lines with slope $-1/c_1$. The utility function u_2 likewise represents preferences for perfect substitutes, where the indifference curves have slope $-1/c_2$. The strictly increasing transformation f does not affect the indifference curves for u_2, but merely serves to determine the utility level associated with each indifference curve.

Notice that the utility $u(x)$ at bundle $x = (x_1, x_2) \in \mathbb{R}^2_+$ exceeds the value $\bar{u} \in \mathbb{R}$ if and only if $u_1(x) \geq \bar{u}$ and $u_2(x) \geq \bar{u}$. The set of bundles weakly preferred to some $\bar{x} \in \mathbb{R}^2_+$ is therefore the intersection of the corresponding sets for utility functions u_1 and u_2. Using this observation, it is easy to depict the indifference curves for u. The indifference curves have a kink at any $\hat{x} \in \mathbb{R}^2_{++}$ solving $u_1(\hat{x}) = u_2(\hat{x})$. Since the indifference curves for u_1 are everywhere steeper than those for u_2, an indifference curve follows that of u_1 to the northwest of the kink, and that of u_2 to the southeast.[6]

Suppose now that the price vector is $p = (p_1, p_2) \gg 0$ satisfying $c_1 < p_2/p_1 < c_2$. The consumer's budget line will then be less steep than the indifference curves for u_1 and steeper than those for u_2. Suppose that $\hat{x} \in \mathbb{R}^2_{++}$ defines a kink point, and suppose that the consumer's income is $m = p_1\hat{x}_1 + p_2\hat{x}_2$. Then $\hat{x}$ will be the consumer's demanded bundle. When we marginally increase either of p_1 or p_2, the budget line will tilt inwards, and the optimal demanded bundle will stay on the kink curve. The equation describing the kink curve is $\hat{x}_1 + c_1\hat{x}_2 = f(\hat{x}_1 + c_2\hat{x}_2)$, and implicit differentiation gives $dx_1[1 - f'(\hat{x}_1 + c_2\hat{x}_2)] = dx_2[c_2 f'(\hat{x}_1 + c_2\hat{x}_2) - c_1]$. Our key observation is that this kink curve will have a negative slope if either $f'(\hat{x}_1 + c_2\hat{x}_2) > 1$ or $f'(\hat{x}_1 + c_2\hat{x}_2) < c_1/c_2$.

In the first case, when $f'(\hat{x}_1 + c_2\hat{x}_2) > 1$, we find that $dx_2/dx_1 > -1/c_2$. Thus the kink curve is flatter than the indifference curves, similar to the situation in Figure 2.1. When the budget line moves inwards in response to an increase of p_2, the demanded bundle follows the kink curve to the northwest, and good 2 has the Giffen property.

In the other case, $f'(\hat{x}_1 + c_2\hat{x}_2) < c_1/c_2$, we have $dx_2/dx_1 < -1/c_1$. Then the kink curve is steeper than the indifference curves, and a similar logic leads to the conclusion that good 1 is a Giffen good.

In general, the strictly increasing f can be constructed to have any slope between zero and infinity at $\hat{x}_1 + c_2\hat{x}_2$ when $\hat{x}$ is a kink point. At one extreme, $f'(\hat{x}_1 + c_2\hat{x}_2) = 0$, the kink curve has $dx_2/dx_1 = -1/c_1$, so that its tangent is the indifference curve for utility function u_1. At the other extreme, the kink curve's tangent is the u_2-indifference curves with slope

[6] Some indifference curves may have no kink. Thus, the indifference curve for $\bar{u} \in \mathbb{R}$ is the $\bar{u}$-indifference curve for u_1 if and only if $\bar{u} \leq f(\bar{u})$, and is the $\bar{u}$-indifference curve for u_2 if and only if $f(c_2\bar{u}/c_1) \leq \bar{u}$.

$-1/c_2$. All intermediate slopes, pointing outwards through the indifference curves at $\hat{x}$, can be attained through the construction of f. The following examples exploit various forms of f.

2.3. Example 2

Quite similar in spirit to Example 1, let $f(u) = Au + B$ where $B > 0$ and $0 < A < c_1/c_2$. The kink equation is $\hat{x}_1 + c_1\hat{x}_2 = A(\hat{x}_1 + c_2\hat{x}_2) + B$, or $\hat{x}_2 = [B - (1-A)\hat{x}_1] / [c_1 - Ac_2]$. Notice that the kink curve is steeper than the u_1-indifference curves, for the slopes satisfy $(1-A)/(c_1 - Ac_2) > 1/c_1$ by $c_2 > c_1$. When $c_1 < p_2/p_1 < c_2$ and $Bp_1/(1-A) < m < Bp_2/(c_1 - Ac_2)$, the consumer optimally chooses at a kink. In this region, the demand function is the unique solution to the kink equation and the budget constraint, and the local solution for good 1 is given by

$$x_1(p_1, p_2, m) = \frac{(c_1 - Ac_2)\, m - Bp_2}{(c_1 - Ac_2)\, p_1 - (1-A)\, p_2}.$$

We see that good 1 is a Giffen good in this region. This demand curve is quite flexible through the parameters, so it could prove useful in empirical investigations.

2.4. Example 3

Examples 1 and 2 are particularly simple since f is linear. On the other hand, it might be desirable to have $u_1(0) = u_2(0) = 0$. This property is satisfied in the next examples.

Suppose that $f(u) = Au^2$ with $A > 0$. One kink point sits at the diagonal bundle $\hat{x} = (t, t)$ where $(1 + c_1)\, t = A(1 + c_2)^2 t^2$, i.e. where $t = (1 + c_1) / \left[A(1 + c_2)^2\right]$. The slope of f at this bundle is $f'((1 + c_1) / [A(1 + c_2)]) = 2(1 + c_1)/(1 + c_2) > 1$ provided that $1 + 2c_1 > c_2$. Since $f'(\hat{x}_1 + c_2\hat{x}_2) > 1$, good 1 is locally a Giffen good when $c_1 < p_2/p_1 < c_2$ and m is close to $(p_1 + p_2)(1 + c_1) / \left[A(1 + c_2)^2\right]$.

Again, it would be straightforward, although tedious, to derive the Marshallian demand function for this example. In the most interesting region, the demand is the unique solution to the quadratic kink equation $\hat{x}_1 + c_1\hat{x}_2 = A(\hat{x}_1 + c_2\hat{x}_2)^2$ and the linear budget constraint $p_1x_1 + p_2x_2 = m$.

Swapping the roles of the two goods, one could consider $f(u) = A\sqrt{u}$. The analysis is very similar to the above, expect that good 2 would now be the Giffen good.

2.5. Example 4

For a more extreme version of example 3, consider $f(u) = AB^3 + A(u - B)^3$ where $A, B > 0$. It is easy to verify that f is strictly increasing with $f(0) = 0$ and $f'(B) = 0$.

Suppose that $c_1/c_2 < AB^2 < 1$. Then the bundle

$$\hat{x} = \frac{B}{c_2 - c_1}\left(c_2 AB^2 - c_1, 1 - AB^2\right)$$

is a kink point where $f'(\hat{x}_1 + c_2\hat{x}_2) = f'(B) = 0$. Thus, the kink curve has slope $dx_2/dx_1 = -1/c_1$, so that its tangent is the indifference curve for u_1. Good 1 is a Giffen good near this situation.

At the opposite extreme, $f(u) = AB^{1/3} + A(u - B)^{1/3}$ where $A, B > 0$. Again, f has the desired properties, but is now infinitely steep at B. Under the suitable parameter restriction, the kink curve can attain slope $dx_2/dx_1 = -1/c_2$, and good 2 is a Giffen good.

2.6. Example 5

A choice of f that slopes variably up and down at kink points can provide local Giffen properties at many places. At some places good 1 may be the Giffen good, while at other places good 2 can play out the Giffen role.

A particular instance is $f(u) = A(u + \sin(u))$. This strictly increasing function has $f(u) = Au$ at all $n\pi$ where $n \in \mathbb{N}$. The function attains its smallest slope of 0 whenever u is $\pi + n2\pi$ for some $n \in \mathbb{N}$, while its greatest slope of $2A$ is attained at all $n2\pi$, $n \in \mathbb{N}$. Suppose that $c_1/c_2 < A < 1 < 2A$. For any $n \in \mathbb{N}$, there is an indifference curve kink at the bundle

$$\hat{x}^n = \frac{n\pi}{c_2 - c_1}\left(c_2 A - c_1, 1 - A\right).$$

For any $n \in \mathbb{N}$, we obtain $f'(\hat{x}_1^n + c_2\hat{x}_2^n) = f'(n\pi)$. Thus, for any even n we have $f'(n\pi) = 2A > 1$, so that good 2 locally has the Giffen property when $c_1 < p_2/p_1 < c_2$ and m is close to $p_1\hat{x}_1^n + p_2\hat{x}_2^n$. For any odd n, we instead have $f'(n\pi) = 0 < c_1/c_2$, so that good 1 is the local Giffen good.

2.7. Cobb-Douglas Example

Examples 1 through 5 inherited two inessential properties from the preferences for perfect substitutes. First, preferences are not strictly convex since the indifference curves have linear segments. Second, when the price ratio is extreme, the optimal consumption is on the boundary of the consumption set. These two properties are avoided in the following example.

With consumption set $\mathbb{R}_+^2$, consider the utility function $u(x) = \min\{u_1(x), u_2(x)\}$ where $u_1(x_1, x_2) = x_1^c x_2^{1-c}$ and $u_2(x_1, x_2) = f\left(x_1^{1-c} x_2^c\right)$. Here $1/2 < c < 1$ is a parameter, and f is an arbitrary strictly increasing function. So, u_1 and u_2 are now familiar representations of Cobb-Douglas preferences (see e.g. Varian (1999), Chapter 4).

At bundle (x_1, x_2), the slope of the indifference curve for u_1 is $-(cx_2)/((1-c)x_1)$, while the slope for the indifference curve for u_2 is $-((1-c)x_2)/(cx_1)$. Since $c/(1-c) > (1-c)/c$ we infer that the u_1 indifference curves are everywhere steeper. Thus, the indifference curves for u are quite similar to those of Example 1. The curves have a kink at any $x \in \mathbb{R}^2_{++}$ solving $u_1(x) = u_2(x)$, and an indifference curve follows that of u_1 to the northwest of the kink, and that of u_2 to the southeast.

Suppose that $f(1) = 1$. When the consumer has income $m = 2$ and faces the given price vector $p = (1, 1)$, the optimal choice is at the kink point $\hat{x} = (1, 1)$. Judicious choices of f again permit us to construct kink curves which are locally between two extremes, one with slope $-c/(1-c)$ following the u_1 indifference curve through $(1, 1)$, the other with slope $-(1-c)/c$ following the u_2 indifference curve through $(1, 1)$. With the negative slope of the kink curve, we can obtain the Giffen property, as before.

2.8. CES Approximation

The examples given so far crucially exploit a kink in the indifference curve, that arises from a non-differentiability of the utility function.[7] One way to escape this, is through approximation with kink-free functions preserving the Giffen property, following the procedure of Moffatt (2002, see his Figure 2). A more direct avenue[8] is to employ the fact that the function $\min\{u_1, u_2\}$ is approximated by the constant elasticity of substitution function $(u_1^\rho + u_2^\rho)^{1/\rho}$ as $\rho \to -\infty$. Consider the specifications of u_1 and u_2 from Examples 1, 2 or 3, in the latter case focusing on $f(u) = A\sqrt{u}$. Notice that $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ are now both concave functions of (x_1, x_2) and that $(u_1^\rho(x_1, x_2) + u_2^\rho(x_1, x_2))^{1/\rho}$ is therefore concave in (x_1, x_2). This utility function therefore represents preferences with all the standard properties, continuity, monotonicity and convexity. Suppose that parameters are such that the kink curve has a negative slope near the kink bundle $\hat{x}$ of our original example. One of the goods then has the Giffen property $dx_\ell/dp_\ell > 0$ in the old example. When $-\rho$ is sufficiently large, all indifference curves near $\hat{x}$ are sufficiently close to those of $\min\{u_1(x_1, x_2), u_2(x_1, x_2)\}$, that the slope of the demand curve is again positive in the CES example.

[7] The literature seems not to consider differentiability a salient property of utility functions.

[8] When employed to our earlier examples, this method may not always secure that the utility function u is quasi-concave.

3. More than Two Goods

3.1. Example 6

With consumption set $\mathbb{R}^L_+$, let $u(x) = \min\{c_1 \cdot x, \ldots, c_k \cdot x, f(c_{k+1} \cdot x), \ldots, f(c_L \cdot x)\}$. Here $1 \leq k < L$, the vectors $c_1, \ldots, c_L \in \mathbb{R}^L_+$ are defined by $c_j = (1, \ldots, 1, 2, 1, \ldots, 1)$ with the 2 in the j'th coordinate, and $f(u) = 1 + (u-1)^3$.

The diagonal bundle $\hat{x} = (1/(L+1), \ldots, 1/(L+1))$ solves $1 = c_1 \cdot x = \cdots = c_k \cdot x = f(c_{k+1} \cdot x) = \cdots = f(c_L \cdot x)$ and therefore sits at a kink (isolated corner) of the indifference surface. It is clearly the demanded bundle when $p = (1, \ldots, 1)$ and $m = L/(L+1)$. Take any $j \leq k$, and suppose that we marginally increase the price p_j to $1 + dp_j > 1$. Similar to the effect in Example 2, the demanded bundle will move inwards along the kink curve. By the symmetry of the kink equations, it should be clear that $dx_1 = \cdots = dx_k$ and $dx_{k+1} = \cdots = dx_L$. Since f moves very slowly away from 1, the demand change dx must solve $c_1 \cdot dx = \cdots = c_k \cdot dx = 0$, and since we move inwards along the kink curve, $c_{k+1} \cdot dx = \cdots = c_L \cdot dx < 0$. The first set of equations give $(k+1)\, dx_1 + (L-k)\, dx_L = 0$. The inequalities give $k dx_1 + (L-k+1)\, dx_L < 0$, and so we can infer that $dx_1 > 0$ and $dx_L < 0$. Since $dx_j = dx_1 > 0$, we have the Giffen property for good j, that $dx_j/dp_j > 0$.

We conclude that every good $1, \ldots, k$ is a Giffen good in this situation. It is of course impossible to have all L goods be Giffen goods if Walras' Law is satisfied, for Giffen implies inferiority, and not all L goods can be inferior.

3.2. Theory

A consumer has consumption set $\mathbb{R}^L_+$ and utility function $u(x) = \min\{u_1(x), \ldots, u_K(x)\}$, where each u_k is a function from $\mathbb{R}^L_+$ to $\mathbb{R}$. Consider the following list of properties of utility functions (see also Section 3.B of Mas-Colell et al.). Property (vi) is also known as convexity of the underlying preference relation.

(i) Continuity: u is a continuous function.

(ii) Monotonicity: if $y \gg x$, then $u(y) > u(x)$.

(iii) Strong Monotonicity: if $y \geq x$ and $y \neq x$ then $u(y) > u(x)$.

(iv) Weak Monotonicity: if $y \geq x$ then $u(y) \geq u(x)$.

(v) Concavity: u is a concave function.

(vi) Quasi-Concavity: for any $\bar{u} \in \mathbb{R}$, the upper contour set $\left\{x \in \mathbb{R}^L_+ : u(x) \geq \bar{u}\right\}$ is convex.

(vii) Strict Quasi-Concavity: for any $\bar{u} \in \mathbb{R}$, any $x, y \in \mathbb{R}_+^L$, and any $\alpha \in (0,1)$, if $u(x) \geq \bar{u}$ and $u(y) \geq \bar{u}$, then $u(\alpha x + (1-\alpha) y) \geq \bar{u}$.

(viii) Homogeneity of Degree 1: u maps $\mathbb{R}_+^L$ into $\mathbb{R}_+$, and $u(\alpha x) = \alpha u(x)$ for all $\alpha \in \mathbb{R}_+$.

It is easy to verify that each of those 8 properties, one by one, is inherited by u from $u_1, \ldots, u_K$:

Proposition 1 *Let $j \in \{i, \ldots, viii\}$. Suppose that for every $k = 1, \ldots, K$, u_k satisfies property (j). Then u satisfies (j).*

Proof: (i) min is a continuous function, and continuity is preserved by the composition of functions. (ii) and (iii) Suppose y and x are as assumed in the property. By definition of u there exists some k, k' (possibly $k = k'$) such that $u(y) = u_k(y)$ and $u(x) = u_{k'}(x) \leq u_k(x)$. Given that u_k satisfies the property, then $u(x) \leq u_k(x) < u_k(y) = u(y)$. (iv) Similar to (ii) and (iii), except that the assumption is $u_k(x) \leq u_k(y)$, which suffices for the conclusion. (v) Let $x, y \in \mathbb{R}_+^L$ and $\alpha \in [0,1]$. We verify Jensen's inequality: $u(\alpha x + (1-\alpha) y) = \min\{u_1(\alpha x + (1-\alpha) y), \ldots, u_K(\alpha x + (1-\alpha) y)\} \geq \min\{\alpha u_1(x) + (1-\alpha) u_1(y), \ldots, \alpha u_K(x) + (1-\alpha) u_K(y)\} \geq \alpha u(x) + (1-\alpha) u(y)$ where the first inequality uses the concavity of the u_k with min being increasing in its arguments, and the second uses concavity of min. (vi) Suppose that $x \in \mathbb{R}_+^L$ and $\bar{u} \in \mathbb{R}$. From the definition of u, $u(x) \geq \bar{u}$ if and only if for every k, $u_k(x) \geq \bar{u}$. Thus the upper contour set for u is the intersection of the K sets for $u_1, \ldots, u_K$. Since the intersection of convex sets is convex, (vi) follows. (vii) Suppose that $\bar{u}, x, y, \alpha$ are given as stated, and that $u(x), u(y) \geq \bar{u}$. By the definition of u, there exists some k such that $u(\alpha x + (1-\alpha) y) = u_k(\alpha x + (1-\alpha) y)$. As noticed in the proof of (vi), we must have $u_k(x), u_k(y) \geq \bar{u}$. When u_k satisfies (vii), it follows that $u(\alpha x + (1-\alpha) y) = u_k(\alpha x + (1-\alpha) y) > \bar{u}$ as desired. (viii) We obtain $u(\alpha x) = \min\{u_1(\alpha x), \ldots, u_K(\alpha x)\} = \min\{\alpha u_1(x), \ldots, \alpha u_K(x)\} = \alpha \min\{u_1(x), \ldots, u_K(x)\} = \alpha u(x)$. □

If the consumption set allows good 1 to vary throughout all of $\mathbb{R}$, a straightforward exercise proves that the Proposition also applies to this property of quasi-linearity with respect to good 1: $u(x + \alpha e_1) = u(x) + \alpha$ for all $\alpha \in \mathbb{R}$, where $e_1 = (1, 0, \ldots, 0)$.

One key property of preferences, playing a central role in the welfare theorems, is local non-satiation: for any $x \in \mathbb{R}_+^L$ and any $\varepsilon > 0$ there exists some $y \in \mathbb{R}_+^L$ with $u(y) > u(x)$ and $||y - x|| < \varepsilon$. This property is not inherited. Consider the following example: $u_1(x_1, x_2) = x_1 - x_2$ and $u_2(x_1, x_2) = x_2 - x_1$. It is simple to see that these two utility functions satisfy local non-satiation. Yet $u(x) = \min\{u_1(x), u_2(x)\}$ is always non-positive, and achieves its satiation utility level 0 on the diagonal where $x_1 = x_2$. Since there is no $y \in \mathbb{R}_+^2$ with $u(y) > 0 = u(1,1)$, u fails local non-satiation. Monotonicity is stronger than local non-satiation, so if every $u_1, \ldots, u_K$ satisfies monotonicity, the Proposition shows that it is inherited by u, and then it is ensured that u satisfies local non-satiation.

The Proposition can be generalized to handle the minimum of an infinite family of utility functions. Suppose thus that $U(x) = \min_{k \in K} u(x,k)$ where K is a compact set and u is continuous. These assumptions guarantee that the minimum is achieved, but also imply that U is continuous by the Theorem of the Maximum. It is a straightforward exercise to see that the proof of the Proposition can be modified to show that the other 7 properties (ii),...,(viii) of $u(\cdot, k)$ are again inherited by U.

4. References

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