Simplicial Manifolds, Bistellar Flips and a 16-Vertex Triangulation of the Poincaré Homology 3-Sphere

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CONTENTS

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We present a computer program based on bistellar operations that provides a useful tool for the construction of simplicial manifolds with few vertices. As an example, we obtain a 16-vertex triangulation of the Poincaré homology 3-sphere; we construct an infinite series of non-PL d-dimensional spheres with $d + 13$ vertices for $d \geq 5$; and we show that if a d-manifold, with $d \geq 5$, admits any triangulation on n vertices, it admits a noncombinatorial triangulation on $n + 12$ vertices.

1. INTRODUCTION

In the early days of topology, manifolds were often studied via triangulations. The combinatorial structure makes the computation of various invariants possible, and theorems can be proved based on the assumption of a suitable triangulation. See, for example, [Kuiper 1979; Moise 1977; Stillwell 1993] for accounts of some main lines in the historical development. Since the manifolds themselves, and not their combinatorial structure, are the real objects of interest in topology, there was a growing desire to get away from triangulations. In the 1930's and 40's algebraic tools gradually replaced the combinatorial ones, and to the extent that from this time on there still was an interest in decomposing a manifold, the more economical CW complexes gained popularity.

While triangulations have always remained of interest to discrete geometers and geometric and PL topologists, the emergence of computers has subtly changed the general situation. It is now possible (at least in principle) to study compact manifolds and compute their invariants on a machine. But a fundamental question naturally arises: *How do you present the manifold to a computer?* It is clear that some finite combinatorial encoding must be used. A decomposition as a CW complex may be elegant and also economical in terms of the number of cells,

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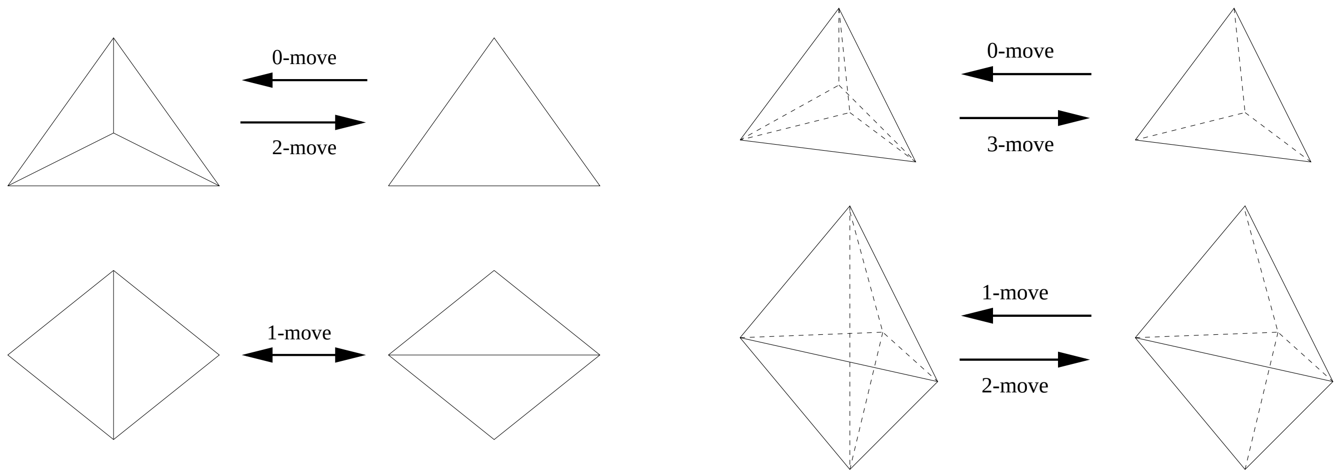

FIGURE 1. Bistellar moves for $d = 2$ (left) and $d = 3$ (right).

but it is in general difficult to explain the attaching maps to a computer. One needs something like a regular CW complex, where the attaching maps are determined by the combinatorics of inclusion of closed cells. However, the conceptually easiest presentation is as a simplicial complex, say, given as the list of its facets (maximal faces). Such an encoding is clear and simple, as long as it is not too large. Thus, the matter of the *size* of a triangulation has taken on practical significance. It is of interest to say something about the number of vertices, or the total number of faces, of a triangulation, and also to explicitly construct minimal or otherwise optimal triangulations.

For earlier work on the topic of minimal triangulations see [Altshuler and Steinberg 1974; 1976; Barnette and Gannon 1976; Brehm and Kühnel 1987; Brehm and Światkowski 1993; Kühnel 1990; 1995; Kühnel and Banchoff 1983; Walkup 1970]. For algorithmic approaches to recognition problems for manifolds see the papers [Matveev 1998; Nabutovsky 1996; Thompson 1994].

The work reported in this paper grew out of a desire to have a *computer tool for experimentation with triangulations.* We had three purposes in mind:

1. to be able to start with some triangulation of a manifold and let the computer search for smaller triangulations;
2. to be able to determine heuristically the homeomorphism type of a manifold and, in particular, to recognize (combinatorial) spheres; and
3. to be able to search for counterexamples to conjectures, where such examples might be hard to find due to their size or complexity.

Since to determine the homeomorphism type of a manifold is a delicate and much studied matter, the second point needs immediate clarification. What we have in mind is a procedure for heuristically comparing a given test manifold with reference manifolds having similar invariants from a library of standard manifolds on few vertices, with no guarantee for success. In future work the combinatorial ideas of this paper can hopefully be expanded and combined with algorithms for computing topological invariants (not only homology, but also fundamental group, characteristic classes, intersection forms, multiplicative structure of cohomology, ...) to create a truly versatile tool for manipulation and identification of manifolds.

A computer program, BISTELLAR [Lutz 1999a], was written which repeatedly modifies a triangulation by local so called "bistellar operations". Such operations for dimensions 2 and 3 are illustrated in Figure 1; we defer the formal definition to Section 2. The program accepts as input a simplicial manifold M (or any pure simplicial complex) presented via the list of its facets. It then searches through other triangulations of M via bistellar moves, using randomness controlled by a "simulated annealing" type strategy, to be explained in Section 3.

The program has turned out to be quite useful for the first two purposes. For reasons that will be explained later (searching for counterexamples to the

"g-conjecture for spheres"), we needed non-PL triangulations of the d-sphere ($d \geq 5$) of manageable size. As a stepping stone in the construction we gave BISTELLAR the task to compute a small triangulation of what Rolfsen [Rolfsen 1976, p. 308] calls "the ubiquitous Poincaré homology sphere". As reported in Section 5 the program produced a triangulation on 16 vertices which seems to be the smallest known triangulation of this manifold. It follows from work of Walkup [1970] that any triangulation must have at least 11 vertices. Thus, it is at the moment impossible to say where between 11 and 16 the truth about the optimal number of vertices lies. However, after having run our program over millions of triangulations, we are prepared to believe that 16 vertices might in fact be best possible for this manifold.

The 16-vertex triangulation of the Poincaré space is the starting point for a proof that there exist non-PL triangulations of the d-sphere on $d+13$ vertices for all $d \geq 5$. This is in turn used to show that if an arbitrary d-manifold admits some triangulation on n vertices, then it admits a non-PL triangulation on $n+12$ vertices ($d \geq 5$). Also, the $(d+13)$-vertex non-PL spheres complement earlier theorems of Barnette and Gannon [1976] and Brehm and Kühnel [1987]; see Section 6.

The search for minimal triangulations using our program has been continued by one of us (Lutz), and has led to several new results. They will be presented elsewhere [Köhler and Lutz 1999; Kühnel and Lutz 1999; Lutz 1999d], but we summarize the main findings.

Combinatorial triangulations were found for

- $S^2 \times S^2$ on 11 vertices,
- $S^3 \times S^2$ on 12 vertices,
- $S^3 \times S^3$ on 13 vertices,
- $(S^2 \times S^2) \# (S^2 \times S^2)$ on 12 vertices,
- $\mathbb{RP}^4$ on 16 vertices.

In all these cases, the theoretically minimal numbers of vertices for combinatorial triangulations of these manifolds are achieved.

The triangulations of $S^3 \times S^2$ on 12 and of $S^3 \times S^3$ on 13 vertices are of particular interest, since they attain the minimal numbers of vertices that any (nonspherical) combinatorial 5- or 6-manifold can have. They therefore establish that the lower bound given in [Brehm and Kühnel 1987] for the number of vertices of combinatorial d-manifolds is sharp in dimensions 5 and 6. For a statement of this bound see Theorem 8 and the sentence following it.

An extended version of the program, BISTELLAR_EQUIVALENT [Lutz 1999b], was used to determine the homeomorphism type of a large number of manifolds, including all triangulated 3-manifolds that have a vertex-transitive automorphism group on $n \leq 15$ vertices [Köhler and Lutz 1999; Lutz 1999d]. The idea behind this is to first construct reference triangulations of interesting manifolds with few vertices. If then a test object has the same homology as a particular reference manifold (this can be checked with the computer program HOMOLOGY by Heckenbach [1997]), it was possible in many cases to find a *bistellar equivalence* between the two manifolds, and thus to show that they are PL homeomorphic. For this we first searched for a small triangulation of the test object, and then applied further bistellar flips until, eventually, we were able to show that the modified test object is combinatorially isomorphic to the reference manifold.

Naturally, this works particularly well for manifolds with a unique minimal triangulation, such as PL d-spheres that can be minimally triangulated as the boundary complex of the $(d+1)$-dimensional simplex. Therefore the program can be used, at least as a heuristic, to determine whether a given simplicial complex is a combinatorial manifold (i.e., whether all vertex links are PL spheres). Other manifolds that have a unique minimal triangulation are, for example, the twisted sphere product (or 3-dimensional Klein bottle) $S^2 \underline{\times} S^1$ [Altshuler and Steinberg 1974; 1976; Walkup 1970] and the complex projective plane $\mathbb{CP}^2$ [Kühnel and Banchoff 1983], in both cases on 9 vertices.

The program has not yet achieved any success for the third purpose, that of finding counterexamples. At the end of Section 2 we report on some experiments of this kind.

The paper is structured as follows. In the next section we review some definitions and some general facts about triangulations of manifolds, bistellar flips and the counting of faces. Section 3 presents the program. In Section 4 we discuss the Poincaré homology 3-sphere and construct some highly symmetric triangulations for input into BISTELLAR. Section 5 presents the 16-vertex triangulation that was found. In Section 6 we derive via multiple suspensions the non-PL d-spheres on few vertices, and discuss how their existence relates to the existing theoretical bounds for such objects. In Section 7 we construct a highly symmetric triangulation of $\mathbb{RP}^3$ using the same general technique as in Section 4.

2. BACKGROUND AND DEFINITIONS

We collect here some definitions and discuss a bit more the background to this paper, including some general facts concerning triangulations of manifolds. For the general notions of topology we refer to [Stillwell 1993] and for PL topology to [Glaser 1970; Hudson 1969; Rourke and Sanderson 1972].

All manifolds in this paper are compact, connected and closed. Since PL concepts play such a role here, we recall the following definitions. A *PL sphere* is a simplicial complex which is piecewise linearly homeomorphic to the boundary of a simplex. A *combinatorial manifold* (or *PL manifold*) is a triangulation of a topological manifold such that the link at every vertex is a PL sphere.

For $d \neq 4$, a triangulation of the d-sphere is PL in the first sense if and only if it is a PL manifold in the second sense. For $d \leq 3$ this follows from the work of Moise [1952] and for $d \geq 5$ from the work of Kirby and Siebenmann [1977]; namely, there is a unique PL structure for spheres in these dimensions. For $d = 4$ this question is not fully understood: Is a combinatorial manifold homeomorphic to the 4-sphere necessarily a PL sphere? Since in dimension 4 the category of PL manifolds is equivalent to the smooth category, the question is equivalent to: Does there exist an "exotic" 4-sphere? (We are grateful to M. Kreck for clarifying this distinction.)

It was shown by Radó [1925] that all 2-manifolds and by Moise [1952] that all 3-manifolds can be triangulated; see also [Moise 1977; Stillwell 1993]. Since the link of a vertex in a triangulated 2-manifold is a polygon and the link of a vertex in a triangulated 3-manifold is a 2-sphere (and therefore PL), all 2- and 3-dimensional manifolds are PL.

The situation is much more subtle in four dimensions. Freedman constructed in 1982 a nondifferentiable analogue of the complex projective plane (see [Freedman and Luo 1989; Freedman and Quinn 1990, Sections 8.3 and 10.1]), and this *fake* $\mathbb{CP}^2$ provides an example of a 4-manifold that cannot be triangulated as a combinatorial manifold. By combining work of Casson with that of Freedman (see [Akbulut and McCarthy 1990, p. xvi]) one obtains examples of topological 4-manifolds *that cannot be triangulated at all.* For expositions of these triangulation questions and related matters see, for instance, [Kirby and Siebenmann 1977, Annex 2 and 3; Kuiper 1979; Lashof 1965; Marin 1988; Moise 1977; Stillwell 1993].

In 1963 Milnor listed seven problems that he considered the toughest and most important in geometric topology (see [Lashof 1965]). Among them is the question whether every topological manifold can be triangulated, now known to have a negative answer. Also on the list is the double suspension problem that asks whether the double suspension of a homology 3-sphere is a topological sphere. This problem was settled by Edwards [Edwards 1975] in 1974 for the double suspension of the Mazur homology 3-sphere which he proved is a topological 5-sphere (see [Daverman 1986, Chapter 12]). The theorem has later been generalized:

Theorem 1 [Cannon 1979]. *The double suspension $\mathcal{S}^2 H^d$ of any d-dimensional homology sphere H^d is homeomorphic to S^{d+2}.*

It follows that $\mathcal{S}^2 H^d$, although homeomorphic to S^{d+2}, has a non-PL structure, since H^d appears as the link of some 1-simplex in $\mathcal{S}^2 H^d$. This fact will be of importance in Section 6.

We now specialize the discussion to the concepts and tools that will be needed in this paper.

Definition 2 [Pachner 1987]. Let M be a simplicial d-manifold (or any pure d-dimensional simplicial complex), and let A be a $(d-i)$-face of M, where $0 \leq i \leq d$. If $\mathrm{link}_M(A)$ is the boundary $\mathrm{Bd}\, B$ of an i-simplex B that is not a face of M, the operation Φ_A on M defined by

$$\Phi_A(M) := (M \setminus (A * \mathrm{Bd}\, B)) \cup ((\mathrm{Bd}\, A) * B)$$

is called a *bistellar i-move.*

We also say *bistellar operations* or *bistellar flips* for bistellar moves. Bistellar i-moves with $i > \lfloor d/2 \rfloor$

are also called *reverse* $(d-i)$*-moves*. Note that a 0-move adds a new vertex to a triangulation, while a reverse 0-move deletes a vertex; see Figure 1. Two pure simplicial complexes are *bistellarly equivalent* if there exists a finite sequence of bistellar operations leading from one triangulation to the other (and vice versa).

It is easy to see that bistellar equivalence implies being PL homeomorphic, for any simplicial manifolds. For combinatorial triangulations the converse is also true.

Theorem 3 [Pachner 1987, Theorem 1]. *Two combinatorial manifolds are bistellarly equivalent if and only if they are PL homeomorphic.*

Define the *bistellar flip graph* of a triangulable manifold M to have as nodes the triangulations of M (or, more precisely, their isomorphism classes up to relabeling the vertices), and an edge between two nodes if one triangulation can be obtained via a single bistellar flip from the other (and vice versa). If the dimension of M is at most 3, then this graph is connected, as shown by the work of Moise [Moise 1952] together with Theorem 3. We will see in Section 6 that if $d \geq 5$ then this graph has infinitely many connected components. Of course, the manifolds within each connected component of the bistellar flip graph are pairwise PL homeomorphic. If M can be triangulated as a combinatorial manifold, then by Pachner's theorem the (infinite) space of all combinatorial triangulations of M is divided into equivalence classes of pairwise PL homeomorphic triangulations which coincide with connected components of the bistellar flip graph. For a discussion of Pachner's theorem in a topological environment see [Lickorish 1997].

We now consider counting faces of *all* dimensions, not just vertices (dimension zero). For more details and references to this area see the survey [Billera and Björner 1997], and for triangulations of spheres and polytopes [Stanley 1985].

Let f_i be the number of i-dimensional faces of a triangulated d-manifold M (with $f_{-1}=1$), and define numbers h_i by

$$\sum_{i=0}^{d+1} h_i\, x^{d+1-i} = \sum_{i=0}^{d+1} f_{i-1}(x-1)^{d+1-i}. \qquad (2\text{–}1)$$

The sequence $(f_0,\ldots,f_d)$ is called the *f-vector* of M, and $(h_0,\ldots,h_{d+1})$ its *h-vector*. The corresponding *g-vector* $(g_0,\ldots,g_{\lfloor (d+1)/2\rfloor})$ is defined by $g_0=1$ and $g_i = h_i - h_{i-1}$, for $i \geq 1$.

It was shown by Klee [1964] for any triangulated manifold M that the face numbers $(f_0,\ldots,f_{\lfloor (d-1)/2\rfloor})$ determine the remaining numbers $(f_{\lfloor (d+1)/2\rfloor},\ldots,f_d)$ via linear relations. From (2–1) we see that this means that $(h_0,\ldots,h_{\lfloor (d+1)/2\rfloor})$ determine the complete f-vector; therefore so does $(g_0,\ldots,g_{\lfloor (d+1)/2\rfloor})$. Thus the g-vector of a triangulated manifold contains complete information about its f-vector.

The relevance of this for our program is the following.

Theorem 4 [Pachner 1987, p. 83]. *Suppose that M' is obtained from M by a bistellar k-move, where $0 \leq k \leq \lfloor (d-1)/2\rfloor$. Then*

$$g_{k+1}(M') = g_{k+1}(M) + 1,$$
$$g_i(M') = g_i(M) \quad \text{for all } i \neq k+1.$$

Also, if d is even and $k = d/2$, then $g_i(M') = g_i(M)$ for all i.

This means that it is very easy to follow and control the successive f-vectors during a sequence of bistellar flips. In our program we compute and store the initial g-vector, which is then updated with a $+1$ (or -1) in position $k+1$ for each k-move (or reverse k-move).

Remark. In the case of odd-dimensional manifolds the result implies that the bistellar flip graph is bipartite — it can be colored by the sum (mod 2) of the entries of the g-vector. In even dimensions, $d/2$-moves do not change the g-vector and sometimes even lead to a combinatorially isomorphic triangulation of a manifold, that is, the bistellar flip graph may have loops.)

The linear relations of Klee take on a particularly attractive form if M triangulates a sphere (the Dehn–Sommerville relations):

$$h_i = h_{d+1-i}. \qquad (2\text{–}2)$$

If furthermore M is *polytopal* (i.e., combinatorially isomorphic to the boundary complex of a simplicial convex polytope), then by a theorem of Stanley [Stanley 1980]

$$(g_0, \ldots, g_{\lfloor (d+1)/2 \rfloor}) \text{ is an M-sequence.} \tag{2–3}$$

The combinatorial condition of being an *M-sequence* (M for Macaulay) is defined as follows, showing that it can easily be tested by machine. For integers $k, n \geq 1$ there is a unique way of writing

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_i}{i}$$

so that $a_k > a_{k-1} > \cdots > a_i \geq i \geq 1$. Then define

$$\partial^k(n) = \binom{a_k - 1}{k-1} + \binom{a_{k-1} - 1}{k-2} + \cdots + \binom{a_i - 1}{i-1}.$$

Also let $\partial^k(0) = 0$. A sequence $(n_0, n_1, \ldots)$ of nonnegative integers is an M-sequence if

$$n_0 = 1 \quad \text{and} \quad \partial^k(n_k) \leq n_{k-1} \text{ for all } k \geq 2.$$

A nontrivial consequence of (2–3) is that $g_i \geq 0$ for polytopal spheres. The *"g-theorem"* states that the conditions (2–2) and (2–3) together characterize the f-vectors of polytopal spheres. The sufficiency of these conditions was proved by Billera and Lee [1981].

The conjecture to which we wanted BISTELLAR to search for counterexamples is the *"g-conjecture for spheres"*, which states that condition (2–3) is valid for *all* triangulated spheres, not just polytopal ones. If correct, this would imply a characterization of the f-vectors of spheres.

The g-conjecture can be deduced from known results for all d-spheres up to dimension 4, but is open for $d \geq 5$. Attempts during the last 20 years to prove it have so far been without success. It therefore seemed to us that the possibility of its falsity should be considered and tested.

In order to look for counterexamples we started with non-PL triangulations of the 5- and 6-sphere and let the bistellar flip program search through thousands of triangulations. This purpose is what originally made us look for small triangulations of the Poincaré 3-sphere and its suspensions; see Section 6 for a description of the spheres we used to start the computer search. The bistellar flip program guarantees by Theorem 3 that all triangulations visited during the search are non-PL, and, in particular, that they are not polytopal. At each step the g-vector is updated, as described in Theorem 4, and tested for being an M-sequence. The parameters for the program can be set to put priority on creating a g-vector that is not an M-sequence (if possible), for example a g-vector with some negative entry.

So, what was the result? No counterexamples to the g-conjecture were found. Although no conclusions can be drawn, we hope that this is an indication that the conjecture is correct.

3. THE BISTELLAR FLIP PROGRAM

The computer program that will now be presented performs walks on the bistellar flip graph of triangulations of a manifold M. By necessity we must restrict attention to some connected component of this graph. For a particular triangulation of M from this component (the input) we want to perform bistellar modifications with the objective to obtain "small" (hopefully even minimal), or otherwise sought-after, triangulations of M (within the component). As an objective function that we want to optimize, we could take for example the total number of faces of a triangulation. Nevertheless, the sum G of the entries of the g-vector seems to be a more appropriate objective function, since any up-move — that is, an i-move with $0 \leq i \leq \lfloor (d-1)/2 \rfloor$ — increases G by one and any down-move (reverse up-move) decreases G by one, so that we have good control over G. (If d is even, then $d/2$-moves do not change G.) In addition to the goal of minimizing the objective function G we perform moves according to *priority rules*. Reverse 0-moves are given the highest priority as they delete a vertex, then come reverse 1-moves, reverse 2-moves, etc. If no further reverse moves are available, this might be due to the fact that we have achieved a global minimum for G within our component of triangulations. But we can as well have gotten stuck in some local minimum.

A concept that is very useful in such situations is *simulated annealing* [Kirkpatrick et al. 1983]. In a continuous version of simulated annealing (see [Rinnooy Kan and Timmer 1989], for instance) one wants to find a global minimum $x_* \in \mathbb{R}^n$ for a real valued objective function $f : \mathbb{R}^n \to \mathbb{R}$, i.e., $x_* \in \mathbb{R}^n$ such that $f(x_*) \leq f(x)$ for all $x \in \mathbb{R}^n$. Starting at some initial point y one moves to a randomly picked neighboring point y' if $\Delta f = f(y') - f(y) \leq 0$. If

$\Delta f > 0$, then we move "uphill" to y' with probability $\exp(-\Delta f/\beta)$ or otherwise stay put at y. In the next step a new neighboring point y'' of y' (or of y if we have not moved) is chosen at random and so on. The *cooling parameter* $\beta > 0$ describes how likely it is to move "uphill" and is usually decreased with time (the number of steps).

We now describe an appropriate annealing-type strategy for bistellar flips. As soon as we are trapped in a "local" minimum, we perform an up-move. (Up-moves are also performed according to priority rules, such as "perform a $(k{+}1)$-move before a k-move".) Sometimes, this already paves the way for further reverse moves that lead away from the local minimum. But we might also fall back into the same local minimum in the following *round*. After a certain number of up-moves has become necessary (we call this the *relaxation parameter*) we start "heating up" the function G, i.e., for a number of steps given by the *heating parameter* we perform only up-moves (as long as this is possible), with the exception that we usually do not perform 0-moves, since this would blow up the size of the complex too quickly. Then we let the system relax until we have to heat up again. If there is more than one option for moves of a certain priority, we pick one of these options randomly and then execute the move.

An Implementation of the Bistellar Flip Program

We start with some triangulation of a d-manifold, represented by the list of its facets, and determine all its faces and compute its f- and g-vector. Next, we check for every $(d-i)$-face of the triangulation whether it is contained in precisely $i+1$ facets. The collection of these faces (together with their respective links) form the *raw options* for bistellar i-moves. If we want to consider *proper options* for i-moves, then we include only those raw options for i-moves for which in addition the links satisfy the condition of being the boundary of an i-simplex that is *not* a face of the triangulation. This last condition is easy to check.

When we determine the raw options at the beginning, we have to check for all f_i i-faces how often they are included in one of the f_d facets. This amounts to $f_i \cdot f_d$ operations. Nevertheless, in the following rounds we do not have to recompute the raw options from scratch, since with any bistellar flip we simply cut out a ball locally and replace it by another ball. All raw options for faces in the interior of the ball that we remove have to be deleted and raw options for the faces in the interior of the new ball have to be included. Raw options for faces on the common boundary of the balls might also change. But altogether, there is only a constant number of faces involved in updating the raw options. Finally, to find out which of the raw options of a given priority are proper options, we have to test the condition on links mentioned above.

We wrote the program BISTELLAR in GAP, as all required operations for sets and lists are available in this computer algebra package [Schönert et al. 1996]. See sidebar on the next page for an excerpt describing our flip strategy for 3-manifolds. For complete information about BISTELLAR, see [Lutz 1999a].

In higher dimensions, the strategy for the options can easily be adapted, although it takes time and experiments to figure out reasonable parameters for heating and relaxation. (This is a common problem with simulated annealing algorithms.)

4. THE UBIQUITOUS POINCARÉ HOMOLOGY 3-SPHERE

The first example of a nonsimply connected manifold having the same homology as the ordinary 3-sphere was found by Poincaré [1904]. It was constructed from two solid double tori identified along their boundary surfaces of genus 2. For this and other constructions of this space, see [Rolfsen 1976, p. 244–250 and 308–311; Stillwell 1993, p. 263–266; Weber and Seifert 1933, p. 245]. This manifold, whose existence prompted the 3-dimensional Poincaré conjecture, has had an enormous influence on the subsequent development of topology. It is discussed in many places; in addition to the sources already mentioned, see, for example, [Dehn 1910; Kirby and Scharlemann 1979; Kneser 1929; Threlfall and Seifert 1931]. We particularly mention [Kirby and Scharlemann 1979], where eight constructions of this space are given and proved to be equivalent. Also, several of the given references discuss the fact that the fundamental group of the Poincaré homology 3-sphere is the binary icosahedral group, of order 120.

```
## initial settings ##
InputFacets;
Compute_RawOptions;
Compute_f_and_g_vector;
g_min:=g;

## parameters ##
rounds:=1;
relaxation:=0;
heating:=0;

while rounds <= 50000 do

  ## strategy for options ##
  options:=[];
  if heating > 0 then
    Include_MoveOptions(1);
    if options = [] then
      Include_ReverseMoveOptions(1);
      heating:=0; fi;
    heating:=heating-1;
  else
    Include_ReverseMoveOptions(0);
    if options = [] then
      Include_ReverseMoveOptions(1);
      if options = [] then
        Include_MoveOptions(1);
        if options = [] then
          Include_MoveOptions(0); fi;
        relaxation:=relaxation+1;
        if relaxation = 10 then
          heating:=15;
          relaxation:=0; fi; fi; fi; fi;

  ## perform Move or ReverseMove ##
  ChooseOptionAtRandom;
  ExecuteOption;
  Update_RawOptions;
  Update_f_and_g_vector;
  Print(rounds,"  ",g,"\n");
  if g < g_min then
    g_min:=g;
    Print("f-vector = ",f,"\n");
    Print(facets,"\n"); fi;

  rounds:=rounds+1;
od;
```

Excerpt from the BISTELLAR program in dimension 3, showing our version of the simulated annealing algorithm for performing walks in the bistellar flip graph of triangulations of a 3-manifold.

Triangulations of the Poincaré homology 3-sphere on 17 and 18 vertices were constructed by Brehm. This is mentioned in the proof of Proposition 3.28 of [Kühnel 1995, p. 55], but no details are given. The first task for our bistellar flip program was to try to improve on this.

In order to have a starting triangulation for the program at hand, we first construct a "small" triangulation of the Poincaré homology 3-sphere. For this, we consider the description of the Poincaré sphere as the *spherical dodecahedron space* which is the cell decomposition of the solid dodecahedron where opposite pentagons on the boundary are identified by a coherent twist of $\pi/5$ radians; see [Threlfall and Seifert 1931] or [Weber and Seifert 1933].

We triangulate the boundary of the dodecahedron by introducing a midpoint for every pair of identified opposite pentagons (see Figure 2).

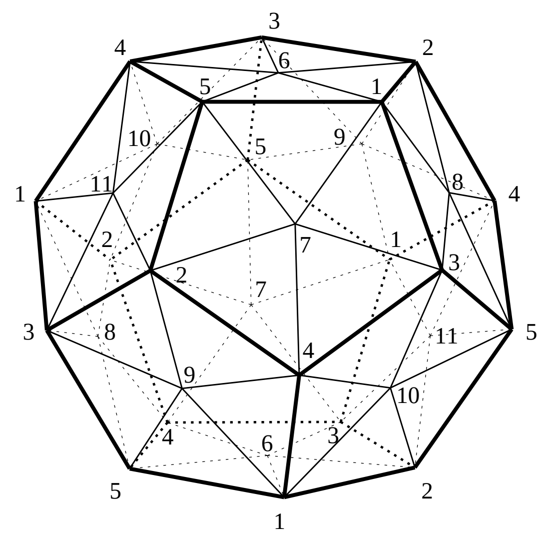

FIGURE 2. A_5-invariant triangulation of the Poincaré 3-sphere.

Into the interior of the dodecahedron we place an icosahedron in such a way that every vertex of the icosahedron corresponds to a copy of a midpoint of a pentagon. For every vertex of the icosahedron we form the cone over the respective pentagon. For every edge of the icosahedron we include the tetrahedron that is determined by this edge and the edge that separates the two corresponding neighboring pentagons. Similarly, for any triangle on the boundary of the icosahedron we take the tetrahedron that is made up by the triangle and the intersection-vertex of the three corresponding neighboring pentagons. Finally, we triangulate the interior of the icosahedron by introducing a center point and we take the cone over the boundary of the icosahedron with respect to the center point. The resulting triangulation of the Poincaré homology 3-sphere has $5 + 6 + 12 + 1 = 24$ vertices and is invariant under the 60-element group A_5 of rotations of the icosahedron and the dodecahedron.

Instead of an icosahedron, we could also place a bipyramid over a pentagon into the interior of the dodecahedron. In this case, the north and south pole of the bipyramid are joined to the dark shaded subcomplexes of Figure 3. Then take one vertex of the equatorial pentagon of the bipyramid and let it correspond to the light shaded subcomplex of Figure 3. By rotations of the cyclic group $\mathbb{Z}_5$ we obtain four additional equatorial subcomplexes, and the seven subcomplexes that we have described cover the boundary of the dodecahedron.

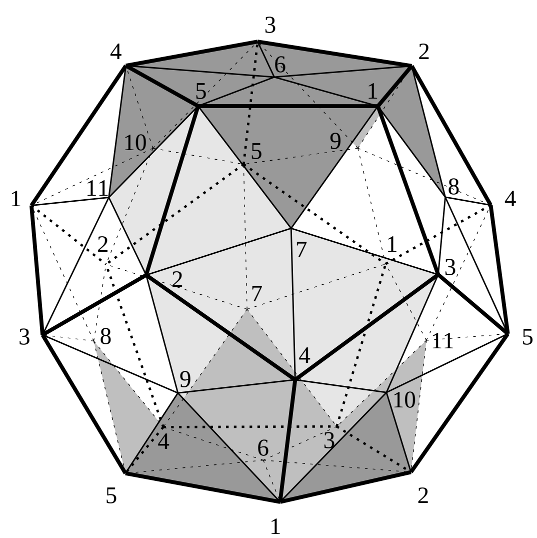

FIGURE 3. $\mathbb{Z}_5$-invariant triangulation of the Poincaré 3-sphere.

Now, triangulate the space between the bipyramid and the (identified) boundary of the dodecahedron similarly as before. For the interior of the bipyramid we introduce an edge connecting north and south pole and then slice the bipyramid like an orange. This provides us with a $\mathbb{Z}_5$-invariant 18-vertex triangulation of the Poincaré sphere. As was mentioned, such a triangulation was previously found by Brehm. By some modification of the identified boundary it is not too difficult to obtain nonsymmetric 17-vertex triangulations, but we were unable to reach 16 vertices by hand.

5. A NON-SYMMETRIC TRIANGULATION Σ^3_{16} ON 16 VERTICES

We applied the bistellar flip program to both the above 18-vertex and the 24-vertex triangulation. After some running time we obtained a 16-vertex triangulation.

Theorem 5. *There exists a triangulation* (*without any symmetries*) *of the Poincaré homology* 3-*sphere on* 16 *vertices with* f-*vector* $f = (16, 106, 180, 90)$.

Proof. The list of facets

1 2 4 9	1 7 8 11	2 6 10 14	4 6 7 11	6 7 12 13
1 2 4 15	1 7 11 12	2 6 12 15	4 6 10 11	6 10 11 12
1 2 6 14	1 8 10 13	2 7 9 13	4 6 10 14	6 12 13 15
1 2 6 15	1 9 11 12	2 7 9 14	4 7 11 15	7 8 10 14
1 2 9 14	1 9 11 14	2 7 10 14	4 8 9 12	7 8 11 15
1 3 4 12	1 10 13 15	2 8 11 15	4 8 9 13	7 8 14 15
1 3 4 15	2 3 5 10	2 8 12 15	4 8 10 13	7 9 14 15
1 3 7 10	2 3 5 11	3 4 5 14	4 8 10 14	8 12 14 15
1 3 7 12	2 3 7 10	3 4 5 15	4 8 12 14	9 10 11 12
1 3 10 15	2 3 7 13	3 4 12 14	4 10 11 13	9 10 11 16
1 4 9 12	2 3 11 13	3 5 10 15	5 6 7 13	9 10 15 16
1 5 6 13	2 4 9 13	3 5 11 14	5 7 9 13	9 11 14 16
1 5 6 14	2 4 11 13	3 7 12 13	5 7 9 15	9 14 15 16
1 5 8 11	2 4 11 15	3 11 13 14	5 8 9 12	10 11 13 16
1 5 8 13	2 5 8 11	3 12 13 14	5 8 9 13	10 13 15 16
1 5 11 14	2 5 8 12	4 5 6 7	5 9 10 12	11 13 14 16
1 6 13 15	2 5 10 12	4 5 6 14	5 9 10 15	12 13 14 15
1 7 8 10	2 6 10 12	4 5 7 15	6 7 11 12	13 14 15 16

determines a 3-dimensional (pure) simplicial complex Σ^3_{16} on 16 vertices with f-vector

$$f = (16, 106, 180, 90).$$

Since this simplicial complex was obtained by means of bistellar flips from a triangulation of the Poincaré sphere, it is PL homeomorphic to this space.

Alternatively, we can assemble the 90 tetrahedra in the interior of the dodecahedron. Once again, we obtain a triangulation of the solid dodecahedron where opposite pentagons on the boundary are identified by a coherent twist of $\pi/5$ radians. Figure 4 shows the corresponding triangulation of the boundary with the respective identifications. Vertices 1–11 lie on the boundary of the dodecahedron, whereas vertices 12–16 lie in the interior.

If a combinatorial manifold has a (combinatorial) symmetry, then the links of the vertices that are

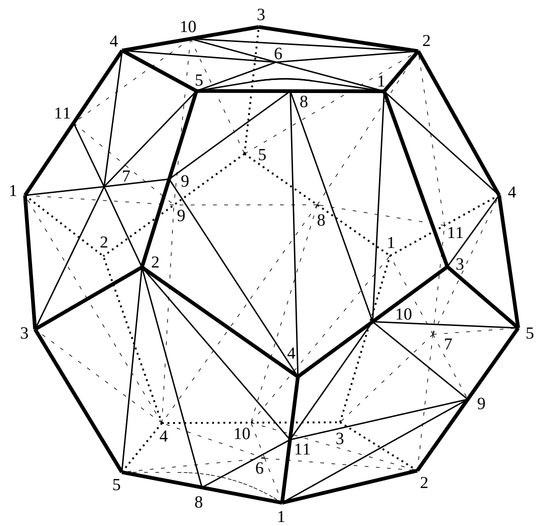

FIGURE 4. 16-vertex triangulation of the Poincaré sphere.

mapped onto each other must be combinatorially equivalent. For Σ^3_{16} the links of the vertices $\{3,6\}$, $\{10,13,14\}$ and $\{2,4,5,7,12\}$ are pairwise combinatorially equivalent within each group, and there are no other such equivalences. Thus, the automorphism group of Σ^3_{16} is a subgroup of $S_2 \times S_3 \times S_5$. However, none of these 1440 permutations, apart from the identity, is in fact a symmetry, and therefore Σ^3_{16} has trivial automorphism group. □

Is there a 15-vertex triangulation of the Poincaré homology 3-sphere? It follows from [Walkup 1970, Theorem 4] that at least 11 vertices are needed. (We are grateful to R. Forman for pointing this out to us.) We let our bistellar flip program run for up to 10^6 moves with changing relaxation and heating parameters. From time to time the triangulation Σ^3_{16} appeared or other triangulations on 16 vertices with larger f-vectors, but never any smaller triangulation or any nonequivalent triangulation with the same f-vector.

Conjecture 6. *The triangulation* Σ^3_{16} *of the Poincaré homology* 3*-sphere has the component-wise minimal* f*-vector* $f = (16, 106, 180, 90)$ *for a triangulation of this manifold and is the unique triangulation with this* f*-vector.*

The boundary of the identified dodecahedron is a $\mathbb{Z}$-acyclic space with the same fundamental group as the Poincaré homology 3-sphere [Bredon 1972, p. 57]. In particular, this 2-dimensional space is not contractible. What is the minimal number of vertices of a simplicial complex that is $\mathbb{Z}$-acyclic but not contractible?

By taking the restriction of Σ^3_{16} to the boundary of the identified dodecahedron we obtain a triangulation on 11 vertices. The bistellar flip program brought this number down to 10. The corresponding f-vector is $f = (10, 40, 31)$. Subsequently another triangulation on 10 vertices with $f = (10, 40, 31)$, shown in Figure 5, was found by hand. Its facets are

1 2 4	1 4 9	2 3 7	3 5 6	4 5 8
1 2 5	1 5 7	2 3 8	3 5 9	4 7 9
1 3 6	1 5 10	2 4 6	3 7 9	4 7 10
1 3 8	1 6 7	2 4 10	3 7 10	5 8 9
1 3 10	1 6 9	2 6 7	4 5 6	5 8 10
1 4 8	2 3 5	2 6 8	4 5 7	6 8 9
		2 8 10		

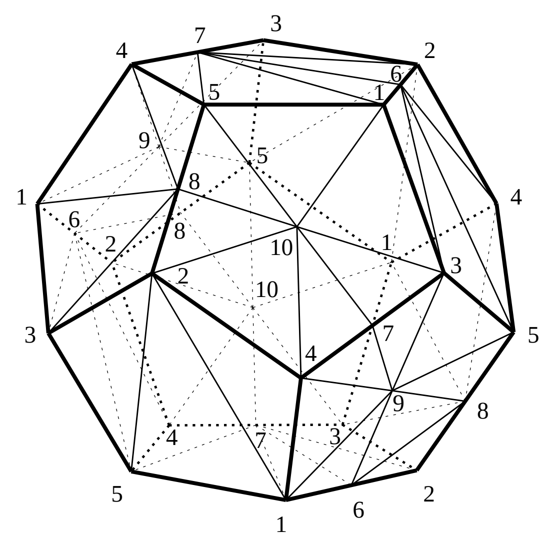

FIGURE 5. A $\mathbb{Z}$-acyclic noncontractible complex on 10 vertices.

We do not know if 10 vertices is best possible for a complex with these properties.

Remark. Taking instead the restriction of Σ^3_{24} (described in Section 4; see Figure 2) to the boundary of the identified dodecahedron we obtain a triangulation on 11 vertices, on which A_5 acts transitively on facets and without stationary points. Its nerve complex provides an 11-dimensional A_5-invariant vertex-transitive $\mathbb{Z}$-acyclic simplicial complex on 30 vertices [Lutz 1999c].

6. A SERIES OF NON-PL d-SPHERES ON d+13 VERTICES FOR d ≥ 5

It follows from Theorem 1 that if we suspend Σ^3_{16} twice, then we obtain a non-PL 5-sphere. If we suspend further, we obtain non-PL spheres of higher dimensions.

Theorem 7. *Let $d \geq 5$. Then there are non-PL triangulations of the d-dimensional sphere on $d+13$ vertices.*

Proof. We first show that for $d \geq 5$ there exist particularly simple non-PL triangulations of the d-dimensional sphere on $d+14$ vertices. For this, we suspend Σ^3_{16} $(d-3)$-times, i.e., we form $(d-3)$-times the join product of Σ^3_{16} with S^0. By the associativity of the join product with respect to the PL-structure [Rourke and Sanderson 1972, 2.24(1)],

$$\big((\cdots((\Sigma^3_{16} * S^0) * S^0) * \cdots * S^0) * S^0\big) = \Sigma^3_{16} * (S^0 * S^0 * \cdots * S^0 * S^0) = \Sigma^3_{16} * S^{d-4}.$$

If we take for S^{d-4} the boundary complex of the $(d-3)$-simplex, then the latter simplicial complex has $16 + (d-2)$ vertices. Note also that it has $90 \cdot (d-2)$ facets, and that the list of its facets is easily compiled by concatenation from the list in Section 5 of the 90 facets of Σ^3_{16} with the list of all $(d-3)$-subsets of a $(d-2)$-set.

An improvement of the number of vertices by one can be obtained if we use *Datta's trick* to construct one-point suspensions of triangulated manifolds M. (We thank W. Kühnel for pointing out this trick to us.) The Datta construction is as follows. Suspend M by using two vertices w_1 and w_2. Then pick a vertex v of M and replace the collection of facets that contain v by the facets that we obtain from the $(d-1)$-facets of the link of v by adding as an extra vertex either w_1 if w_2 is already contained in the respective $(d-1)$-facet, or otherwise w_2 if w_1 is already contained. The reverse procedure to this operation is called *starring a vertex in "an edge"* in [Bagchi and Datta 1998, Def. 9]. The authors of that paper remark that this generalized bistellar operation does not change the PL homeomorphism type of the suspension if M is a manifold (or a pseudomanifold). If we take $(d-3)$-times the one point Datta-suspension of Σ^3_{16}, then we obtain a non-PL d-sphere with $d+13$ vertices. □

Theorem 7 complements the following two results, which show that triangulated manifolds with "few" vertices must be PL spheres.

Theorem 8. *Let M be a triangulated d-manifold on n vertices.*

(a) [Barnette and Gannon 1976] *If $n < d+6$ and $d \geq 5$, then M is a PL sphere.*
(b) [Brehm and Kühnel 1987] *If $n < 3\lceil d/2\rceil + 3$ and M is combinatorial, then M is a PL sphere.*

Brehm and Kühnel [1987] also show that if $n = 3d/2+3$, then M is either a PL d-sphere or a "manifold like a projective plane" (the latter case can occur only for $d = 2$, 4, 8 or 16). The following consequence of Theorem 7 shows that the assumption "combinatorial" can not be removed from the Brehm–Kühnel theorem.

Corollary 9. *There exist non-PL d-spheres with $n \leq 3d/2+3$ vertices for $d \geq 19$.*

Question 10. *Are there non-PL d-spheres for $d \geq 5$ with less than $d+13$ vertices?*

We tried on this question with BISTELLAR for $d = 5$. Starting with the (ordinary) double suspension with 20 vertices of the 16-vertex triangulation of the Poincaré homology 3-sphere, we were able to get down to 18 vertices, but not further. The f-vector of the smallest non-PL 5-sphere that we found is $f = (18, 139, 503, 904, 783, 261)$.

We next show that for $d \geq 5$ there exists to any triangulation of a d-manifold M a non-PL triangulation of M with few additional vertices.

Theorem 11. *Let M be a topological d-manifold, for $d \geq 5$, that can be triangulated with n vertices. Then there are non-PL triangulations of M with $n+12$ vertices.*

Proof: Let M be a simplicial d-manifold with n vertices and $d \geq 5$. If the triangulation of M is non-PL, then nothing has to be done. So assume that M is combinatorial. Let (by Theorem 7) Σ^d be a simplicial non-PL sphere on $d+13$ vertices. Then there exists a vertex v of Σ^d for which the corresponding link is not a combinatorial sphere. Choose a facet of Σ^d that is not contained in the star of v and delete this facet from Σ^d. Also delete some facet from M and glue the remaining complexes together along the boundaries of the deleted simplices. The resulting manifold is the connected sum $\Sigma^d \# M$. Topologically, $\Sigma^d \# M$ is homeomorphic to M, but on the PL level it provides a non-PL triangulation of M, since $\mathrm{link}_{\Sigma^d}(v) = \mathrm{link}_M(v)$. We count the vertices of $\Sigma^d \# M$. The complexes M and Σ^d contribute n and $d+13$ vertices respectively. By the identification of the boundaries of the two d-simplices, we loose $d+1$ vertices. Thus, $\Sigma^d \# M$ has $n+(d+13)-(d+1) = n+12$ vertices. □

Finally, we prove the result on connected components of the bistellar flip graph referred to in Section 2.

Theorem 12. *Let M be a triangulable manifold of dimension $d \geq 5$. Then there are infinitely many connected components of the bistellar flip graph of M.*

Proof. Let H be any homology 3-sphere with nontrivial fundamental group $\pi_1(H)$, such as the Poincaré homology 3-sphere. We construct in three steps infinitely many triangulations of M that cannot pairwise be reached from one another by bistellar flips.

First, we form k-fold connected sums of H. These connected sums are again homology spheres, nevertheless they are pairwise nonhomeomorphic for different values of k. This is due to the fact that the fundamental group of a connected sum $M \# N$ of two manifolds M and N, with (nontrivial) fundamental groups $\pi_1(M)$ and $\pi_1(N)$ respectively, is the free product $\pi_1(M) * \pi_1(N)$. Thus the connected sums $H^{\#k}$ and $H^{\#l}$ have distinct fundamental groups if $k \neq l$.

In the second step, we take for $k \neq l$ the join products of the boundary complex of a $(d-3)$-simplex with $H^{\#k}$ and $H^{\#l}$. The resulting simplicial complexes, S^d_k respectively S^d_l, are non-PL spheres (as in the proof of Theorem 7) that have the homology spheres $H^{\#k}$ and $H^{\#l}$ sitting in their respective triangulations as the links of some $(d-4)$-faces. From the combinatorics of the join construction it is easy to see that the links of $(d-4)$-faces in S^d_k are all nonhomeomorphic to $H^{\#l}$, and the links of $(d-4)$-faces in S^d_l are all nonhomeomorphic to $H^{\#k}$. Now, focus on a copy of $H^{\#k}$ that sits in S^d_k as the link of a $(d-4)$-face F. If we apply any bistellar flip to S^d_k, then this operation may alter but not delete this copy of $H^{\#k}$. This is so, because the definition of bistellar flips shows that the face F, or any subface of F, cannot be the pivot face of a bistellar move, and the link of F will itself be altered at most by a bistellar move and thus its homeomorphism type is preserved. The same argument used in reverse shows that the bistellar flip will not produce $H^{\#l}$ as the link of some $(d-4)$-faces in S^d_k. It thus follows that S^d_l cannot be reached from S^d_k via bistellar flips, and vice versa.

Finally, we will use the infinite number of examples of pairwise nonbistellarly equivalent triangulations of d-spheres S^d_k to obtain an infinite number of pairwise nonbistellarly equivalent triangulations of M. For this, let Φ be the set of those spheres S^d_k such that $H^{\#k}$ is not homeomorphic to the link of any of the $(d-4)$-faces of M. The set Φ is infinite, since there are only finitely many links in M. Then, just as in the proof of Theorem 11, form connected sums $S^d_k \# M$ of the spheres $S^d_k \in \Phi$ with M in a way that guarantees that $H^{\#k}$ remains as the link of some $(d-4)$-face of $S^d_k \# M$. By the same argument as in the second step, $S^d_k \# M$ and $S^d_l \# M$ cannot be reached from one another via bistellar flips. □

7. AN A_5-INVARIANT TRIANGULATION of $\mathbb{RP}^3$ WITH 29 VERTICES

The idea of coherent twists on the dodecahedron can be used to create other interesting 3-manifolds besides the spherical dodecahedron space. For instance, Weber and Seifert [1933] constructed a *hyperbolic dodecahedron space*, a manifold with homology $H_* = (\mathbb{Z}, \mathbb{Z}_5^3, 0, \mathbb{Z})$, by again identifying the boundary of the solid dodecahedron, this time with a coherent twist of $3\pi/5$ instead of $\pi/5$ radians.

If we twist by $5\pi/5$, we obtain $\mathbb{RP}^3$. Figure 6 gives a triangulation of the identified boundary for the latter manifold (where the identified boundary is the nonorientable surface $\mathbb{RP}^2$).

As done previously for the spherical dodecahedron space, we place an icosahedron with additional center point into the interior of the dodecahedron. This yields an A_5-invariant triangulation of $\mathbb{RP}^3$ with 29 vertices. Moreover, there is also an A_5-invariant triangulation of $\mathbb{RP}^3$ on $6+12+1$ vertices that is defined by placing an icosahedron with center point

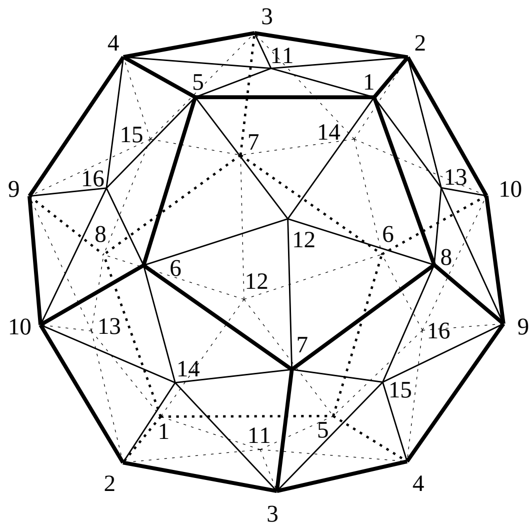

FIGURE 6. 29-vertex triangulation of $\mathbb{RP}^3$.

into the interior of an outer icosahedron with identifications on the boundary by reflection at the origin. For a vertex-minimal triangulation of $\mathbb{RP}^3$ on 11 vertices see [Brehm and Światkowski 1993; Köhler and Lutz 1999; Walkup 1970].

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