

SIMPLICES OF MAXIMAL VOLUME IN HYPERBOLIC n -SPACE

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1. Introduction

Consider hyperbolic n -space H^n represented as the Poincaré disk model

$$H^n \sim D^n = \{\mathbf{x} \in \mathbf{R}^n \mid \|\mathbf{x}\| < 1\}$$

with the Riemannian metric

$$ds^2 = \frac{4}{(1-r^2)^2} \sum_{i=1}^n (dx_i)^2 \quad \text{where} \quad r^2 = \sum_{i=1}^n x_i^2.$$

The geodesics in H^n are the circles orthogonal to the “sphere at infinity”

$$\partial H^n = \{\mathbf{x} \in \mathbf{R}^n \mid \|\mathbf{x}\| = 1\} = S^{n-1}.$$

An n -simplex in H^n with vertices $\mathbf{v}_0, \dots, \mathbf{v}_n \in H^n \cup \partial H^n$ is the closed subset of H^n bounded by the $n+1$ spheres which contain all the vertices except one and which are orthogonal to S^{n-1} . A simplex is called *ideal* if all the vertices are on the sphere at infinity. It is easy to see that the volume of a hyperbolic n -simplex is finite also if some of the vertices are on the sphere at infinity. A simplex is called *regular* if any permutation of its vertices can be induced by an isometry of H^n . This makes sense also for ideal simplices since any isometry of H^n can be extended continuously to $H^n \cup \partial H^n$. There is, up to isometry, only one ideal regular n -simplex in H^n .

The main result of the present paper is the following theorem which was conjectured by Thurston ([6], section 6.1).

THEOREM 1. *In hyperbolic n -space, for $n \geq 2$, a simplex is of maximal volume if and only if it is ideal and regular.*

Since any hyperbolic n -simplex is contained in an ideal one it suffices, when proving Theorem 1, to consider ideal simplices. We shall use the notation $\tau[n]$ for an arbitrary ideal n -simplex in H^n , while $\tau_0[n]$ always denotes a regular $\tau[n]$.

For $n=2$ any $\tau[2]$ is regular and has area equal to π , so in this case the theorem is trivially true.

For $n=3$ one has Lobatcheffsky's volume formula, [1]. For the form of it given below see e.g. Milnor [3]. In any $\tau[3]$ opposite dihedral angles are equal, and if α, β, γ are the three dihedral angles at one vertex then $\alpha + \beta + \gamma = \pi$, and the volume is given by

$$V(\tau[3]) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma)$$

where

$$\Lambda(\sigma) = - \int_0^\sigma \log(2 \sin u) du.$$

As shown in [3] this formula implies Theorem 1 for $n=3$.

The motivation for the present study is a very elegant proof, due to Gromov, of Mostow's rigidity theorem, [5], for oriented closed hyperbolic 3-manifolds. The theorem states that for $n \geq 3$ two oriented, closed, hyperbolic n -manifolds which are homotopy equivalent are automatically isometric. It is clear that Gromov's proof (as presented in Thurston's lecture notes [6], section 6.3) works also for $n > 3$ once one knows that ideal simplices of maximal volume in H^n are automatically regular.

For the convenience of the reader we give here a very brief outline of Gromov's argument.

Let $f: M \rightarrow N$ be a homotopy equivalence between closed, oriented hyperbolic n -manifolds with $n \geq 3$. To prove that M and N are isometric one notes that they are orbit spaces $\Gamma \backslash H^n$ and $\Theta \backslash H^n$, respectively, for discrete isometry groups Γ and Θ on hyperbolic n -space H^n . Also, f induces an isomorphism $f_*: \Gamma \rightarrow \Theta$ and it lifts to a map $\tilde{f}: H^n \rightarrow H^n$ which is equivariant with respect to $f_*: \Gamma \rightarrow \Theta$. The first step now consists in showing that \tilde{f} "induces" a continuous map $f^\infty: S^{n-1} \rightarrow S^{n-1}$ on the sphere at infinity; f^∞ is also equivariant with respect to f_* . In the second step one utilizes Gromov's norm to prove that f^∞ has the following property:

(1.1) Whenever $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n \in S^{n-1}$ span an ideal hyperbolic simplex of maximal volume then so do $f^\infty(\mathbf{v}_0), f^\infty(\mathbf{v}_1), \dots, f^\infty(\mathbf{v}_n)$.

At this point Theorem 1 enters. It is used simply to translate (1.1) into

(1.2) Whenever $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n \in S^{n-1}$ span a regular, ideal, hyperbolic simplex, then so do $f^\infty(\mathbf{v}_0), f^\infty(\mathbf{v}_1), \dots, f^\infty(\mathbf{v}_n)$.

The fourth step then consists in proving that any continuous map $f^\infty: S^{n-1} \rightarrow S^{n-1}$ satisfying (1.2) is the "restriction" to S^{n-1} of a unique isometry g of H^n (when $n \geq 3$). Since this g is still equivariant with respect to $f_*: \Gamma \rightarrow \Theta$ it induces the desired isometry $M \rightarrow N$.

The proof of Theorem 1 avoids explicit computation of the volumes $V(\tau_0[n])$. Nevertheless the methods involved can be used to give an asymptotic estimate of $V(\tau_0[n])$ for $n \rightarrow \infty$. We found that

$$\lim_{n \rightarrow \infty} \frac{V(\tau_0[n])}{V(\sigma_0[n])} = \sqrt{e}$$

where $\sigma_0[n]$ is a regular euclidean n -simplex with vertices on the unit sphere. This asymptotic formula has been known to Milnor for some time [4], but since his proof is less direct than ours, we find it worthwhile to present our proof here (cf. section 4).

2. Recollections about hyperbolic n -space

Besides the Poincaré disk model of H^n we shall use two other models, namely the projective model and the half space model. The *projective model* can be obtained from the Poincaré disk model by use of the map

$$p: \mathbf{x} \rightarrow \frac{2}{1 + \|\mathbf{x}\|^2} \mathbf{x}, \quad \|\mathbf{x}\| < 1.$$

Note that $p(H^n) = D^n$ and that p can be extended continuously to $H^n \cup \partial H^n$ by putting $p(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in S^{n-1}$. The induced metric on $p(H^n)$ is

$$ds^2 = (1 - r^2)^{-1} \sum_i (dx_i)^2 + (1 - r^2)^{-2} \sum_{i,j} x_i x_j dx_i dx_j,$$

and the associated volume form is

$$dV = (1 - r^2)^{-(n+1)/2} dx_1 \dots dx_n.$$

The advantage of the projective model is that geodesics become straight lines in the euclidean geometry on D^n . Hence, if $\tau[n]$ is an ideal hyperbolic n -simplex with vertices $\mathbf{v}_0, \dots, \mathbf{v}_n$ on S^{n-1} then $p(\tau[n])$ is simply the euclidean n -simplex with the same vertices. Therefore, the volume of $\tau[n]$ is given by the formula

$$V(\tau[n]) = \int_{p(\tau[n])} (1 - r^2)^{-(n+1)/2} d\mathbf{r}. \quad (2.1)$$

Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis in \mathbf{R}^n . The *half space model* of H^n can be obtained from the Poincaré disk model by use of the map

$$h: \mathbf{x} \mapsto \frac{1}{\|\mathbf{x} - \mathbf{e}_n\|^2} (2x_1, 2x_2, \dots, 2x_{n-1}, 1 - \|\mathbf{x}\|^2), \quad \|\mathbf{x}\| < 1.$$

Note that $h(H^n)$ is the half space $\{x \in \mathbf{R}^n \mid x_n > 0\}$. Moreover, h can be extended to the sphere at infinity by using the same formula except for $x = e_n$ where one puts $h(e_n) = \infty$. Then $h(\partial H^n) = \mathbf{R}^{n-1} \cup \{\infty\}$ with $\mathbf{R}^{n-1} = \{x \in \mathbf{R}^n \mid x_n = 0\}$. The induced metric on $h(H^n)$ is

$$ds^2 = x_n^{-2} \sum_i (dx_i)^2,$$

and the associated volume form is

$$dV = x_n^{-n} dx_1 \dots dx_n.$$

The geodesics in $h(H^n)$ are half circles and half lines orthogonal to \mathbf{R}^{n-1} .

Let $\tau[n]$ be an ideal simplex with vertices v_0, \dots, v_n . It is no loss of generality to assume that $v_0 = e_n$, and hence $h(v_0) = \infty$. The isometries of $h(H^n)$ fixing ∞ on the boundary form the group generated by (a) translations parallel to \mathbf{R}^{n-1} , (b) rotations leaving the x_n -axis pointwise fixed, and (c) multiplications by positive scalars. Hence, by replacing $\tau[n]$ by an isometric n -simplex one can achieve that $h(v_0) = \infty$ and $h(v_i) \in S^{n-2} \subseteq \mathbf{R}^{n-1}$ ($i = 1, 2, \dots, n$). Let $\varepsilon(\tau[n])$ be the euclidean $(n-1)$ -simplex in \mathbf{R}^{n-1} spanned by $h(v_1), \dots, h(v_n)$. Then $h(\tau[n]) - \{\infty\}$ consists of those points of $\varepsilon(\tau[n]) \times [0, \infty[$ which are outside the unit disk in \mathbf{R}^n . Thus, putting $\varrho = (x_1^2 + \dots + x_{n-1}^2)^{1/2}$ and $d\mathbf{p} = dx_1 \dots dx_{n-1}$, one gets

$$\begin{aligned} V(\tau[n]) &= \int_{\varepsilon(\tau[n])} \left(\int_{(1-\varrho^2)^{1/2}}^{\infty} x^{-n} dx \right) d\mathbf{p} \\ &= \frac{1}{n-1} \int_{\varepsilon(\tau[n])} (1-\varrho^2)^{-(n-1)/2} d\mathbf{p}. \end{aligned} \tag{2.2}$$

Let us finally note, that $\tau[n]$ is regular if and only if $\varepsilon(\tau[n])$ is euclidean regular.

3. Proof of Theorem 1

The proof of Theorem 1 relies on an interplay between the formulas (2.1) and (2.2). The fact that (2.1) expresses $V(\tau[n])$ as an integral over an n -dimensional euclidean simplex while (2.2) expresses $V(\tau[n])$ as an integral over an $(n-1)$ -dimensional euclidean simplex makes it possible to compare volumes of ideal simplices in H^{n+1} with volumes of ideal simplices in H^n , and finally to prove the main theorem by induction on n .

We start by giving an estimate for the growth of $V(\tau_0[n])$ which will be used in the proof but which is also of interest in itself. Recall that $\tau_0[n]$ denotes a *regular* ideal n -simplex in H^n .

PROPOSITION 2. *For all $n \geq 2$ one has*

$$\frac{n-1}{n^2} \leq \frac{V(\tau_0[n+1])}{V(\tau_0[n])} \leq \frac{1}{n}. \tag{3.1}$$

Remark. The upper bound was noted by Thurston ([6], section 6.1).

Proof. Let $\sigma_0[n]$ be any regular euclidean n -simplex with vertices on S^{n-1} . We shall prove the following three formulas

$$\int_{\sigma_0[n]} (1-r^2)^{-(n+1)/2} d\mathbf{r} = V(\tau_0[n]) \quad (3.2)$$

$$\int_{\sigma_0[n]} (1-r^2)^{-n/2} d\mathbf{r} = nV(\tau_0[n+1]) \quad (3.3)$$

$$\int_{\sigma_0[n]} (1-r^2)^{-(n-1)/2} d\mathbf{r} = \frac{n-1}{n} V(\tau_0[n]). \quad (3.4)$$

Clearly these three formulas imply that

$$\frac{n-1}{n} V(\tau_0[n]) \leq nV(\tau_0[n+1]) \leq V(\tau_0[n])$$

which is equivalent to (3.1).

Since all ideal, regular n -simplices in H^n are isometric we can assume that $p(\tau_0[n])$ is euclidean regular. Hence (2.1) implies (3.2). Next (2.2) implies (3.3) because regularity of $\tau_0[n+1]$ assures regularity of the euclidean n -simplex $\varepsilon(\tau_0[n+1])$. It remains to prove (3.4).

We shall apply Gauss' divergence formula

$$\int_{\sigma_0[n]} \operatorname{div} \mathbf{V}(\mathbf{r}) d\mathbf{r} = \int_{\partial\sigma_0[n]} \mathbf{V} \cdot \mathbf{n} dS \quad (3.5)$$

to the vector field

$$\mathbf{V}(\mathbf{r}) = (1-r^2)^{-(n-1)/2} \mathbf{r}, \quad \|\mathbf{r}\| < 1.$$

Here, of course, \mathbf{n} is the outward pointing normal to the boundary $\partial\sigma_0[n]$. An easy computation shows that

$$\operatorname{div} \mathbf{V}(\mathbf{r}) = (1-r^2)^{-(n-1)/2} + (n-1)(1-r^2)^{-(n+1)/2}.$$

For simplicity put

$$\varphi_n(\alpha) = \int_{\sigma_0[n]} (1-r^2)^{-\alpha} d\mathbf{r}. \quad (3.6)$$

Then the left hand side of (3.5) becomes

$$\varphi_n\left(\frac{n-1}{2}\right) + (n-1)\varphi_n\left(\frac{n+1}{2}\right).$$

To compute the right hand side of (3.5) we note that $\partial\sigma_0[n]$ consists of $(n+1)$ regular $(n-1)$ -simplices $\partial_i\sigma_0[n]$, $i=0, 1, \dots, n$. On $\partial_i\sigma_0[n]$ one has

$$\begin{aligned} 1-r^2 &= \varrho_n^2 - \varrho^2, \quad \mathbf{r} \in \partial_i\sigma_0[n] \\ \mathbf{r} \cdot \mathbf{n} &= 1/n \end{aligned}$$

where $\varrho_n = (1-n^{-2})^{\frac{1}{2}}$ is the radius of the circumscribed $(n-2)$ -sphere for $\partial_i\sigma_0[n]$, and ϱ denotes the distance from the center of $\partial_i\sigma_0[n]$ to the point $\mathbf{r} \in \partial_i\sigma_0[n]$. Therefore the right hand side of (3.5) becomes

$$\frac{n+1}{n} \int_{\partial_0\sigma_0[n]} (\varrho_n^2 - \varrho^2)^{-(n-1)/2} dS.$$

Since $\partial_0\sigma_0[n]$ is isometric to $\varrho_n \cdot \sigma_0[n-1]$ this integral transforms into

$$\frac{n+1}{n} \int_{\sigma_0[n-1]} (\varrho_n^2 - \varrho_n^2 r^2)^{-(n-1)/2} \varrho_n^{n-1} d\mathbf{r} = \frac{n+1}{n} \int_{\sigma_0[n-1]} (1-r^2)^{-(n-1)/2} d\mathbf{r} = \frac{n+1}{n} \varphi_{n-1} \left(\frac{n-1}{2} \right).$$

Thus we have proved

$$\varphi_n \left(\frac{n-1}{2} \right) + (n-1) \varphi_n \left(\frac{n+1}{2} \right) = \frac{n+1}{n} \varphi_{n-1} \left(\frac{n-1}{2} \right). \quad (3.7)$$

By (3.2) and (3.3) $\varphi_n((n+1)/2) = V(\tau_0[n])$ and $\varphi_{n-1}((n-1)/2) = (n-1)V(\tau_0[n])$. Hence

$$\varphi_n \left(\frac{n-1}{2} \right) = \frac{n-1}{n} V(\tau_0[n])$$

which proves (3.4).

LEMMA 3. *Let $f:]0, 1] \rightarrow \mathbf{R}$ be continuous and concave. Let \mathbf{c} be the center of mass of an arbitrary euclidean n -simplex $\sigma[n]$ with vertices on S^{n-1} , and put $c = \|\mathbf{c}\|$. Then*

$$V(\sigma[n])^{-1} \int_{\sigma[n]} f(1-r^2) d\mathbf{r} \leq V(\sigma_0[n])^{-1} \int_{\sigma_0[n]} f((1-c^2)(1-r^2)) d\mathbf{r}$$

whenever both of these improper integrals converge. Moreover, if f is strictly concave then equality holds if and only if $\sigma[n]$ is regular.

Proof. Let the left and right hand side of the inequality be A and B respectively. Let $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n$ be the vertices of $\sigma[n]$. We have the standard n -simplex

$$\Delta[n] = \{(t_0, t_1, \dots, t_n) \mid t_i \geq 0, \sum t_i = 1\} \subseteq \mathbf{R}^{n+1}.$$

Under the homeomorphism $(t_0, t_1, \dots, t_n) \rightarrow \sum t_i \mathbf{v}_i$ of $\Delta[n]$ with $\sigma[n]$ the measure $V(\sigma[n]^{-1}) dx$ on $\sigma[n]$ transforms into a measure μ on $\Delta[n]$ which is just the "Lebesgue measure" normalized to have $\mu(\Delta[n])=1$. Hence

$$A = \int_{\Delta[n]} f(1 - \|\sum_i t_i \mathbf{v}_i\|^2) d\mu.$$

Since μ is invariant under the transformation $t_i \rightarrow t_{\pi(i)}$ for any permutation π of $0, 1, \dots, n$ we also have

$$A = \int_{\Delta[n]} f(1 - \|\sum_i t_{\pi(i)} \mathbf{v}_i\|^2) d\mu, \quad \text{for any } \pi.$$

If E denotes the formation of mean values over all such π , then

$$A = E \left(\int_{\Delta[n]} f(1 - \|\sum_i t_{\pi(i)} \mathbf{v}_i\|^2) d\mu \right).$$

The concavity of f then implies that

$$A \leq \int_{\Delta[n]} f(E(1 - \|\sum_i t_{\pi(i)} \mathbf{v}_i\|^2)) d\mu. \quad (3.8)$$

The mean value involved here can easily be computed from the following formulas

$$\begin{aligned} \|\sum_i t_{\pi(i)} \mathbf{v}_i\|^2 &= \sum_{i \neq j} t_{\pi(i)} t_{\pi(j)} (\mathbf{v}_i, \mathbf{v}_j) + \sum_i t_i^2 \\ E(t_{\pi(i)} t_{\pi(j)}) &= \frac{1}{n(n+1)} \sum_{k \neq l} t_k t_l = \frac{1}{n(n+1)} (1 - \sum_i t_i^2), \quad i \neq j \\ \sum_{i \neq j} (\mathbf{v}_i, \mathbf{v}_j) &= \|\sum_i \mathbf{v}_i\|^2 - \sum_i \|\mathbf{v}_i\|^2 = (n+1)^2 c^2 - (n+1). \end{aligned}$$

Here, of course, (\cdot, \cdot) is the euclidean inner product. One gets

$$A \leq \int_{\Delta[n]} f\left(\frac{n+1}{n} (1 - c^2) (1 - \sum_i t_i^2)\right) d\mu. \quad (3.9)$$

If $\sigma[n]$ is regular then equality holds in (3.9). Therefore, if one applies (3.9) to $\sigma_0[n]$ and to $g(x) = f((1 - c^2)x)$ one gets

$$B = \int_{\Delta[n]} f\left(\frac{n+1}{n} (1 - c^2) (1 - \sum_i t_i^2)\right) d\mu. \quad (3.10)$$

Here we have used that the center of mass for $\sigma_0[n]$ is $\mathbf{0}$. This finishes the proof of $A \leq B$.

If equality holds in Lemma 3 then we also have equality in (3.8). In case of strict concavity this is possible only when

$$\left\| \sum t_{\pi(i)} \mathbf{v}_i \right\|^2 = \left\| \sum t_i \mathbf{v}_i \right\|^2$$

for all $(t_0, t_1, \dots, t_n) \in \Delta[n]$ and all permutations π . Letting $t_0 = t_1 = \frac{1}{2}$, $t_i = 0$ for $i > 1$ it follows that

$$\|\mathbf{v}_1 + \mathbf{v}_2\| = \|\mathbf{v}_i + \mathbf{v}_j\| \quad \text{for all } i \neq j.$$

Since $\|\mathbf{v}_i - \mathbf{v}_j\|^2 = 4 - \|\mathbf{v}_i + \mathbf{v}_j\|^2$ we see that

$$\|\mathbf{v}_1 - \mathbf{v}_2\| = \|\mathbf{v}_i - \mathbf{v}_j\| \quad \text{for all } i \neq j,$$

and that guarantees the euclidean regularity of $\sigma[n]$.

End of proof of Theorem 1. Assume inductively that the theorem holds for some $n \geq 3$, and consider an arbitrary $\tau[n+1]$. Put

$$f(t) = t^{-n/2} - K_n t^{-(n+1)/2}, \quad 0 < t \leq 1$$

where $K_n = nV(\tau_0[n+1])/V(\tau_0[n])$. An elementary computation shows that f is strictly concave on $]0, 1]$ if and only if $K_n \geq n(n+2)/(n+1)(n+3)$. On the other hand Proposition 2 guarantees that $K_n \geq (n-1)/n$ which exceeds $n(n+2)/(n+1)(n+3)$ for $n \geq 3$. Lemma 3 can, therefore, be applied to f and the euclidean n -simplex $\sigma[n] = \varepsilon(\tau[n+1])$ (cf. section 2). Using also (2.2) (for $n+1$) and (2.1) and letting $\tau[n] = p^{-1}(\sigma[n])$ one gets

$$\begin{aligned} nV(\tau[n+1]) - K_n V(\tau[n]) &\leq \int_{\sigma_0[n]} f((1-c^2)(1-r^2)) \, d\mathbf{r} \\ &= (1-c^2)^{-n/2} nV(\tau_0[n+1]) - K_n (1-c^2)^{-(n+1)/2} V(\tau_0[n]) \\ &\leq (1-c^2)^{-n/2} (nV(\tau_0[n+1]) - K_n V(\tau_0[n])) \\ &= 0. \end{aligned} \tag{3.11}$$

By the inductive hypothesis $V(\tau[n]) \leq V(\tau_0[n])$ so (3.11) implies

$$nV(\tau[n+1]) \leq K_n V(\tau_0[n]) = nV(\tau_0[n+1]) \tag{3.12}$$

which shows that $V(\tau_0[n+1])$ is maximal.

If equality holds in (3.12) then also in (3.11). By Lemma 3 this implies that $\varepsilon(\tau[n+1])$ is euclidean regular. But then $\tau[n+1]$ is hyperbolically regular.

4. An asymptotic formula for $V(\tau_0[n])$

From section 1 we have

$$V(\tau_0[2]) = \pi = 3.14159\dots$$

$$V(\tau_0[3]) = 3 \int_0^{\pi/3} -\log(2 \sin \theta) d\theta = 1.01494\dots$$

We mention, without giving details, that it is possible to compute $V(\tau_0[4])$ using the generalized Gauss formula (cf. Klein [1], p. 205). We found

$$V(\tau_0[4]) = \frac{10\pi}{3} \arcsin \frac{1}{3} - \frac{\pi^2}{3} = 0.26889\dots$$

It seems to be very difficult to obtain simple expressions for $V(\tau_0[n])$ when $n \geq 5$. However, we have the following asymptotic formula for $V(\tau_0[n])$ (recall that $\sigma_0[n]$ is a regular euclidean n -simplex with vertices on S^{n-1}).

THEOREM 4.

$$\lim_{n \rightarrow \infty} \frac{V(\tau_0[n])}{V(\sigma_0[n])} = \sqrt{e}. \quad (4.1)$$

The proof of Theorem 4 relies on an investigation of the functions

$$\varphi_n(\alpha) = \int_{\sigma_0[n]} (1-r^2)^{-\alpha} d\mathbf{r}, \quad n=1, 2, \dots$$

When $n \geq 2$ $\varphi_n(\alpha)$ is defined for $\alpha \leq (n+1)/2$ (in fact $\varphi_n(\alpha) < \infty$ if and only if $\alpha < n$ but we shall not need this). Moreover φ_n is monotonically increasing, and being an integral of a logarithmically convex function φ_n is itself logarithmically convex, i.e. $\alpha \rightarrow \log \varphi_n(\alpha)$ is a convex function.

LEMMA 5.

$$\varphi_n(-1)/\varphi_n(0) = \frac{n+1}{n+2}, \quad n \geq 1 \quad (4.2)$$

$$\varphi_n\left(\frac{n-1}{2}\right) / \varphi_n\left(\frac{n+1}{2}\right) = \frac{n-1}{n}, \quad n \geq 2. \quad (4.3)$$

Proof. Formula (4.3) is an immediate consequence of (3.2) and (3.4). To prove (4.2) consider a regular euclidean simplex $\sigma_0[n]$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n$ on the unit sphere.

Regularity implies that the inner product $(\mathbf{v}_i, \mathbf{v}_j)$ equals $-1/n$ for $i \neq j$. Let $d\mu$ be the normalized ‘‘Lebesgue measure’’ on $\Delta[n] = \{(t_0, t_1, \dots, t_n) \mid t_i \geq 0, \sum t_i = 1\} \subseteq \mathbf{R}^{n+1}$. Arguing as in the proof of Lemma 3 we get

$$\begin{aligned} \varphi_n(-1)/\varphi_n(0) &= V(\sigma_0[n])^{-1} \int_{\sigma_0[n]} (1-r^2) dr \\ &= \int_{\Delta[n]} (1 - \|\sum_i t_i \mathbf{v}_i\|^2) d\mu \\ &= \int_{\Delta[n]} (1 - \sum_i t_i^2 \|\mathbf{v}_i\|^2 - \sum_{i \neq j} t_i t_j (\mathbf{v}_i, \mathbf{v}_j)) d\mu \\ &= \int_{\Delta[n]} \left(1 - \sum_i t_i^2 + \frac{1}{n} \sum_{i \neq j} t_i t_j\right) d\mu \\ &= \frac{n+1}{n} \int_{\Delta[n]} (1 - \sum_i t_i^2) d\mu. \end{aligned}$$

Since $\mu(\Delta[n])=1$ and since $\int_{\Delta[n]} t_i^2 d\mu$ is independent of $i=0, 1, \dots, n$ we get

$$\int_{\Delta[n]} (1 - \sum_i t_i^2) d\mu = 1 - (n+1) \int_{\Delta[n]} t_n^2 d\mu.$$

The map $(t_0, \dots, t_n) \rightarrow (t_1, \dots, t_n)$ is an affine isomorphism of $\Delta[n]$ onto $\{t \in \mathbf{R}^n \mid t_i \geq 0, \sum_{i=1}^n t_i \leq 1\}$ which transforms $d\mu$ into the measure $n! dt_1 \dots dt_n$.

Hence,

$$\begin{aligned} \int_{\Delta[n]} t_n^2 d\mu &= n! \int_{t_i \geq 0, \sum_{i=1}^n t_i \leq 1} t_n^2 dt_1 \dots dt_n \\ &= n! \int_0^1 \left(\int_{t_i \geq 0, \sum_{i=1}^{n-1} t_i \leq 1-t_n} dt_1 \dots dt_{n-1} \right) t_n^2 dt_n \\ &= n! \int_0^1 \frac{1}{(n-1)!} (1-t_n)^{n-1} t_n^2 dt_n \\ &= \frac{2}{(n+1)(n+2)}. \end{aligned}$$

And thus

$$\varphi_n(-1)/\varphi_n(0) = \frac{n+1}{n} \left(1 - (n+1) \int_{\Delta[n]} t_n^2 d\mu\right) = \frac{n+1}{n+2}.$$

Proof of Theorem 4. Let $n \geq 2$. Using the logarithmic convexity of φ_n one gets

$$\left(\frac{\varphi_n(0)}{\varphi_n(-1)}\right)^{(n-1)/2} \leq \left(\frac{\varphi_n((n-1)/2)}{\varphi_n(0)}\right) \leq \left(\frac{\varphi_n((n+1)/2)}{\varphi_n((n-1)/2)}\right)^{(n-1)/2}.$$

Since $\varphi_n(0) = V(\sigma_0[n])$ and $\varphi_n((n-1)/2) = ((n-1)/n)V(\tau_0[n])$ (by (3.4)) we get, by applying Lemma 5, that

$$\left(\frac{n+2}{n+1}\right)^{(n-1)/2} \leq \frac{n-1}{n} \frac{V(\tau_0[n])}{V(\sigma_0[n])} \leq \left(\frac{n}{n-1}\right)^{(n-1)/2}.$$

Since

$$\lim_{n \rightarrow \infty} \left(\frac{n+2}{n+1}\right)^{(n-1)/2} = \lim_{n \rightarrow \infty} \left(\frac{n}{n-1}\right)^{(n-1)/2} = \sqrt{e}$$

this proves Theorem 4.

Remark. Using the fact that the edglength of $\sigma_0[n]$ is $(2(1+1/n))^{1/2}$ the volume of $\sigma_0[n]$ can easily be computed to be

$$V(\sigma_0[n]) = \frac{\sqrt{n+1}}{n!} \left(1 + \frac{1}{n}\right)^{n/2}$$

which is asymptotically equal to $\sqrt{n}/n! \sqrt{e}$ for $n \rightarrow \infty$. Hence, by Theorem 4

$$V(\tau_0[n]) \sim \frac{\sqrt{n}}{n!} e.$$

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