Simple waves and a characteristic decomposition of the two dimensional compressible Euler equations

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Abstract

We present a characteristic decomposition of the potential flow equation in the self-similar plane. The decomposition allows for a proof that any wave adjacent to a constant state is a simple wave for the adiabatic Euler system. This result is a generalization of the well-known result on 2-d steady potential flow and a recent similar result on the pressure gradient system

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1 Introduction

The one-dimensional wave equation

$$u_{tt} - c^2 u_{xx} = 0 \tag{1}$$

with constant speed c has an interesting decomposition

$$(\partial_t + c\partial_x)(\partial_t - c\partial_x)u = 0, \tag{2}$$

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or

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0 \tag{3}$$

known from elementary text books. One can rewrite them as

$$\partial_+\partial_- u = 0, \qquad \text{or } \partial_-\partial_+ u = 0, \tag{4}$$

where $\partial_\pm = \partial_t \pm c\partial_x$. Sometimes, the same fact is written in Riemann invariants

$$\partial_t R + c\partial_x R = 0, \quad \partial_t S - c\partial_x S = 0 \tag{5}$$

for the Riemann invariants

$$R := \partial_t - c\partial_x u, \quad S := \partial_t + c\partial_x u. \tag{6}$$

For a pair of system of hyperbolic conservation laws

$$\begin{bmatrix} u \\ v \end{bmatrix}_t + \begin{bmatrix} f(u,v) \\ g(u,v) \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{7}$$

it is known that a pair of Riemann invariants exist so that the system can be rewritten as

$$\begin{cases} \partial_t R + \lambda_1(u,v)\partial_x R = 0, \\ \partial_t S + \lambda_2(u,v)\partial_x S = 0, \end{cases} \tag{8}$$

where (R, S) are the Riemann invariants and the λ's are the two eigenvalues of the system.

These decompositions and Riemann invariants are useful in the construction of solutions, for example, the construction of D'Alembert formula, and proof of development of singularities ([4]). An example of the system is the system of isentropic irrotational steady two-dimensional Euler equations for compressible ideal gases

$$\begin{cases} (c^2 - u^2)u_x - uv(u_y + v_x) + (c^2 - v^2)v_y &= 0, \\ u_y - v_x &= 0 \end{cases} \tag{9}$$

supplemented by Bernoulli's law

$$\frac{c^2}{\gamma - 1} + \frac{u^2 + v^2}{2} = k^2 \tag{10}$$

where $\gamma > 1$ is the gas constant while $k > 0$ is an integration constant. This system has two unknowns (u, v), and by the existence of Riemann invariants, any solution adjacent to a constant state is a simple wave. A simple wave means a solution (u, v) that depends on a single parameter rather than the pair parameters (x, y). Since there is the lack of the explicit expressions, the concept of Riemann invariants plays a limited role in a much broader sense, e.g., to treat the full Euler equations.

In recent years, the pressure gradient system

$$\begin{cases} u_t + p_x &= 0 \\ v_t + p_y &= 0 \\ E_t + (up)_x + (vp)_y &= 0, \end{cases} \tag{11}$$

where $E = p + (u^2 + v^2)/2$, has been known to have "simple waves" adjacent to a constant state (u, v, p) in the self-similar variable plane $(\xi, \eta) = (x/t, y/t)$. This system has three equations and no Riemann invariants have been found. But the equation for the unknown variable p in the (ξ, η) plane

$$(p - \xi^2)p_{\xi\xi} - 2\xi\eta p_{\xi\eta} + (p - \eta^2)p_{\eta\eta} + \frac{(\xi p_\xi + \eta p_\eta)^2}{p} - 2(\xi p_\xi + \eta p_\eta) = 0 \tag{12}$$

allows for a decomposition

$$\partial_+\partial_- p = m_+\partial_- p, \quad m_+ := \frac{r^4\lambda_+}{2p^2}p_r, \tag{13}$$

where (r, θ) denotes the polar coordinates of the (ξ, η) plane and

$$\partial_+ = \partial_\theta + \lambda_+^{-1}\partial_r; \partial_- = \partial_\theta + \lambda_-^{-1}\partial_r; \lambda_\pm = \pm\sqrt{\frac{p}{r^2(r^2 - p)}}. \tag{14}$$

For convenience of verification we state that the p equation in polar coordinates takes the form

$$(p - r^2)p_{rr} + \frac{p}{r^2}p_{\theta\theta} + \frac{p}{r}p_r + \frac{1}{p}(rp_r)^2 - 2rp_r = 0. \tag{15}$$

The characteristics are defined by

$$\frac{d\theta}{dr} = \lambda_\pm. \tag{16}$$

In addition, we know that

$$\partial_\pm\lambda_\pm = n_\pm\partial_\pm p \tag{17}$$

for some nice factors $n_\pm$. These facts allow for expressions

$$\partial_\mp(\partial_\pm\lambda_\pm) = (\partial_\pm\lambda_\pm)f_\pm \tag{18}$$

for some nice factors $f_\pm$. This decomposition leads directly to the fact that

Proposition 1. *A state adjacent to a constant state for the pressure gradient system must be a simple wave in which p is constant along characteristics of a plus (or minus) family.*

These lead to the desire to consider the pseudo-steady isentropic irrotational Euler system which has three equations with source terms,

$$\begin{cases} (\rho U)_\xi + (\rho V)_\eta = -2\rho, \\ (\rho U)_\xi + (\rho V)_\eta = -3\rho U, \\ (\rho U)_\xi + (\rho V)_\eta = -3\rho V, \end{cases} \tag{19}$$

where $(\xi, \eta) = (x/t, y/t)$, and $(U, V) = (u - \xi, v - \eta)$ is the pseudo-velocity. It turns out that we are unable to find explicit forms of the Riemann invariants, but decompositions similar to $\partial_+\partial_-\lambda_- = m\partial_-\lambda_-$ hold for some m, presented in Section 4.

We use the characteristic decomposition of Section 4 to establish in Section 5 that adjacent to a constant state a wave must be a simple wave for the pseudo-steady irrotational isentropic Euler system. A simple wave for this case is such that one family of wave characteristics are straight lines and the physical quantities velocity, speed of sound, pressure, and density are constant along the wave characteristics. Further, using the fact that entropy and vorticity are constant along the pseudo-flow characteristics (the pseudo-flow lines), our irrotational result extends to the adiabatic full Euler system, see Section 5.

2 Two-by-two system

Consider a 2×2 hyperbolic system in the Riemann invariants

$$\begin{cases} \partial_t R + \lambda_1 \partial_x R = 0, \\ \partial_t S + \lambda_2 \partial_x S = 0. \end{cases} \tag{20}$$

So we find that

$$\partial_2\lambda_2 := (\partial_t + \lambda_2\partial_x)\lambda_2 = \lambda_{2,R}\partial_2 R, \tag{21}$$

where $\lambda_{2,R} := \partial_R\lambda_2$. We go on to find

$$\partial_1\partial_2 R = \frac{\partial_1\lambda_2 - \partial_2\lambda_1}{\lambda_2 - \lambda_1}\partial_2 R, \tag{22}$$

and so

$$\partial_1\left(\frac{1}{\lambda_{2,R}}\partial_2\lambda_2\right) = \frac{\partial_1\lambda_2 - \partial_2\lambda_1}{\lambda_2 - \lambda_1}\frac{\partial_2\lambda_2}{\lambda_{2,R}} \tag{23}$$

which implies

$$\partial_1\partial_2\lambda_2 = \left(\frac{\partial_1\lambda_2 - \partial_2\lambda_1}{\lambda_2 - \lambda_1} + \frac{\partial_1\lambda_{2,R}}{\lambda_{2,R}}\right)\partial_2\lambda_2. \tag{24}$$

The elegant form is undermined by the dependence on the Riemann invariant R via the term $\lambda_{2,R}$. It is not useful when the explicit form of the Riemann invariants are not known. But we think it is worth mentioning. For example, (24) can be used directly to show that all characteristics in a wave adjacent to a constant state are straight and thus such a wave is a simple wave.

3 Steady Euler system

Let us build explicitly the characteristic decomposition for the steady Euler system for isentropic irrotational flow (9)(10) in the absence of the explicit form of the Riemann invariants. The same technique can be extended to the case of pseudo-steady case in Section 4. We write the system in the form

$$\begin{bmatrix} u \\ v \end{bmatrix}_x + \begin{bmatrix} \frac{-2uv}{c^2-u^2} & \frac{c^2-v^2}{c^2-u^2} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_y = 0. \tag{25}$$

The matrix has eigenvalues

$$\lambda_\pm = \frac{uv \pm \sqrt{c^2(u^2+v^2-c^2)}}{u^2-c^2} \quad (= \frac{dy}{dx}) \tag{26}$$

which are solutions to the characteristic equation

$$\lambda^2 + \frac{2uv}{c^2-u^2}\lambda + \frac{c^2-v^2}{c^2-u^2} = 0. \tag{27}$$

We have the left eigenvectors

$$\ell_\pm = [1, \lambda_\mp], \tag{28}$$

where we have used the relation

$$\lambda_\pm \lambda_\mp = \frac{c^2-v^2}{c^2-u^2}. \tag{29}$$

The characteristic form of the system is therefore

$$\ell_\pm \begin{bmatrix} u \\ v \end{bmatrix}_x + \lambda_\pm \ell_\pm \begin{bmatrix} u \\ v \end{bmatrix}_y = 0, \tag{30}$$

which is equivalent to

$$\partial_\pm u + \lambda_\mp \partial_\pm v = 0. \tag{31}$$

We then have

$$\partial_- \lambda_- = \partial_x \lambda_- + \lambda_- \partial_y \lambda_- = \partial_u \lambda_- \partial_- u + \partial_v \lambda_- \partial_- v = \left(\partial_u \lambda_- - \partial_v \lambda_- / \lambda_+\right) \partial_- u. \tag{32}$$

We shall ignore the similar calculation for $\partial_+\lambda_+$ for simplicity of notation.

Now that the term $\partial_-\lambda_-$ differs from $\partial_- u$ by a lower-order factor, we shall focus our attention on $\partial_- u$. First we see that we can derive a second-order equation for u, i.e.,

$$u_{yy} - \left(\frac{2uv}{c^2 - v^2}\, u_y - \frac{c^2 - u^2}{c^2 - v^2}\, u_x \right)_x = 0, \tag{33}$$

or equivalently

$$u_{xx} - \frac{2uv}{c^2 - u^2} u_{xy} + \frac{c^2 - v^2}{c^2 - u^2} u_{yy} = \frac{c^2 - v^2}{c^2 - u^2} \left(\left(\frac{2uv}{c^2 - v^2} \right)_x u_y - \left(\frac{c^2 - u^2}{c^2 - v^2} \right)_x u_x \right). \tag{34}$$

We now compute the ordered derivative $\partial_+\partial_- u$ to find

$$\partial_+\partial_- u = u_{xx} + (\lambda_+ + \lambda_-) u_{xy} + \lambda_+\lambda_- u_{yy} + \partial_+\lambda_-\, u_y. \tag{35}$$

We find that

$$\partial_+\lambda_- = \partial_+ u (\partial_u \lambda_- - \partial_v \lambda_- / \lambda_-). \tag{36}$$

Thus we obtain

$$\begin{aligned} \partial_+\partial_- u \quad = \quad & \tfrac{c^2 - v^2}{c^2 - u^2} \left(\left(\tfrac{2uv}{c^2 - v^2} \right)_x u_y - \left(\tfrac{c^2 - u^2}{c^2 - v^2} \right)_x u_x \right) \\ & + (u_x + \lambda_+ u_y) u_y (\partial_u \lambda_- - \partial_v \lambda_- / \lambda_-). \end{aligned} \tag{37}$$

We notice that the above right-hand side is a quadratic form in (u_x, u_y), once we substitute v_x by u_y. So we compute further. We use the Bernoulli's law to find

$$(c^2)_x = -(\gamma - 1)(u u_x + v u_y). \tag{38}$$

So we find

$$\begin{aligned} \left(\tfrac{2uv}{c^2 - v^2} \right)_x \quad &= \quad \tfrac{2}{(c^2 - v^2)^2} [v u_x (c^2 - u^2 - v^2 + \gamma u^2) + u u_y (c^2 + \gamma v^2)] \\ \left(\tfrac{c^2 - u^2}{c^2 - v^2} \right)_x \quad &= \quad \tfrac{-1}{(c^2 - v^2)^2} [u u_x (2c^2 - v^2 - u^2 + \gamma u^2 - \gamma v^2) \\ & \qquad - v u_y (2c^2 - v^2 + \gamma v^2 - u^2 - \gamma u^2)]. \end{aligned} \tag{39}$$

We now compute the factor $\partial_u \lambda_- - \lambda_-^{-1} \partial_v \lambda_-$. We use Bernoulli's law to find

$$(c^2)_u = -(\gamma - 1) u, \quad (c^2)_v = -(\gamma - 1) v. \tag{40}$$

We use the characteristic equation

$$(c^2 - u^2)\lambda^2 + 2uv\lambda + c^2 - v^2 = 0 \tag{41}$$

to obtain

$$\begin{aligned}\partial_u\lambda_- &= \tfrac{\lambda_-^2(\gamma+1)u-2v\lambda_-+(\gamma-1)u}{2\lambda_-(c^2-u^2)+2uv},\\ \partial_v\lambda_- &= \tfrac{\lambda_-^2(\gamma-1)v-2u\lambda_-+(\gamma+1)v}{2\lambda_-(c^2-u^2)+2uv}.\end{aligned} \tag{42}$$

We then simply compute to find

$$\partial_u\lambda_- - \lambda_-^{-1}\partial_v\lambda_- = \frac{\gamma+1}{2\lambda_-(c^2-u^2)+2uv}\,\frac{(u\lambda_- - v)^3}{c^2\lambda_-}. \tag{43}$$

Coming back to our equation for $\partial_+\partial_- u$, we have

$$\begin{aligned}&(c^2-u^2)(c^2-v^2)\partial_+\partial_- u\\ = \;& u_x^2 u(2c^2-u^2-v^2+\gamma u^2-\gamma v^2)\\ &+u_x u_y(-vu^2-v^3+3\gamma vu^2-\gamma v^3+Q)\\ &+u_y^2[2u(c^2+\gamma v^2)+\lambda_+ Q]\end{aligned} \tag{44}$$

where we have introduced the notation

$$Q := \frac{(c^2-u^2)(c^2-v^2)}{2\lambda_-(c^2-u^2)+2uv}\,\frac{\gamma+1}{c^2\lambda_-}\,(u\lambda_- - v)^3 = \frac{(\gamma+1)(c^2-v^2)(uH-vc)^3}{2(c^2-u^2)H(cH-uv)}, \tag{45}$$

where

$$H := \sqrt{u^2+v^2-c^2}. \tag{46}$$

We then factorize the quadratic form to find finally

$$\partial_+\partial_- u = \frac{u(2c^2-u^2-v^2+\gamma u^2-\gamma v^2)}{(c^2-u^2)(c^2-v^2)}(\partial_x u + \alpha\partial_y u)\partial_- u, \tag{47}$$

where

$$\alpha = \frac{2u(c^2+\gamma v^2)+\lambda_+ Q}{\lambda_- u(2c^2-u^2-v^2+\gamma u^2-\gamma v^2)}. \tag{48}$$

Proposition 2 *There holds the identity*

$$\partial_+\partial_- u = m(\partial_x u + \alpha\partial_y u)\partial_- u, \tag{49}$$

for some functions $\alpha(u,v)$ *given in* (48) *and some* m *given by*

$$m = \frac{u(2c^2-u^2-v^2+\gamma u^2-\gamma v^2)}{(c^2-u^2)(c^2-v^2)}. \tag{50}$$

We use the relation

$$\partial_- u = \partial_- \lambda_- / (\partial_u \lambda_- - \partial_v \lambda_- \lambda_+^{-1}) \tag{51}$$

to go back to $\partial_+ \partial_- \lambda_-$. We find

$$\partial_u \lambda_- - \partial_v \lambda_- \lambda_+^{-1} = -\frac{[4c^2 + (\gamma - 3)(u^2 + v^2)][v\lambda_-(c^2 + u^2) + (c^2 - v^2)u]}{(c^2 - v^2)(c^2 - u^2)[2\lambda_-(c^2 - u^2) + 2uv]} =: G. \tag{52}$$

So we have

$$\partial_+ (\frac{1}{G} \partial_- \lambda_-) = m \, \partial_\alpha u \, \frac{\partial_- \lambda_-}{G}, \tag{53}$$

or

$$\partial_+ \partial_- \lambda_- = (m \partial_\alpha u + \partial_+ (\ln |G|)) \partial_- \lambda_-. \tag{54}$$

Proposition 3 *There holds the identity*

$$\partial_+ \partial_- \lambda_- = m \partial_- \lambda_- \tag{55}$$

for some $m = m(u, v)(\partial_x u + \beta(u, v) \partial_y u)$. A similar identity holds for $\partial_- \partial_+ \lambda_+$.

We remark that in the application on simple waves, the equation for u is sufficient and the equations for $\lambda_\pm$ are not needed.

4 Pseudo-steady Euler

We consider the two-dimensional isentropic irrotational ideal flow in the self-similar plane $(\xi, \eta) = (x/t, y/t)$. There holds the Bernoulli's law

$$\frac{c^2}{\gamma - 1} + \frac{U^2 + V^2}{2} = -\varphi \tag{56}$$

where c is the speed of sound, $(U, V) = (u - \xi, v - \eta)$ are the pseudo-velocity, while (u, v) is the physical velocity, and φ is the pseudo-potential such that

$$\varphi_\xi = U, \qquad \varphi_\eta = V. \tag{57}$$

The equations of motion can be written as

$$\begin{cases} (c^2 - U^2) U_\xi - UV(U_\eta + V_\xi) + (c^2 - V^2) V_\eta = -2c^2 + U^2 + V^2 \\ V_\xi - U_\eta = 0. \end{cases} \tag{58}$$

We can rewrite the equations of motion in a new form

$$\begin{bmatrix} u \\ v \end{bmatrix}_\xi + \begin{bmatrix} \frac{-2UV}{c^2-U^2} & \frac{c^2-V^2}{c^2-U^2} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_\eta = 0 \tag{59}$$

to draw as much parallelism to the steady case as possible. We emphasize the mixed use of the variables (U, V) and (u, v), i.e., (U, V) is used in the coefficients while (u, v) is used in differentiation. This way we obtain zero on the right-hand side for the system.

The eigenvalues are similar as before:

$$\frac{d\eta}{d\xi} = \Lambda_\pm = \frac{UV \pm \sqrt{c^2(U^2 + V^2 - c^2)}}{U^2 - c^2}. \tag{60}$$

The left eigenvectors are

$$\ell_\pm = [1, \Lambda_\mp]. \tag{61}$$

And we have similarly

$$\partial_\pm u + \Lambda_\mp \partial_\pm v = 0. \tag{62}$$

Our $\Lambda_\pm$ now depend on more than (U, V). But, let us regard $\Lambda_\pm$ as a simple function of three variables $\Lambda_\pm = \Lambda_\pm(U, V, c^2)$ as given in (60). Thus we need to build differentiation laws for c^2. We can directly obtain

$$\left(\frac{c^2}{\gamma - 1}\right)_\xi + Uu_\xi + Vv_\xi = 0, \quad \left(\frac{c^2}{\gamma - 1}\right)_\eta + Uu_\eta + Vv_\eta = 0. \tag{63}$$

We have

$$\partial_\pm c^2 = -(\gamma - 1)(U\partial_\pm u + V\partial_\pm v). \tag{64}$$

So we move on to compute

$$\begin{aligned} \partial_\pm \Lambda_\pm &= \partial_U \Lambda_\pm \partial_\pm U + \partial_V \Lambda_\pm \partial_\pm V + \partial_{c^2} \Lambda_\pm \partial_\pm c^2 \\ &= \partial_U \Lambda_\pm (\partial_\pm u - 1) + \partial_V \Lambda_\pm (\partial_\pm v - \Lambda_\pm) + \partial_{c^2} \Lambda_\pm \partial_\pm c^2 \\ &= \partial_U \Lambda_\pm \partial_\pm u + \partial_V \Lambda_\pm \partial_\pm v + \partial_{c^2} \Lambda_\pm \partial_\pm c^2 - \partial_U \Lambda_\pm - \partial_V \Lambda_\pm \Lambda_\pm. \end{aligned} \tag{65}$$

We need to handle the term $\partial_U \Lambda_\pm + \partial_V \Lambda_\pm \Lambda_\pm$. We show it is zero. Recalling that

$$(c^2 - U^2)\Lambda^2 + 2UV\Lambda + c^2 - V^2 = 0, \tag{66}$$

and regarding that Λ depends on the three quantities (U, V, c^2) independently, we can easily find

$$\Lambda_U = \frac{\Lambda(U\Lambda - V)}{\Lambda(c^2 - U^2) + UV}, \quad \Lambda_V = -\frac{U\Lambda - V}{\Lambda(c^2 - U^2) + UV}. \tag{67}$$

Thus

$$\Lambda_U + \Lambda\Lambda_V = 0. \tag{68}$$

Therefore we end up with

$$\partial_\pm \Lambda_\pm = \left[\partial_U \Lambda_\pm - \Lambda_\mp^{-1} \partial_V \Lambda_\pm - (\gamma-1)\partial_{c^2}\Lambda_\pm (U - \Lambda_\mp^{-1} V)\right] \partial_\pm u. \tag{69}$$

Thus, if one of the quantities (u, v, c^2, Λ_-) is a constant along Λ_-, so is all the rest. So far the properties are very similar to the steady case.

We derive an equation for $\partial_- u$. We have a similar second-order equation for u

$$u_{\eta\eta} = \left(\frac{2UV}{c^2 - V^2}\, u_\eta - \frac{c^2 - U^2}{c^2 - V^2}\, u_\xi\right)_\xi. \tag{70}$$

We have similarly

$$\begin{aligned} \partial_+\partial_- u &= u_{\xi\xi} + (\Lambda_+ + \Lambda_-) u_{\xi\eta} + \Lambda_-\Lambda_+ u_{\eta\eta} + \partial_+\Lambda_- u_\eta \\ &= \tfrac{c^2-V^2}{c^2-U^2}\left[\left(\tfrac{2UV}{c^2-V^2}\right)_\xi u_\eta - \left(\tfrac{c^2-U^2}{c^2-V^2}\right)_\xi u_\xi\right] + \partial_+\Lambda_- u_\eta. \end{aligned} \tag{71}$$

We compute

$$\begin{aligned} \partial_+\Lambda_- &= \partial_U\Lambda_-\partial_+U + \partial_V\Lambda_-\partial_+V + \partial_{c^2}\Lambda_-\partial_+c^2 \\ &= [\partial_U\Lambda_- - \tfrac{1}{\Lambda_-}\partial_V\Lambda_- - (\gamma-1)\partial_{c^2}\Lambda_-(U - \tfrac{1}{\Lambda_-}V)]\partial_+u \\ &\quad -(\partial_U\Lambda_- + \Lambda_+\partial_V\Lambda_-). \end{aligned} \tag{72}$$

We continue to find

$$\begin{aligned} \partial_+\partial_- u = {} & \tfrac{c^2-V^2}{c^2-U^2}\Big\{\tfrac{u_\eta}{(c^2-V^2)^2}\left[2V_\xi U(c^2-V^2) + 2VU_\xi(c^2-V^2) - 2VU((c^2)_\xi - 2VV_\xi)\right] \\ & -\tfrac{u_\xi}{(c^2-V^2)^2}\left[((c^2)_\xi - 2UU_\xi)(c^2-V^2) - (c^2-U^2)((c^2)_\xi - 2VV_\xi)\right]\Big\} \\ & +\partial_+u\, u_\eta\left[\partial_U\Lambda_- - \tfrac{1}{\Lambda_-}\partial_V\Lambda_- - (\gamma-1)\partial_{c^2}\Lambda_-(U - \tfrac{1}{\Lambda_-}V)\right] \\ & -u_\eta(\partial_U\Lambda_- + \Lambda_+\partial_V\Lambda_-). \end{aligned} \tag{73}$$

We apply the rule $U_\xi = u_\xi - 1, V_\xi = v_\xi = u_\eta$ to find

$$\begin{aligned} \partial_+\partial_- u = {} & \tfrac{c^2-V^2}{c^2-U^2}\Big\{\tfrac{u_\eta}{(c^2-V^2)^2}\left[2u_\eta U(c^2-V^2) + 2Vu_\xi(c^2-V^2)\right. \\ & \left. +2VU((\gamma-1)Uu_\xi + (\gamma+1)Vu_\eta)\right] \\ & -\tfrac{u_\xi}{(c^2-V^2)^2}\left[-((\gamma+1)Uu_\xi + (\gamma-1)Vu_\eta)(c^2-V^2)\right. \\ & \left. +(c^2-U^2)((\gamma-1)Uu_\xi + (\gamma+1)Vu_\eta)\right]\Big\} \\ & +\partial_+u\, u_\eta\left[\partial_U\Lambda_- - \tfrac{1}{\Lambda_-}\partial_V\Lambda_- - (\gamma-1)\partial_{c^2}\Lambda_-(U - \tfrac{1}{\Lambda_-}V)\right] \\ & \tfrac{-2Vv_\xi - 2Uu_\xi}{c^2-U^2} - u_\eta(\partial_U\Lambda_- + \Lambda_+\partial_V\Lambda_-). \end{aligned} \tag{74}$$

We note that there appear terms which are linear in the derivatives of (u, v) in addition to the pure quadratic form as in the steady case. The pure quadratic form is identical to the steady case, so we do not need to handle it further. The linear form can be handled as follows. First we use the derivatives (Λ_U, Λ_V) to compute

$$\partial_U \Lambda_- + \Lambda_+ \partial_V \Lambda_- = \partial_U \Lambda_- + \frac{1}{\Lambda_-} \frac{c^2 - V^2}{c^2 - U^2} \partial_V \Lambda_- = \frac{2(U\Lambda_- - V)}{c^2 - U^2}. \tag{75}$$

Then we have

$$\frac{-2Vv_\xi - 2Uu_\xi}{c^2 - U^2} - u_\eta(\partial_U \Lambda_- + \Lambda_+ \partial_V \Lambda_-) = -\frac{2U}{c^2 - U^2} \partial_- u. \tag{76}$$

Thus the linear form is also in the direction of Λ_-. Combining the steps we end up with

Theorem 4. *There holds*

$$\partial_+ \partial_- u = \frac{U(2c^2 - U^2 - V^2 + \gamma U^2 - \gamma V^2)}{(c^2 - U^2)(c^2 - V^2)} (\partial_\xi u + A \partial_\eta u) \partial_- u - \frac{2U}{c^2 - U^2} \partial_- u, \tag{77}$$

where

$$A := \frac{2U(c^2 + \gamma V^2) + \Lambda_+ \tilde{Q}}{\Lambda_- U(2c^2 - U^2 - V^2 + \gamma U^2 - \gamma V^2)}, \tag{78}$$

and

$$\tilde{Q} := \frac{(c^2 - U^2)(c^2 - V^2)}{2\Lambda_-(c^2 - U^2) + 2UV} \frac{\gamma + 1}{c^2 \Lambda_-} (U\Lambda_- - V)^3. \tag{79}$$

We then have

Theorem 5. *There holds*

$$\partial_+(\partial_- \Lambda_-) = m \, \partial_- \Lambda_-. \tag{80}$$

Similarly there holds $\partial_-(\partial_+ \Lambda_+) = n \, \partial_+ \Lambda_+$.

5 Application: Simple waves

For a system of hyperbolic conservation laws in one-space dimension, a centered rarefaction wave is a simple wave, in which one family of characteristics are straight lines and the dependent variables are constant along a characteristic. See any text book on systems of conservation laws, e. g., Courant and Friedrichs [1]and others' [8][2].

Simple waves for the two-dimensional steady Euler system are similar, i.e., one family of characteristics are straight and the velocity are constant along the characteristics.

For the the two-dimensional self-similar pressure gradient system, see [3], simple waves can be defined similarly, i.e., one nonlinear family of characteristics are straight and the pressure is constant along them. We note that we do not require the velocity to be constant. This way, by the characteristic decomposition, we find that *a wave adjacent to a constant state is a simple wave.*

In the construction of solutions to the two-dimensional Riemann problem for the Euler system, see any of the sources [6][8][7], it is important to know how to construct solutions adjacent to a constant state in addition to the constructions of the interaction of rarefaction waves ([5]), subsonic solutions, and transonic shock waves. The characteristic decomposition $\partial_1\partial_2\lambda_2 = m\partial_2\lambda_2$ allows us to conclude that

Theorem 6. *Adjacent to a constant state in the self-similar plane of the potential flow system is a simple wave in which the physical variables* (u, v, c) *are constant along a family of characteristics which are straight lines.*

5.1 Simple waves for full Euler

Consider the adiabatic Euler system for an ideal fluid

$$\begin{cases} \rho_t + \nabla\cdot(\rho u) = 0, \\ (\rho u)_t + \nabla\cdot(\rho u \otimes u + pI) = 0, \\ (\rho E)_t + \nabla\cdot(\rho E u + pu) = 0, \end{cases} \tag{81}$$

where

$$E := \frac{1}{2}|u|^2 + e,$$

where e is the internal energy. For a polytropic gas, there holds

$$e = \frac{1}{\gamma-1}\frac{p}{\rho},$$

where $\gamma > 1$. In the self-similar plane and for smooth solutions, the system takes the form:

$$\begin{cases} \frac{1}{\rho}\partial_s\rho + u_\xi + v_\eta = 0, \\ \partial_s u + \frac{1}{\rho}p_\xi = 0, \\ \partial_s v + \frac{1}{\rho}p_\eta = 0, \\ \frac{1}{\gamma p}\partial_s p + u_\xi + v_\eta = 0 \end{cases} \tag{82}$$

where

$$\partial_s := (u-\xi)\partial_\xi + (v-\eta)\partial_\eta$$

which we call pseudo-flow directions, as opposed to the other two characteristic directions, called (pseudo-)wave characteristics. We easily derive

$$\partial_s(p\rho^{-\gamma}) = 0, \tag{83}$$

and

$$\omega_t + (u\omega)_x + (v\omega)_y + (\frac{p_y}{\rho})_x - (\frac{p_x}{\rho})_y = 0 \tag{84}$$

for the vorticity $\omega := v_x - u_y$. So entropy $p\rho^{-\gamma}$ is constant along the pseudo-flow lines. For a region Ω whose pseudo-flow lines come from a constant state, we see that the entropy is constant in the region. For isentropic region, vorticity has zero source of production since $(\frac{p_y}{\rho})_x - (\frac{p_x}{\rho})_y = 0$. Thus vorticity $t\omega = v_\xi - u_\eta$ (setting $t = 1$ then) satisfies

$$\partial_s(\omega/\rho) = 0. \tag{85}$$

Hence, for a region whose pseudo-flow lines come from a constant state, the vorticity must be zero everywhere. So the region is irrotational and isentropic. Thus our formulas for the potential flow apply. We have

Theorem 7. *Adjacent to a constant state in the self-similar plane of the adiabatic Euler system is a simple wave in which the physical variables (u, v, c, p, ρ) are constant along a family of wave characteristics which are straight lines, provided that the region is such that its pseudo-flow characteristics extend into the state of constant.*

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