SIMPLEST EXP-FUNCTION METHOD FOR EXACT SOLUTIONS OF MIKHAUILOV-NOVIKOV-WANG EQUATIONS

by

Sheng ZHANG[a,c*], Caihong YOU[a], and Bo XU[b,c]

[a] School of Mathematics and Physics, Bohai University, Jinzhou, China
[b] School of Education and Sports, Bohai University, Jinzhou, China
[c] Department of Mathematics, Hohhot Minzu College, Hohhot, China

Original scientific paper
https://doi.org/10.2298/TSCI1904381Z

In this paper, the simplest exp-function method which combines the exp-function method with a direct algorithm is used to exactly solve the Mikhauilov-Novikov-Wang equations. As a result, two explicit and exact solutions are obtained. It is shown that the simplest exp-function method provides a simpler but more effective mathematical tool for constructing exact solutions of non-linear evolution equations in fluids.

Key words: *Mikhauilov-Novikov-Wang equations, simplest exp-function method, exact solution*

Introduction

Non-linear phenomena of the real world involved in many fields like physics, biology, economics, chemistry, mechanics, fluid dynamics, and engineering are often described by non-linear evolution equations (NLEE). To gain more insight into these physical phenomena, researchers usually resort to solutions of such NLEE. Since the exp-function method was proposed by He and Wu [1], more and more exact solutions of NLEE have been obtained, such as those in [2-8]. In 2016, Zhang *et al.* [9] present a simple and effective algorithm of the exp-function method for dealing with the so-called middle expression expansion problem (MEEP) appeared in the process of computation. Comparatively speaking, the MEEP of the exp-function method results from the rational ansätz [9]:

$$u=\frac{\sum_{n=-f}^{g}a_n\exp(n\xi)}{\sum_{m=-p}^{q}b_m\exp(m\xi)},\quad \xi=\sum_{i=1}^{s}k_ix_i+wt \tag{1}$$

supposed for a given NLEE with independent variables t, x_1, x_2, …, x_s, and dependent variable u:

$$F(u,u_t,u_{x_1},u_{x_2},\cdots,u_{x_m},u_{x_1t},u_{x_2t}\cdots,u_{x_st},u_{tt},u_{x_1x_1},u_{x_2x_2},\cdots,u_{x_sx_s},\cdots)=0 \tag{2}$$

In eq. (1), a_n and b_m are undetermined constants, f, p, g, and q are determined by proceeding the homogeneous balance between the highest order linear term and the highest order non-linear term in eq. (2). As pointed out in [1, 2, 9] that the final solution obtained by the exp-function method does not strongly depend on the choices of values of f, p, g, and q and $f = p = g = q = 1$ is the simplest choice. In the direct algorithm [9] of the exp-function method, Zhang *et al.* [9] introduced a general eq.:

* Corresponding author, e-mail: szhangchina@126.com

$$u=\sum_{j=0}^{n}u_j(t,x_1,x_2,\cdots,x_s)\varphi^{j-n},\quad \varphi=1+\mathrm{e}^{\xi} \tag{3}$$

where the non-negative integer n is determined by balancing the highest order linear term with the highest order non-linear term in eq. (2), $u_j=u_j(t,x_1,x_2,\cdots,x_s)$, $j=0,1,2,\cdots,n$, are coefficient functions to be determined by the system of PDE resulted from substituting eq. (3) into eq. (2). It is shown [9] that the eq. (3) has an advantage over the one in eq. (1) to deal with the so-called the MEEP when using the exp-function method to transform a given NLEE into the overdetermined system of algebraic or differential equations. Besides, if ξ is a non-linear function of the indicated independent variables then the eq. (3) can be used to construct non-traveling wave solutions of given NLEE like the Jimbo-Miwa [9].

In this paper, we would like to extend the approach [9] (the simplest exp-function method for short) to the new Mikhauilov-Novikov-Wang equations [10-12]:

$$u_t=18v_{xxx}-36(vu)_x-u_{7x}+49u_xu_{xxxx}+14uu_{5x}+84u_{xx}u_{xxx}-70u_x^3-\\ -252uu_xu_{xx}-56u^2u_{xxx}+\frac{224}{3}u^3u_x \tag{4}$$

$$v_t=-36vv_x+v_xv_{xxxx}+3vu_{5x}-12v_xuu_{xx}-72vu_xu_{xx}-36vuu_{xxx}-\\ -6v_xu_x^2+\frac{32}{3}u^3v_x+96vu^2u_x \tag{5}$$

As far as we know, eqs. (4) and (5) have not be solved exactly. In fact, it is difficult to solve eqs. (4) and (5) by the exiting analytical methods. In the next sections, we shall consider two special cases of eqs. (4) and (5).

First special case

In order to solve eqs. (4) and (5) conveniently, we first consider case when $v = 0$ [13]:

$$u_t=-u_{7x}+49u_xu_{xxxx}+14uu_{5x}+84u_{xx}u_{xxx}-70u_x^3-252uu_xu_{xx}-56u^2u_{xxx}+\frac{224}{3}u^3u_x \tag{6}$$

Balancing the highest order linear term u_{7x} with the highest order non-linear term u^3u_x, we have $n+7=4n+1$, *i. e.*, $n=2$, and then suppose that:

$$u=u_0(x,t)\varphi^{-2}+u_1(x,t)\varphi^{-1}+u_2(x,t),\quad \varphi=1+\mathrm{e}^{\xi},\quad \xi=kx+ct \tag{7}$$

Substituting eq. (7) into eq. (6) and then equating each coefficient of the same order power of $\varphi^i\,(i=0,-1,-2,\cdots,-9)$ to zero yields a set of PDE:

$$\varphi^{-9}:\quad -40320k^7\mathrm{e}^{7\xi}u_0+33936k^5\mathrm{e}^{5\xi}u_0^2-4928k^3\mathrm{e}^{3\xi}u_0^3+\frac{448}{3}k\mathrm{e}^{\xi}u_0^4=0 \tag{8}$$

$$\varphi^{-8}:\quad 105840k^7\mathrm{e}^{6\xi}u_0-44016k^5\mathrm{e}^{4\xi}u_0^2+2016k^3\mathrm{e}^{2\xi}u_0^3-5040k^7\mathrm{e}^{7\xi}u_1+27048k^5\mathrm{e}^{5\xi}u_0u_1-\\ -9408k^3\mathrm{e}^{3\xi}u_0^2u_1+\frac{1568}{3}k\mathrm{e}^{\xi}u_0^3u_1+35280k^6\mathrm{e}^{6\xi}u_{0x}-40824k^4\mathrm{e}^{4\xi}u_0u_{0x}+\\ +5376k^2\mathrm{e}^{2\xi}u_0^2u_{0x}-\frac{224}{3}u_0^3u_{0x}=0 \tag{9}$$

$$\varphi^{-7}: \quad -100800k^7\mathrm{e}^{5\xi}u_0 + 16548k^5\mathrm{e}^{3\xi}u_0^2 - 112k^3\mathrm{e}^{\xi}u_0^3 + 15120k^7\mathrm{e}^{6\xi}u_1 -$$
$$-39816k^5\mathrm{e}^{4\xi}u_0u_1 + 4368k^3\mathrm{e}^{2\xi}u_0^2u_1 + 3864k^5\mathrm{e}^{5\xi}u_1^2 - 5460k^3\mathrm{e}^{3\xi}u_0u_1^2 + 672k\mathrm{e}^{\xi}u_0^2u_1^2 +$$
$$+10080k^5\mathrm{e}^{5\xi}u_0u_2 - 5712k^3\mathrm{e}^{3\xi}u_0^2u_2 + 448k\mathrm{e}^{\xi}u_0^3u_2 - 75600k^6\mathrm{e}^{5\xi}u_{0x} + 36288k^4\mathrm{e}^{3\xi}u_0u_{0x} -$$
$$-840k^2\mathrm{e}^{\xi}u_0^2u_{0x} - 19320k^4\mathrm{e}^{4\xi}u_1u_{0x} + 7896k^2\mathrm{e}^{2\xi}u_0u_1u_{0x} - 224u_0^2u_1u_{0x} + 10752k^3\mathrm{e}^{3\xi}u_{0x}^2 -$$
$$-1428k\mathrm{e}^{\xi}u_0u_{0x}^2 + 5040k^6\mathrm{e}^{6\xi}u_{1x} - 16968k^4\mathrm{e}^{4\xi}u_0u_{1x} + 3696k^2\mathrm{e}^{2\xi}u_0^2u_{1x} - \frac{224}{3}u_0^3u_{1x} -$$
$$15120k^5\mathrm{e}^{5\xi}u_{0xx} + 11928k^3\mathrm{e}^{3\xi}u_0u_{0xx} - 840k\mathrm{e}^{\xi}u_0^2u_{0xx} = 0 \tag{10}$$

$$\varphi^{-6}: \quad 42000k^7\mathrm{e}^{4\xi}u_0 - 1792k^5\mathrm{e}^{2\xi}u_0^2 - 16800k^7\mathrm{e}^{5\xi}u_1 + 17290k^5\mathrm{e}^{3\xi}u_0u_1 -$$
$$-280k^3\mathrm{e}^{\xi}u_0^2u_1 - 6636k^5\mathrm{e}^{4\xi}u_1^2 + 2940k^3\mathrm{e}^{2\xi}u_0u_1^2 - 910k^3\mathrm{e}^{3\xi}u_1^3 + \frac{1120}{3}k\mathrm{e}^{\xi}u_0u_1^3 -$$
$$-16800k^5\mathrm{e}^{4\xi}u_0u_2 + 3024k^3\mathrm{e}^{2\xi}u_0^2u_2 + 1680k^5\mathrm{e}^{5\xi}u_1u_2 - 5880k^3\mathrm{e}^{3\xi}u_0u_1u_2 +$$
$$+1120k\mathrm{e}^{\xi}u_0^2u_1u_2 + 54600k^6\mathrm{e}^{4\xi}u_{0x} - 7462k^4\mathrm{e}^{2\xi}u_0u_{0x} + 19908k^4\mathrm{e}^{3\xi}u_1u_{0x} -$$
$$-1428k^2\mathrm{e}^{\xi}u_0u_1u_{0x} + 2730k^2\mathrm{e}^{2\xi}u_1^2u_{0x} - 224u_0u_1^2u_{0x} - 8400k^4\mathrm{e}^{4\xi}u_2u_{0x} + 5544k^2\mathrm{e}^{2\xi}u_0u_2u_{0x} -$$
$$-224u_0^2u_2u_{0x} - 5544k^3\mathrm{e}^{2\xi}u_{0x}^2 - 1218k\mathrm{e}^{\xi}u_1u_{0x}^2 + 70u_{0x}^3 - 12600k^6\mathrm{e}^{5\xi}u_{1x} + 17472k^4\mathrm{e}^{3\xi}u_0u_{1x} -$$
$$-672k^2\mathrm{e}^{\xi}u_0^2u_{1x} - 6048k^4\mathrm{e}^{4\xi}u_1u_{1x} + 5040k^2\mathrm{e}^{2\xi}u_0u_1u_{1x} - 224u_0^2u_1u_{1x} + 10920k^3\mathrm{e}^{3\xi}u_{0x}u_{1x} -$$
$$-2352k\mathrm{e}^{\xi}u_0u_{0x}u_{1x} - 5880k^4\mathrm{e}^{4\xi}u_0u_{2x} + 2352k^2\mathrm{e}^{2\xi}u_0^2u_{2x} - \frac{224}{3}u_0^3u_{2x} + 25200k^5\mathrm{e}^{4\xi}u_{0xx} -$$
$$-6216k^3\mathrm{e}^{2\xi}u_0u_{0xx} + 6636k^3\mathrm{e}^{3\xi}u_1u_{0xx} - 1428k\mathrm{e}^{\xi}u_0u_1u_{0xx} - 5292k^2\mathrm{e}^{2\xi}u_{0x}u_{0xx} + 252u_0u_{0x}u_{0xx} -$$
$$-2520k^5\mathrm{e}^{5\xi}u_{1xx} + 5544k^3\mathrm{e}^{3\xi}u_0u_{1xx} - 672k\mathrm{e}^{\xi}u_0^2u_{1xx} + 4200k^4\mathrm{e}^{4\xi}u_{0xxx} -$$
$$-2128k^2\mathrm{e}^{2\xi}u_0u_{0xxx} + 56u_0^2u_{0xxx} = 0 \tag{11}$$

$$\varphi^{-5}: \quad -7224k^7\mathrm{e}^{3\xi}u_0 + 28k^5\mathrm{e}^{\xi}u_0^2 + 8400k^7\mathrm{e}^{4\xi}u_1 - 2212k^5\mathrm{e}^{2\xi}u_0u_1 + 3458k^5\mathrm{e}^{3\xi}u_1^2 -$$
$$-224k^3\mathrm{e}^{\xi}u_0u_1^2 + 588k^3\mathrm{e}^{2\xi}u_1^3 + \frac{224}{3}k\mathrm{e}^{\xi}u_1^4 + 8400k^5\mathrm{e}^{3\xi}u_0u_2 - 224k^3\mathrm{e}^{\xi}u_0^2u_2 -$$
$$-3360k^5\mathrm{e}^{4\xi}u_1u_2 + 3696k^3\mathrm{e}^{2\xi}u_0u_1u_2 - 1176k^3\mathrm{e}^{3\xi}u_1^2u_2 + 896k\mathrm{e}^{\xi}u_0u_1^2u_2 - 1344k^3\mathrm{e}^{3\xi}u_0u_2^2 +$$
$$+448k\mathrm{e}^{\xi}u_0^2u_2^2 - 15120k^6\mathrm{e}^{3\xi}u_{0x} + 238k^4\mathrm{e}^{\xi}u_0u_{0x} - 4858k^4\mathrm{e}^{2\xi}u_1u_{0x} -$$
$$-588k^2\mathrm{e}^{\xi}u_1^2u_{0x} - \frac{224}{3}u_1^3u_{0x} + 10080k^4\mathrm{e}^{3\xi}u_2u_{0x} - 1176k^2\mathrm{e}^{\xi}u_0u_2u_{0x} +$$
$$+3528k^2\mathrm{e}^{2\xi}u_1u_2u_{0x} - 448u_0u_1u_2u_{0x} + 392k^3\mathrm{e}^{\xi}u_{0x}^2 - 1008k\mathrm{e}^{\xi}u_2u_{0x}^2 + 10920k^6\mathrm{e}^{4\xi}u_{1x} -$$
$$-4270k^4\mathrm{e}^{2\xi}u_0u_{1x} + 7476k^4\mathrm{e}^{3\xi}u_1u_{1x} - 1092k^2\mathrm{e}^{\xi}u_0u_1u_{1x} + 1554k^2\mathrm{e}^{2\xi}u_1^2u_{1x} - 224u_0u_1^2u_{1x} -$$

$$-1680k^4e^{4\xi}u_2u_{1x}+3192k^2e^{2\xi}u_0u_2u_{1x}-224u_0^2u_2u_{1x}-6720k^3e^{2\xi}u_{0x}u_{1x}-1932ke^{\xi}u_1u_{0x}u_{1x}+$$
$$+210u_{0x}^2u_{1x}+2184k^3e^{3\xi}u_{1x}^2-924ke^{\xi}u_0u_{1x}^2+7056k^4e^{3\xi}u_0u_{2x}-504k^2e^{\xi}u_0^2u_{2x}-$$
$$-1176k^4e^{4\xi}u_1u_{2x}+2856k^2e^{2\xi}u_0u_1u_{2x}-224u_0^2u_1u_{2x}+4704k^3e^{3\xi}u_{0x}u_{2x}-$$
$$-1848ke^{\xi}u_0u_{0x}u_{2x}-12600k^5e^{3\xi}u_{0xx}+448k^3e^{\xi}u_0u_{0xx}-4116k^3e^{2\xi}u_1u_{0xx}-$$
$$-588ke^{\xi}u_1^2u_{0xx}+3360k^3e^{3\xi}u_2u_{0xx}-1176ke^{\xi}u_0u_2u_{0xx}+1092k^2e^{\xi}u_{0x}u_{0xx}+$$
$$+252u_1u_{0x}u_{0xx}-3276k^2e^{2\xi}u_{1x}u_{0xx}+252u_0u_{1x}u_{0xx}+504ke^{\xi}u_{0xx}^2+$$
$$+5040k^5e^{4\xi}u_{1xx}-3444k^3e^{2\xi}u_0u_{1xx}+2436k^3e^{3\xi}u_1u_{1xx}-1092ke^{\xi}u_0u_1u_{1xx}-3108k^2e^{2\xi}u_{0x}u_{1xx}+$$
$$+252u_0u_{0x}u_{1xx}+2016k^3e^{3\xi}u_0u_{2xx}-504ke^{\xi}u_0^2u_{2xx}-5040k^4e^{3\xi}u_{0xxx}+448k^2e^{\xi}u_0u_{0xxx}-$$
$$-1400k^2e^{2\xi}u_1u_{0xxx}+112u_0u_1u_{0xxx}+728ke^{\xi}u_{0x}u_{0xxx}+840k^4e^{4\xi}u_{1xxx}-1176k^2e^{2\xi}u_0u_{1xxx}+$$
$$+56u_0^2u_{1xxx}-840k^3e^{3\xi}u_{0xxxx}+238ke^{\xi}u_0u_{0xxxx}=0 \qquad (12)$$

$$\varphi^{-4}:\quad 378k^7e^{2\xi}u_0-1806k^7e^{3\xi}u_1+42k^5e^{\xi}u_0u_1-553k^5e^{2\xi}u_1^2-56k^3e^{\xi}u_1^3-$$
$$-1260k^5e^{2\xi}u_0u_2+2100k^5e^{3\xi}u_1u_2-336k^3e^{\xi}u_0u_1u_2+924k^3e^{2\xi}u_1^2u_2+224ke^{\xi}u_1^3u_2+$$
$$+1008k^3e^{2\xi}u_0u_2^2-336k^3e^{3\xi}u_1u_2^2+672ke^{\xi}u_0u_1u_2^2+1320k^6e^{2\xi}u_{0x}+189k^4e^{\xi}u_1u_{0x}-$$
$$-2940k^4e^{2\xi}u_2u_{0x}-924k^2e^{\xi}u_1u_2u_{0x}-224u_1^2u_2u_{0x}+1008k^2e^{2\xi}u_2^2u_{0x}-$$
$$-224u_0u_2^2u_{0x}-3780k^6e^{3\xi}u_{1x}+168k^4e^{\xi}u_0u_{1x}-2282k^4e^{2\xi}u_1u_{1x}-420k^2e^{\xi}u_1^2u_{1x}-\frac{224}{3}u_1^3u_{1x}+$$
$$+2520k^4e^{3\xi}u_2u_{1x}-840k^2e^{\xi}u_0u_2u_{1x}+1680k^2e^{2\xi}u_1u_2u_{1x}-448u_0u_1u_2u_{1x}+588k^3e^{\xi}u_{0x}u_{1x}-$$
$$-1512ke^{\xi}u_2u_{0x}u_{1x}-1680k^3e^{2\xi}u_{1x}^2-714ke^{\xi}u_1u_{1x}^2+210u_{0x}u_{1x}^2-2058k^4e^{2\xi}u_0u_{2x}+$$
$$+1764k^4e^{3\xi}u_1u_{2x}-756k^2e^{\xi}u_0u_1u_{2x}+714k^2e^{2\xi}u_1^2u_{2x}-224u_0u_1^2u_{2x}+1512k^2e^{2\xi}u_0u_2u_{2x}-$$
$$-224u_0^2u_2u_{2x}-3528k^3e^{2\xi}u_{0x}u_{2x}-1428ke^{\xi}u_1u_{0x}u_{2x}+210u_{0x}^2u_{2x}+1176k^3e^{3\xi}u_{1x}u_{2x}-$$
$$-1344ke^{\xi}u_0u_{1x}u_{2x}+1890k^5e^{2\xi}u_{0xx}+364k^3e^{\xi}u_1u_{0xx}-2520k^3e^{2\xi}u_2u_{0xx}-924ke^{\xi}u_1u_2u_{0xx}+$$
$$+252u_2u_{0x}u_{0xx}+840k^2e^{\xi}u_{1x}u_{0xx}+252u_1u_{1x}u_{0xx}-1764k^2e^{2\xi}u_{2x}u_{0xx}+$$
$$+252u_0u_{2x}u_{0xx}-3150k^5e^{3\xi}u_{1xx}+308k^3e^{\xi}u_0u_{1xx}-1890k^3e^{3\xi}u_1u_{1xx}-420ke^{\xi}u_1^2u_{1xx}+$$
$$+840k^3e^{3\xi}u_2u_{1xx}-840ke^{\xi}u_0u_2u_{1xx}+798k^2e^{\xi}u_{0x}u_{1xx}+252u_1u_{0x}u_{1xx}-1596k^2e^{2\xi}u_{1x}u_{1xx}+$$
$$+252u_0u_{1x}u_{1xx}+756ke^{\xi}u_{0xx}u_{1xx}-1512k^3e^{2\xi}u_0u_{2xx}+504k^3e^{3\xi}u_1u_{2xx}-756ke^{\xi}u_0u_1u_{2xx}-$$
$$-1512k^2e^{2\xi}u_{0x}u_{2xx}+252u_0u_{0x}u_{2xx}+1470k^4e^{2\xi}u_{0xxx}+364k^2e^{\xi}u_1u_{0xxx}+56u_1^2u_{0xxx}-$$
$$-840k^2e^{2\xi}u_2u_{0xxx}+112u_0u_2u_{0xxx}+560ke^{\xi}u_{1x}u_{0xxx}-84u_{0xx}u_{0xxx}-1260k^4e^{3\xi}u_{1xxx}+$$

$$+308k^2e^{\xi}u_0u_{1xxx}-644k^2e^{2\xi}u_1u_{1xxx}+112u_0u_1u_{1xxx}+532ke^{\xi}u_{0x}u_{1xxx}-504k^2e^{2\xi}u_0u_{2xxx}+$$
$$+56u_0^2u_{2xxx}+630k^3e^{2\xi}u_{0xxxx}+189ke^{\xi}u_1u_{0xxxx}-49u_{0x}u_{0xxxx}-210k^3e^{3\xi}u_{1xxxx}+168ke^{\xi}u_0u_{1xxxx}+$$
$$+126k^2e^{2\xi}u_{0,5x}-14u_0u_{0,5x}=0 \qquad (13)$$

$$\varphi^{-3}:\quad -2ce^{\xi}u_0-2k^7e^{\xi}u_0+126k^7e^{2\xi}u_1+14k^5e^{\xi}u_1^2+28k^5e^{\xi}u_0u_2-420k^5e^{2\xi}u_1u_2-$$
$$-112k^3e^{\xi}u_1^2u_2-112k^3e^{\xi}u_0u_2^2+336k^3e^{2\xi}u_1u_2^2+224ke^{\xi}u_1^2u_2^2+\frac{448}{3}ke^{\xi}u_0u_2^3-$$
$$-14k^6e^{\xi}u_{0x}+140k^4e^{\xi}u_2u_{0x}-336k^2e^{\xi}u_2^2u_{0x}-224u_1u_2^2u_{0x}+434k^6e^{2\xi}u_{1x}+$$
$$+119k^4e^{\xi}u_1u_{1x}-980k^4e^{2\xi}u_2u_{1x}-588k^2e^{\xi}u_2u_1u_{1x}-224u_1^2u_2u_{1x}+$$
$$+336k^2e^{2\xi}u_2^2u_{1x}-224u_0u_2^2u_{1x}+196k^3e^{\xi}u_{1x}^2-504ke^{\xi}u_2u_{1x}^2+70u_{1x}^3+98k^4e^{\xi}u_0u_{2x}-$$
$$-686k^4e^{2\xi}u_1u_{2x}-252k^2e^{\xi}u_1^2u_{2x}-\frac{224}{3}u_1^3u_{2x}-504k^2e^{\xi}u_0u_2u_{2x}+504k^2e^{2\xi}u_1u_2u_{2x}-$$
$$-448u_0u_1u_2u_{2x}+392k^3e^{\xi}u_{0x}u_{2x}-1008ke^{\xi}u_2u_{0x}u_{2x}-1176k^3e^{2\xi}u_{1x}u_{2x}-924ke^{\xi}u_1u_{1x}u_{2x}+$$
$$+420u_{0x}u_{1x}u_{2x}-420ke^{\xi}u_0u_{2x}^2-42k^5e^{\xi}u_{0xx}+280k^3e^{\xi}u_2u_{0xx}-336ke^{\xi}u_2^2u_{0xx}+252u_2u_{1x}u_{0xx}+$$
$$+588k^2e^{\xi}u_{2x}u_{0xx}+252u_1u_{2x}u_{0xx}+630k^5e^{2\xi}u_{1xx}+224k^3e^{\xi}u_1u_{1xx}-840k^3e^{2\xi}u_2u_{1xx}-$$
$$-588ke^{\xi}u_1u_2u_{1xx}+252u_2u_{0x}u_{1xx}+546k^2e^{\xi}u_{1x}u_{1xx}+252u_1u_{1x}u_{1xx}-588k^2e^{2\xi}u_{2x}u_{1xx}+$$
$$+252u_0u_{2x}u_{1xx}+252ke^{\xi}u_{1xx}^2+168k^3e^{\xi}u_0u_{2xx}-504k^3e^{2\xi}u_1u_{2xx}-252ke^{\xi}u_1^2u_{2xx}-$$
$$-504ke^{\xi}u_0u_2u_{2xx}+504k^2e^{\xi}u_{0x}u_{2xx}+252u_1u_{0x}u_{2xx}-504k^2e^{2\xi}u_{1x}u_{2xx}+252u_0u_{1x}u_{2xx}+$$
$$+504ke^{\xi}u_{0xx}u_{2xx}-70k^4e^{\xi}u_{0xxx}+280k^2e^{\xi}u_2u_{0xxx}+112u_1u_2u_{0xxx}+392ke^{\xi}u_{2x}u_{0xxx}-$$
$$-84u_{1xx}u_{0xxx}+490k^4e^{2\xi}u_{1xxx}+224k^2e^{\xi}u_1u_{1xxx}+56u_1^2u_{1xxx}-280k^2e^{2\xi}u_2u_{1xxx}+$$
$$+112u_0u_2u_{1xxx}+364ke^{\xi}u_{1x}u_{1xxx}-84u_{0xx}u_{1xxx}+168k^2e^{\xi}u_0u_{2xxx}-168k^2e^{2\xi}u_1u_{2xxx}+$$
$$+112u_0u_1u_{2xxx}+336ke^{\xi}u_{0x}u_{2xxx}-70k^3e^{\xi}u_{0xxxx}+140ke^{\xi}u_2u_{0xxxx}-49u_{1x}u_{0xxxx}+$$
$$+210k^3e^{2\xi}u_{1xxxx}+119ke^{\xi}u_1u_{1xxxx}-49u_{0x}u_{1xxxx}+98ke^{\xi}u_0u_{2xxxx}-42k^2e^{\xi}u_{0,5x}-$$
$$-14u_1u_{0,5x}+42k^2e^{2\xi}u_{1,5x}-14u_0u_{1,5x}-14ke^{\xi}u_{0,6x}=0 \qquad (14)$$

$$\varphi^{-2}:-ce^{\xi}u_1-k^7e^{\xi}u_1+14k^5e^{\xi}u_1u_2-56k^3e^{\xi}u_1u_2^2+\frac{224}{3}ke^{\xi}u_1u_2^3+u_{0t}-\frac{224}{3}u_2^3u_{0x}-7k^6e^{\xi}u_{1x}+$$

$$+70k^4e^{\xi}u_2u_{1x}-168k^2e^{\xi}u_2^2u_{1x}-224u_1u_2^2u_{1x}+49k^4e^{\xi}u_1u_{2x}-252k^2e^{\xi}u_1u_2u_{2x}-224u_1^2u_2u_{2x}-$$
$$-224u_0u_2^2u_{2x}+196k^3e^{\xi}u_{1x}u_{2x}-504ke^{\xi}u_2u_{1x}u_{2x}+210u_{1x}^2u_{2x}-210ke^{\xi}u_1u_{2x}^2+210u_{0x}u_{2x}^2+$$
$$+252u_2u_{2x}u_{0xx}-21k^5e^{\xi}u_{1xx}+140k^3e^{\xi}u_2u_{1xx}-168ke^{\xi}u_2^2u_{1xx}+252u_2u_{1x}u_{1xx}+294k^2e^{\xi}u_{2x}u_{1xx}+$$

$$+252u_1u_{2x}u_{1xx}+84k^3\mathrm{e}^{\xi}u_1u_{2xx}-252k\mathrm{e}^{\xi}u_1u_2u_{2xx}+252u_2u_{0x}u_{2xx}+252k^2\mathrm{e}^{\xi}u_{1x}u_{2xx}+$$
$$+252u_0u_{2x}u_{2xx}+252k\mathrm{e}^{\xi}u_{1xx}u_{2xx}+56u_2^2u_{0xxx}-252u_1u_{1x}u_{2xx}-84u_{2xx}u_{0xxx}-$$
$$-35k^4\mathrm{e}^{\xi}u_{1xxx}+140k^2\mathrm{e}^{\xi}u_2u_{1xxx}+112u_1u_2u_{1xxx}+196k\mathrm{e}^{\xi}u_{2x}u_{1xxx}-84u_{1xx}u_{1xxx}+84k^2\mathrm{e}^{\xi}u_1u_{2xxx}+$$
$$+56u_1^2u_{2xxx}+112u_0u_2u_{2xxx}+168k\mathrm{e}^{\xi}u_{1x}u_{2xxx}-84u_{0xx}u_{2xxx}-49u_{2x}u_{0xxxx}-35k^3\mathrm{e}^{\xi}u_{1xxxx}+$$
$$+70k\mathrm{e}^{\xi}u_2u_{1xxxx}-49u_{1x}u_{1xxxx}+49k\mathrm{e}^{\xi}u_1u_{2xxxx}-49u_{0x}u_{2xxxx}-14u_2u_{0,5x}-$$
$$-21k^2\mathrm{e}^{\xi}u_{1,5x}-14u_1u_{1,5x}-14u_0u_{2,5x}-7k\mathrm{e}^{\xi}u_{1,6x}+u_{0,7x}=0 \tag{15}$$

$$\varphi^{-1}:\quad u_{1t}-\frac{224}{3}u_2^3u_{1x}-224u_1u_2^2u_{2x}+210u_{1x}u_{2x}^2+252u_2u_{2x}u_{1xx}+252u_2u_{1x}u_{2xx}+$$
$$+252u_1u_{2x}u_{2xx}+56u_2^2u_{1xxx}-84u_{2xx}u_{1xxx}+112u_1u_2u_{2xxx}-84u_{1xx}u_{2xxx}-49u_{2x}u_{1xxxx}-$$
$$-49u_{1x}u_{2xxxx}-14u_2u_{1xxxxx}-14u_1u_{2xxxxx}+u_{1xxxxxxx}=0 \tag{16}$$

$$\varphi^{0}:\quad u_{2t}-\frac{224}{3}u_2^3u_{2x}+70u_{2x}^3+252u_2u_{2x}u_{2xx}+56u_2^2u_{2xxx}-84u_{2xx}u_{2xxx}-$$
$$-49u_{2x}u_{2xxxx}-14u_2u_{2xxxxx}+u_{2xxxxxxx}=0 \tag{17}$$

Solving eqs. (8)-(17), we have:

$$u_0=\frac{3k^2}{2}\mathrm{e}^{2\xi},\quad u_1=-\frac{3k^2}{2}\mathrm{e}^{\xi},\quad u_2=\frac{k^2}{8},\quad c=\frac{k^7}{48} \tag{18}$$

and hence obtain an exact solution of the MNW eqs. (4) and (5) when $v=0$:

$$u=\frac{3k^2\mathrm{e}^{2\xi}}{2(1+\mathrm{e}^{\xi})^2}-\frac{3k^2\mathrm{e}^{\xi}}{2(1+\mathrm{e}^{\xi})}+\frac{1}{8}k^2,\quad \xi=kx+\frac{k^7}{48}t \tag{19}$$

Second special case

In this section, we solve eqs. (4) and (5) in the case when $v=1$:

$$u_t=-36u_x-u_{7x}+49u_xu_{xxxx}+14uu_{5x}+84u_{xx}u_{xxx}-70u_x^3-$$
$$-252uu_xu_{xx}-56u^2u_{xxx}+\frac{224}{3}u^3u_x \tag{20}$$

$$0=3u_{5x}-72u_xu_{xx}-36uu_{xxx}+96u^2u_x \tag{21}$$

Proceeding the homogeneous balance between u_{7x} and u^3u_x, we easily obtain $n=2$. Then we suppose that:

$$u=u_0(x,t)\varphi^{-2}+u_1(x,t)\varphi^{-1}+u_2(x,t),\quad \varphi=1+\mathrm{e}^{\xi},\quad \xi=kx+ct \tag{22}$$

Similarly, we substitute eq. (22) into eqs. (20) and (21) and then equate each coefficient of the same order power of $\varphi^i\,(i=0,-1,-2,\cdots,-9)$ to zero, then a set of PDE are obtained. Solving the set of PDE, we have:

$$u_0=\frac{3k^2}{2}\mathrm{e}^{2\xi},\quad u_1=-\frac{3k^2}{2}\mathrm{e}^{\xi},\quad u_2=\frac{k^2}{8},\quad c=\frac{k(k^6-1728)}{48} \tag{23}$$

and hence obtain another exact solution of the MNW eqs. (4) and (5) when $v=1$:

$$u=\frac{3k^2\mathrm{e}^{2\xi}}{2(1+\mathrm{e}^{\xi})^2}-\frac{3k^2\mathrm{e}^{\xi}}{2(1+\mathrm{e}^{\xi})}+\frac{1}{8}k^2,\quad \xi=kx+\frac{k(k^6-1728)}{48}t \tag{24}$$

Conclusion

In summary, we have extended the simplest exp-function method to the Mikhauilov-Novikov-Wang eqs. (4) and (5) for $v=0$ and $v=1$, respectively. As a result, two explicit and exact solutions are obtained. It is shown that the general eq. (3) can easily deal with the so-called MEEP of the exp-function method to some extent. Therefore, we can conclude that the simplest exp-function method provides a simpler but more effective mathematical tool for constructing exact solutions of NLEE in fluids. Fractional-order differential calculus and fractal derivative models and their applications have attached much attention [13-25], extending the simplest exp-function method to fractal calculus and fractional calculus is worthy of study. Mikhauilov-Novikov-Wang equations with fractional derivatives or fractal derivatives have not been studied in any open literature.

Acknowledgment

This work was supported by the Natural Science Foundation of China (11547005), the Natural Science Foundation of Liaoning Province of China (20170540007), the Natural Science Foundation of Education Department of Liaoning Province of China (LZ2017002) and Innovative Talents Support Program in Colleges and Universities of Liaoning Province (LR2016021).

References

[1] He, J.-H., Wu, X. H., Exp-Function Method for Non-Linear Wave Equations, *Chaos, Solitons and Fractals*, *30* (2006), 3, pp. 700-708
[2] He, J.-H., Abdou, M. A., New Periodic Solutions for Non-Linear Evolution Equations Using Exp-Function Method, *Chaos, Solitons and Fractals*, *34* (2006), 5, pp. 1421-1429
[3] He, J.-H., Exp-Function Method for Fractional Differential Equations, *Journal of Non-Linear Sciences and Numerical Simulation*, *14* (2013), 6, pp. 363-366
[4] Ebaid, A., Exact Solitary Wave Solutions for Some Non-Linear Evolution Equations via Exp-Function Method, *Physics Letters A*, *365* (2007), 3, pp. 213-219
[5] Zhang, S., Application of Exp-Function Method to a KdV Equation with Variable Coefficients, *Physics Letters A*, *365* (2007), 5-6, 448-453
[6] Zhang S., Application of Exp-Function Method to High-Dimensional Non-Linear Evolution Equation, *Chaos, Solitons and Fractals*, *38* (2008), 1, pp. 270-276
[7] Zhang, S., Zhang, H. Q., An Exp-Function Method for New *N*-soliton Solutions with Arbitrary Functions of a (2+1)-Dimensional vcBK System, *Computers and Mathematics with Applications*, *61* (2011), 8, pp. 1923-1930
[8] Zhang, S., *et al.*, Multi-Wave Solutions for a Non-Isospectral KdV-Type Equation with Variable Coefficients, *Thermal Science*, *16* (2012), 5, pp. 1576-1579
[9] Zhang, S., *et al.*, A Direct Algorithm of Exp-Function Method for Non-Linear Evolution Equations in Fluids, *Thermal Science*, *20* (2016), 3, pp. 881-884

[10] Shan, X. Y., Zhu. J. Y., The Mikhauilov-Novikov-Wang Hierarchy and its Hamiltonian Structures, *Acta Physica Polonica B*, *43* (2012), 10, 1953-1963
[11] Zhang, S., Zhang, L. Y., New Periodic Wave Solutions for MNW Hierarchy with the Aid of Computerized Symbolic Computation, *Advances in Computer Science Research*, *71* (2017), 1, pp. 1233-1237
[12] Zhang, S., Tian, C., A New Mathematical Model and its Solvability Test on Computer, *Advances in Engineering Research*, *97* (2016), 1, pp. 375-379
[13] Zhang, S., *et al.*, Variable Separation for Time Fractional Advection-Dispersion Equation with Initial and Boundary Conditions, *Thermal Science*, *20* (2016), 3, pp. 789-792
[14] Wang, K. L., Liu, S. Y., He's Fractional Derivative for Non-Linear Fractional Heat Transfer Equation, *Thermal Science*, *20* (2016), 3, pp. 793-796
[15] Wang, J., Hu, Y., On Chain Rule in Fractional Calculus, *Thermal Science*, *20*, (2016), 3, pp. 803-806
[16] Yang, X. J., Fractional Derivatives of Constant and Variable Orders Applied to Anomalous Relaxation Models in Heat-Transfer Problems, *Thermal Science*, *21* (2017), 3, pp. 1161-1171
[17] Zhang, S., Hong, S. Y., Variable Separation Method for a Non-Linear Time Fractional Partial Differential Equation with Forcing Term, *Journal of Computational and Applied Mathematics*, *339* (2018), 1, pp. 297-305
[18] Zhang, S., *et al.*, Exact Solutions of Time Fractional Heat-Like and Wave-Like Equations with Variable Coefficients, *Thermal Science*, *20* (2016), Suppl. 3, pp. S689-S693
[19] He, J.-H. A Tutorial Review on Fractal Spacetime and Fractional Calculus, *International Journal of Theoretical Physics*, *53* (2014), 11, pp. 3698-3718
[20] He, J.-H. Fractal Calculus and its Geometrical Explanation, *Results in Physics*, *10* (2018), Sept., pp. 272-276
[21] Wang, Y., An, J. Y., Amplitude-Frequency Relationship to a Fractional Duffing Oscillator Arising in Microphysics and Tsunami Motion, *Journal of Low Frequency Noise, Vibration & Active Control,* Online first, https://doi.org/10.1177/1461348418795813
[22] Li, X. X., *et al.* A Fractal Modification of the Surface Coverage Model for an Electrochemical Arsenic Sensor, *Electrochimica Acta, 296* (2019), Feb., pp. 491-493
[23] Wang, Q. L., *et al.*, Fractal Calculus and its Application to Explanation of Biomechanism of Polar Bear Hairs, *Fractals, 26* (2018), 6, ID 1850086
[24] Wang, Y., Deng, Q. G., Fractal Derivative Model for Tsunami Travelling, *Fractals, 27* (2019), 2, ID 1950017
[25] Hu, Y., He, J.-H, On Fractal Space-Time and Fractional Calculus, *Thermal Science*, *20* (2016), 3, pp. 773-777

Paper submitted: April 27, 2018
Paper revised: November 23, 2018
Paper accepted: November 23, 2018