SIMPLICIAL NONPOSITIVE CURVATURE

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ABSTRACT

We introduce a family of conditions on a simplicial complex that we call local k-largeness ($k \geq 6$ is an integer). They are simply stated, combinatorial and easily checkable. One of our themes is that local 6-largeness is a good analogue of the non-positive curvature: locally 6-large spaces have many properties similar to non-positively curved ones. However, local 6-largeness neither implies nor is implied by non-positive curvature of the standard metric. One can think of these results as a higher dimensional version of small cancellation theory. On the other hand, we show that k-largeness implies non-positive curvature if k is sufficiently large. We also show that locally k-large spaces exist in every dimension. We use this to answer questions raised by D. Burago, M. Gromov and I. Leary.

Introduction

Spaces of non-positive curvature have been intensively investigated over the past 50 years. More recently non-riemannian metric spaces, for which non-positive or negative curvature is defined by comparison inequalities, the so-called CAT(0) or CAT(−1) spaces, have been studied, mainly in geometric group theory [BH].

Many CAT(0) spaces are obtained by combinatorial constructions. These constitute a significant part of small cancellation theory [LS], which deals mostly with 2-dimensional complexes. Cubical complexes are the main source of high dimensional CAT(0) spaces. The crucial observation which permits their study is Gromov's lemma: a cubical complex with its standard piecewise euclidean metric is CAT(0) if and only if the links of its vertices are flag simplicial complexes. The flag property is an easily checkable, purely combinatorial condition.

It is natural to ask if something similar holds for simplicial complexes:

(1) can one formulate the CAT(0) property of the standard piecewise euclidean metric on a simplicial complex in combinatorial terms;
(2) is there a *simple* combinatorial condition implying CAT(0);
(3) is there a simple condition implying Gromov hyperbolicity.

We do not answer the first question but we provide a satisfactory answers to the other two. Namely, in Section 1 we introduce the notion of a locally k-large simplicial complex, where $k \geq 4$ is an integer. It is defined in terms of links in the complex by very simple combinatorial means. We show in Sections 15 and 16 that,

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for every n, there is an explicit constant $k(n)$ such that if X^n is a locally $k(n)$-large, n-dimensional simplicial complex, then its standard piecewise euclidean metric is CAT(0). Taking a slightly bigger constant $k(n)$ we conclude that X^n admits a CAT(-1) metric. We also show (Section 2) that the universal covers of locally 7-large complexes are Gromov hyperbolic. These facts are well known in dimension 2, where our definition of locally 6- and 7-large coincides with the CAT(0) and, respectively the CAT(-1) property of the standard piecewise constant-curvature metrics.

We claim that "locally 6-large" is the right simplicial analogue of non-positive curvature. This condition neither implies nor is implied by the CAT(0) property of the standard metric, but shares many of its consequences. We describe some of them later in this introduction. The results are proved using combinatorial (but metrically inspired) concepts. This is very much in the spirit of small cancellation theory. The novelty is that our approach works in any dimension.

Let us point out that the flag condition from Gromov's lemma is equivalent to the "4-large" property. Also, Siebenmann's "flag-no-square" condition appearing in the study of CAT(-1) property of cubical complexes is equivalent to "5-large".

Finer properties of high dimensional locally 6-large simplicial complexes seem to be fairly different from the properties one sees when studying non-positively curved manifolds. Manifolds of dimension greater than 2 do not admit locally 6-large triangulations. As we show in [JS2], the fundamental groups of many aspherical manifolds cannot be embedded into the fundamental groups of locally 6-large complexes. Still, high dimensional locally 6-large spaces abound. We construct a great many very symmetric examples by developing certain simplices of groups. In particular, we can obtain in this way compact orientable locally 6-large pseudomanifolds of any dimension.

We now briefly describe the contents of the paper, which naturally splits into five parts.

In the first part (Sections 1 and 2) we introduce the concepts of a locally k-large simplicial complex, a k-systolic simplicial complex, and a k-systolic group. Here we briefly recall these concepts (see also the remark after Lemma 1.3). Given an integer $k \geq 4$, a simplicial complex is locally k-large if every cycle consisting of less than k edges in any of its links has some two consecutive edges contained in a 2-simplex of this link. A simplicial complex is k-systolic if it is locally k-large, connected and simply connected. A group is k-systolic if it acts simplicially, properly discontinously and cocompactly on a k-systolic complex. A simplicial complex (or a group) is systolic if it is 6-systolic.

In Section 1 we give a useful criterion for k-largeness ($k \geq 6$) in terms of links and lengths of homotopically nontrivial loops (Corollary 1.5). This is done with a simplification argument on simplicial disc diagrams reminiscent of small cancellation theory arguments. Similar reasoning allows us to establish in Section 2 the following result.

Theorem **A** *(See Theorem 2.1 and Corollary 2.2 in the text).*

(1) Let X *be a 7-systolic simplicial complex. Then the 1-skeleton* $X^{(1)}$ *of* X *with its standard geodesic metric is hyperbolic in the sense of Gromov.*
(2) Any 7-systolic group is word-hyperbolic.

The main idea exploited in the second part of the paper (Sections 3–6) is that of local convexity. We introduce it in Section 3 under the name of local 3-convexity. It allows us to define "small extensions" (Sections 4 and 5). These may be viewed as an analogue of the exponential map with a built-in divergence property for trajectories. Using small extensions we show the following three results.

Theorem **B** *(See Theorem 4.1.1 in the text). — The universal cover of a finite dimensional connected locally 6-large simplicial complex is contractible. In particular, any finite dimensional systolic simplicial complex is contractible.*

This is an analogue of the classical Cartan–Hadamard theorem.

Theorem **C** *(See Theorem 4.1.2 in the text). — Let* $f : Q \to X$ *be a locally 3-convex map of a connected simplicial complex* Q *to a finite dimensional connected locally 6-large simplicial complex* X*. Then the induced homomorphism* $f_* : \pi_1 Q \to \pi_1 X$ *of fundamental groups is injective.*

Note that Theorem C applies to the inclusion maps of locally 3-convex subcomplexes $Q \subset X$. The analogous statement in riemannian geometry asserts that the fundamental group of a locally geodesically convex subset in a complete non-positively curved manifold injects into the fundamental group of the ambient space (this is also true for locally CAT(0) geodesic metric spaces).

Theorem **D** *(See Theorem 6.1 in the text). — Every connected locally 6-large simplicial complex of groups is developable.*

Theorem D will be crucial for the constructions in the last part of the paper. It is analogous to the classical result asserting that non-positively curved complexes of groups are developable.

The results in part three of the paper (Sections 7–13) are based on a certain convexity property of balls in systolic complexes, described in Section 7 (Corollary 7.9). The main result in this part is the following.

Theorem **E** *(See Theorem 13.1 in the text). — Let* G *be a systolic group, i.e. a group acting simplicially, properly discontinuously and cocompactly on a systolic simplicial complex. Then* G *is biautomatic.*

Many corollaries of Theorem E can be obtained by using the well-developed theory of biautomatic groups [ECHLPT]. In particular, systolic groups satisfy quadratic isoperimetric inequalities, their abelian subgroups are undistorted, their solvable subgroups are virtually abelian, etc.

Theorem E is the culmination of a series of results concerning systolic complexes, which have independent interest. For example, in Section 8 we define a simplicial analogue of the projection map onto a convex subset. We also show that this map does not increase distances (Fact 8.2). In Section 9 we introduce the concept of directed geodesics and show their existence and uniqueness (Corollary 9.7). Finally, we establish in Sections 11–12 the two-sided fellow traveller property for directed geodesics, the main ingredient in the proof of Theorem E.

To prove Theorem E one needs, besides properties of directed geodesics, an argument which enables the passage from the space on which the group acts to the group itself, especially in the case where the group action has nontrivial stabilizers. The argument we use in this paper has been expanded and applied in other situations by the second author [S].

Part four of the paper (Sections 14–16) addresses the relationship between the k-systolic and CAT(κ) properties. We have

Theorem **F** *(See Theorem 14.1 in the text). — Let* Π *be a finite set of isometry classes of metric simplices of constant curvature* 1, 0 *or* -1. *Then there is an integer* $k \geq 6$, *depending only on* Π, *such that:*

(1) if X *is a piecewise spherical k-large complex with* Shapes(X) $\subset \Pi$ *then* X *is* CAT(1)*;*
(2) if X *is piecewise euclidean (respectively, piecewise hyperbolic), locally k-large and* Shapes(X) $\subset \Pi$ *then* X *is non-positively curved (respectively, has curvature* $\kappa \leq -1$*);*
(3) if, in addition to the assumptions of (2), X *is simply connected, then it is* CAT(0) *(respectively,* CAT(-1)*).*

We offer two proofs of Theorem F. The first one (in Section 14) covers the general case, but the estimates for the systolic constants are not explicit. The second one (Section 15) yields potentially explicit constants, but covers only metrics for which the simplices have all angles acute. In Section 16 we work out explicit estimates for the standard piecewise euclidean metric (based on the second proof), and obtain the following.

Theorem **G** *(See Theorem 16.1 in the text). — Let k be an integer such that*

$$k \geq \frac{7\pi\sqrt{2}}{2} \cdot n + 2.$$

Then any k-systolic simplicial complex X *with* $\dim X \leq n$ *is* CAT(0) *with respect to the standard piecewise euclidean metric.*

The last part of the paper (Sections 17–20) deals with constructions of k-large complexes of high dimensions. The complexes we obtain arise as developments of appropriate simplices of groups. The constructions are based on the second important idea of the paper, the notion of extra-tilability of simplices of groups (Section 18). Extra-tilability matches with local convexity of balls in systolic spaces in an interesting way, and allows us to construct subgroups with large fundamental domains. As a consequence, we obtain large compact quotients of universal covers of simplices of groups, which in turn allows us to use induction in the constructions.

The key result in this part is Theorem H below. The technical notions occurring in its statement, which are standard in the theory of complexes of groups, are recalled in Section 17.

Theorem **H** *(See Proposition 19.1 in the text). — Let* Δ *be a simplex and suppose that for any codimension-1 face s of* Δ *we are given a finite group* A_s*. Then for any* $k \geq 6$ *there exists a simplex of finite groups* $\mathcal{G} = (\{G_\sigma\}, \{\psi_{\sigma\tau}\})$ *and a locally injective and surjective morphism* $m : \mathcal{G} \to F$ *to a finite group* F *such that* $G_\Delta = \{1\}$*,* $G_s = A_s$ *for any codimension-1 face s of* Δ*, and the development* $D(\mathcal{G}, m)$ *associated with the morphism m is a (finite and) k-large simplicial complex.*

As an application of Theorems F and H we obtain the following.

Theorem **J** *(See Corollary 19.3.2,3 in the text).*

(i) For each natural number n there exists an n-dimensional compact simplicial orientable pseudomanifold whose universal cover is CAT(0) *with respect to the standard piecewise euclidean metric.*
(ii) For each natural number n and each real number $d > 0$ *there exists an n-dimensional compact simplicial orientable pseudomanifold whose universal cover is* CAT(−1) *with respect to the piecewise hyperbolic metric for which the simplices are regular hyperbolic with edge lengths d.*

Theorem J answers a question of D. Burago and collaborators [Bu, BuFKK], motivated by their investigations of billiards. The result can be extended from simplices to more general domains. We present the exposition of this more general result in [JS3].

As a step in the proof of Theorem J one gets the existence of k-large compact orientable pseudomanifolds of arbitrary dimension n, for any $k \geq 6$. It is interesting to compare this with our earlier paper [JS1], where we establish the existence of hyperbolic Coxeter groups of arbitrary virtual cohomological dimension. The existence of such (right-angled) groups is reduced in [JS1] to the existence in arbitrary dimension of compact orientable pseudomanifolds which satisfy the flag-no-square condition (they occur as nerves of the corresponding right-angled Coxeter groups). Since the flag-no-square condition is equivalent to 5-largeness, we obtain in the present paper compact orientable pseudomanifolds which satisfy even stronger conditions, with a significantly different construction than that in [JS1].

The result from [JS1] mentioned above can also be compared with another result from the present paper, Theorem K, which can be deduced from Theorem H.

Theorem **K** *(See Corollary 19.3.1 in the text). — For each natural number n there exists a developable simplex of groups whose fundamental group is Gromov-hyperbolic, virtually torsion-free, and has virtual cohomological dimension n.*

A less immediate consequence of Theorem H, below, answers a question of M. Gromov. Normal simplicial pseudomanifolds occurring in the statement of this result form a natural class containing, among others, all triangulations of manifolds. By a branched covering we mean a simplicial map which is a covering outside the codimension-2 skeleton.

Theorem **L** *(See Theorem 20.1 in the text). — Let* X *be a compact connected normal simplicial pseudomanifold equipped with a piecewise euclidean (respectively, piecewise hyperbolic) metric. Then* X *has a compact branched covering* Y *which is non-positively curved (respectively, has curvature* $\kappa \leq -1$*) with respect to the induced piecewise constant curvature metric.*

We apply the same methods to answer a question of Ian Leary concerning homotopy types of classifying spaces for proper G-bundles of Gromov hyperbolic groups G (see [QGGT, Question 1.24]). We refer to [LN] for the background on the following result.

Theorem **M** *(See Corollary 20.4 in the text). — Any finite complex* K *is homotopy equivalent to the classifying space for proper* G*-bundles of a* CAT(−1) *(hence Gromov hyperbolic) group* G.

We started to work on the present paper in 2000. The initial aim was to construct hyperbolic Coxeter groups of arbitrarily large virtual cohomological dimension via 5-large pseudomanifolds. After proving first few results on 6-large spaces, we found a shortcut – retractible and extra retractible complexes of groups – which we eventually used in [JS1]. Our further study of the subject was motivated by the question of D. Burago et al. on existence of simplicial pseudomanifolds whose standard piecewise flat metrics are nonpositively curved. This led us to the question about the relationship between the k-large and CAT(κ) conditions.

Since 2002 we gave several lectures on the subject (at the conferences in Luminy in 2002, in Durham in 2003, and in several other places). At the Luminy conference, M. Gromov asked the question about ramified covers (see Theorem L), and gave us significant moral support with the rest of the project. In the Spring of 2003 one of us had first discussions with Dani Wise which were very useful. We did not circulate a preprint, and in late 2003, we have learned that Frederic Haglund has independently obtained some of our results (roughly, those in Sections 1–8 and 17–19). Part of his work is described in [H].

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1. k-large and k-systolic simplicial complexes

In this section we define and study first properties and examples of k-large and k-systolic simplicial complexes.

We allow that simplicial complexes are not locally finite. In Sections 1–3, if not explicitly assumed otherwise, we also allow that they are not finite dimensional, i.e. may contain simplices of arbitrarily large finite dimension. (We call such complexes *infinite dimensional.*) Starting from Section 4, we assume that simplicial complexes are finite dimensional. It is possible that some of the results in Sections 4–12 extend to infinite dimensional case, but this would require arguments different from ours.

If a simplicial complex is finite dimensional, its topology is induced by the standard piecewise euclidean metric.

We express topological properties of infinite dimensional simplicial complexes in terms of their appropriate finite dimensional skeleta.

We are most interested in simplicial complexes equipped with proper discontinuous and cocompact group actions by simplicial automorphisms. Those are finite dimensional and locally finite.

Let X be a simplicial complex, and σ a simplex in X. The *link* of X at σ, denoted X_σ, is a subcomplex of X consisting of all simplices that are disjoint from σ and which together with σ span a simplex of X. The *residue* of σ in X, $\mathrm{Res}(\sigma, X)$, is the union of all simplices of X that contain σ. It is also called the (closed) star of σ. The residue $\mathrm{Res}(\sigma, X)$ is naturally the join of σ and the link X_σ.

A subcomplex K in X is called *full* (in X) if any simplex of X spanned by a set of vertices in K is a simplex of K. If K is full in X, then K_σ is full in X_σ for any simplex σ in K. A similar property holds also for residues.

A simplicial complex X is *flag* if any finite set of vertices, which are pairwise connected by edges of X, spans a simplex of X. Clearly, a full subcomplex in a flag complex is flag. Note also that X is flag if and only if for any simplex σ the link X_σ is full in X. Flag simplicial complexes arise naturally in the study of CAT(0) property of cubical complexes [Gr-HG,BH].

A *cycle* in a simplicial complex X is a subcomplex γ of X isomorphic to a triangulation of S^1. Denote by $|\gamma|$ the length of γ, i.e. the number of 1-simplices in γ. A *full cycle* in X is a cycle that is full as subcomplex of X. Define the *systole* of X to be

$$sys(X) = \min\{|\gamma| : \gamma \text{ is a full cycle in } X\}.$$

In particular, we have $sys(X) \geq 3$ for any simplicial complex X, and if there is no full cycle in X, $sys(X) = \infty$. This definition is somewhat reminiscent of the notion of systole in riemannian geometry, hence the name.

1.1. *Definition. — Given a natural number $k \geq 4$, a simplicial complex* X *is*

- k-large *if $sys(X) \geq k$ and $sys(X_\sigma) \geq k$ for each simplex σ of* X*;*
- locally k-large *if the residue of every simplex of* X *is k-large;*
- k-systolic *if it is connected, simply connected and locally k-large.*

(Here we use convention that an infinite dimensional simplicial complex is connected when its 1-skeleton is connected, and it is simply connected when its 2-skeleton is simply connected. To be consistent with the latter, by the universal cover of an infinite dimensional flag complex we mean the flag completion of the universal cover of its 2-skeleton.)

A group acting properly discontinously and cocompactly, by automorphisms, on a k-systolic simplicial complex, is called a *k-systolic group.*

A 6-systolic complex or a group is called *systolic*. 6-systolic complexes and groups are the main objects of study in this paper. Since the word "six-systolic" is somewhat hard to pronounce, we abbreviate it to "systolic".

Some easy properties of the above introduced classes of simplicial complexes are gathered in Fact 1.2. The proofs are immediate hence we omit them.

1.2. *Fact.*

(0) A complex is locally k-large if and only if the link of every nonempty simplex has the systole at least k.
(1) A (locally) k-large complex is (locally) m-large for $k \geq m$.
(2) A full subcomplex in a (locally) k-large complex is (locally) k-large.
(3) A simplicial complex is 4-large if and only if it is flag.
(4) For $k > 4$, X is k-large if and only if it is flag and $sys(X) \geq k$.

Note that, in view of property (4) above, a simplicial complex X is 5-large if it is a "flag-no-square" complex, or verifies "Siebenmann no square condition", a condition which arises in the study of CAT(-1) property of cubical complexes [Gr-HG].

The next result will be used in the proof of Lemma 1.7 for the purpose analogous to reducing van Kampen diagrams in small cancellation theory.

1.3. *Lemma. — Suppose that* X *is k-large and* S^1_m *denotes the triangulation of* S^1 *with m 1-cells. If* $m < k$ *then any simplicial map* $f : S^1_m \to X$ *extends to a simplicial map from the disc* D^2*, triangulated so that triangulation on the boundary is* S^1_m *and so that there are no interior vertices in* D^2.

Proof. — We will use induction with respect to m. For $m = 3$ the statement follows from flagness of X. Suppose $m > 3$. Then there are some non-consecutive vertices u, w of S^1_m whose images under f either coincide or are connected with an edge of X. Split S^1_m into polygonal paths A and B with endpoints u, w. Consider new triangulations of S^1, denoted S^1_A and S^1_B, obtained by adding the edge (u, w) to A and B, respectively. Note that, by the choice of u and w, restrictions of the map f to A and B extend uniquely to the simplicial maps $f_A : S^1_A \to X$ and $f_B : S^1_B \to X$.

Since the vertices u, w are non-consecutive in S^1_m, the triangulations S^1_A, S^1_B consist of fewer than m edges. By the inductive assumption, there are triangulations D_A, D_B of the 2-disk, and their simplicial maps F_A, F_B to X extending the maps f_A, f_B, as required in the lemma. We then get extension of f as required by gluing D_A to D_B along (u, w) and taking as F the union of the maps F_A, F_B.

Remark. — It follows easily from the above lemma that every cycle of length less than k in a k-large simplicial complex X has some two consecutive edges contained in a common 2-simplex of X. In view of the fact that links of links of a simplicial complex X are themselves links of X, this gives the following characterization of k-systolicity: a simplicial complex X is k-systolic if it is connected, simply connected, and every cycle of length less than k in any link of X has some two consecutive edges contained in a 2-simplex of this link.

There are 4-systolic (respectively 5-systolic) complexes that are not 4-large (respectively, 5-large). For example, take two octahedra (respectively, icosahedra), delete the interior of a triangle from each copy and glue the resulting boundaries. However, for $k \geq 6$ we have

1.4. *Proposition.* — *If* X *is a k-systolic simplicial complex with* $k \geq 6$ *then* X *is k-large.*

Before proving the above proposition, we derive its corollary which will be useful for our later constructions of k-large complexes in Sections 18–19.

A *homotopical systole* of a simplicial complex X is the minimal length of a cycle that is homotopically nontrivial in X. (If X is infinite dimensional, we say that a cycle is homotopically nontrivial in X if it is homotopically nontrivial in the 2-skeleton of X.) We denote homotopical systole of X by $sys_h(X)$.

1.5. *Corollary.* — *Let* $k \geq 6$. *A simplicial complex* X *is k-large if and only if it is locally k-large and* $sys_h(X) \geq k$.

Proof. — One of the implications follows from Proposition 1.4 by noting that if X is locally k-large then there is no full homotopically trivial cycle of length less than k in X (because, by Proposition 1.4, there is no such cycle in the universal cover of X). The second implication follows by observing that the shortest homotopically nontrivial cycle in any simplicial complex X is full.

Proof of Proposition 1.4. — We need to show that $sys(X) \geq k$. Consider a full cycle γ in X. A *filling* of γ is a continuous map $f : \Delta \to X$ such that Δ is the 2-disc and the restriction $f|_{\partial\Delta}$ is a homeomorphism on γ. Since X is simply connected, there is a filling $f_0 : \Delta_0 \to X$. Using relative Simplicial Approximation Theorem we can also arrange that Δ_0 is a simplicial disc and f_0 is a simplicial map (which is a simplicial homeomorphism on the boundary). Recall that a simplicial map is *nondegenerate* if it is injective on each simplex of the triangulation.

To proceed with the proof we need two lemmas, the first of which is related to van Kampen Lemma from the small cancellation theory. The elementary proofs of both lemmas are deferred until the end of this section.

1.6. *Lemma.* — *Let* X *be a simplicial complex, and* γ *a homotopically trivial cycle in* X. *Then there exists a filling* f *of* γ, $f : \Delta_1 \to X$, *which is a nondegenerate simplicial map from a simplicial 2-disc* Δ_1, *and which maps the boundary of* Δ_1 *isomorphically on* γ.

1.7. *Lemma.* — *Let* X, γ, *satisfy the assumptions of Lemma 1.6, and* X *is locally* k*-large. Then there exists a nondegenerate simplicial filling* $f : \Delta_2 \to X$ *of* γ, *such that every interior vertex of* Δ_2 *is contained in at least* k *triangles. Any filling of* γ *with the minimal number of triangles has this property. If moreover* γ *is a full subcomplex in* X, *then every boundary vertex of* Δ_2 *is contained in at least two triangles, and there is at least one internal vertex in* Δ_2.

To conclude the proof of Proposition 1.4 we use the Gauss–Bonnet theorem. Let $\chi(v)$ denote the number of triangles containing vertex v. Then

$$1 = \chi(\Delta_2) = \frac{1}{6}\left[\sum_{v\in B}(3 - \chi(v)) + \sum_{v\in I}(6 - \chi(v))\right],$$

where B denotes the set of vertices on the boundary and I the set of vertices in the interior of Δ_2. Since the second sum is at most $6 - k$, and the terms of the first sum are at most 1, we conclude that $|\gamma| = \#B \geq k$, and hence $sys(X) \geq k$.

1.8. *Examples and non-examples of k-large and k-systolic complexes, for* $k \geq 6$.

(1) A graph X is k-large if and only if $sys(X) \geq k$. It is k-systolic if and only if it is a tree.
(2) Trees are the examples of complexes that are k-large for any k. We will call such complexes ∞-*large*. Connected ∞-large complexes are necessarily ∞-systolic, since they are easily seen to be simply connected.
(3) Let Y be a triangulation of Euclidean or hyperbolic plane by congruent equilateral triangles with angles $2\pi/m$. Then Y is m-systolic. Let X be a simplicial surface obtained as a quotient of Y. If $6 \leq k \leq m$ then X is k-large if and only if $sys_h(X) \geq k$. By residual finiteness of the automorphism group of Y, this gives lots of k-large surfaces.
(4) Let X be the Cayley complex of a group with triangular presentation (i.e. presentation with all relations of length 3). Then X is systolic if and only if the group (presentation) satisfies the $C(3)-T(6)$ small cancellation condition. More generally, a connected and simply connected 2-dimensional simplicial complex X is systolic if and only if it is a so called $C(3)-T(6)$ simplicial complex [LS]. For example, buildings of type $\widetilde{A}_2$, viewed as simplicial complexes, are systolic.
(5) Using the combinatorial Gauss–Bonnet theorem, one sees that a triangulation of the 2-sphere is never k-large, for any $k \geq 6$. It follows that no

triangulation of a manifold M with $\dim M \geq 3$ is 6-large, since 2-spheres occur as links of some simplices in M.

(6) As we show later in this paper, for any $k \geq 6$ there exist k-large simplicial pseudomanifolds in any dimension (see the next example). Moreover, any finite simplicial pseudomanifold admits a finite k-large branched cover, for any $k \geq 6$.

(7) We briefly describe an example of a 3-dimensional systolic pseudomanifold, which is the simplest new example obtained with the method presented later in this paper. For any integer $m \geq 3$ consider the Coxeter group

$$W_m = \langle s_1, s_2, s_3 \,|\, s_1^2, s_2^2, s_3^2, (s_1 s_2)^m, (s_1 s_3)^m, (s_2 s_3)^m \rangle$$

and its Coxeter complex Σ_m being the triangulation of the plane with $2m$ triangles around every vertex. By residual finiteness of W_m, there is a torsion free normal subgroup $N < W_m$ of finite index, such that the quotient surface $N \backslash \Sigma_m$ is $2m$-large. We show that there exists a simply connected 3-dimensional pseudo-manifold $X = X(m, N)$ with all vertex links isomorphic to $N \backslash \Sigma_m$. It is obtained as the universal development of the 3-simplex of groups with groups Z_2 at 2-faces, with dihedral groups D_m (of order $2m$) at edges, and with the quotient groups W_m/N at vertices. Developability of this simplex of groups is addressed later in this paper, see Theorem D.

(8) There is a characterization of finite ∞-large simplicial complexes, due to G. A. Dirac [D]. It says that the class of finite ∞-large simplicial complexes is precisely the smallest class $\mathscr{C}$ such that:
 (i) single simplex of arbitrary dimension belongs to $\mathscr{C}$;
 (ii) if X is obtained from some complexes $X_1, X_2 \in \mathscr{C}$ by gluing them along a single simplex, then $X \in \mathscr{C}$.

The proof of this fact may be also found in [GLR].

Proof of Lemma 1.6. — We introduce a class of complexes and maps more general than simplicial ones. An *almost simplicial 2-complex* is a cell complex whose cells are simplices glued to lower dimensional skeleta through nondegenerate maps. More precisely, we allow multiple edges and loops in the 1-skeleton, and we require that the interior of each boundary edge of a 2-cell is glued to the 1-skeleton homeomorphically on the interior of some 1-cell. A *simplicial map* from an almost simplicial 2-complex to a simplicial complex is determined by its values at the vertices in the same way as an ordinary simplicial map (for example, a loop is necessarily mapped to a vertex).

Suppose γ is a closed embedded (contractible) polygonal curve in a simplicial complex X, and suppose $f_0 : \Delta_0 \to X$ is a simplicial filling of γ. We will first modify it to a nondegenerate simplicial filling $f_0' : \Delta_0' \to X$ with Δ_0' almost simplicial. This will be done in a sequence of modifications as follows. Suppose e is an edge in Δ_0 which is mapped by f_0 to a vertex. Then there are two 2-cells in Δ_0 adjacent to e. Delete (the interior of the union of) these two cells from Δ_0 and glue the four resulting free edges in pairs, so that the two distinct vertices of e are identified. This gives an almost simplicial disc Δ' with the simplicial map f' to X induced from f_0 (and is the reason for introducing almost simplicial triangulations).

We wish to repeat the same modification procedure with the new triangulation, but now, due to the fact that the triangulation is almost simplicial, we need to consider two more cases.

The first possibility is that e is a loop. It then bounds a sub-disc D of Δ'. There is also a 2-cell C outside D adjacent to e. If all the edges of C are loops, then we have a nested family of discs bounded by them; take e^* to be the outermost loop and repeat the argument with e^* in place of e. Eventually we arrive at the situation where the two remaining edges of C are embedded. Now delete from Δ' the interior of the union of D and C, and glue the two resulting free edges to get a new almost simplicial disc Δ' with the induced simplicial map f' to X.

The second possibility is that e is adjacent on both sides to the same 2-cell C of Δ'. Then e is not a loop, and plays the role of two out of three boundary edges of C. The remaining third edge is necessarily a loop; thus we are in the situation as in the previous case, and we perform the modification as above.

Since a modification reduces the number of 2-cells in Δ', we eventually obtain an almost simplicial filling $f_0' : \Delta_0' \to X$ which is nondegenerate (since it is nondegenerate on the 1-skeleton of Δ_0').

The next step is to further modify the filling so that it remains nondegenerate but becomes simplicial. Note that, since f_0' is nondegenerate, Δ_0' has no loop edges. It is then sufficient to eliminate multiple edges (i.e. edges sharing both endpoints), while keeping induced maps to X nondegenerate, as an almost simplicial disc without loops and multiple edges is simplicial.

Consider a pair e_1, e_2 of edges in Δ_0' with common endpoints. Their union bounds a subdisc D of Δ_0'. Remove the interior of D from Δ_0' and glue the resulting two free edges with each other, getting new Δ_0' with new nondegenerate simplicial map f' to X induced from the previous one. Again, the procedure terminates, since the number of 2-cells in Δ_0' decreases. The final result $f_1 : \Delta_1 \to X$ is a nondegenerate simplicial filling, as required.

Notice that the procedure we describe does not change the map f on the boundary.

Proof of Lemma 1.7. — Take a filling produced in Lemma 1.6 and suppose v is an interior vertex of Δ_1 contained in less than k triangles. First we shall construct a filling $f_1' : \Delta_1' \to X$ of γ, with Δ_1' having one less interior vertex than Δ_1. We delete the interior of subdisc $\mathrm{Res}(v, \Delta_1)$, replace it with some triangulation given by Lemma 1.3, and define f_1' so that it coincides with f_1 on $\Delta_1 \setminus int[\mathrm{Res}(v, \Delta_1)]$.

The resulting filling is in general not nondegenerate, but the triangulation does have fewer simplices. Now we apply to it procedure used in the proof of Lemma 1.6, which produces a nondegenerate simplicial map with still fewer simplices.

Iteration of this procedure terminates after finitely many steps yielding a simplicial disc Δ_2 and a map $f_2 : \Delta_2 \to X$ which establishes the first part of Lemma 1.7.

Now, each boundary vertex is contained in at least 2 triangles and there is at least one interior vertex, since otherwise the boundary $\partial\Delta_2$ is not full in Δ_2 and thus γ is not full in X. This completes the proof of Lemma 1.7.

2. 7-systolic implies hyperbolic

One of the main themes of this paper is that k-systolic complexes with $k \geq 6$ resemble to a large extent CAT(0) spaces, though there are no obvious CAT(0) metrics on them. As a first step in this direction we show in this section that 7-systolic complexes and groups are hyperbolic in the sense of Gromov. This solves a problem pointed out by M. Gromov [Gr-AI, Remark (a) on p. 176] to find a purely combinatorial condition for simplicial complexes of arbitrary dimension that yields hyperbolicity. For an exposition of the theory of hyperbolic metric spaces and groups see [BH, GdelaH].

2.1. *Theorem.* — *Let* X *be a 7-systolic simplicial complex. Then the 1-skeleton* $X^{(1)}$ *of* X *with its standard geodesic metric is hyperbolic in the sense of Gromov. More precisely, any geodesic triangle in* $X^{(1)}$*, with vertices at vertices of* X*, is* δ*-thin with* $\delta = \frac{5}{2}$.

Remark. — Note that, strictly speaking, to prove hyperbolicity of a graph one needs to show uniform thinness of all geodesic triangles, not only those with vertices at vertices of the graph. However, uniform thinness of the latter triangles, say with constant δ, easily implies uniform thinness of geodesic m-gons with $m \leq 6$ and with vertices at vertices of the graph, with constant 4δ. (By thinness of an m-gon we mean that any of its edges remains close to the union of all other edges.) This is true since we can "decompose" an m-gon into $m-2 \leq 4$ geodesic triangles, without adding new vertices. Since an arbitrary geodesic triangle in a graph determines a geodesic m-gon as above, with edges of length 1 near some vertices of the triangle, it inherits thinness from this m-gon, with constant $4\delta + 1$. In particular, the 1-skeleton of a 7-systolic complex is δ-hyperbolic with $\delta = 11$. This estimate is by no means optimal.

Since a 7-systolic group is quasi-isometric to (the 1-skeleton of) the corresponding 7-systolic simplicial complex on which it acts, Theorem 2.1 implies the following.

2.2. *Corollary. — A 7-systolic group is word-hyperbolic*

Proof of Theorem 2.1. — Take any three vertices x, y, z in X, and join them by three geodesics $\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$ in $X^{(1)}$ to obtain a triangle γ. We need to show that every point on the side γ_{xy} is distance at most $\frac{5}{2}$ in $X^{(1)}$ from the union of the remaining two sides.

Clearly γ_{xy} is embedded. Without loss of generality we can assume that γ is embedded (i.e. geodesics $\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$ intersect only at their endpoints) in view of the following

2.3. *Lemma. — Suppose that two vertices $x, y \in X$ are joined by two geodesics $\gamma_{xy}, \gamma^*_{xy}$ in $X^{(1)}$. Then for any vertex a on γ_{xy} there is a vertex a^* on γ^*_{xy}, so that a, a^* are joined by an edge in X. In particular, any point on the geodesic γ_{xy} is distance at most $\frac{3}{2}$ in $X^{(1)}$ from the geodesic γ^*_{xy}.*

Proof of Lemma 2.3. — Without loss of generality we can assume $\gamma_{xy}, \gamma^*_{xy}$ are disjoint (except at the endpoints). Lemma 1.6 produces a filling in X of the digon formed by $\gamma_{xy}, \gamma^*_{xy}$, so that each vertex on the boundary is contained in at least 2 triangles, possibly with the exception of x, y. Suppose Lemma 2.3 is false. Then (the filling of) the digon has at least one internal vertex.

Apply the Gauss–Bonnet formula as in the proof of Proposition 1.4 to the digon. In the first sum at most two terms can be equal 2; the second sum is strictly negative. Thus, if there are k negative terms in the first sum, there are also at least $k + 3$ positive terms. Hence on one of the geodesics, say γ_{xy}, there are n vertices with negative contribution to the Gauss–Bonnet sum and at least $n + 2$ vertices with positive contribution. Thus negative vertices cannot separate positive ones, and we have two positive vertices that are either consecutive or separated only by several zero vertices (i.e. vertices with zero contribution to the Gauss–Bonnet sum). But this contradicts the fact that γ_{xy} is a geodesic in $X^{(1)}$, hence the lemma.

Coming back to the proof of Theorem 2.1, take a filling of γ in X constructed as in Lemma 1.6. The domain of the filling map is a disc Δ triangulated so that each vertex in the interior is contained in at least 7 triangles, and each vertex at the boundary, with possible exception of points x, y, z, is contained in at least two triangles.

A vertex on the boundary is called *positive, negative or zero vertex* if $3 - \chi(v)$ is positive, negative or zero respectively. Let p (respectively n) denote the number of positive (respectively negative) vertices other than x, y, z at the boundary $\partial\Delta$. Since γ_{xy} is a geodesic in $X^{(1)}$, any two positive vertices in the interior of γ_{xy} are separated by a negative one. Apply the Gauss–Bonnet formula to the disc Δ. The three terms in the first sum corresponding to the vertices x, y, z are ≤ 2 and the remaining terms of this sum are at most 1. If the second sum is ≤ -4 then $p \geq n+4$, and hence on one of the sides of the triangle there are at least 2 more positive vertices than negative. Thus there are two positive vertices which are not separated by a negative one, a contradiction. Hence Δ has at most three internal vertices.

We claim that a geodesic triangle γ having a filling Δ as above with at most three internal vertices is $\frac{5}{2}$-thin To prove this, take first a vertex v on γ_{xy} whose distance from both x, y is bigger than 2. If its distance in $\Delta^{(1)}$ from the union of remaining sides is also bigger than 2, there are at least 6 vertices in Δ which are distance 2 from v. Only two of these vertices are on γ_{xy}, so at least four of them are internal in Δ, a contradiction. Thus the distance of v from the remaining two sides is at most 2. It follows easily that the distance in $\Delta^{(1)}$ of any point on the side γ_{xy} from the union of remaining two sides is at most $\frac{5}{2}$.

Triangles in the range of the filling map are thinner than in the source, which concludes the proof.

3. 3-convexity in simplicial complexes

In this section we introduce the notion of 3-convexity and study its basic properties. It is inspired by the notion of convexity in spheres, or by the notion of π-convexity in CAT(1) spaces (compare Fact 15.4). The notion will play the key role in our later developments.

Given a simplicial complex X and its subcomplex Q, a *cycle in the pair* (X, Q) is a polygonal path γ in the 1-skeleton of X with endpoints contained in Q and without self-intersections, except a possible coincidence of the endpoints. A cycle γ as above is *full* in (X, Q) if its simplicial span in X is contained in the union $\gamma \cup Q$. A subcomplex Q in a simplicial complex X is 3-*convex* if Q is full in X and every full cycle in (X, Q) of length less than 3 (i.e. consisting of less than 3 edges) is contained in Q.

Remark. — 3-convexity can be expressed equivalently as follows. A subcomplex Q in a simplicial complex X is 3-convex if and only if it is full and, given any geodesic γ of length 2 in the 1-skeleton of X, with both endpoints in Q, the mid-point of γ (and thus the whole of γ) is also contained in Q.

3.1. *Examples of 3-convex subcomplexes.*

(1) Each simplicial complex X is 3-convex in itself.
(2) Let X be a flag simplicial complex. Then any simplex $\Delta \subset X$ is 3-convex in X. To see this, consider a cycle γ of length 2 in (X, Δ), with its mid-vertex not in Δ. We need to show that γ is not full in (X, Δ). Since both endpoints of γ are in Δ, they are connected with an edge of X. Since X is flag, this implies that the three vertices of γ span a 2-simplex of X. Since this 2-simplex is not a face of Δ, the simplicial span of γ is not contained in $\gamma \cup \Delta$, which completes the proof.
(3) Let X be a 5-large simplicial complex. Then the residue $Y = \text{Res}(\sigma, X)$ of any simplex σ is 3-convex in X. Indeed, Y is a full subcomplex of X, because X is flag. Moreover, any cycle of length 2 in (X, Y), with its mid-vertex not in Y, and with the endpoints connected with an edge of X, is not full in (X, Y) by the argument as in the previous example. Thus it remains to exclude existence of a cycle γ of length 2 in (X, Y), not contained in Y, with the endpoints not connected with an edge of X. Since the endpoints p, q of such γ are at distance 2 and contained in the residue of σ, they both are not in σ. Moreover, there is a vertex w of σ at distance 2 from the mid-vertex m of γ, since otherwise (due to flagness of X) m is in the residue. But then we get a full cycle of length 4 in X, passing through vertices p, m, q, w, p, which contradicts 5-largeness.
(4) By a *clique* in a flag simplicial complex X we mean the subcomplex spanned by any set of vertices that are pairwise connected with edges of X. The clique spanned by a finite set is a simplex, and no other cliques occur in finite dimensional complexes. An infinite dimensional complex may contain an *infinite clique*, i.e. the one spanned by an infinite vertex set. Clearly, infinite cliques are not simplices, though some of their properties are analogous to those of simplices. For example, the argument as in (2) above shows that any clique in a flag simplicial complex X is 3-convex in X.

Facts 3.2–3.4 below follow easily from the definitions.

3.2. *Fact.*

(1) The intersection of any family of 3-convex subcomplexes is a 3-convex subcomplex.
(2) If Q is 3-convex in X and L is 3-convex in Q then L is 3-convex in X.

(3) Let X be a flag simplicial complex and Q its 3-convex subcomplex. Then for any simplex σ of Q the link Q_σ is 3-convex in the link X_σ.

A subcomplex Q is *locally 3-convex* in X if for every nonempty simplex σ of Q the link Q_σ is 3-convex in the link X_σ. Note that this definition allows equalities $Q_\sigma = X_\sigma$ at simplices σ of Q.

3.3. *Fact.*

(1) Any 3-convex subcomplex in a flag complex X is a locally 3-convex subcomplex in X.
(2) The intersection of any family of locally 3-convex subcomplexes is a locally 3-convex subcomplex.

We now turn to convexity properties in k-large and locally k-large complexes. Since a full subcomplex of a k-large complex is k-large, we have

3.4. *Fact.*

(1) A 3-convex subcomplex of a k-large simplicial complex is k-large.
(2) A locally 3-convex subcomplex in a locally k-large simplicial complex is locally k-large.

A cycle γ is *homotopically trivial* in (X, Q) if there is a path η in Q connecting the endpoints of γ such that the loop $\gamma \cup \eta$ is contractible in X. In this definition we allow that η reduces to a point (when the endpoints of γ coincide). A cycle is *homotopically non-trivial* if it is not homotopically trivial. Note that a cycle connecting distinct components of Q is always homotopically non-trivial.

A *relative homotopical systole* for the pair (X, Q) of a simplicial complex and its subcomplex, denoted $sys_h(X, Q)$, is the length of the shortest homotopically non-trivial cycle in (X, Q). The next proposition shows that in locally 6-large simplicial complexes, 3-convexity can be characterized in terms of local 3-convexity and the relative homotopical systole.

3.5. *Proposition. — Let* X *be a locally 6-large simplicial complex and let* Q *be a full subcomplex of* X.

(1) If Q *is locally 3-convex in* X *and* $sys_h(X, Q) \geq 3$ *then* Q *is 3-convex.*
(2) The converse implication holds provided X *is flag.*

Proof. — To prove (2), take the shortest cycle γ homotopically nontrivial in (X, Q) and note that it intersects Q only at its endpoints. The length $|\gamma|$ of γ cannot be 1 since Q is full. If $|\gamma| = 2$ then γ is not full in (X, Q) due to

3-convexity of Q. Then either the endpoints of γ span an edge not contained in Q, which contradicts the fullness of Q, or otherwise the three vertices of γ span a 2-simplex in X, contradicting the fact that γ is homotopically nontrivial in (X, Q). Hence $sys_h(X, Q) \geq 3$. Since X is flag and Q is 3-convex, it is also locally 3-convex (Fact 3.3.1), and part (2) follows.

To prove part (1), suppose we have a length d full cycle γ in (X, Q), intersecting Q only at its endpoints. We have to prove that $d \geq 3$. If γ is homotopically nontrivial in (X, Q) we are done, since $sys_h(X, Q) \geq 3$. We therefore assume that γ is homotopically trivial in (X, Q). Thus there is a polygonal path η contained in Q, with the same endpoints as γ, such that the union $\gamma \cup \eta$ is a contractible loop in X. Moreover, η can be chosen so that it is of minimal length. In particular the closed polygonal path $\gamma \cup \eta$ is embedded in X. Beware that if the endpoints of γ coincide then η reduces to a single vertex.

By Lemma 1.6, there is a simplicial disc D filling the loop $\gamma \cup \eta$ in X. Among all choices of η and D, we pick one for which D has the smallest number of triangles (that may affect the choice of η). By Lemma 1.7 the interior vertices of D are contained in at least 6 triangles of D.

Every interior vertex of γ (viewed as the boundary vertex of D) is contained in at least two triangles of D, since γ is full in (X, Q). Every interior vertex v of η (viewed as the boundary vertex of D) is contained in at least 3 triangles of D. Indeed, if v is contained in one triangle of D, (the image of) the triangle is in Q (since Q is full), and η is not of minimal length. If v is contained in two triangles of D, they are both in Q by local 3-convexity and by minimality of η, and then D is not minimal. Finally, initial and terminal vertices of γ (which may coincide) are contained in at least one triangle.

Denote, as in Section 1, by $\chi(v)$ the number of triangles in D containing v. Let I, G, E denote the sets of interior vertices in D, γ and η respectively. Suppose that the endpoints of γ do not coincide, and denote them by a, b. Applying the inequalities we just established and the Gauss–Bonnet theorem we get

$$\begin{aligned} 1 &= \frac{1}{6}\Big[\sum_{v\in I}(6-\chi(v)) + \sum_{v\in G}(3-\chi(v)) \\ &\qquad + \sum_{v\in E}(3-\chi(v)) + 3 - \chi(a) + 3 - \chi(b)\Big] \\ &\leq \frac{1}{6}(0 + d - 1 + 0 + 4). \end{aligned}$$

Thus $3 \leq d$ as required.

Dealing similarly with the remaining case, in which the endpoints of γ coincide, we get even sharper estimate $4 \leq d$. Hence the proposition.

Proposition 3.5 allows to decide inductively if a subcomplex in a 6-large complex is 3-convex, by referring to 3-convexity of its links. The next three results apply this idea and give some criteria for 3-convexity. By *diameter* of a complex we mean the maximum distance between its vertices in the 1-skeleton of the complex.

3.6. *Lemma. — Let* Q *be a full locally 3-convex subcomplex in a 6-large complex* X *and suppose that* Q *is connected and* $diam(Q) \leq sys_h(X) - 3$. *Then* Q *is 3-convex in* X.

Proof. — With Proposition 3.5, it suffices to prove that $sys_h(X, Q) \geq 3$. Let γ be a homotopically nontrivial cycle in (X, Q). By the assumptions of the lemma, there is a polygonal path η of length $\leq sys_h(X) - 3$ contained in Q and with the same endpoints as γ. Moreover, the closed path $\gamma \cup \eta$ is homotopically nontrivial in X, and thus the length of this path is at least $sys_h(X)$ by Corollary 1.5. But this means that the length of γ is at least 3, which finishes the proof.

3.7. *Lemma. — Let* Q *be a full connected finite dimensional subcomplex in a 6-large simplicial complex* X. *Suppose that* $diam(Q) \leq sys_h(X) - 3$ *and that for each simplex* σ *of* Q *either* $Q_\sigma = X_\sigma$ *or* Q_σ *is connected with* $diam(Q_\sigma) \leq 3$. *Then* Q *is 3-convex in* X.

Proof. — Induction over the dimension of Q using Lemma 3.6.

Lemmas 3.6 and 3.7 immediately imply the following corollary in which a subcomplex Q may have infinite dimension.

3.8. *Corollary. — Let* Q *be a full connected subcomplex in a 6-large simplicial complex* X. *Suppose that* $diam(Q) \leq sys_h(X) - 3$ *and that for each simplex* σ *of* Q *either* $Q_\sigma = X_\sigma$ *or* Q_σ *is a finite dimensional connected complex with diameter* $diam(Q_\sigma) \leq 3$. *Then* Q *is 3-convex in* X.

4. Locally 3-convex maps and their applications

In this section we introduce the concept of a locally 3-convex map. We also use this concept to prove that finite dimensional 6-large simplicial complexes are aspherical, or equivalently, finite dimensional systolic complexes are contractible (Theorem 4.1.1, or Theorem B of Introduction). This result may be viewed as an analogue (for simplicial nonpositive curvature) of the Cartan–Hadamard theorem.

Although the notion of a locally 3-convex map makes sense in infinite dimensional case, its application in our arguments requires the additional assumption about finite dimension.

Given a nondegenerate simplicial map $f : Q \to X$ and a simplex $\sigma \in Q$, the *induced map on links* $f_\sigma : Q_\sigma \to X_{f(\sigma)}$ is the map obtained by restricting f to the link Q_σ (the image of this restriction is necessarily contained in the link $X_{f(\sigma)}$). We will say that a nondegenerate simplicial map $f : Q \to X$ is *locally injective*, if for any simplex $\sigma \subset Q$ the induced map f_σ is injective. Let X be a locally 6-large simplicial complex and Q an arbitrary simplicial complex. A nondegenerate locally injective simplicial map $f : Q \to X$ is *locally 3-convex*, if for each simplex $\sigma \subset Q$ the image $f_\sigma(Q_\sigma))$ is 3-convex in $X_{f(\sigma)}$ (in particular, $f_\sigma(Q_\sigma)$ may be the whole of $X_{f(\sigma)}$). Note that if $Q \subset X$ is a locally 3-convex subcomplex then the inclusion map is clearly locally 3-convex.

4.1. *Theorem. — Let* X *be a finite dimensional connected locally 6-large simplicial complex.*

(1) The universal cover $\widetilde{X}$ *of* X *is contractible. In particular, any finite dimensional systolic simplicial complex is contractible.*
(2) Suppose that Q *is a connected simplicial complex and* $f : Q \to X$ *is a locally 3-convex simplicial map. Then the induced homomorphism* $f_* : \pi_1 Q \to \pi_1 X$ *of fundamental groups is injective.*

To prove Theorem 4.1 we will use the fact that locally 3-convex maps can be extended to covering maps. We formulate this fact more precisely as Proposition 4.2 below, and then show how it implies the theorem. The proof of Proposition 4.2 occupies the last part of this section and it uses a technical result, Lemma 4.3, the proof of which we defer until Section 5.

4.2. *Proposition. — Suppose that* X *is a finite dimensional locally 6-large simplicial complex and let* $f : Q \to X$ *be a locally 3-convex map. Then* f *extends to a covering map* $f_e : Q_e \to X$ *in such a way that* Q *is a deformation retract of* Q_e.

Proof of Theorem 4.1. — A function $f : \{v\} \to X$ that sends a vertex v to a vertex of X is clearly locally 3-convex. By Proposition 4.2, it extends to a covering map $f_e : Y \to X$, where Y is contractible. This proves part (1).

To prove (2), note that by Proposition 4.2 the map f extends to a covering map $f_e : Q_e \to X$ such that the inclusion $Q \subset Q_e$ is a homotopy equivalence. Since a covering map induces a monomorphism of fundamental groups, the theorem follows.

The proof of Proposition 4.2 requires some preparations. Given a locally 3-convex map $f : Q \to X$, define

$$\partial_f Q := \{\sigma \in Q \,|\, f_\sigma : Q_\sigma \to X_{f(\sigma)} \text{ is not an isomorphism}\},$$

and observe that $\partial_f Q$ is a simplicial subcomplex of Q. $\partial_f Q$ can be thought of as a kind of boundary of Q relative to f, hence the notation. For a subcomplex

K of a simplicial complex L, denote by $N_L(K)$ the subcomplex of L being the union of all (closed) simplices that intersect K.

A *small extension* of a locally 3-convex map $f : Q \to X$ is a map $Ef : EQ \to X$ satisfying the following conditions:

(E1) EQ is a simplicial complex containing Q as a subcomplex and $N_{EQ}(Q) = EQ$;
(E2) Ef is a nondegenerate simplicial map that extends f;
(E3) for each simplex $\tau \in EQ$ that intersects Q the map $(Ef)_\tau : (EQ)_\tau \to X_{f(\tau)}$ is an isomorphism;
(E4) Ef is locally 3-convex;
(E5) Q is a deformation retract in EQ.

4.3. *Lemma. — Every locally 3-convex map* $f : Q \to X$ *to a finite dimensional locally 6-large simplicial complex* X *admits a small extension.*

We defer the proof of the lemma until Section 5 but show now how it implies Proposition 4.2.

Proof of Proposition 4.2. — Put $E^0 f = f$ and $E^0 Q = Q$. Define recursively a sequence of small extensions $E^j f : E^j Q \to X$ by $E^{j+1}Q = E(E^j Q)$ and $E^{j+1} f = E(E^j f)$. Put $Q_e := \bigcup_{j=0}^{\infty} E^j Q$ and $f_e := \bigcup_{j=0}^{\infty} E^j f$, thus getting a map $f_e : Q_e \to X$. Since by property (E3) of a small extension the induced map $(f_e)_\tau : (Q_e)_\tau \to X_{f_e(\tau)}$ is an isomorphism for each simplex $\tau \in Q_e$, it follows that f_e is a covering map. By property (E5), Q is contained in Q_e as a deformation retract, hence the proposition.

5. Existence of small extensions

This section is entirely devoted to the proof of Lemma 4.3.

We start with some definitions and notation. Given a finite dimensional locally 6-large simplicial complex X and a locally 3-convex map $f : Q \to X$, define the following family of pairs of simplices

$$\mathscr{E}_f := \{(\sigma, \tau) \in \partial_f Q \times X : \tau \subset X_{f(\sigma)}, \tau \cap f_\sigma(Q_\sigma) = \emptyset\}.$$

The motivation for considering the family $\mathscr{E}_f$ is as follows. Suppose we are given a small extension $Ef : EQ \to X$ of f. Then to any pair $(\sigma, \tau) \in \mathscr{E}_f$ there corresponds a simplex $(Ef)_\sigma^{-1}(\tau) \in (EQ)_\sigma \subset EQ$, which we denote shortly τ^σ. Moreover, we have $Ef(\tau^\sigma) = \tau$. This shows that pairs from $\mathscr{E}_f$ represent "germs" of the extension of f to Ef. In fact, we will construct a small extension Ef making use of the set $\mathscr{E}_f$. For this we also need the smaller family

$$\mathscr{E}_f^{max} := \{(\sigma, \tau) \in \mathscr{E}_f : \text{ there is no } \rho \supset \sigma \text{ with } (\rho, \tau) \in \mathscr{E}_f\}.$$

As we will see later, the elements of the set $\mathscr{E}_f^{max}$ will correspond bijectively, through the map $(\sigma, \tau) \to \tau^\sigma$, to the simplices disjoint with Q in the constructed small extension domain EQ.

The next lemma collects basic properties of the families $\mathscr{E}_f$ and $\mathscr{E}_f^{max}$.

5.1. *Lemma.*

(1) If $(\sigma, \tau) \in \mathscr{E}_f$ and $\rho \subset \sigma$ then $(\rho, \tau) \in \mathscr{E}_f$.
(2) If $(\sigma_i, \tau) \in \mathscr{E}_f$ for $i = 1, 2$ and $\sigma_1 \cap \sigma_2 \neq \emptyset$ then there is $\sigma \in Q$ containing both σ_1 and σ_2 such that $(\sigma, \tau) \in \mathscr{E}_f$.
(3) If $(\sigma_i, \tau) \in \mathscr{E}_f^{max}$ for $i = 1, 2$ and if $\sigma_1 \neq \sigma_2$ then $\sigma_1 \cap \sigma_2 = \emptyset$.
(4) Given $(\sigma, \tau) \in \mathscr{E}_f$, there exists a unique simplex $\pi_{\sigma,\tau} \subset \partial_f Q$ such that $\sigma \subset \pi_{\sigma,\tau}$ and $(\pi_{\sigma,\tau}, \tau) \in \mathscr{E}_f^{max}$.
(5) If $(\sigma, \tau) \in \mathscr{E}_f$ and $\rho \subset \sigma$ then $\pi_{\rho,\tau} = \pi_{\sigma,\tau}$.

In the proofs of Lemma 5.1 and of the remaining results in this section we will often use the following.

Notation.

(1) Given a simplex σ and its face ρ, we denote by $\sigma - \rho$ the face of σ spanned by all the vertices not contained in ρ.
(2) Given simplices σ, τ in a simplicial complex K, denote by $\sigma * \tau$ the simplex of K spanned by the union of the vertex sets of σ and τ. Note that in general such a simplex in K may not exist. We will speak of simplices of this form only when they exist.

Proof of Lemma 5.1. — To prove (1), consider first the case when τ is a 0-simplex (i.e. a vertex). Let $(\sigma, v) \in \mathscr{E}_f$, where v is a vertex, and let $\rho \subset \sigma$. If $(\rho, v) \notin \mathscr{E}_f$, it follows that $v \in f_\rho(Q_\rho)$. We also have $\sigma - \rho \subset f_\rho(Q_\rho)$, because $\sigma \subset Q$. On the other hand, the simplex $f(\sigma - \rho) * v \subset X_{f(\rho)}$ is not contained in $f_\rho(Q_\rho)$, because the simplex $f(\sigma) * v \subset X$ is not contained in $f(\sigma) * f_\sigma(Q_\sigma)$. This contradicts fullness of $f_\rho(Q_\rho) \subset X_{f(\rho)}$ (which holds by local 3-convexity of f). Thus the assertion follows in this case.

To deal with the other cases, suppose now that $(\sigma, \tau) \in \mathscr{E}_f$ and $\dim \tau \geq 1$. For any vertex v of τ we clearly have $(\sigma, v) \in \mathscr{E}_f$. It follows from what we have just proved for vertices that if $\rho \subset \sigma$ then $v \notin f_\rho(Q_\rho)$ for any vertex $v \in \tau$. Then clearly $\tau \cap f_\rho(Q_\rho) = \emptyset$ and thus $(\rho, \tau) \in \mathscr{E}_f$. This finishes the proof of (1).

To prove (2), we first show that the union of the vertices of σ_1 and σ_2 spans a simplex of Q. Put $\rho = \sigma_1 \cap \sigma_2$. Since Q_ρ is flag (because the isomorphic complex $f_\rho(Q_\rho)$ is 3-convex, and hence full, in $X_{f(\rho)}$ which is 6-large and hence flag), it is sufficient to show that there is an edge in Q_ρ between any two vertices $v_1 \in \sigma_1 - \rho$ and $v_2 \in \sigma_2 - \rho$. For an arbitrary vertex $t \in \tau$ we get polygonal path $f(v_1)tf(v_2)$ in $X_{f(\rho)}$, intersecting $f_\rho(Q_\rho)$ only at its endpoints. By 3-convexity of $f_\rho(Q_\rho)$ in $X_{f(\rho)}$, this path cannot be full in $(X_{f(\rho)}, f_\rho(Q_\rho))$, and hence there is an edge in $X_{f(\rho)}$ between $f(v_1)$ and $f(v_2)$. By the fact that $f_\rho(Q_\rho)$ is full in $X_{f(\rho)}$, this edge is in $f_\rho(Q_\rho)$, and thus v_1v_2 is an edge in Q_ρ.

Let σ be the simplex of Q spanned by the union of σ_1 and σ_2. We now show that $\tau \in X_{f(\sigma)}$ or equivalently that $f(\sigma)$ and τ span a simplex of X. For this it is sufficient to show that the three simplices τ, $f(\sigma - \sigma_1)$ and $f(\sigma - \sigma_2)$ span a simplex of $X_{f(\rho)}$. The latter follows from the fact that $X_{f(\rho)}$ is flag (since X is locally 6-large) and from the easy observation that the three simplices span the simplices of $X_{f(\sigma_1 \cap \sigma_2)}$ pairwise.

It remains to show that $\tau \cap f_\sigma(Q_\sigma) = \emptyset$, but this follows from the inclusion $f_\sigma(Q_\sigma) \subset f_{\sigma_1}(Q_{\sigma_1})$ and the assumption that $(\sigma_1, \tau) \in \mathcal{E}_f$. Thus we get $(\sigma, \tau) \in \mathcal{E}_f$, which completes the proof of (2).

Part (3) is a direct consequence of part (2). In view of the assumption that X is finite dimensional, (4) and (5) follow easily from (3).

We now start the construction of a small extension. Together with verification of conditions (E1)–(E5) from the definition, this construction occupies the rest of this section.

Simplicial complex EQ. — As the vertex set of EQ take the (disjoint) union of the vertex set of Q and the set $\{(\sigma, v) \in \mathcal{E}_f^{max} : v \text{ is a vertex}\}$. For any pair $(\sigma, \tau) \in \mathcal{E}_f$ let $\delta_{\sigma,\tau}$ be the simplex spanned by the set consisting of all vertices in σ and all vertices of form $(\pi_{\sigma,t}, t)$, where t is a vertex of τ. Define EQ to be the union of Q and the simplices $\delta_{\sigma,\tau}$ for all $(\sigma, \tau) \in \mathcal{E}_f$.

It is immediate from the above description that $Q \subset EQ$ and $N_{EQ}(Q) = EQ$, i.e. that the constructed complex EQ satisfies condition (E1) in the definition of a small extension. The next fact collects some more detailed properties of the complex EQ, useful for later arguments in this section.

5.2. *Fact.*

(1) The simplices of EQ with all vertices in Q are exactly the simplices of Q. In other words, Q is a full subcomplex in EQ.
(2) The simplices of EQ with part of vertices in Q and part of vertices outside Q are exactly the simplices $\delta_{\sigma,\tau} : (\sigma, \tau) \in \mathcal{E}_f$. Moreover, for distinct pairs $(\sigma, \tau) \in \mathcal{E}_f$ the corresponding simplices $\delta_{\sigma,\tau}$ are distinct.
(3) The simplices of EQ disjoint with Q are exactly the simplices $\delta_{\sigma,\tau} - \sigma$: $(\sigma, \tau) \in \mathcal{E}_f^{max}$.

(4) If $\sigma_1 \subset \sigma_2$ and $(\sigma_i, \tau) \in \mathcal{E}_f$ for $i = 1, 2$ then the corresponding simplices $\delta_{\sigma_i,\tau} - \sigma_i$ coincide.

(5) For distinct pairs $(\sigma, \tau) \in \mathcal{E}_f^{max}$ the corresponding simplices $\delta_{\sigma,\tau} - \sigma$ are distinct. Moreover, if $(\sigma_i, \tau) \in \mathcal{E}_f^{max}$ for $i = 1, 2$ and $\sigma_1 \neq \sigma_2$ (which by Lemma 5.1.3 means that these simplices σ_i are disjoint) then the corresponding simplices $\delta_{\sigma_i,\tau}$ are also disjoint.

(6) Complex EQ is the union of Q and the family of (closed) simplices $\delta_{\sigma,\tau} : (\sigma, \tau) \in \mathcal{E}_f^{max}$.

Proof. — All parts except (5) follow easily from the description of EQ. To prove (5), suppose that $(\sigma_i, \tau_i) : i = 1, 2$ are distinct pairs from $\mathcal{E}_f^{max}$. If $\tau_1 \neq \tau_2$ then the sets of vertices of the simplices $\delta_{\sigma_i,\tau_i} - \sigma_i : i = 1, 2$ are easily seen to be distinct. If $\tau_1 = \tau_2$ then $\sigma_1 \neq \sigma_2$, and we are in the assumptions of the second statement in (5). Since we know that then $\sigma_1 \cap \sigma_2 = \emptyset$, it is sufficient to show that the simplices $\delta_{\sigma_i,\tau_i} - \sigma_i$ are disjoint for $i = 1, 2$. For brevity, put $\tau := \tau_1 = \tau_2$, and let $t \in \tau$ be a vertex. We will show that the vertex $(\pi_{\sigma_2,t}, t) \in \delta_{\sigma_2,\tau} - \sigma_2$ is not a vertex of the simplex $\delta_{\sigma_1,\tau} - \sigma_1$, which is clearly sufficient for completing the proof of (5). The vertices in $\delta_{\sigma_1,\tau_1} - \sigma_1$ other than $(\pi_{\sigma_1,t}, t)$ are distinct from $(\pi_{\sigma_2,t}, t)$, since their projections to X differ from t. It thus remains to show that $(\pi_{\sigma_1,t}, t) \neq (\pi_{\sigma_2,t}, t)$, i.e. that $\pi_{\sigma_1,t} \neq \pi_{\sigma_2,t}$. Suppose that the latter is not true and $\pi_{\sigma_1,t} = \pi_{\sigma_2,t}$. Then $\sigma_1 * \sigma_2$ is a simplex of $\partial_f Q$, since both σ_1 and σ_2 are contained in $\pi_{\sigma_1,t}$. We then have $f(\sigma_1) * (\tau - t) \subset X_t$, $f(\sigma_2) * (\tau - t) \subset X_t$ and $f(\sigma_1) * f(\sigma_2) = f(\sigma_1 * \sigma_2) \subset X_t$. Since the link X_t is flag (because X is locally 6-large), it follows that $f(\sigma_1 * \sigma_2) * (\tau - t) \subset X_t$, and hence $(\sigma_1 * \sigma_2, \tau) \in \mathcal{E}_f$. This contradicts any of the assumptions $(\sigma_i, \tau_i) \in \mathcal{E}_f^{max}$ thus completing the proof.

Simplicial map $Ef : EQ \to X$. — Define Ef by putting first $Ef|_Q = f$ and $Ef((\sigma, v)) = v$ for all vertices (σ, v), and then extending simplicially. Observe that since in this way the vertices of any simplex $\delta_{\sigma,\tau}$ are mapped bijectively to the vertices of the simplex $f(\sigma) * \tau \subset X$, the simplicial map $Ef : EQ \to X$ is both well defined and nondegenerate, hence it fulfills condition (E2) of a small extension.

Passing to condition (E3), note that if $g : K \to L$ is a nondegenerate simplicial map, and if for some vertex $v \in K$ the induced map $g_v : K_v \to L_{g(v)}$ is an isomorphism, then for any simplex $\sigma \subset K$ containing v the map $g_\sigma : K_\sigma \to L_{g(\sigma)}$ is also an isomorphism. It is then sufficient to prove that $(Ef)_v : (EQ)_v \to X_{f(v)}$ is an isomorphism for any vertex $v \in Q$. This fact is immediate for all vertices v of Q not contained in $\partial_f Q$, since for them we have $(EQ)_v = Q_v$ and $(Ef)_v = f_v$. It remains to prove this fact for vertices $v \in \partial_f Q$.

A nondegenerate simplicial map is an isomorphism if it is bijective on the vertex sets and surjective. We now check those two properties for the map $(Ef)_v$ with any vertex $v \in \partial_f Q$.

Given $v \in \partial_f Q$, the simplices of EQ that contain v are either contained in Q or have a form $\delta_{\sigma,\tau}$ with $(\sigma,\tau) \in \mathcal{E}_f$ and $v \in \sigma$. Thus, the vertices of $(EQ)_v$ are either contained in Q_v or are the vertices other than v in 1-simplices $\delta_{v,w}$ (for all $(v,w) \in \mathcal{E}_f$ with w a vertex). The latter vertices are the vertices $(\pi_{v,w}, w) \in \mathcal{E}_f^{max}$. Vertices of Q_v are mapped by $(Ef)_v$ bijectively on the vertices of $f_v(Q_v)$, while the vertices $(\pi_{v,w}, w)$ are mapped bijectively to the vertices $w \in X_{f(v)}$ not contained in $f_v(Q_v)$. Thus the map $(Ef)_v : (EQ)_v \to X_{f(v)}$ is bijective on the vertex sets.

To prove surjectivity of the map $(Ef)_v$, choose any simplex ρ in the link $X_{f(v)}$. We need to show that ρ is in the image of $(Ef)_v$. If $\rho \subset f_v(Q_v)$, there is nothing to show. Otherwise, put $\rho_0 := \rho \cap f_v(Q_v)$. Since, by local 3-convexity of f, $f_v(Q_v)$ is a full subcomplex of $X_{f(v)}$, ρ_0 is either empty or a single proper face of ρ. We then clearly have $(v, \rho - \rho_0) \in \mathcal{E}_f$, and we deduce that $(v * f^{-1}(\rho_0), \rho - \rho_0) \in \mathcal{E}_f$. Since clearly $Ef((\delta_{v*f^{-1}(\rho_0), \rho-\rho_0})) = \rho * f(v)$, it follows that ρ is in the image of $(Ef)_v$ as required.

Local 3-convexity of Ef. — Since, according to (E3), the map $(Ef)_\delta : (EQ)_\delta \to X_{f(\delta)}$ is an isomorphism for any simplex $\delta \subset EQ$ that intersects Q, the local 3-convexity condition for Ef is fulfilled at such simplices. Thus to establish (E4), it remains to check that for any simplex ρ in EQ disjoint with Q the induced map $(Ef)_\rho : (EQ)_\rho \to X_{f(\rho)}$ is injective and the subcomplex $(Ef)_\rho((EQ)_\rho)$ is 3-convex in the link $X_{Ef(\rho)}$. For this we need the following.

5.3. *Lemma.* — *Given a simplex ρ in* EQ *disjoint with* Q*, let $\rho = \delta_{\sigma,\tau} - \sigma$ for the appropriate $(\sigma,\tau) \in \mathcal{E}_f^{max}$ (as in Fact 5.2.3). Then $\sigma \subset (EQ)_\rho$ and $N_{(EQ)_\rho}(\sigma) = (EQ)_\rho$.*

The proof of Lemma 5.3 requires the following.

5.4. *Claim.* — Under assumptions of Lemma 5.3, the residue $\mathrm{Res}(\rho, EQ)$ is equal to the union U of the simplices δ_{σ_0,τ_0} such that $(\sigma_0,\tau_0) \in \mathcal{E}_f$, $\sigma_0 \subset \sigma$ and $\tau \subset \tau_0$.

Proof. — The inclusion $U \subset \mathrm{Res}(\rho, EQ)$ is easy in view of Fact 5.2.4. To get the converse inclusion, denote by π an arbitrary simplex in EQ that contains ρ. By the construction of EQ, π is contained in a simplex $\delta_{\sigma',\tau'}$ for some $(\sigma',\tau') \in \mathcal{E}_f$. Looking at vertices not contained in Q in $\delta_{\sigma,\tau}$ and $\delta_{\sigma',\tau'}$, we conclude that $\tau \subset \tau'$. Then $(\sigma',\tau) \in \mathcal{E}_f$ and consequently $(\pi_{\sigma',\tau}, \tau) \in \mathcal{E}_f^{max}$. Since we have also $(\sigma,\tau) \in \mathcal{E}_f^{max}$, Lemma 5.1.3 implies that either $\pi_{\sigma',\tau} = \sigma$ or $\pi_{\sigma',\tau} \cap \sigma = \emptyset$. In the first of these two cases we have $\sigma' \subset \pi_{\sigma',\tau} = \sigma$ and thus $\rho \subset \delta_{\sigma',\tau'}$, $(\sigma',\tau') \in \mathcal{E}_f$, $\tau \subset \tau'$ and $\sigma' \subset \sigma$. Hence $\pi \subset U$. The case of $\pi_{\sigma',\tau} \cap \sigma = \emptyset$ is in fact impossible, since if it holds then the argument as in the proof of the second statement in Fact 5.2.5 shows that the simplices $\delta_{\sigma,\tau}$ and $\delta_{\pi_{\sigma',\tau},\tau}$ are disjoint, and thus cannot both contain ρ. Hence the claim.

For later application in Section 7 we state here an observation immediately implied by Claim 5.4.

5.5. *Corollary.* — *Under assumptions of Lemma 5.3, the intersection* $\mathrm{Res}(\rho, \mathrm{EQ}) \cap \mathrm{Q}$ *is equal to* σ.

Proof of Lemma 5.3. — A simplex δ_{σ_0,τ_0} as in the claim determines the simplex $\delta_{\sigma_0,\tau_0} - \rho$ in the link $(\mathrm{EQ})_\rho$. The claim implies that $(\mathrm{EQ})_\rho$ is the union of such simplices $\delta_{\sigma_0,\tau_0} - \rho$. Since any such simplex shares a face with the simplex σ, namely the face σ_0, it follows that $\mathrm{N}_{(\mathrm{EQ})_\rho}(\sigma) = (\mathrm{EQ})_\rho$, as expected.

We are now ready to prove that the map $\mathrm{E}f$ is locally injective, a first step in showing its local 3-convexity. The next lemma establishes much stronger local property of $\mathrm{E}f$ which will be referred to in later parts of the paper.

5.6. *Proposition.* — *Given a simplex* $\rho = \delta_{\sigma,\tau} - \sigma$ *with* $(\sigma, \tau) \in \mathcal{E}_f^{max}$, *the induced map* $(\mathrm{E}f)_\rho$ *maps the link* $(\mathrm{EQ})_\rho$ *isomorphically onto the subcomplex* $\mathrm{N}_{\mathrm{X}_{\mathrm{E}f(\rho)}}(f(\sigma))$ *in the link* $\mathrm{X}_{\mathrm{E}f(\rho)}$. *In particular, this map is injective.*

Proof. — The proof relies on the following general observation which we state without proof.

Claim. — Let K be a simplicial complex, $\pi \subset \mathrm{K}$ a simplex, and suppose that $\mathrm{N}_\mathrm{K}(\pi) = \mathrm{K}$. Furthermore, let L be a flag simplicial complex and $h : \mathrm{K} \to \mathrm{L}$ a nondegenerate simplicial map. If for any simplex $\alpha \subset \pi$ the induced map $h_\alpha : \mathrm{K}_\alpha \to \mathrm{L}_{h(\alpha)}$ is an isomorphism, then h maps K isomorphically on the subcomplex $\mathrm{N}_\mathrm{L}(h(\pi))$.

We now check that putting $\mathrm{K} = (\mathrm{EQ})_\rho$, $\pi = \sigma$, $\mathrm{L} = \mathrm{X}_{\mathrm{E}f(\rho)}$ and $h = (\mathrm{E}f)_\rho$, all the assumptions in the claim are satisfied. The fact that $\mathrm{N}_\mathrm{K}(\pi) = \mathrm{K}$ follows from Lemma 5.3. The map $h = (\mathrm{E}f)_\rho$ is nondegenerate because, by condition (E2), so is $\mathrm{E}f$. The complex $\mathrm{L} = \mathrm{X}_{\mathrm{E}f(\rho)}$ is flag because X is locally 6-large. It remains to check the properties of the induced maps $h_\alpha = ((\mathrm{E}f)_\rho)_\alpha : ((\mathrm{EQ})_\rho)_\alpha \to (\mathrm{X}_{\mathrm{E}f(\rho)})_{(\mathrm{E}f)_\rho(\alpha)}$.

Observe that we have the identifications $((\mathrm{EQ})_\rho)_\alpha = (\mathrm{EQ})_{\rho*\alpha}$, $(\mathrm{X}_{\mathrm{E}f(\rho)})_{(\mathrm{E}f)_\rho(\alpha)} = \mathrm{X}_{\mathrm{E}f(\rho*\alpha)}$ and $((\mathrm{E}f)_\rho)_\alpha = (\mathrm{E}f)_{\rho*\alpha}$. The fact that $((\mathrm{E}f)_\rho)_\alpha$ is an isomorphism follows then from the already proved property (E3) for $\mathrm{E}f$, by realizing that the simplex $\rho * \alpha$ intersects Q at α. Thus, by applying the claim, the proposition follows.

In order to prove that the map $\mathrm{E}f$ is locally 3-convex it now remains to prove that, under notation of Proposition 5.6, the image complex $(\mathrm{E}f)_\rho((\mathrm{EQ})_\rho)$

is 3-convex in the link $X_{f(\rho)}$. We do this by referring to Lemma 3.7. By Proposition 5.6, the image complex $(Ef)_\rho((EQ)_\rho)$ is the neighbourhood of some simplex in $X_{(Ef)_\rho}$. The fact that any subcomplex of this form is full in the corresponding 6-large complex follows by arguments similar to those in Example 3.1.3 (we omit them). Proposition 5.6 implies also that the subcomplex $(Ef)_\rho((EQ)_\rho)$ is connected and that $diam[(Ef)_\rho((EQ)_\rho)] \leq 3$. Since the links of the complex $(Ef)_\rho((EQ)_\rho)$ are isomorphic to the complexes $(Ef)_{\rho'}((EQ)_{\rho'})$ for appropriate simplices $\rho' \supset \rho$, it follows that $(Ef)_\rho((EQ)_\rho)$ satisfies the assumptions of Lemma 3.7, which completes the proof of property (E4) for Ef.

Deformation retraction. — Put

$$Q^i := Q \cup \bigcup \{\delta_{\sigma,\tau} : (\sigma, \tau) \in \mathcal{E}_f^{max}, \dim \tau < i\}.$$

Let $\dim X = n$. Then the dimension of any simplex τ such that $(\sigma, \tau) \in \mathcal{E}_f^{max}$ is not greater than $n - 1$. Consequently, by Fact 5.2.6, we get

$$Q = Q^0 \subset Q^1 \subset ... \subset Q^n = EQ.$$

We will show that Q^i is a deformation retract of Q^{i+1} for $i = 0, 1, ..., n-1$, which clearly implies that Q is a deformation retract of EQ.

5.7. *Lemma.* — *Let* $(\sigma, \tau) \in \mathcal{E}_f^{max}$ *and* $\dim \tau = i$. *Then, denoting* $\tau^\sigma = \delta_{\sigma,\tau} - \sigma$, *we have*

(1) $\delta_{\sigma,\tau} \cap Q^i = \sigma * \partial\tau^\sigma$, *where* $\partial\tau^\sigma$ *is the ordinary boundary subcomplex of the simplex* τ^σ*;*
(2) $\delta_{\sigma,\tau} \setminus Q^i$ *is a connected component in* $Q^{i+1} \setminus Q^i$.

Proof. — By definition, Q^i is a subcomplex of EQ consisting of all those simplices of EQ which have at most i vertices outside Q. Thus $\delta_{\sigma,\tau} \cap Q^i$ consists of those faces of $\delta_{\sigma,\tau}$ which have at most i vertices outside Q. Since $\delta_{\sigma,\tau} = \sigma * \tau^\sigma$, $\delta_{\sigma,\tau} \cap Q = \sigma$ and $\dim \tau^\sigma = \dim \tau = i$, this easily implies (1).

To prove (2), it is sufficient to show that for any $(\sigma', \tau') \in \mathcal{E}_f^{max}$ with $\dim \tau' = i$, distinct from (σ, τ), we have $(\delta_{\sigma,\tau} \setminus Q^i) \cap (\delta_{\sigma',\tau'} \setminus Q^i) = \emptyset$. Suppose this is not true and consequently $\delta_{\sigma,\tau} \cap \delta_{\sigma',\tau'}$ is not contained in $\sigma * \partial\tau^\sigma$. Then $\tau^\sigma \subset \delta_{\sigma,\tau} \cap \delta_{\sigma',\tau'}$, and in fact τ^σ has to be a face in $(\tau')^{\sigma'}$, because the vertices of τ^σ are all outside Q. Since $\dim \tau^\sigma = \dim(\tau')^{\sigma'}$ (they are both equal to i), we have $\tau^\sigma = (\tau')^{\sigma'}$. In view of Fact 5.2.5 this implies that $(\sigma, \tau) = (\sigma', \tau')$, which contradicts the assumption that these pairs are distinct. Thus the lemma follows.

To finish the proof that Q^i is a deformation retract of Q^{i+1} observe that, in view of Lemma 5.7.2, deformation retraction of Q^{i+1} onto Q^i can be composed out of independently performed deformation retractions of simplices $\delta_{\sigma,\tau}$ (for

$(\sigma, \tau) \in \mathcal{E}_f^{max}$ and $\dim \tau = i$) onto their intersections with Q^i. The existence of the latter deformation retractions is implied by Lemma 5.7.1 and the elementary fact that $\sigma * \partial\tau$ is a deformation retract of $\sigma * \tau$. Since this gives the last condition (E5) from the definition of a small extension, the proof of Lemma 4.3 is completed.

6. Locally 6-large simplicial complexes of groups

In this section we sketch the necessary background for and the proof of the following.

6.1. *Theorem. — Every connected, locally 6-large, finite dimensional simplicial complex of groups is developable.*

Theorem 6.1 allows to construct locally 6-large simplicial complexes by means of complexes of groups. We will extensively exploit this possibility in our constructions in Sections 18–20.

The proof of Theorem 6.1 is based on a version of Proposition 4.2 for locally 3-convex maps to locally 6-large simplicial complexes of groups, and it is very similar to the proof of Theorem 4.1.1.

We refer the reader to [BH] for details related to the notion of a complex of groups. We refer also to Section 17 of this paper for an easier exposition of a special case, namely a simplex of groups.

For a simplicial complex X, let $\mathcal{X}$ be the *scwol* (small category without loops, as defined in [BH, p. 520]) related to the barycentric subdivision of X, defined as follows. A vertex set $\mathcal{V} = \mathcal{V}(\mathcal{X})$ of $\mathcal{X}$ consists of simplices σ of X and a set $\mathcal{E} = \mathcal{E}(\mathcal{X})$ of directed edges of $\mathcal{X}$ consists of pairs $a = (\tau, \sigma)$ such that σ is a proper face of τ (i.e. $\sigma \subset \tau$ and $\sigma \neq \tau$).

A *complex of groups* $G(\mathcal{X}) = (\{G_\sigma\}, \{\psi_{\sigma\tau}\}, \{g_{\sigma\tau\rho}\})$ over a simplicial complex X is given by the following data (cf. [BH, p. 535, Definition 2.1]):

(1) for each $\sigma \in \mathcal{V}$ a group G_σ called the *local group* at σ;
(2) for each $(\tau, \sigma) \in \mathcal{E}$ an injective homomorphism $\psi_{\sigma\tau} : G_\tau \to G_\sigma$;
(3) for each triple $\sigma \subset \tau \subset \rho$ of simplices with $\sigma \neq \tau \neq \rho$ a *twisting element* $g_{\sigma\tau\rho} \in G_\sigma$

with the following compatibility conditions:

(**i**) $$\mathrm{Ad}(g_{\sigma\tau\rho})\psi_{\sigma\rho} = \psi_{\sigma\tau}\psi_{\tau\rho},$$

where $\mathrm{Ad}(g_{\sigma\tau\rho})$ is the conjugation by $g_{\sigma\tau\rho}$ in G_σ, and

(**ii**) $$\psi_{\sigma\tau}(g_{\tau\rho\pi})g_{\sigma\tau\pi} = g_{\sigma\tau\rho}g_{\sigma\rho\pi}$$

for each $\sigma \subset \tau \subset \rho \subset \pi$ with $\sigma \neq \tau \neq \rho \neq \pi$.

Remark. — For many purposes (e.g. for our considerations in Sections 17–20) it is sufficient to deal with the so called *simple* complexes of groups, for which all the twisting elements are trivial. We may then speak of a complex of groups $G(\mathcal{X}) = (\{G_\sigma\}, \{\psi_{\sigma\tau}\})$ consisting of local groups G_σ and injective homomorphisms $\psi_{\sigma\tau}$. Since the compatibility condition (**i**) reads then as $\psi_{\sigma\rho} = \psi_{\sigma\tau}\psi_{\tau\rho}$, we may view the homomorphisms $\psi_{\sigma\tau}$ as inclusions of subgroups.

Let $G(\mathcal{X})$ be a complex of groups over a simplicial complex X, and let σ be a simplex of X. For any simplex $\tau \in X_\sigma$ put $G^\sigma_\tau := \psi_{\sigma(\tau*\sigma)}(G_{\tau*\sigma}) \subset G_\sigma$. A *link* of $G(\mathcal{X})$ at σ, denoted $L(G(\mathcal{X}), \sigma)$ is a complex defined by

$$L(G(\mathcal{X}), \sigma) := \Big[\bigcup_{\tau \in X_\sigma} \tau \times \big(G_\sigma / G^\sigma_\tau\big)\Big] / \sim,$$

where the equivalence relation $\sim$ is determined by the maps $(\tau_1, g_1 G^\sigma_{\tau_1}) \to (\tau_2, g_2 G^\sigma_{\tau_2})$ induced by inclusions on first coordinates, for all simplices $\tau_1 \subset \tau_2 \in X_\sigma$ and for all $g_1, g_2 \in G_\sigma$ such that $g_1 G^\sigma_{\tau_1} = g_2 g^{-1}_{\sigma(\tau_1*\sigma)(\tau_2*\sigma)} G^\sigma_{\tau_1}$ (cf. [BH, p. 564, Section 4.20]).

Remark. — Simplices (τ, gG^σ_τ) map injectively into the link $L(G(\mathcal{X}), \sigma)$. Nevertheless, $L(G(\mathcal{X}), \sigma)$ needn't be a simplicial complex in the strict sense, since it may contain double edges.

Link $L(G(\mathcal{X}), \sigma)$ carries a natural action of the group G_σ, defined by $g(x, g' G^\sigma_\tau) = (x, gg' G^\sigma_\tau)$. There is a G_σ-invariant map $p_\sigma : L(G(\mathcal{X}), \sigma) \to X_\sigma$ defined by $p_\sigma(x, gG^\sigma_\tau) = x$, which is nondegenerate (i.e. injective on each simplex) and induces an isomorphism $G_\sigma \backslash L(G(\mathcal{X}), \sigma) \to X_\sigma$.

6.2. *Definition.* — *A complex of groups* $G(\mathcal{X})$ *over a simplicial complex* X *is* locally 6-large, *if for each simplex* σ *of* X *the link* $L(G(\mathcal{X}), \sigma)$ *is a 6-large simplicial complex.*

The above definition makes the statement of Theorem 6.1 precise. Our method of proof requires the notion of a locally 3-convex map to a locally 6-large complex of groups.

Let Q be a simplicial complex and $G(\mathcal{X})$ a locally 6-large complex of groups over a simplicial complex X. A *map* of Q to $G(\mathcal{X})$ consists of a nondegenerate simplicial map $f : Q \to X$ (which induces in the obvious way maps $\mathcal{V}(\mathcal{Q}) \to \mathcal{V}(\mathcal{X})$ and $\mathcal{E}(\mathcal{Q}) \to \mathcal{E}(\mathcal{X})$, denoted also by f, for the associated scwols $\mathcal{Q}$ and $\mathcal{X}$), and a family $\phi(\tau, \sigma) : (\tau, \sigma) \in \mathcal{E}(\mathcal{Q})$ of elements $\phi(\tau, \sigma) \in G_{f(\sigma)}$, such that

$$\phi(\rho, \sigma) = \phi(\tau, \sigma)\psi_{f(\sigma)f(\tau)}(\phi(\rho, \tau))g_{f(\sigma)f(\tau)f(\rho)} \text{ for } \sigma \subset \tau \subset \rho.$$

Remark. — The above notion of map to a simplicial complex of groups is a special case of the notion of morphism for complexes of groups, cf. [BH, p. 536, Definition 2.4]. It is obtained by viewing a simplicial complex Q as the trivial simplicial complex of groups over Q (i.e. a complex with trivial local groups, homomorphisms and twisting elements).

For any simplex $\sigma \in Q$ a map $(f,\phi) : Q \to G(\mathscr{X})$ induces the map $(f,\phi)_\sigma : Q_\sigma \to L(G(\mathscr{X}), f(\sigma))$ of links, defined by

$$(f,\phi)_\sigma(\tau) = \big(f(\tau), \phi(\sigma, \sigma * \tau) G_{f(\tau)}^{f(\sigma)}\big)$$

(compare [BH, p. 565, Proposition 4.23]).

6.3. *Definition.* — *Let* $G(\mathscr{X})$ *be a locally 6-large simplicial complex of groups. A map* $(f,\phi) : Q \to G(\mathscr{X})$ *is* locally 3-convex *if for each simplex* $\sigma \in Q$ *the induced map* $(f,\phi)_\sigma : Q_\sigma \to L(G(\mathscr{X}), f(\sigma))$ *is injective and the image* $(f,\phi)_\sigma(Q_\sigma)$ *is 3-convex in the link* $L(G(\mathscr{X}), f(\sigma))$. *A map* $(f,\phi) : Q \to G(\mathscr{X})$ *is a* covering, *if for each simplex* $\sigma \in Q$ *the induced map* $(f,\phi)_\sigma : Q_\sigma \to L(G(\mathscr{X}), f(\sigma))$ *is an isomorphism.*

We now state a result that generalizes Proposition 4.2 to the case of locally 3-convex maps to simplicial complexes of groups.

6.4. *Proposition.* — *Let* $(f,\phi) : Q \to G(\mathscr{X})$ *be a locally 3-convex map of a simplicial complex* Q *to a locally 6-large finite dimensional simplicial complex of groups* $G(\mathscr{X})$. *Then* (f,ϕ) *extends to a covering map* $(f_e, \phi_e) : Q_e \to G(\mathscr{X})$ *in such a way that* Q *is a deformation retract of* Q_e.

The proof of the above proposition goes along the same lines as the proof of Proposition 4.2. The objects $\partial_f Q$ and $\mathscr{E}_f$ occurring in the latter proof (especially in the construction of a small extension for a convex map f in Section 5) have to be replaced by the objects $\partial_{(f,\phi)}Q$ and $\mathscr{E}_{(f,\phi)}$ defined in an analogous way as follows. $\partial_{(f,\phi)}Q$ is the subcomplex of Q consisting of all those simplices $\sigma \subset Q$ for which the induced map $(f,\phi)_\sigma : Q_\sigma \to L(G(\mathscr{X}), f(\sigma))$ is not an isomorphism. $\mathscr{E}_{(f,\phi)}$ is the set of all pairs (σ, τ) such that $\sigma \subset \partial_{(f,\phi)}Q$, $\tau \subset L(G(\mathscr{X}), f(\sigma))$ and $\tau \cap (f,\phi)_\sigma(Q_\sigma) = \emptyset$. We omit details.

Proof of Theorem 6.1. — Let $G(\mathscr{X})$ be a simplicial complex of groups over a connected finite dimensional simplicial complex X and suppose it is locally 6-large. We have to show that $G(\mathscr{X})$ is developable.

Denote by $\{v\}$ the simplicial complex consisting of a single vertex v. A map $i : \{v\} \to X$ that sends v to any vertex of X may be viewed as a locally 3-convex map of $\{v\}$ to $G(\mathscr{X})$ (the family $\phi(\tau,\sigma) : (\tau,\sigma) \in \mathscr{E}(\{v\})$ is then empty). By

Proposition 6.4, the map i extends to a covering map $(h, \psi) : Y \to G(\mathscr{X})$, with Y that retracts on v and thus is contractible. In particular, (h, ψ) is the universal covering of $G(\mathscr{X})$.

Let Γ be the group of deck-transformations of the covering (h, ψ). The elements of Γ are the simplicial automorphisms $\gamma : Y \to Y$ which satisfy the following two conditions:

(1) the map $h \circ \gamma : Y \to X$ and the family $\psi \circ \gamma(\tau, \sigma) : (\tau, \sigma) \in \mathscr{E}(\mathscr{Y})$ describe a well defined map $(h \circ \gamma, \psi \circ \gamma)$ from Y to $G(\mathscr{X})$;

(2) γ preserves the projection h, i.e. $h \circ \gamma = h$.

By the properties of the universal covering, $G(\mathscr{X})$ is isomorphic to the complex of groups associated to the action of Γ on Y and hence it is developable. This finishes the proof.

7. Systolic complexes and their convex subcomplexes

Recall that a simplicial complex X is *systolic* if it is locally 6-large, connected and simply connected. In this section, as in the remaining part of the paper, we assume that X is finite dimensional. We start the systematic study of systolic complexes, by introducing the notions of convexity and strong convexity, and deriving their basic properties.

7.1. *Definition. — A subcomplex* Q *in a systolic complex* X *is* convex *if it is connected and locally 3-convex.*

Note that, by Fact 3.3.1, any connected 3-convex subcomplex of X is convex. In particular, any simplex and any residue in a systolic complex is convex (see Example 3.1.2 and 3.1.3).

7.2. *Lemma. — Let* Q *be a convex subcomplex of a systolic complex* X. *Then*

(1) Q *is contractible;*
(2) Q *is full in* X*;*
(3) Q *is 3-convex in* X*;*
(4) if X *is k-systolic (for some $k \geq 6$) then* Q *is k-systolic.*

Proof. — In view of contractibility of X (Theorem 4.1.1), (1) follows from Proposition 4.2 applied to the inclusion map $Q \to X$. By Proposition 4.2 (and its proof), X is isomorphic to the complex Q_e obtained from Q by the infinite sequence of small extensions. Moreover, the full simplicial span of Q in X is clearly contained in EQ. Together with Fact 5.2.1 (which says that Q is full in EQ), this implies (2). By contractibility of X and connectedness of Q, there is no homotopically nontrivial cycle in (X, Q) and thus $sys_h(X, Q) = \infty$. Together with Proposition 3.5.1, this implies (3). In view of Fact 1.2.2, part (4) follows from (1) and (2).

The next lemma describes small extensions of (the inclusion maps of) convex subcomplexes.

7.3. *Lemma.* — *Let* $f : Q \to X$ *be the inclusion map of a convex subcomplex* Q *in a systolic complex* X. *Then any small extension* $Ef : EQ \to X$ *maps* EQ *isomorphically to the subcomplex* $N_X(Q) \subset X$. *Thus* EQ *can be identified with the subcomplex* $N_X(Q)$ *and* Ef *with the inclusion map* $N_X(Q) \to X$.

Proof. — According to Proposition 4.2 and its proof, a small extension $Ef :$ $EQ \to X$ can be further extended to a covering map $\tilde{f} : Y \to X$, in such a way that Q is a deformation retract of Y. One easily observes that then $EQ = N_Y(Q)$ and $Ef = \tilde{f}|_{EQ}$. Since X is simply connected and Y connected, the covering map $\tilde{f}$ is an isomorphism. Hence the lemma.

7.4. *Corollary.* — *Let* Q *be a convex subcomplex in a systolic complex* X. *Then*

(1) the subcomplex $N_X(Q)$ *is also convex in* X*;*
(2) Q *is a deformation retract of the neighborhood* $N_X(Q)$.

Proof. — In view of Lemma 7.3, it follows from condition (E4) of a small extension that the neighborhood $N_X(Q)$ is locally 3-convex in X. Since it is also connected, part (1) of the corollary follows. Part (2) follows similarly from condition (E5).

Given a convex subcomplex Q in a systolic complex X, define a system $B_n = B_n(Q, X)$ of combinatorial *balls* in X of radii n centered at Q as $B_0 := Q$ and $B_{n+1} := N_X(B_n)$ for $n \geq 0$. From Corollary 7.4 and Lemma 7.2 we get

7.5. *Corollary.* — *Let* Q *be a convex subcomplex in a systolic complex* X. *Then for any natural n the ball* $B_n(Q, X)$ *is a convex subcomplex in* X. *Any ball* $B_n(Q, X)$ *is full in* X *and contractible. It is also a deformation retract of the ball* $B_m(Q, X)$ *for any* $m > n$. *Finally, if* X *is k-systolic (for some* $k \geq 6$*) then the ball* $B_n(Q, X)$ *is also k-systolic.*

For $n \geq 1$, the *sphere* of radius n centered at a convex subcomplex Q is the full subcomplex $S_n(Q, X)$ in X spanned by the vertices at combinatorial distance n from Q.

For a convex subcomplex $Q \subset X$ the *boundary* ∂Q is a subcomplex consisting of all simplices $\sigma \subset Q$ with $Q_\sigma \neq X_\sigma$. If $f : Q \to X$ denotes the inclusion map, we have $\partial Q = \partial_f Q$.

7.6. *Lemma. — Let* Q *be a convex subcomplex in a systolic complex* X *and let* $n \geq 1$ *be a natural number. Then*

(1) $B_n(Q, X)$ *is the full subcomplex of* X *spanned by the set of all vertices of* X *at combinatorial distance* $\leq n$ *from* Q*;*
(2) $S_n(Q, X) \subset B_n(Q, X)$*;*
(3) $S_n(Q, X)$ *is equal to the union of those simplices in the ball* $B_n(Q, X)$ *which are disjoint with* $B_{n-1}(Q, X)$*;*
(4) $\partial B_n(Q, X) \subset S_n(Q, X)$.

Proof. — Observe that, by definition of balls, the vertex set of the ball $B_n(Q, X)$ is exactly the set of all vertices of X at combinatorial distance $\leq n$ from Q. Since, by Corollary 7.5, the ball $B_n(Q, X)$ is full in X, this proves (1). Parts (2) and (3) follow from (1). In view of Lemma 7.3, part (4) follows from property (E3) of small extension.

The next result will be often used in later sections, especially in establishing properties of projection maps (onto convex subsets) and directed geodesics.

7.7. *Lemma. — For any convex subcomplex* Q *in a systolic complex* X *and for any simplex* $\sigma \subset N_X(Q)$ *disjoint from* Q*, the intersection* $Q \cap \mathrm{Res}(\sigma, X)$ *is a single simplex of* Q*. Moreover, if* σ' *is a face of* σ*, then* $Q \cap \mathrm{Res}(\sigma, X)$ *is a face of* $Q \cap \mathrm{Res}(\sigma', X)$.

Proof. — In view of Lemma 7.3, it follows from Corollary 5.5 that the intersection $Q \cap \mathrm{Res}(\sigma, N_X(Q))$ is a single simplex of Q. Since it is clear from the definition of the neighborhood that $\mathrm{Res}(\sigma, X) \cap Q = \mathrm{Res}(\sigma, N_X(Q)) \cap Q$, the first assertion follows. The second assertion is clear due to reversed inclusion between residues of a simplex and its face.

7.8. *Lemma. — Let* $Q \subset X$ *be a convex subcomplex and let* ρ *be a simplex in* $N_X(Q)$ *disjoint with* Q*. Let* $\sigma = Q \cap \mathrm{Res}(\sigma, X)$ *be the simplex as in Lemma 7.7. Then the link of the subcomplex* $N_X(Q)$ *at* ρ *has the form of a simplicial neighborhood of a single simplex, namely* $[N_X(Q)]_\rho = N_{X_\rho}(\sigma)$.

Proof. — In view of Lemma 7.3, it follows from Lemma 5.3 that $[N_X(Q)]_\rho = N_{[N_X(Q)]_\rho}(\sigma)$. But, since $\sigma \subset Q$, we have $N_X(\sigma) = N_{N_X(Q)}(\sigma)$ and hence also $N_{[N_X(Q)]_\rho}(\sigma) = N_{X_\rho}(\sigma)$, which finishes the proof.

For later applications, we state the specializations of Lemmas 7.7 and 7.8 to the case of balls.

7.9. *Corollary.* — *Let* X *be a systolic simplicial complex,* Q *a convex subcomplex in* X*, and* $n \geq 1$ *a natural number. Then for any simplex* ρ *of the sphere* $S_n(Q, X)$

(1) the intersection $B_{n-1}(Q, X) \cap \mathrm{Res}(\rho, X)$ *is a single (nonempty) simplex of* X*;*
(2) if $\sigma = B_{n-1}(Q, X) \cap \mathrm{Res}(\rho, X)$ *is the simplex as in (1), we have* $[B_n(Q, X)]_\rho = N_{X_\rho}(\sigma)$.

We now turn to the notion of strong convexity, a sharper variant of convexity. This notion is intimately related with the concept of extra-tilability playing a central role in our constructions in Sections 18–19. The definition is inspired by the property of neighborhoods of convex subcomplexes described in Lemma 7.8.

7.10. *Definition.* — *A subcomplex* Q *in a systolic simplicial complex* X *is* strongly convex *if it is connected and for any simplex* ρ *of* Q *the link* Q_ρ *coincides either with the whole link* X_ρ *or with the neighborhood* $N_{X_\rho}(\sigma)$ *of some simplex* $\sigma \subset X_\rho$.

Since the neighborhoods of simplices have diameters ≤ 3, Lemma 3.7 implies that links of a strongly convex subcomplex are 3-convex in the corresponding links of X. In particular, this gives the following.

7.11. *Corollary.* — *Every strongly convex subcomplex is convex.*

Next corollary is an immediate consequence of Lemmas 7.6.4 and 7.8, and of Corollary 7.5.

7.12. *Corollary.* — *Let* Q *be a convex subcomplex in a systolic simplicial complex* X. *Then the neighborhood* $N_X(Q)$ *is strongly convex in* X. *Moreover, for any natural* $n \geq 1$ *the ball* $B_n(Q, X)$ *is strongly convex in* X.

8. Projections onto convex subcomplexes

In this section we define and study a natural map from a finite dimensional systolic complex to its convex subcomplex, which we call projection. This map resembles the projection of a CAT(0) space to its convex subset along the shortest geodesics connecting points of the space with the subset. We introduce also projection rays which are analogues of the above geodesics.

Given a simplicial complex K, we denote by K′ its first barycentric subdivision. For a simplex $\sigma \subset K$, we denote by b_σ the barycenter of σ, a vertex in K′.

We denote by $dist_K$ the combinatorial distance (in the 1-skeleton of K) between the vertices of K. We also use a simplified notation B_nQ, S_nQ for balls $B_n(Q, X)$ and spheres $S_n(Q, X)$, X being fixed throughout the whole section. In particular, under this convention, the neighborhood $N_X(Q)$ is denoted by B_1Q.

Given a convex subcomplex Q in a systolic complex X, define an *elementary projection* $\pi_Q : (B_1Q)' \to Q'$ between the barycentrically subdivided complexes by putting

$$\pi_Q(b_\sigma) = \begin{cases} b_{\sigma\cap Q} & \text{if } \sigma \cap Q \neq \emptyset \\ b_\tau & \text{if } \sigma \cap Q = \emptyset, \text{ where } \tau = \mathrm{Res}(\sigma, X) \cap Q \end{cases}$$

and extending simplicially. By Lemmas 7.2.3 and 7.7, π_Q is a well defined simplicial map. It is also clear that π_Q restricted to Q' is the identity on Q', i.e. π_Q is a retraction.

Remark. — One verifies that, viewing B_1Q as a small extension domain EQ for the inclusion map $Q \to X$, the elementary projection π_Q coincides with the deformation retraction $EQ \to Q$ constructed in Section 5.

8.1. *Lemma.* — *Let* Q *be a convex subcomplex in a systolic complex* X, *and let* $\sigma \subset (B_1Q)'$ *be a simplex not contained in* Q'. *Then* $\pi_Q(\sigma) \subset (\partial Q)'$.

Proof. — Since $(\partial Q)'$ is a full subcomplex in Q', it is sufficient to prove the lemma for vertices. A vertex in $(B_1Q)'$ not contained in Q' has the form b_τ for some simplex $\tau \subset B_1Q$ not contained in Q. Let $\rho \subset Q$ be the simplex given by $\pi_Q(b_\tau) = b_\rho$. By the definition of π_Q, if τ intersects Q then $\tau - \rho \in X_\rho$ and if τ is disjoint with Q then $\tau \in X_\rho$. In any case it follows that $Q_\rho \neq X_\rho$, hence $\rho \subset \partial Q$ and $b_\rho \in (\partial Q)'$.

Denote by $P^n_Q : (B_nQ)' \to Q'$ the composition map $\pi_{B_{n-1}Q} \circ \pi_{B_{n-2}Q} \circ \ldots \circ \pi_{B_1Q} \circ \pi_Q$ and observe that P^n_Q extends P^m_Q if $n > m$. Denote then by $P_Q : X' \to Q'$ the union $\bigcup_n P^n_Q$ and call it the *projection to* Q. The first two parts of the next fact follow from the properties of elementary extensions. Part (3) is true for any simplicial map between two simplicial complexes.

8.2. *Fact.* — The projection P_Q satisfies the following properties:

(1) $P_Q|_{Q'} = id_{Q'}$;
(2) if σ is a simplex of X' not contained in Q' then $P_Q(\sigma) \subset \partial Q$;
(3) $dist_{Q'}(P_Q(v), P_Q(w)) \leq dist_{X'}(v, w)$ for any vertices $v, w \in X'$.

We turn to defining projection rays. To do this, we need the following variants of projection maps, between face posets rather than barycentric subdivisions. For any convex subcomplex $Q \subset X$, and any simplex σ of X, put $\widehat{P}_Q(\sigma) = \tau$ if and only if $P_Q(b_\sigma) = b_\tau$, and similarly put $\widehat{\pi}_Q(\sigma) = \tau$ if and only if $\pi_Q(b_\sigma) = b_\tau$.

Let $Q \subset X$ be a convex subcomplex and let $\sigma \subset S_n Q$. The *projection ray* from σ to Q is the sequence $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_n$ of simplices in X given by $\sigma_k = \widehat{\pi}_{B_{n-k+1}Q}(\sigma_{k-1})$ for $k = 1, \ldots, n$. Equivalently, this sequence is given by $\sigma_k = \widehat{P}_{B_{n-k}Q}(\sigma_0)$.

Now we list obvious properties of projection rays.

8.3. *Fact.*

(1) Any two consecutive simplices σ_k, σ_{k+1} in a projection ray are disjoint and span a simplex of X.
(2) If σ_k and σ_m are simplices in a projection ray then for any vertices $v \in \sigma_k$ and $w \in \sigma_m$ we have $dist_X(v, w) = |k - m|$.
(3) If $\sigma_0, \ldots, \sigma_n$ is a projection ray on Q then $\sigma_k, \sigma_{k+1}, \ldots, \sigma_n$, for any $0 < k < n$, is also a projection ray on Q.

A less obvious property, giving an intrinsic characterization of projection rays, is

8.4. *Lemma. — If $\sigma_0, \ldots, \sigma_n$ is a projection ray on Q then $\sigma_0, \ldots, \sigma_k$, for any $0 < k \leq n$, is a projection ray on σ_k, where we view σ_k as a convex subcomplex of* X.

Proof. — Note first that $B_m\sigma_k \subset B_{n-k+m}Q$ for any $0 \leq m < k$. Since $\mathrm{Res}(\sigma_{k-m-1}, X)$ contains σ_{k-m}, we have

$$\sigma_{k-m} \subset \mathrm{Res}(\sigma_{k-m-1}, X) \cap B_m\sigma_k \subset \mathrm{Res}(\sigma_{k-m-1}, X) \cap B_{n-k+m}Q = \sigma_{k-m}.$$

Thus all the inclusions above are equalities, so in particular

$$\mathrm{Res}(\sigma_{k-m-1}, X) \cap B_m\sigma_k = \sigma_{k-m},$$

hence the lemma.

8.5. *Corollary. — A projection ray in a systolic complex is uniquely determined by its initial and final simplex.*

8.6. *Lemma. — Let σ and τ be two simplices in a systolic complex* X *such that $dist_X(v, w) = n$ for all vertices $v \in \sigma$ and $w \in \tau$. Then there is a face $\rho \subset \tau$ such that σ is connected to ρ by a projection ray of form $\sigma, \sigma_1, \ldots, \sigma_{n-1}, \rho$.*

Proof. — The required projection ray corresponds to the projection $\widehat{P}_\tau$ on the subcomplex $\tau \subset X$, with $\rho = \widehat{P}_\tau(\sigma)$.

9. Directed geodesics

In this section we introduce the notion of a directed geodesic in a locally 6-large simplicial complex. It is a sequence of simplices satisfying certain condition that involves triples of consecutive simplices in the sequence, and is in this sense local. The adjective "directed" tells that a directed geodesic is in general not symmetric, i.e. it fails to be a directed geodesics after reversing the order. We show that in systolic complexes the local notion of a directed geodesic coinsides with the global notion of a projection ray (introduced in Section 7). In Sections 10–12 we study further global properties of directed geodesics in systolic complexes.

9.1. *Definition. — A sequence* (σ_n) *of simplices in a locally 6-large simplicial complex* X *is a* directed geodesic *if it satisfies the following properties:*

(1) any two consecutive simplices σ_i, σ_{i+1} *in the sequence are disjoint and span a simplex of* X*;*
(2) for any three consecutive simplices $\sigma_i, \sigma_{i+1}, \sigma_{i+2}$ *in the sequence we have*

$$\mathrm{Res}(\sigma_i, X_{\sigma_{i+1}}) \cap B_1(\sigma_{i+2}, X_{\sigma_{i+1}}) = \emptyset.$$

Observe the lack of symmetry in condition (2), and the local nature of the definition. It is clear that images of directed geodesics under covering maps, or their lifts under such maps, are again directed geodesics. The next lemma shows an alternative and simpler way to define directed geodesics in systolic complexes.

9.2. *Lemma. — If* X *is a systolic complex then condition (2) in the definition of a directed geodesic (Definition 9.1) can be replaced with the following condition:*

$$(\mathbf{2'}) \qquad \mathrm{Res}(\sigma_i, X) \cap B_1(\sigma_{i+2}, X) = \sigma_{i+1}.$$

Proof. — Since $\mathrm{Res}(\sigma_i, X_{\sigma_{i+1}}) = \mathrm{Res}(\sigma_i, X) \cap X_{\sigma_{i+1}}$ and $B_1(\sigma_{i+2}, X_{\sigma_{i+1}}) = B_1(\sigma_{i+2}, X) \cap X_{\sigma_{i+1}}$, we get the inclusion

$$\sigma_{i+1} * [\mathrm{Res}(\sigma_i, X_{\sigma_{i+1}}) \cap B_1(\sigma_{i+2}, X_{\sigma_{i+1}})] \subset \mathrm{Res}(\sigma_i, X) \cap B_1(\sigma_{i+2}, X)$$

(where $\sigma * \emptyset = \sigma$). Hence (2') implies (2). To prove the converse, suppose that $\mathrm{Res}(\sigma_i, X) \cap B_1(\sigma_{i+2}, X)$ contains a vertex v not in σ_{i+1}. Then (2) implies that v is not in the link $X_{\sigma_{i+1}}$, and hence also not in the residue $\mathrm{Res}(\sigma_{i+1}, X)$. Moreover, both σ_i and σ_{i+2} are contained in $\mathrm{Res}(v, X) \cap \mathrm{Res}(\sigma_{i+1}, X)$, which is a simplex according to Lemma 7.7. Thus σ_i and σ_{i+2} span a simplex, but this is impossible due to condition (2) and the fact that X is flag.

Existence of many finite directed geodesics is implied by the following two results. We will not deal with infinite directed geodesics in this paper.

9.3. *Lemma.* — *Each projection ray in a systolic simplicial complex is a directed geodesic.*

Proof. — By Fact 8.3.1, a projection ray $\sigma_0, \ldots, \sigma_n$ satisfies condition (1) of Definition 9.1. In view of Lemma 9.2, it is now sufficient to check condition (2') from this lemma. To do this, note that any subsequence $\sigma_i, \sigma_{i+1}, \sigma_{i+2}$ is a projection ray on σ_{i+2} (see Fact 8.3.3 and Lemma 8.4). By the definition of a projection ray and by Lemma 7.7 we get $\mathrm{Res}(\sigma_i, X) \cap B_1(\sigma_{i+2}, X) = \sigma_{i+1}$, and the lemma follows.

9.4. *Corollary.* — *Any sequence of simplices in a locally 6-large complex* X *that lifts to a projection ray in the universal cover of* X *is a directed geodesic.*

We now turn to proving that (lifts of) directed geodesics coincide with projection rays. We start with a preparatory result.

9.5. *Lemma.* — *Let* X *be a systolic complex,* Q *its convex subcomplex, and suppose that* σ *is a simplex in the sphere* $S_1(Q, X)$. *Denote by* τ *the simplex* $\mathrm{Res}(\sigma, X) \cap Q$. *Then* $[B_1(\tau, X)]_\sigma = [B_1(X, Q)]_\sigma$.

Proof. — Since $\tau \subset Q$, it is clear that $[B_1(\tau, X)]_\sigma \subset [B_1(X, Q)]_\sigma$. To prove the converse inclusion, note that since all the involved complexes are full in X, it is sufficient to show that if v is a vertex in $[B_1(X, Q)]_\sigma$ then $v \in [B_1(\tau, X)]_\sigma$. Let v be any vertex of $[B_1(X, Q)]_\sigma$. If $v \in Q$ then $v \in \mathrm{Res}(\sigma, X) \cap Q = \tau$, and hence $v \in [B_1(\tau, X)]_\sigma$. If $v \notin Q$ then $\sigma * v \subset S_1(Q, X)$ and thus $\mathrm{Res}(\sigma * v, X) \cap Q \neq \emptyset$. Moreover, by Lemma 7.7 we have $\mathrm{Res}(\sigma * v, X) \cap Q \subset \mathrm{Res}(\sigma, X) \cap Q = \tau$, and hence $v \in B_1(\tau, X)$. Since the ball $B_1(\tau, X)$ is full in X, we get that $\sigma * v \subset B_1(\tau, X)$ and thus again $v \in [B_1(\tau, X)]_\sigma$, hence the lemma.

9.6. *Proposition.* — *A directed geodesic* $\sigma_0, \ldots, \sigma_n$ *in a systolic complex is a projection ray on its final simplex* σ_n.

Proof. — By Lemma 9.2 we have $\mathrm{Res}(\sigma_{n-2}, X) \cap B_1(\sigma_n, X) = \sigma_{n-1}$, so σ_{n-2}, σ_{n-1}, σ_n is a projection ray on σ_n. Suppose inductively that for some $1 \leq k \leq n-2$ the sequence $\sigma_k, \sigma_{k+1}, \ldots, \sigma_n$ is a projection ray on σ_n. We will prove that the sequence $\sigma_{k-1}, \sigma_k, \ldots, \sigma_n$ is also a projection ray on σ_n. To do this, we need to show that (1) σ_{k-1} is disjoint with the ball $B_{n-k}(\sigma_n, X)$ and (2) $\mathrm{Res}(\sigma_{k-1}, X) \cap B_{n-k}(\sigma_n, X) = \sigma_k$.

By Lemma 9.5 we have

$$[B_1(\sigma_{k+1}, X)]_{\sigma_k} = [B_1(B_{n-k-1}(\sigma_n, X), X)]_{\sigma_k} = [B_{n-k}(\sigma_n, X)]_{\sigma_k}.$$

We then get

$$\sigma_{k-1} \cap [B_{n-k}(\sigma_n, X)]_{\sigma_k} = \sigma_{k-1} \cap [B_1(\sigma_{k+1}, X)]_{\sigma_k} = \emptyset,$$

where the last equality follows from the definition of directed geodesic applied to the simplices $\sigma_{k-1}, \sigma_k, \sigma_{k+1}$. Since the ball $B_{n-k}(\sigma_n, X)$ is full in X and $\sigma_{k-1} * \sigma_k$ is a simplex of X, this implies (1). Moreover, since X is flag and balls in X are full, we get

$$\begin{aligned} B_1(\sigma_k, X) \cap B_1(\sigma_{k+1}, X) &= \sigma_k * [B_1(\sigma_{k+1}, X)]_{\sigma_k} = \sigma_k * [B_{n-k}(\sigma_n, X)]_{\sigma_k} \\ &= B_1(\sigma_k, X) \cap B_{n-k}(\sigma_n, X). \end{aligned} \tag{9.6.1}$$

By Lemma 7.7, the intersection $\text{Res}(\sigma_{k-1}, X) \cap B_{n-k}(\sigma_n, X)$ is a simplex containing σ_k, so in particular this intersection is contained in the ball $B_1(\sigma_k, X)$. Consequently, by applying (9.6.1) we have

$$\begin{aligned} \text{Res}(\sigma_{k-1}, X) \cap B_{n-k}(\sigma_n, X) &= \text{Res}(\sigma_{k-1}, X) \cap B_{n-k}(\sigma_n, X) \cap B_1(\sigma_k, X) \\ &= \text{Res}(\sigma_{k-1}, X) \cap B_1(\sigma_{k+1}, X) \cap B_1(\sigma_k, X) \\ &= \sigma_k, \end{aligned}$$

where the last equality follows from Lemma 9.2. This shows that $\sigma_{k-1}, \sigma_k, \dots, \sigma_n$ is a projection ray on σ_n, hence the proposition.

Proposition 9.6 and Lemma 9.3 show that the sets of finite directed geodesics and of projection rays coincide. As a consequence of Corollary 8.5 and Lemma 8.6 we obtain therefore the following.

9.7. *Corollary. — Given vertices v, w in a systolic complex there is exactly one directed geodesic from v to w.*

As an easy consequence of Fact 8.3.2 we get also the following.

9.8. *Corollary. — Let v, w be two vertices in a systolic complex X such that $dist_X(v, w) = n$. Then the directed geodesic from v to w consists of $n + 1$ simplices.*

10. Directed geodesics and convexity

In this section we study the behavior of directed geodesics with respect to convex subcomplexes in systolic complexes. We also obtain several more properties of convex subcomplexes.

In the proofs in this section we will often use (without referring explicitly to) both assertions of Lemma 7.7.

10.1. *Lemma.* — *Let* Q *be a convex subcomplex in a systolic complex* X. *Let* $B_n = B_n(Q, X)$ *and* $S_n = S_n(Q, X)$ *be the systems of balls and spheres in* X *centered at* Q. *For any directed geodesic* $\sigma_1, \sigma_2, \sigma_3$ *and for any* $n \geq 0$:

(1) *if* $\sigma_1 \subset B_n$ *and* $\sigma_2 \subset S_{n+1}$ *then* $\sigma_3 \subset S_{n+2}$;
(2) *if* $\sigma_1 \subset B_n$ *and* σ_2 *intersects* B_n *but is not contained in* B_n, *then* $\sigma_3 \cap B_n = \emptyset$;
(3) *if* σ_1 *intersects* B_n *but is not contained in* B_n, *and if* $\sigma_2 \cap B_n = \emptyset$, *then* σ_3 *is not contained in* B_{n+1}.

Proof. — To prove (1), observe that $\sigma_3 \cap B_n = \emptyset$, since otherwise both simplices $\sigma_3 \cap B_n$ and σ_1 are faces of the simplex $\mathrm{Res}(\sigma_2, X) \cap B_n$, and this contradicts condition (2') from Lemma 9.2. Suppose that $\tau = \sigma_3 \cap S_{n+1}$ is not empty. It is a face of σ since, by definition, the sphere S_{n+1} is full in X. Note that both simplices σ_1 and $\mathrm{Res}(\sigma_2 * \tau, X) \cap B_n$ are faces of the simplex $\mathrm{Res}(\sigma_2, X) \cap B_n$. It follows that the intersection $B_1(\tau, X) \cap \mathrm{Res}(\sigma_1, X)$ contains the join $\sigma_2 * [\mathrm{Res}(\sigma_2 * \tau, X) \cap B_n]$, and hence is larger than σ_2 (here we use flagness of X). Thus the same is true for the intersection $B_1(\sigma_3, X) \cap \mathrm{Res}(\sigma_1, X)$, contradicting condition (2') of Lemma 9.2. This implies that σ_3 is disjoint with both B_n and S_{n+1}, hence it is contained in S_{n+2}.

To prove (2), suppose that the intersection $\tau := \sigma_3 \cap B_n$ is not empty. It is then a face of σ_3 (because B_n is full) and we denote it by τ. Similarly, using the fact that spheres are full in X, denote by ρ the simplex $S_{n+1} \cap \sigma_2$. Observe that both σ_1 and τ are faces of the simplex $\mathrm{Res}(\rho, X) \cap B_n$, which clearly contradicts condition (2') of Lemma 9.2.

To prove (3), note that by the assumptions we get that $\sigma_2 \subset S_{n+1}$.

If $\tau = \sigma_3 \cap B_n \neq \emptyset$ then both τ and σ_1 are the faces of the simplex $\mathrm{Res}(\sigma_2, X) \cap B_n$, contradicting condition (2') of Lemma 9.2. If $\sigma_3 \subset S_{n+1}$ then the simplex $\mathrm{Res}(\sigma_2 * \sigma_3, X) \cap B_n$ and the simplex $\sigma_1 \cap B_n$ are faces of the simplex $\mathrm{Res}(\sigma_2, X) \cap B_n$, which again contradicts (2').

Since σ_3 is disjoint with B_n and not contained in S_{n+1}, it is not contained in B_{n+1}, hence the lemma.

Remark. — The following uniform interpretation of the three parts of Lemma 10.1 provides the idea for proving the next result. A simplex σ_1 is *closer* to Q than a simplex σ_2 if any of the assumptions from parts (1)–(3) is satisfied. The lemma says that if σ_1 is closer than σ_2 then σ_2 is closer than σ_3.

10.2. *Proposition.* — *Let* Q *be a convex subcomplex in a systolic complex* X, *and let* $\sigma_0, \ldots, \sigma_n$ *be a directed geodesic in* X *such that* $\sigma_0 \subset Q$ *and* $\sigma_n \subset Q$. *Then for each* $0 < i < n$ *we have* $\sigma_i \subset Q$.

Proof. — Suppose that some of the simplices in the directed geodesic is not contained in Q. Then there is i such that $\sigma_i \subset Q$ and σ_{i+1} is not contained in Q. Applying Lemma 10.1 inductively, we get that σ_k is not contained in Q for all $k > i$. This contradicts the assumption that $\sigma_n \subset Q$, hence the proposition.

10.3. *Lemma. — The intersection of any family of convex subcomplexes in a given systolic complex is a convex subcomplex.*

Proof. — Since any convex subcomplex is locally 3-convex, it follows from Fact 3.3.2 that the intersection of convex subcomplexes is locally 3-convex. It remains to show that this intersection is connected.

Let v, w be any two vertices in the intersection. By Lemma 8.6, these vertices are connected by a projection ray. Since, according to Lemma 9.3, this projection ray is a directed geodesic, it follows from Proposition 10.2 that all its simplices are contained in the intersection. Consequently, since the intersection of full subcomplexes is full, there is a path in (the 1-skeleton of) the intersection between v and w, hence connectivity.

10.4. *Lemma. — For each subcomplex* K *of a systolic complex* X *there is the smallest convex subcomplex conv*(K) *in* X *that contains* K *(we will call it the convex hull of* K *in* X*). Moreover, if* K *is bounded (with respect to the combinatorial distance), its convex hull is also bounded.*

Proof. — Since K is contained in at least one convex subcomplex of X, namely in X itself, we define *conv*(K) to be the intersection of all convex subcomplexes in X containing K. According to Lemma 10.3, this intersection is convex. If K is bounded, it is contained in some ball in X centered at a vertex. Since, by Corollary 7.5, this ball is convex, the convex hull of K is clearly bounded.

11. Fellow traveller property

In this section we prove that directed geodesics in a systolic complex satisfy fellow traveller property. We show this property in a setting suitable for applications in Section 13, where we prove that systolic groups are biautomatic.

Let X be a systolic simplicial complex and let v, w be vertices in X. An *allowable geodesic* from v to w in the 1-skeleton $X^{(1)}$ is an infinite sequence $(u_i)_{i=0}^{\infty}$ of vertices of X such that if $v = \sigma_0, \sigma_1, ..., \sigma_n = w$ is the directed geodesic in X from v to w then

(1) $u_i \in \sigma_i$ for $0 \leq i \leq n$ (in particular, $u_0 = v$ and $u_n = w$);
(2) $u_i = u_n = w$ for $i > n$.

Fact 8.3.1 (together with Proposition 9.6) implies that the sequence of vertices in an allowable geodesic, before it becomes constant, forms a polygonal path in the 1-skeleton $X^{(1)}$. Moreover, Fact 8.3.2 implies the following.

11.1. *Fact.* — If $dist_X(v, w) = n$ and $(u_i)_{i=0}^{\infty}$ is an allowable geodesic from v to w, then for $0 \leq j < k \leq n$ we have $dist_X(u_j, u_k) = k - j$, i.e. the subsequence $(u_i)_{i=0}^{n}$ determines a geodesic in $X^{(1)}$.

We will prove the following variant of the fellow traveller property.

11.2. *Proposition.* — *Let* X *be a systolic complex and suppose that* $(u_i)_{i=0}^{\infty}$ *and* $(t_i)_{i=0}^{\infty}$ *are allowable geodesics in* $X^{(1)}$ *from* v *to* w *and from* p *to* q *respectively. Then for each* $i \geq 0$ *we have*

$$dist_X(u_i, t_i) \leq 3 \cdot \max[dist_X(v, p), dist_X(w, q)] + 1.$$

Remark. — Note that the fellow traveller property does not in general hold for arbitrary geodesics in the 1-skeleton of a systolic complex, as can be easily observed for example in the triangulation of the euclidean plane by congruent equilateral triangles.

The proof of Proposition 11.2 is based on Lemma 11.3, the first part of which we prove at the end of this section, and the second in Section 12. In this lemma we use a convention that if $\sigma_0, ..., \sigma_n$ is a directed geodesic then it extends to the infinite sequence $(\sigma_i)_{i=0}^{\infty}$ by putting $\sigma_i = \sigma_n$ for $i > n$. We denote by X' the first barycentric subdivision of a simplicial complex X and by b_σ the barycenter of a simplex $\sigma \subset X$ (which is a vertex in X').

11.3. *Lemma.* — *Let* X *be a systolic complex and let* $(\sigma_i)_{i=0}^{n}$, $(\tau_i)_{i=0}^{m}$ *be directed geodesics in* X.

(1) If $\sigma_n = \tau_m$ *then* $dist_{X'}(b_{\sigma_i}, b_{\tau_i}) \leq 2 \cdot dist_{X'}(b_{\sigma_0}, b_{\tau_0})$ *for each* $i \geq 0$.

(2) If $\sigma_0 = \tau_0$ *is a vertex then* $dist_{X'}(b_{\sigma_i}, b_{\tau_i}) \leq dist_{X'}(b_{\sigma_n}, b_{\tau_m})$ *for each* $i \geq 0$.

Proof of Proposition 11.2 (assuming Lemma 11.3). — Let $(\sigma_i)_{i=0}^{n}$, $(\tau_i)_{i=0}^{m}$ and $(\rho_i)_{i=0}^{l}$ be the directed geodesics in X from v to w, from p to q and from p to w respectively. By Lemma 11.3, for each $i \geq 0$ we have $dist_{X'}(b_{\sigma_i}, b_{\rho_i}) \leq 2 \cdot dist_{X'}(b_{\sigma_0}, b\rho_0)$ and $dist_{X'}(b_{\rho_i}, b_{\tau_i}) \leq dist_{X'}(b_{\rho_l}, b_{\tau_m})$. It clearly implies that for each $i \geq 0$

$$dist_{X'}(b_{\sigma_i}, b_{\tau_i}) \leq 3 \cdot \max[dist_{X'}(b_{\sigma_0}, b_{\tau_0}), dist_{X'}(b_{\sigma_n}, b_{\tau_m})].$$

Since for any vertices x, y belonging to simplices α, β in X respectively we have

$$2 \cdot dist_X(x, y) \leq dist_{X'}(b_\alpha, b_\beta) + 2 \quad \text{and} \quad dist_{X'}(x, y) = 2 \cdot dist_X(x, y),$$

the proposition follows.

Proof of Lemma 11.3.1. — Under assumptions of the lemma, $\sigma_0 \in S_n(\sigma_n, X)$ and $\tau_0 \in S_m(\tau_m, X) = S_m(\sigma_n, X)$. Suppose $n \geq m$. Then $dist_{X'}(b_{\sigma_0}, b_{\tau_0}) \geq 2n - 2m$. On the other hand, applying Fact 8.2 to the convex subcomplex $Q = B_{m-i}(\sigma_n, X) = B_{m-i}(\tau_m, X)$ (or $Q = \sigma_n = \tau_m$ if $i > m$) we get $dist_{X'}(b_{\sigma_{i+n-m}}, b_{\tau_i}) \leq dist_{X'}(b_{\sigma_0}, b_{\tau_0})$. This implies the following estimate:

$$\begin{aligned} dist_{X'}(b_{\sigma_i}, b_{\tau_i}) &\leq dist_{X'}(b_{\sigma_i}, b_{\sigma_{i+n-m}}) + dist_{X'}(b_{\sigma_{i+n-m}}, b_{\tau_i}) \\ &\leq dist_{X'}(b_{\sigma_0}, b_{\tau_0}) + (2n - 2m) \leq 2 \cdot dist_{X'}(b_{\sigma_0}, b_{\tau_0}), \end{aligned}$$

which finishes the proof.

12. Inverse fellow traveller property

In this section we study properties of the family of all directed geodesics started at a fixed vertex p in a finite dimensional systolic complex X. In particular, we obtain the proof of Lemma 11.3.2.

Let X be a systolic complex and let p be a vertex in X. We say that a simplex $\tau \subset X$ is *accessible* from p if there exists a directed geodesic from p to τ. By Fact 8.3.2, to be accessible from p, a simplex τ must be contained in some sphere $S_n(p, X)$. Not all simplices from such spheres are accessible from p. However, it follows from Corollary 9.7 that every vertex in X distinct from p is accessible from p. Let $\tau \subset S_{n+1}(\sigma, X)$ be a simplex accessible from p. Denote by $c_p(\tau)$ the simplex that precedes τ in the directed geodesic from p to τ. More precisely, if $\sigma_0, \sigma_1, ..., \sigma_n, \sigma_{n+1}$ is the directed geodesic from p to τ (i.e. $\sigma_0 = p$ and $\sigma_{n+1} = \tau$) then we put $c_p(\tau) := \sigma_n$.

We use the notation concerning barycentric subdivisions as in the previous section.

12.1. *Proposition.* — *For any systolic complex* X, *any vertex* p *in* X *and any* $n \geq 0$ *there is a simplicial map* $c_p^n : [B_{n+1}(p, X)]' \to [B_n(p, X)]'$ *satisfying the following properties:*

(1) c_p^n *restricted to* $[B_n(p, X)]'$ *is the identity;*
(2) $c_p^n(b_\tau) = b_{c_p(\tau)}$ *for any simplex* $\tau \subset S_{n+1}(\sigma, X)$ *that is accessible from* p.

The proof of Proposition 12.1 requires several preparatory results. Before getting to them we first give the proof of Lemma 11.3.2 based on the proposition.

Proof of Lemma 11.3.2. — Let (σ_i) and (τ_i) be the sequences as in the lemma. Consider the maps $C^i_{\sigma_0} : X' \to [B_i(\sigma_0, X)]'$ given by

$$C^i_{\sigma_0} := \bigcup_{k=1}^{\infty} c^{i+k}_{\sigma_0} \circ c^{i+k-1}_{\sigma_0} \circ ... \circ c^i_{\sigma_0}$$

and note that we have $C^i_{\sigma_0}(b_{\sigma_n}) = b_{\sigma_i}$ and $C^i_{\sigma_0}(b_{\tau_m}) = b_{\tau_i}$. Since the maps $C^i_{\sigma_0}$ are simplicial map, they do not increase combinatorial distances, hence the lemma.

The next series of results prepares the background for proving Proposition 12.1.

12.2. *Lemma.* — *If τ is accessible from p and ϱ is a face of τ, then ϱ is accessible from p and $c_p(\varrho) = c_p(\tau)$.*

Proof. — Let $p, \sigma_1, \ldots, \sigma_{n-1}, \tau$ be the directed geodesic from p to τ. It is sufficient to show that $p, \sigma_1, \ldots, \sigma_{n-1}, \varrho$ is also a directed geodesic. To do this, we only need to check the condition for directed geodesic at the final triple $\sigma_{n-2}, \sigma_{n-1}, \varrho$. It follows easily by observing that $B_1(\varrho, X_{\sigma_{n-1}}) \subset B_1(\tau, X_{\sigma_{n-1}})$.

12.3. *Corollary.* — *Let τ_1, τ_2 be simplices that are accessible from p and suppose they intersect. Then their corresponding directed geodesics from p coincide except at the last simplices. In other words, we then have $c_p(\tau_1) = c_p(\tau_2)$.*

12.4. *Lemma.* — *Suppose $e = (v_1, v_2)$ is a 1-simplex in $S_n(p, X)$ not accessible from p and denote by σ_0 the last common simplex in the directed geodesics from p to v_1 and v_2. Denote by $\sigma_0, \sigma^1_1, \sigma^1_2, \ldots, \sigma^1_{n-1}, v_1$ and $\sigma_0, \sigma^2_1, \sigma^2_2, \ldots, \sigma^2_{n-1}, v_2$ the directed geodesics from σ_0 to v_1 and v_2 (which are parts of the corresponding geodesics from p). Suppose that the projection ray from σ_0 on e terminates at v_1 (it terminates at some vertex of e since e is not accessible from p, and hence also not accessible from σ_0). Then (1) $\sigma^2_1 \subset \sigma^1_1$, (2) $\sigma^1_2 \cap \sigma^2_2 = \emptyset$ and (3) σ^1_2, σ^2_2 span a simplex of X.*

Proof. — Note that by our assumptions the directed geodesic $\sigma_0, \sigma^1_1, \sigma^1_2, \ldots, \sigma^1_{n-1}, v_1$ is the projection ray from σ_0 on e. Therefore we have

$$\sigma^1_1 = \operatorname{Res}(\sigma_0, X) \cap B_{n-1}(v_1, X) = \operatorname{Res}(\sigma_0, X) \cap B_{n-1}(e, X) \quad \text{and}$$
$$\sigma^2_1 = \operatorname{Res}(\sigma_0, X) \cap B_{n-1}(v_2, X) \subset \operatorname{Res}(\sigma_0, X) \cap B_{n-1}(e, X),$$

hence (1).

To prove (2), suppose that $\sigma^1_2 \cap \sigma^2_2 = \alpha \neq \emptyset$. Then, according to Lemma 12.2, $\sigma_0, \sigma^1_1, \alpha$ and $\sigma_0, \sigma^2_1, \alpha$ are directed geodesics. Moreover, these geodesics are distinct because $\sigma^1_1 \neq \sigma^2_1$, which contradicts uniqueness (Corollary 8.5 and Proposition 9.6).

To prove (3), note that in view of (1) we have

$$\sigma^1_2 = \operatorname{Res}\big(\sigma^1_1, X\big) \cap B_{n-2}(e, X) \subset \operatorname{Res}\big(\sigma^2_1, X\big) \cap B_{n-2}(e, X) \quad \text{and}$$
$$\sigma^2_2 = \operatorname{Res}\big(\sigma^2_1, X\big) \cap B_{n-2}(v_2, X) \subset \operatorname{Res}\big(\sigma^2_1, X\big) \cap B_{n-2}(e, X),$$

where the first inclusion follows from (1) and second from the fact that $v_2 \subset e$. By Corollary 7.9.1, the intersection $\beta = \operatorname{Res}(\sigma^2_1, X) \cap B_{n-2}(e, X)$ is a simplex in X, and since we have $\sigma^1_2, \sigma^2_2 \subset \beta$, the lemma follows.

12.5. *Lemma.* — *Suppose* $e = (v_1, v_2)$ *is a 1-simplex in* $S_n(p, X)$ *not accessible from* p. *Then the simplices* $c_p(v_1)$ *and* $c_p(v_2)$ *span a simplex of* X.

Proof. — As in the statement of Lemma 12.4, denote by σ_0 the last common simplex in the directed geodesics from p to v_1 and v_2. Denote also by $\sigma_0, \sigma_1^1, \sigma_2^1, \ldots, \sigma_{n-1}^1, v_1$ and $\sigma_0, \sigma_1^2, \sigma_2^2, \ldots, \sigma_{n-1}^2, v_2$ the directed geodesics from σ_0 to v_1 and v_2 (which are parts of the corresponding geodesics from p), and assume (without loss of generality) that the first of them is the projection ray from σ_0 on e.

Claim **1.** — Let $\sigma_1^1 - \sigma_1^2$ be the face of the simplex σ_1^1 spanned by the vertices not contained in σ_1^2. Then $\sigma_1^1 - \sigma_1^2 \subset S_n(v_2, X)$.

To prove Claim 1, note that for any vertex $u \in \sigma_1^1 - \sigma_1^2$ we have the estimate

$$dist_X(u, v_2) \leq dist_X\left(u, \sigma_1^2\right) + dist_X\left(\sigma_1^2, v_2\right) = 1 + (n-1) = n.$$

If $dist_X(u, v_2) = n - 1$ for some $u \in \sigma_1^1 - \sigma_1^2$, then $u \in \text{Res}(\sigma_0, X) \cap B_{n-1}(v_2, X) = \sigma_1^2$, a contradiction.

A similar argument based on Claim 1 and the fact that $\sigma_1^1 - \sigma_1^2$ and σ_2^1 span a simplex in X gives the following.

Claim **2.** — For $k = 2, 3, \ldots, n-1$ we have $\sigma_k^1 \subset S_{n-k+1}(v_2, X)$.

Returning to the proof of Lemma 12.5, we will show that for $k = 1, 2, \ldots, n-1$ the simplices σ_k^1, σ_k^2 span a simplex of X. The assertion holds for $k = 1, 2$ due to Lemma 12.4. Suppose, by induction, that σ_k^1, σ_k^2 span a simplex. Then both σ_{k+1}^1 and σ_k^2 are contained in the intersection $\text{Res}(\sigma_k^1) \cap B_{n-k}(v_2, X)$ which is a simplex of X (the first inclusion is provided by Claim 2). Consequently, both simplices σ_{k+1}^1 and σ_{k+1}^2 are contained in $\text{Res}(\sigma_k^2, X) \cap B_{n-k-1}(e, X)$, which is also a simplex of X, hence σ_{k+1}^1 and σ_{k+1}^2 span a simplex.

This shows that the simplices $c_p(v_1) = \sigma_{n-1}^1$ and $c_p(v_2) = \sigma_{n-1}^2$ span a simplex of X, as required.

12.6. *Lemma.* — *For any simplex* $\tau \subset S_n(p, X)$ *the family* $\{c_p(v) : v$ *is a vertex of* $\tau\}$ *of simplices spans a simplex in* $S_{n-1}(p, X)$.

Proof. — Observe first that any two simplices $c_p(v_1), c_p(v_2)$ from the family span a simplex. If (v_1, v_2) is a 1-simplex not accessible from p, this is due to Lemma 12.5. Otherwise this follows from the equality $c_p(v_1) = c_p(v_2)$ implied by Lemma 12.2.

Since the complex X is flag and the sphere $S_{n-1}(p, X)$ is a full subcomplex, the above observation implies that the whole family spans a simplex of this sphere.

For a simplex $\tau \subset S_n(p, X)$ not accessible from p let $c_p(\tau)$ be the simplicial span of the family of simplices $\{c_p(v) : v \text{ is a vertex of } \tau\}$. By Lemma 12.2, if τ is a simplex accessible from σ then the simplicial span of the set $\{c_p(v) : v \text{ is a vertex of } \tau\}$ equals $c_p(\tau)$ (as defined at the beginning of the section). Thus the above definition of $c_p(\tau)$ applies to all simplices $\tau \subset S_n(p, X)$. In particular, this implies the following.

12.7. *Corollary.* — *If* $\tau \subset S_n(p, X)$ *is a simplex and* ρ *is a face of* τ *then* $c_p(\rho)$ *is a face of* $c_p(\tau)$.

12.8. *Lemma.* — *If* $v \in S_{n-1}(p, X)$ *and* $w \in S_n(p, X)$ *are vertices that span a 1-simplex* e *then the simplex* $c_p(w)$ *and the vertex* v *span a simplex in* $S_{n-1}(p, X)$.

Proof. — Since the intersection $\text{Res}(w, X) \cap S_{n-1}(p, X)$ is a simplex of X (Corollary 7.9.1), the lemma follows by observing that both $c_p(w)$ and v are contained in this intersection.

An argument similar to that in Lemma 12.6 gives the following.

12.9. *Corollary.* — *If* $\tau \subset B_n(p, X)$ *is any simplex then the family of simplices* $\{c_p(v) : v \text{ is a vertex of } \tau \cap S_n(p, X)\} \cup \{\tau \cap B_{n-1}(p, X)\}$ *spans a simplex in* $B_{n-1}(p, X)$.

Proof of Proposition 12.1. — In view of Corollary 12.9, for any simplex $\tau \subset B_{n+1}(p, X)$ the simplicial span of the family $\{c_p(v) : v \text{ is a vertex of } \tau \cap S_{n+1}(p, X)\} \cup \{\tau \cap B_n(p, X)\}$ is a simplex. Denote this simplex by $\widehat{c_p^n}(\tau)$ and note that it is contained in $B_n(p, X)$. Put $c_p^n(b_\tau)$ to be the barycenter of $\widehat{c_p^n}(\tau)$, and note that this defines the simplicial map $[B_{n+1}(p, X)]' \to B_n(p, X)$ (between the first barycentric subdivisions) which we also denote c_p^n. It follows directly from the definition, that c_p^n satisfies assertions (1) and (2) of the proposition.

13. Systolic groups are biautomatic

We refer the reader to [ECHLPT] for the background on biautomatic groups. Biautomaticity implies various algorithmic and geometric properties of a group, in particular semihyperbolicity [AB] and its consequences.

13.1. *Theorem.* — *Let* G *be a group acting simplicially, properly discontinuously and cocompactly on a systolic simplicial complex* X. *Then* G *is biautomatic.*

Proof. — The proof is based on the fact that directed geodesics in X are recognizable in local terms and satisfy fellow traveller property. Specifically, we will construct a finite symmetric subset $\mathscr{A} \subset G$ generating G as a semigroup, and a language $\mathscr{L}$ over $\mathscr{A}$ (whose strings are closely related to some directed geodesics in X) such that

(1) $\mathscr{L}$ is regular;
(2) the canonical map $\mathscr{L} \to G$ is surjective;
(3) $\mathscr{L}$ satisfies the 2-sided fellow traveller property.

To prove that $\mathscr{L}$ is regular, we shall construct a nondeterministic finite state automaton for which $\mathscr{L}$ is the accepted language.

Given a systolic group G acting on the corresponding complex X, put $K = G\backslash X'$, where X' is the barycentric subdivision of X. Since G acts on X' without inversions (i.e. if an element $g \in G$ fixes a simplex of X' then it fixes all vertices in this simplex), K is a multisimplicial complex (simplices are embedded in K but a set of vertices may span more than one simplex). Moreover, since the action of G on X is cocompact, K is finite.

Generating set $\mathscr{A}$. — Choose a set of representatives V_0 for the family of G-orbits in the vertex set $V(X')$ (with respect to the induced action of G on this set). For a vertex $v \in V(X')$ we shall denote by $\bar{v} \in V_0$ the representative of its G-orbit. For any $v \in V(X')$ define the set $\Lambda_v := \{g \in G : v = g\bar{v}\}$ and call it *the set of labels* of v.

13.2. *Fact.* — $\Lambda_v = g \cdot G_{\bar{v}} = G_v \cdot g$ for any $g \in \Lambda_v$, where $G_{\bar{v}}$ and G_v are the stabilizers of the corresponding vertices in G.

Let $E(X')$ be the set of all pairs $(v, w) \in V(X') \times V(X')$ such that v, w span a 1-simplex of X'. For any pair $(v, w) \in E(X')$ put $\Lambda_{v,w} := \Lambda_v^{-1} \cdot \Lambda_w$. Call the family $\Lambda := \{\Lambda_{v,w} : (v, w) \in E(X')\}$ *the multilabelling on* $E(X')$.

13.3. *Lemma.* — *(1)* $\Lambda_{w,v} = \Lambda_{v,w}^{-1}$.
(2) Multilabelling Λ *on* $E(X')$ *is* G*-invariant, i.e.* $\Lambda_{gv,gw} = \Lambda_{v,w}$ *for any* $(v, w) \in E(X')$ *and any* $g \in G$.
(3) For a fixed $v_0 \in V_0$ *the set*

$$\mathscr{A} := [\bigcup \{\Lambda_{v,w} : (v, w) \in E(X')\} \cup G_{v_0}] \setminus \{1\}$$

(where 1 *is the unit of* G*) is a finite symmetric set generating* G *as a semigroup.*

Proof. — Parts (1) and (2) are obvious. To prove (3), observe that by G-invariance multilabelling Λ on $E(X')$ induces the multilabelling on the set E_K of pairs of vertices that span a 1-simplex of K (we will denote this induced labelling also by Λ). Thus the finiteness of $\mathscr{A}$ follows from finiteness of K and from finiteness of the label sets $\Lambda_{v,w}$, as well as from finiteness of the stabilizers of vertices in X′ (implied by proper discontinuity of the action of G on X). The fact that $\mathscr{A}$ is symmetric follows from part (1). It remains to prove that $\mathscr{A}$ generates G as a semigroup.

Let $g \in G$ be arbitrary. Let $v_0, v_1, \ldots, v_n = gv_0$ be the sequence of vertices in a polygonal path in the 1-skeleton of X′. For each v_i choose a label $g_i \in \Lambda_{v_i}$ with the only restriction that $g_n = g$. Put $\lambda_i := g_{i-1}^{-1}g_i$ for $i = 1, \ldots, n$ and note that $g = g_0\lambda_1\lambda_2\ldots\lambda_n$. Since $g_0 \in \Lambda_{v_0} = G_{v_0}$ and $\lambda_i \in \Lambda_{v_{i-1},v_i}$, the lemma follows.

Language $\mathscr{L}$. — Fix V_0 as above, a vertex $v_0 \in V_0$, and take $\mathscr{A}$ to be the generating set as in Lemma 13.3.3. We define a language $\mathscr{L}$ over the alphabet $\mathscr{A}$ by describing, for arbitrary $g \in G$, the set of all strings in $\mathscr{L}$ that are mapped to g through the evaluation map $\mathscr{L} \to G$.

Let $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}, \sigma_n$ be the directed geodesic in X from v_0 to gv_0. It induces the sequence

$$\sigma_0, \sigma_0 * \sigma_1, \sigma_1, \sigma_1 * \sigma_2, \ldots, \sigma_{n-1}, \sigma_{n-1} * \sigma_n, \sigma_n$$

of simplices, and consequently the sequence $b_0, b_1, \ldots, b_{2n}$ of vertices in X′ being the barycenters of the simplices in the previous sequence. Clearly, this sequence corresponds to a polygonal path connecting v_0 to gv_0 in the 1-skeleton of X′. Consider all strings over $\mathscr{A}$ defined in terms of the path $b_0, b_1, \ldots, b_{2n}$ as follows. For $i = 0, 1, \ldots, 2n$ choose a label $g_i \in \Lambda_{b_i}$ arbitrarily, with the only restriction that $g_{2n} = g$. For $i = 1, 2, \ldots, n$ put $\lambda_i := g_{i-1}^{-1}g_i$ and take the string $g_0\lambda_1\ldots\lambda_{2n}$ with omitted occurrences of the unit element of G. Note that for $g = 1$ this construction gives only the nullstring ε. Take as $\mathscr{L}$ the set of all such strings, for all $g \in G$.

It is clear from the description of $\mathscr{L}$ and from the existence of directed geodesics in X between any two vertices, that the evaluation map $\mathscr{L} \to G$ is surjective. To prove fellow traveller property for $\mathscr{L}$, consider the map $\varphi : G \to X$ given by $\varphi(g) := gv_0$ and note that it is a quasi-isometry. Note also that paths in the Cayley graph $C(G, \mathscr{A})$ corresponding to the strings of $\mathscr{L}$ are, by definition, mapped through φ uniformly close to the appropriate directed geodesics in X, where the distance is controlled by the diameter of the (finite due to cocompactness of G) set V_0. Thus the language $\mathscr{L}$ inherits the 2-sided fellow traveller property from the set of directed geodesics in X (Proposition 11.2). We omit straightforward details of this argument.

To get the fact that G is biautomatic, it remains to prove that the language $\mathscr{L}$ is regular.

Finite state automaton. — Consider a nondeterministic finite state automaton M defined as follows. The unique start state in M is the vertex $v \in K$ corresponding to the vertex $v_0 \in X'$ through the quotient map $X' \to K$. Other states are the pairs $(v, h) : h \in G_{v_0}$ and the triples $(u, w, \lambda) : (u, w) \in E(K), \lambda \in \Lambda_{u,w}$. The accept states in M are the state $(v, 1)$ and the states of form (u, w, λ) with $w = v$.

There are three kinds of arrows in M.

(1) For each $h \in G_{v_0}$ there is an arrow labelled h from the start state v to the state (v, h).
(2) For each $u \in V(K)$ such that $(v, u) \in E(K)$ and for each $\lambda \in \Lambda_{v,u}$ there is an arrow labelled λ from each of the states (v, h) to the state (v, u, λ).
(3) The third kind of arrows requires longer description. Suppose u, w, y are the vertices of K such that $(u, w) \in E(K)$ and $(w, y) \in E(K)$, and suppose $\lambda \in \Lambda_{u,w}$ and $\mu \in \Lambda_{w,y}$. Let $\bar{u}, \bar{w}, \bar{y}$ be the representatives in V_0 of the G-orbits of these vertices. Note that then we have $(\lambda^{-1}\bar{u}, \bar{w}) \in E(X')$ and $(\bar{w}, \mu\bar{y}) \in E(X')$. Denote by ρ, σ, τ respectively the simplices in X whose barycenters are $\lambda^{-1}\bar{u}, \bar{w}, \mu\bar{y}$. There is an arrow labelled μ from the state (u, w, λ) to the state (w, y, μ) if and only if one of the following two conditions holds:
 (i) ρ and τ are disjoint and span σ;
 (ii) σ is a proper face in both ρ and τ and

$$\mathrm{Res}(\rho_\sigma, X_\sigma) \cap B_1(\tau_\sigma, X_\sigma) = \emptyset.$$

Denote by $\mathscr{L}_M$ the language accepted by the automaton M. The fact that $\mathscr{L} \subset \mathscr{L}_M$ follows easily from the description of strings in $\mathscr{L}$. To prove the converse inclusion, consider any path of arrows in the automaton M that gives an accepted string of the language $\mathscr{L}_M$. This path is uniquely determined by the corresponding sequence of states, and we denote this sequence by

$$u_0, (u_0, g_0), (u_0, u_1, \lambda_1), \ldots, (u_{n-1}, u_n, \lambda_n),$$

where $u_0 = u_n = v$, $(u_{i-1}, u_i) \in E(K)$, $g_0 \in G_{v_0}$ and $\lambda_i \in \Lambda_{u_{i-1},u_i}$ for $1 \leq i \leq n$. A string in $\mathscr{L}_M$ obtained from this path is $g_0\lambda_1 \ldots \lambda_n$, where the occurrences of the unit $1 \in G$ are omitted.

For each $0 \leq i \leq n$ denote by $g_i \in G$ the product $g_0\lambda_i \ldots \lambda_i$ and by $\bar{u}_i$ the vertex in V_0 representing the G-orbit in $V(X')$ corresponding to u_i. For each such i put $b_i := g_i\bar{u}_i$ and denote by σ_i the simplex of X with barycenter b_i. Observe that any triple b_{i-1}, b_i, b_{i+1} can be expressed as

$$g_i\lambda_i^{-1}\bar{u}_{i-1},\ g_i\bar{u}_i,\ g_i\lambda_{i+1}\bar{u}_{i+1}$$

and thus the triple ρ, σ, τ of simplices with barycenters $\lambda_i^{-1}\bar{u}_{i-1}, \bar{u}_i, \lambda_{i+1}\bar{u}_{i+1}$ is mapped by g_i to the triple $\sigma_{i-1}, \sigma_i, \sigma_{i+1}$. By the description of arrows in M, and by the facts that $\sigma_0 = v_0$ and $\sigma_n = g_n v_0$ are the vertices and that each g_i is a simplicial automorphism of X, we get that if i is odd then σ_{i-1} and σ_{i+1} are disjoint and span σ_i, while if $i > 0$ is even then σ_i is a proper face in both σ_{i-1} and σ_{i+1} and

$$\mathrm{Res}((\sigma_{i-1})_{\sigma_i}, X_{\sigma_i}) \cap B_1((\sigma_{i+1})_{\sigma_i}, X_{\sigma_i}) = \emptyset.$$

In particular, it follows that n is even and that for any even $0 < i < n$ we have

$$\mathrm{Res}(\sigma_{i-2}, X_{\sigma_i}) \cap B_1(\sigma_{i+2}, X_{\sigma_i}) = \emptyset.$$

Thus the sequence $\sigma_0, \sigma_2, \sigma_4, ..., \sigma_n$ is a directed geodesic in X from v_0 to $g_n v_0$, and $\sigma_{2i+1} = \sigma_{2i} * \sigma_{2i+2}$ for any $0 \leq i < n/2$. Since we also have $g_i \in \Lambda_{b_i}$ for $0 \leq i \leq n$, the string $g_0\lambda_1...\lambda_n$ (with occurrences of 1 deleted) has the form as in the description of the language $\mathscr{L}$, i.e. it belongs to $\mathscr{L}$. This proves the regularity of $\mathscr{L}$.

14. Systolic versus CAT(κ)

In this section we discuss the relationship between k-systolic conditions and comparison CAT(κ) conditions for various metrics on simplicial complexes. As a main reference on CAT(κ) spaces we use [BH].

We start with few remarks concerning the standard piecewise euclidean metrics on simplicial complexes. In these metrics each simplex is isometric with the regular euclidean simplex of the same dimension with side lengths equal 1. An easy observation shows that in dimension 2 a simplicial complex X is systolic if and only if it is CAT(0) with respect to the standard piecewise euclidean metric. A local version of this observation says that X is locally 6-large if and only if it is nonpositively curved.

It turns out that the equivalence of the two curvature conditions as above does not hold in higher dimensions. To see a counterexample in dimension 3, recall that the angle α in the regular 3-simplex between a 2-face and a 1-face meeting at a vertex is less than $\pi/3$. Consider a simplicial complex X which is the union of six 3-simplices defined as follows. Consider vertices v_i and 1-simplices e_i with $i \in Z/3Z$ and a 2-complex K given as

$$K = \bigcup_{i \in Z/3Z} (v_i * e_i \cup e_i * v_{i+1}).$$

Take X to be the simplicial cone over K. X is easily seen to be 6-systolic, and on the other hand it is not CAT(0) since the spherical link of X at the cone vertex contains a closed geodesic of length 6α, which is less than 2π. Similar counterexamples can be constructed in any dimension $n \geq 3$. This shows that 6-systolic complexes are not necessarily CAT(0) for the standard piecewise euclidean metric.

The converse implication between the two conditions is also not true in higher dimensions. Consider the n-dimensional simplicial complex Y_n equal to the simplicial join of an $(n-2)$-dimensional simplex σ and the 1-dimensional cycle consisting of five edges. Clearly, Y_n is not 6-systolic, as its link at σ shows. On the other hand, the dihedral angle β_n in the regular n-simplex (between the faces of codimension 1) grows to $\pi/2$ as n grows to infinity. In fact, $\beta_n > 2\pi/5$ for all $n \geq 4$. This implies that Y_n is CAT(0) if $n \geq 4$, so a CAT(0) complex is not necessarily 6-systolic in these dimensions.

A much more subtle question is whether a 6-systolic complex admits any piecewise euclidean metric for which it is CAT(0). We do not have the answer to this question, but suspect it is negative.

An important problem that we study in the remaining part of this section is whether the stronger systolic conditions, i.e. k-systolicity for sufficiently large k, imply CAT(0) or even CAT(-1) condition for piecewise euclidean or piecewise hyperbolic metrics. Given a metric simplicial complex X, denote by Shapes(X) the set of isometry classes of the faces of X. Our main result in this section is the following.

14.1. *Theorem. — Let* Π *be a finite set of isometry classes of metric simplices of constant curvature* 1, 0 *or* -1. *Then there is a natural number* $k \geq 6$, *depending only on* Π, *such that:*

(1) if X *is a piecewise spherical k-large complex with* Shapes(X) $\subset \Pi$ *then* X *is* CAT(1)*;*
(2) if X *is piecewise euclidean (respectively, piecewise hyperbolic), locally k-large and* Shapes(X) $\subset \Pi$ *then* X *is nonpositively curved (respectively, has curvature* $\kappa \leq -1$*);*
(3) if, in addition to the assumptions of (2), X *is simply connected then it is* CAT(0) *(respectively,* CAT(-1)*).*

Remarks.

(1) The above theorem, combined with the constructions of k-systolic complexes in Sections 18 and 19, provides large class of new interesting examples of CAT(1), CAT(0) and CAT(-1) spaces.
(2) The proof of Theorem 14.1 given below does not lead to effective estimates for the number k. In Section 16 we explicitly estimate k for regular piecewise euclidean metrics.

Observe that parts (2) and (3) of the theorem follow directly from part (1) in view of characterization of the curvature bounds in terms of CAT(1) property for spherical links of a complex [BH, Theorems 5.2 and 5.4, p. 206]. We thus concentrate on the proof of part (1).

We start with two auxiliary results for which we need some preparation. Given a closed geodesic γ in a piecewise spherical simplicial complex X with Shapes(X) finite, the *size* of γ is the number of maximal nontrivial subsegments in γ contained in a single simplex of X. Note that this number is always finite since any local geodesic of finite length in X is the concatenation of a finite number of segments, each contained in a simplex ([BH, Corollary 7.29, p. 110]). The following result is a reformulation of [BH, Theorem 7.28, p. 109] or [B, Lemma 1].

14.2. *Theorem.* — *Given a finite set* $\mathscr{S}$ *of isometry classes of spherical simplices, there is a natural number* N *(depending on* $\mathscr{S}$*) such that if a local geodesic* γ *in a piecewise spherical simplicial complex* X *with* Shapes(X) $\subset \mathscr{S}$ *has length less than* 2π *then its size is less than* N.

Recall that a simplicial complex is ∞-*large* if it is *k*-large for any natural *k*. Using Fact 1.2.4 we can also characterize ∞-large simplicial complexes as those which are flag and contain no full cycle. In the proof of Theorem 14.1 we need the following.

14.3. *Proposition.* — *Let* X *be a piecewise spherical* ∞*-large simplicial complex with* Shapes(X) *finite. Then* X *contains no closed local geodesic.*

Remark. — Note that the above proposition implies that any piecewise spherical (with constant curvature 1) ∞-large simplicial complex is CAT(1). The straightforward argument for this uses the following two facts:

(1) a piecewise spherical complex is CAT(1) if and only if neither this complex nor any of its (spherical) links contains a closed geodesic of length less than 2π (compare [BH, Theorem 5.4(7), p. 206]);
(2) links of an ∞-large simplicial complex are ∞-large.

Proposition 14.3 is the direct consequence of the following.

14.4. *Lemma.* — *Let* X *be a piecewise spherical* ∞*-large simplicial complex with* Shapes(X) *finite. Then any local geodesic* γ *in* X *connecting two point of some simplex of* X *is contained in this simplex.*

Proof. — Let K be the full subcomplex of X spanned by those simplices whose interiors are intersected by γ. Since γ has finite size, the subcomplex K is finite. Moreover, γ is a local geodesic in K and K is still ∞-large. We argue by induction with respect to the number of maximal simplices in K. When K is a single simplex the assertion is obvious. Otherwise, by Dirac's characterization of finite ∞-large complexes (see Example 1.8.8), K is obtained from some ∞-large complexes K_1, K_2 by gluing along a single simplex σ. Note that γ has to be contained in K_1 or K_2, since otherwise K_1 or K_2 contains some subsegment of γ with both endpoints in σ but not contained in σ, and this is impossible by inductive assumption. If γ is contained in K_1 or K_2, the same inductive assumption implies the assertion, and the lemma follows.

Proof of Theorem 14.1. — As mentioned before, it is sufficient to prove part (1) of the theorem, i.e. the case of piecewise spherical complexes.

Let $\mathscr{S}$ be the link completion of Π, i.e. the union of Π and the set of isometry classes of all links in simplices representing all classes from Π. Since Π is finite, so is $\mathscr{S}$. Consider all closed geodesics γ of length less than 2π in all piecewise spherical flag simplicial complexes X with Shapes(X) $\subset \mathscr{S}$. For each such geodesic denote by K_γ the full subcomplex in the corresponding complex X spanned by the union of all simplices of X whose interior is intersected by γ. There are only finitely many combinatorial types of complexes K_γ as above because, due to Theorem 14.2, the number of vertices in any such complex is bounded by a universal constant (e.g. by the product of a constant N from Theorem 14.2 for the set $\mathscr{S}$ and the maximal dimension of a simplex with isometry class in $\mathscr{S}$).

Since each of the complexes K_γ contains a closed geodesic, it follows from Proposition 14.3 that it is not ∞-large. In particular, the systole $sys(K_\gamma)$ of any such complex is finite. Put

$$k = \max\{sys(K_\gamma) : K_\gamma \text{ as above}\} + 1$$

and note that the maximum is taken over a finite set (due to finiteness of combinatorial types of complexes K_γ).

We claim that any k-large piecewise spherical simplicial complex Y with Shapes(Y) $\subset \Pi$ is CAT(1). To prove this, observe that due to the definition of k, neither Y nor any of its links contains a closed geodesic of length less than 2π (this implies that Y is CAT(1), as already mentioned before; see [BH, Theorem 5.4(7), p. 206]). If this were not the case, we would have the corresponding subcomplex K_γ with $sys(K_\gamma) < k$ in a complex Z isomorphic either to Y or to some link of Y. Since Shapes(Z) $\subset \mathscr{S}$, we would have K_γ containing a full cycle of length less than k. But, since K_γ is a full subcomplex in Z, the same cycle would be full in the complex Z, contradicting the fact that Y is k-large. This completes the proof.

14.5. *Remark.* — Theorem 14.1 applies in particular to finite dimensional simplicial complexes equipped with the standard piecewise euclidean metrics. Note however that for these metrics the number k in the assertion of the theorem necessarily grows to infinity as the dimension of a complex grows. To see this, recall that if σ is the regular spherical $(2n-1)$-simplex with side lengths $\pi/3$ (i.e. the simplex occurring as the spherical link of the regular euclidean $2n$-simplex at a vertex) then the distance d_n between the barycenters of opposite $(n-1)$-faces in σ converges to 0 as n grows. In fact $d_n = \arccos(\frac{n}{n+1})$. For any $m \geq 3$ consider the simplicial complex X^{2n}_m of dimension $2n$ defined as the simplicial cone over the complex $\cup_{i\in Z/mZ}\tau_i * \tau_{i+1}$, where τ_i is an $(n-1)$-simplex for any $i \in Z/mZ$. Clearly, X^{2n}_m is an m-systolic simplicial complex. If we equip it with the standard piecewise euclidean metric, its spherical link at the cone vertex obviously contains a closed geodesic of length $m \cdot d_n$. A necessary condition for X^{2n}_m to be CAT(0) is that $m \cdot d_n \geq 2\pi$, i.e. that $m \geq 2\pi/d_n$, which justifies our observation.

15. Acute angled complexes

In this section we present another proof of Theorem 14.1, for the restricted case of acute angled complexes. Despite being less general, the proof has two advantages. First, its conclusion in the spherical case is stronger, namely that there is no homotopically trivial closed local geodesic both in the complex and in any of its links. Second, the proof in this section allows explicit and realistic estimates for the number k in the assertion. In Section 16 we give such estimates for standard piecewise euclidean metrics on complexes of any dimension.

A constant curvature simplex (spherical, euclidean or hyperbolic) is *acute angled* if all its dihedral angles (between codimension 1 faces) are less than $\pi/2$. A constant curvature metric simplicial complex is *acute angled* if all its faces are acute angled. Observe that if σ is an acute angled simplex then its links σ_τ at all faces τ are acute angled spherical simplices. Thus, all links of an acute angled complex are acute angled spherical complexes. Hence, as in Section 14, it is clearly sufficient to prove the theorem for (acute angled) spherical complexes.

We start with a few definitions. A *small ball* in a systolic simplicial complex X is a subcomplex of form $B_i(\sigma, X)$ for some simplex σ of X and for some $i \in \{0, 1, 2\}$. Given a real number $r > 0$, we say that a subset A in a geodesic metric space X is *r-convex* if for any two points in A at distance in X less than r, any geodesic in X connecting these two points is contained in A. The proof of Theorem 14.1 presented in this section relies on the following.

15.1. *Proposition.* — *Let* X *be a systolic piecewise spherical acute angled simplicial complex and suppose that*

(0) the set Shapes(X) *is finite;*
(1) all links of X *are* CAT(1)*;*
(2) all the small balls in the links of X *are* π*-convex.*

Then X *does not admit a closed local geodesic. Moreover, for any simplex* ν *in* X *any ball* $B = B_m(\nu, X)$ *is local-geodesically convex (i.e. any local geodesic segment in* X *with its endpoints in* B *is contained in* B*).*

Before giving a proof we present two useful corollaries to Proposition 15.1. Note that, by combining assumption (1) and the first assertion of the proposition we get that X as above is CAT(1). This observation is refined in the first corollary below. The *girth* of the complex X, denoted girth(X), is the infimum of the lengths of homotopically nontrivial paths in X.

15.2. *Corollary. — Let* X *be a locally 6-large piecewise spherical acute angled simplicial complex satisfying assumptions (0), (1) and (2) in Proposition 15.1, and suppose that* $\mathrm{girth}(X) \geq 2\pi$*. Then* X *is* CAT(1).

Proof. — Recall that if all links of a piecewise spherical complex X are CAT(1) and if there is no closed geodesic in X of length less than 2π then X is CAT(1). It remains to check the second assumption in the above statement. By applying Proposition 15.1 to the universal covering of X we conclude that there are no closed homotopically trivial geodesics in X. On the other hand, the length of each homotopically nontrivial closed geodesic in X is not less then $\mathrm{girth}(X) \geq 2\pi$, and the corollary follows.

To prove Theorem 14.1 we will need another result easily implied by Proposition 15.1.

15.3. *Corollary. — Let* X *be as in Corollary 15.2. Put*

$$\delta := \max\{diam(\sigma) : \sigma \in \mathrm{Shapes}(X)\}.$$

Suppose also that $\mathrm{girth}(X) \geq \pi + 5\delta$*. Then any small ball in* X *is* π*-convex.*

Proof. — Fix a small ball B in X. It is sufficient to prove that any geodesic segment in X intersecting B only at its endpoints has length $\geq \pi$. This is true if X is simply connected since it follows from the last assertion of Proposition 15.1 that there is no geodesic segment in X intersecting B only at its endpoints. Thus, in the general case, such a geodesic segment has to be homotopically nontrivial in X/B, and hence its length l can be estimated by

$$l \geq \mathrm{girth}(X) - diam(B) \geq \mathrm{girth}(X) - 5\delta \geq (\pi + 5\delta) - 5\delta = \pi.$$

This finishes the proof.

To prove Proposition 15.1 we need four preparatory results.

15.4. *Fact.* — Let K be a connected subcomplex in a CAT(1) piecewise spherical complex S. Suppose that links of K are π-convex in the corresponding links of S and that *diam*(K) $< \pi$. Then (1) K is π-convex in S and (2) K is CAT(1).

Proof. — Since *diam*(K) $< \pi$, any two points of K are connected by a geodesic segment in K of length less than π. Since K is locally π-convex in S, this segment is a local geodesic in S (compare [BH, Remark 5.7, p. 60] or [CD, Lemma 1.6.5]). Since S is CAT(1), this segment is a geodesic in S ([BH, Proposition 1.4(2), p. 160]) and, since S is π-uniquely geodesic (condition (6) in [BH, Theorem 5.4, p. 206]), it is the unique geodesic in S connecting these two points, hence (1). The same argument shows that K is π-uniquely geodesic, hence (2) (by equivalence of (5) and (6) in [BH, Theorem 5.4, p. 206]).

15.5. *Lemma.* — *Let* $X_0 = \sigma * X$ *and* $Y_0 = \sigma * Y$, *where* Y *is a subcomplex in a simplicial complex* X, σ *is a simplex and* $*$ *denotes the simplicial join. Suppose that* X_0 *is equipped with a piecewise spherical metric with all simplices acute angled, and that the spherical link* $(X_0)_\sigma$ *is* CAT(1) *while the spherical link* $(Y_0)_\sigma$ *is* π*-convex in* $(X_0)_\sigma$. *Then* X_0 *is* CAT(1) *and* Y_0 *is* π*-convex in* X_0.

Proof. — Denote by $*_s$ the operation of spherical join for piecewise spherical complexes. Viewing the simplex σ as embedded in the sphere S^n of dimension $n = \dim \sigma$, we can consider the embedding $i : X_0 \to S^n *_s (X_0)_\sigma$ which is isometric on simplices of X_0. For an appropriate choice of a piecewise spherical simplicial structure on $S^n *_s (X_0)_\sigma$, the map i identifies X_0 as a subcomplex in $S^n *_s (X_0)_\sigma$.

We will prove simultaneously the following three statements by induction on $k = \dim(X_0)$:

(1) X_0 is π-convex in $S^n *_s (X_0)_\sigma$ (here we identify X_0 with its image $i(X_0)$ through the embedding i);
(2) X_0 is CAT(1);
(3) Y_0 is π-convex in X_0.

The statements are clearly true if $k = 1$. The inductive step will be based on the observation that the assumptions in 15.5 are inherited by pairs of spherical links $(X_0)_\tau, (Y_0)_\tau$ for any simplex τ of Y_0. More precisely, denote by $\sigma + \tau$ the smallest simplex in X_0 containing both σ and τ, and by $\sigma - \tau$ the maximal face of σ disjoint with τ (empty, if $\sigma \subset \tau$). Then, for any τ in X_0 we have $(X_0)_\tau = (\sigma - \tau) * (X_0)_{\sigma+\tau}$. Moreover, if τ is contained in Y_0, we also have $(Y_0)_\tau = (\sigma - \tau) * (Y_0)_{\sigma+\tau}$. The metric assumptions of the lemma are satisfied for these links because both CAT(1) and π-convexity are the properties inherited by links.

Fix the pair X_0, Y_0 as in the lemma and suppose inductively that the assertions (1)–(3) are satisfied by the pairs of links as above. View X_0 as a subset in the spherical join $S^n *_s (X_0)_\sigma$. If τ is a simplex of X_0 containing σ, then the metric links $(X_0)_\tau$ and $[S^n *_s (X_0)_\sigma]_\tau$ coincide. Otherwise, the inclusion $(X_0)\tau \subset [S^n *_s (X_0)_\sigma]_\tau$ has the same form as the inclusion $X_0 \subset S^n *_s (X_0)_\sigma$. More precisely, the link $[S^n * (X_0)_\sigma]_\tau$ canonically identifies with the spherical join $S^m *_s (X_0)_{\sigma+\tau}$, where S^m is the sphere of dimension $m = \dim(\sigma + \tau)_\tau$. Moreover, $(X_0)_\tau$ has the form as X_0, with the simplex $(\sigma + \tau)_\tau = \sigma - \tau$ playing the role of σ, and with $[(X_0)_\tau]_{\sigma-\tau} = (X_0)_{\sigma+\tau}$ (metrically). An inclusion of $\sigma - \tau$ in S^m determines then the inclusion of $(X_0)_\tau$ in $S^m *_s (X_0)_{\sigma+\tau}$ which coincides with the inclusion of the metric links at τ of X_0 and $S^n *_s (X_0)_\sigma$.

By combining the above observation with assertion (1) in the inductive assumption we conclude that links of X_0 are π-convex in the corresponding links of the simplicial join $S^n *_s (X_0)_\sigma$. Since this join is CAT(1) (because spherical joins of CAT(1) spaces are CAT(1)) and $diam(X_0) < \pi$ (due to acute angleness), assertions (1) and (2) for X_0 follow from assertions (1) and (2) of Fact 15.4.

To prove that Y_0 is π-convex in X_0, observe that links of Y_0 are π-convex in the corresponding links of X_0. In view of the above described forms of links of X_0 and Y_0 this follows from the statement (3) in the inductive assumption. Since X_0 is already proved to be CAT(1), and the diameter of Y_0 is less than π, Fact 15.4.1 implies statement (3) for the pair X_0, Y_0, and the lemma follows.

Next two preparatory results concern combinatorial properties (related to convexity) of balls in systolic simplicial complexes. We fix the following assumptions and notation for these two results. Let X be a systolic simplicial complex with $\dim(X) = n$ and let ν be a simplex of X. For a ball $B = B_m(\nu, X)$ in X with $m \geq 1$ and with the sphere $S = S_m(\nu, m)$ (as defined in Section 7) consider the sequence $B = B^0 \subset B^1 \subset B^2 \subset \ldots \subset B^n = N_X(B)$ of subcomplexes in X defined recursively by

$$B^i = B^{i-1} \cup \bigcup \{\mathrm{Res}(\sigma, X) : \sigma \subset S,\ \dim(\sigma) = n - i\}.$$

Clearly, we then have $B^n = B_1(B, X) = B_{m+1}(\nu, X)$.

15.6. *Lemma. — Let $\sigma \subset S$ be a simplex of dimension $n - i$. Then*

(1) the link $(B^{i-1})_\sigma$ is a small ball in X_σ;
(2) $B^{i-1} \cap \mathrm{Res}(\sigma, X) = \mathrm{Res}(\sigma, B^{i-1})$.

Proof. — Recall that, by Corollary 7.9.2, $B_\sigma = B_1(\rho, X_\sigma)$ for some simplex $\rho \subset X_\sigma$. Moreover, from the definition of B^{i-1} it follows that $(B^{i-1})_\sigma = B_1(B_\sigma, X_\sigma)$, hence (1).

To prove (2), suppose that τ is a simplex in $B^{i-1} \cap \text{Res}(\sigma, X)$. Let τ_1, τ_2 be the maximal faces of τ contained in B and disjoint from B respectively. The latter is well defined since B is full in X (Corollary 7.5). For the same reason τ_1, τ_2 span τ and that τ_1, σ span a simplex of B. Since $\tau_2 \subset B^{i-1}$, there is a simplex $\rho \subset S$ of dimension at least $n-i+1$ such that $\tau_2 \subset \text{Res}(\rho, X)$. On the other hand, by Corollary 7.9.1, $\text{Res}(\tau_2, X) \cap B$ is a single simplex, and since it contains ρ, its dimension is at least $n - i + 1$. It follows that $[\text{Res}(\tau_2, X) \cap B] * \tau_2$ is a simplex in B^{i-1}. But $\text{Res}(\tau_2, X) \cap B$ also contains σ, hence σ and $\tau = \tau_1 * \tau_2$ span a simplex of B^{i-1}. This gives the inclusion $B^{i-1} \cap \text{Res}(\sigma, X) \subset \text{Res}(\sigma, B^{i-1})$, and since the converse inclusion is obvious, the lemma follows.

15.7. *Lemma.* — *If* σ_1, σ_2 *are two distinct simplices of dimension* $n - i$ *in* S *then* $\text{Res}(\sigma_1, X) \cap \text{Res}(\sigma_2, X) \subset B^{i-1}$.

Proof. — Let $\tau \subset \text{Res}(\sigma_1, X) \cap \text{Res}(\sigma_2, X)$ and suppose that, contrary to the assertion, τ is not contained in B^{i-1}. Since B is full in X (see Corollary 7.5) and $B \subset B^{i-1}$, by passing to a face of τ if necessary, we may (and will) assume that τ is disjoint with B. By convexity of B we know that the intersection $\text{Res}(\tau, X) \cap B$, which contains both σ_1 and σ_2, is then a single simplex (Corollary 7.9.1) which is contained in S and which we denote by σ. It follows that $\dim \sigma > \dim \sigma_1 = \dim \sigma_2 = n-i$, and hence $\text{Res}(\sigma, X) \subset B^{i-1}$. But τ is clearly contained in $\text{Res}(\sigma, X)$, and hence also in B^{i-1}, a contradiction. Hence the lemma.

Proof of Proposition 15.1. — It is sufficient to prove the last assertion in the statement of the proposition, i.e. that balls in X are local-geodesically convex: if there is a closed local geodesic γ in X then any ball intersecting γ and not containing it is not local-geodesically convex.

By the assumption that Shapes(X) is finite we know that a local geodesic in X of finite length is the concatenation of a finite number of segments, each contained in a simplex of X ([BH, Corollary 7.29, p. 110]). Thus, to prove the proposition, it is sufficient to apply recursively the following

Claim. — A local geodesic γ in X that leaves a ball $B = B_m(v, X)$ does not return to B before leaving the ball $B_{m+1}(v, X)$.

Suppose that $\dim(X) = n$. To get the claim it is sufficient to show that, for any $1 \leq i \leq n$, if a local geodesic γ leaves B^{i-1} then it does not return to B^{i-1} before leaving B^i.

Let γ be a local geodesic that leaves B^{i-1}. We may assume that γ is a local geodesic ray in X starting at a point $p \in B^{i-1}$ and locally near p intersecting B^{i-1} only at p. It may happen that γ leaves B_i at the same moment, i.e. that locally near p it intersects B^i only at p. Then our assertion holds. We will then consider the opposite case, when γ remains in B^i near p.

Note that, due to Lemma 15.7, the sets $\mathrm{Res}(\sigma, X) \setminus B^{i-1}$ for all simplices $\sigma \subset S$ with $\dim \sigma = n - i$ are pairwise disjoint. Thus, leaving B^{i-1}, γ enters exactly one of them. Again by Lemma 15.7, it is sufficient to show that γ does not return to B^{i-1} before leaving $\mathrm{Res}(\sigma, X)$.

Now we make use of Lemma 15.5. Put $X_0 = \mathrm{Res}(\sigma, X)$ and $Y_0 = B^{i-1} \cap X_0$. We then have $X_0 = \sigma * X_\sigma$ and, by Lemma 15.6.2, $Y_0 = \sigma * (B^{i-1})_\sigma$ (simplicially). Since, by Lemma 15.6.1, $(B^{i-1})_\sigma$ is a small ball in X_σ, it follows from assumptions of Proposition 15.1 that the pair X_0, Y_0 satisfies both combinatorial and metric assumptions of Lemma 15.5. Thus X_0 is CAT(1) while Y_0 is π-convex in X_0.

Any part of the local geodesic γ passing through X_0 is clearly a local geodesic in X_0. Moreover, since a local geodesic of length less than π in a CAT(1) space is a geodesic, and since $\mathit{diam}(X_0) < \pi$, any local geodesic in X_0 has length less than π. By π-convexity of Y_0, the maximal initial segment γ_0 of γ contained in X_0 (which has length less than π) intersects Y_0 only at the initial point p, and hence it intersects B^{i-1} only at p. Thus, γ does not return to B^{i-1} before leaving $X_0 = \mathrm{Res}(\sigma, X)$, which completes the proof.

Proof of Theorem 14.1 (for acute angled piecewise spherical complexes). — Note first that the theorem clearly holds for complexes X with $\dim X \leq 1$. Moreover, the number k can be chosen so large that additionally the small balls in those complexes are all π-convex. We will prove theorem together with the additional property of π-injectivity for all small balls in X, using induction with respect to $n = \dim X$.

Suppose that the theorem and the assertion that all small balls in X are π-convex holds for all complexes X with $\dim X \leq n$. Let Π be a finite set of (isometry classes of) acute angled spherical simplices, and denote by $L(\Pi)$ the set of (isometry classes of) all links of simplices from Π. Then $L(\Pi)$ is also finite. Let k_1 be a natural number as prescribed by the inductive assumption for complexes X with $\mathrm{Shapes}(X) \subset L(\Pi)$ and $\dim X \leq n$. Let X be a k_1-large complex with $\mathrm{Shapes}(X) \subset \Pi$ and with $\dim(X) = n + 1$. Then, by the inductive assumption, the links of X are CAT(1) and all small balls in those links are π-convex. Thus X satisfies the assumptions of Proposition 15.1, and hence also the assumptions of Corollaries 15.2 and 15.3 except perhaps those concerning girth. To get the inductive step, note that by requiring that $\mathit{sys}_h(X) \geq k$ for sufficiently large $k \geq k_1$ we can assure that girth(X) is as large as we wish. In particular, we can assure that $\mathrm{girth}(X) \geq \max(2\pi, \pi + 5\delta)$, where $\delta = \max\{\mathit{diam}\Delta : \Delta \in \Pi\}$. It follows that if X is k-large (which implies that links of X are k_1-large and $\mathit{sys}_h(X) \geq k$) then X is CAT(1) (Corollary 15.2) and the small balls in X are π-convex (Corollary 15.3). This finishes the inductive proof.

15.8. *Remark*. — In the next section we give explicit estimates of girth(X) in terms of $sys_h(X)$ for piecewise spherical complexes occurring as links in complexes with the standard piecewise euclidean metric. In view of the last part of the above proof, this gives explicit constants k in Theorem 14.1, depending only on dimension, for complexes with the standard piecewise euclidean metric. In principle, such explicit estimates for constants k can be obtained for other finite sets of acute angled shapes as well.

16. Explicit constants

In this section we prove more explicit version of Theorem 14.1.3, for complexes with standard piecewise euclidean metrics. It is obtained by referring to the arguments from Section 15. A large part of the section deals with more general metrics and the obtained results can be used to derive explicit estimates for other classes of piecewise constant curvature acute angled complexes. In the case that we study in detail we get the following.

16.1. *Theorem*. — *Let k be a natural number such that*

$$k \geq \frac{7\pi\sqrt{2}}{2} \cdot n + 2.$$

Then any k-systolic simplicial complex X *with* $\dim X \leq n$ *is* CAT(0) *with respect to the standard piecewise euclidean metric.*

Remark. — The estimate for k in the above theorem is obviously not optimal. It gives $k \geq 34$ for $n = 2$, while $k \geq 6$ is clearly sufficient. For $n = 3$ the theorem gives $n \geq 49$, while a careful application of our methods allows to get $k \geq 11$. The estimate seems also to be far from optimal asymptotically, as $k \to \infty$. We expect that $k \geq C \cdot \sqrt{n}$ for some constant C is sufficient. Note that, since in Remark 14.5 we have $d_n \sim 1/\sqrt{n}$, the latter prediction coincides with the necessary condition observed in this remark. Also, our choice of functions $\varphi^n_{L,j}$ (just before Lemma 16.7, below) is clearly not optimal, and we see potential for improvement of the estimate in making this choice more carefully.

To prove Theorem 16.1 we need a few preparatory results. We formulate first three of them in the framework of piecewise riemannian simplicial complexes (though we are interested in piecewise spherical case only), because it exhibits better the essence of our arguments.

Recall that a *riemannian simplex* is a simplex equipped with a smooth riemannian metric. A *piecewise riemannian simplicial complex* is obtained from riemannian simplices by gluing them together along some of their faces through diffeomorphisms that preserve riemannian metrics restricted to those faces. A piecewise riemannian simplicial complex having the set Shapes(X) of its riemannian simplices finite, is equipped with the metric given by minimizing lengths of piecewise smooth paths. Our aim is the following.

16.2. *Proposition. — Let $\mathcal{S}$ be a finite set of isometry classes of riemannian simplices. Then there exists a constant* $D_{\mathcal{S}} > 0$ *such that if* X *is a metric simplicial complex with* Shapes(X) $\subset \mathcal{S}$ *then* $\text{girth}(X) \geq D_{\mathcal{S}} \cdot (sys_h(X) - 2)$.

In the proof of the above proposition we will need to estimate distances in complexes X in terms of gradients of some piecewise smooth functions. A real valued function $f : X \to R$ is *piecewise smooth* if its restriction $f|_\sigma$ to any simplex $\sigma \subset X$ is smooth. Given such a function, put

$$M_f := \sup\{\max\{\|\nabla(f|_\sigma)(x)\| : x \in \sigma\} : \sigma \subset X\},$$

where ∇ denotes gradient and $\|\cdot\|$ denotes length (for vectors tangent to σ) with respect to the riemannian metric on σ. One of the well known properties of gradient is the following.

16.3. *Lemma. — Let* $f : X \to R$ *be a piecewise smooth function on a connected metric (riemannian) simplicial complex* X. *Then for any points* $p, q \in X$ *we have* $|f(p) - f(q)| \leq M_f \cdot d_X(p, q)$. *In particular, if the supremum* M_f *is finite then*

$$d_X(p, q) \geq \frac{1}{M_f} \cdot |f(p) - f(q)|.$$

Given a connected simplicial complex X and a simplex $\sigma \subset X$, a *distance-like function for* (X, σ) is a piecewise smooth function $f : X \to R$ such that $S_i(\sigma, X) \subset f^{-1}(i)$. (Recall that $S_i(\sigma, X)$ is a subcomplex of X spanned by the set of all vertices in X at polygonal distance i from σ.)

16.4. *Lemma. — Given a finite set $\mathcal{S}$ of isometry classes of riemannian simplices, there is a constant* $0 < M_{\mathcal{S}} < \infty$ *with the following property: For any connected metric simplicial complex* X *with* Shapes(X) $\subset \mathcal{S}$ *and for any simplex* $\sigma \subset X$ *there is a distance-like function* f *for* (X, σ) *with* $M_f \leq M_{\mathcal{S}}$.

Proof. — We will show that distance-like functions for complexes X with Shapes(X) finite can be constructed out of an essentially finite collection $\mathcal{F}_{\mathcal{S}}$ of functions on the simplices from Shapes(X). (By saying that $\mathcal{F}_{\mathcal{S}}$ is essentially finite we mean that it is obtained from some finite sub-collection $\mathcal{F}^0_{\mathcal{S}}$ by adding constants.) This clearly implies the lemma since for such functions the supremum

M_f is taken essentially over a subset in a finite set of numbers, namely the set of maxima of gradient lengths for functions in $\mathscr{F}_{\mathscr{S}}$. A collection $\mathscr{F}_{\mathscr{S}}$ as above can be constructed as follows.

For each 1-simplex E in $\mathscr{S}$ consider all combinations of values 0 and 1 at the vertices of E. For each such combination take a smooth function $\varphi : E \to R$ compatible with the prescribed values at vertices and such that φ is constant if the two values at vertices are equal. Further, for each 2-simplex Δ in $\mathscr{S}$ consider all combinations of values 0 and 1 at the vertices. Given such a combination, for each boundary face of Δ consider the already defined function on the simplex in $\mathscr{S}$ isometric to this face, respecting the prescribed values at vertices. Extend the so obtained function on the boundary of Δ to a smooth function on Δ so that it is a constant function if the prescribed values at the vertices are all equal. By applying this procedure gradually to the simplices in $\mathscr{S}$ of all dimensions we get a finite collection $\mathscr{F}^0_{\mathscr{S}}$ of functions. As $\mathscr{F}_{\mathscr{S}}$ take the set of all functions obtained from the functions in $\mathscr{F}^0_{\mathscr{S}}$ by adding constants from the set of natural numbers (including 0).

For any complex X with Shapes(X) $\subset \mathscr{S}$ and for any simplex $\sigma \subset X$ one can construct a distance-like function f for (X, σ) simplex-wise, out of the functions from $\mathscr{F}_{\mathscr{S}}$, as follows. As values of f at the vertices of X take their polygonal distances from σ. Next, observe that for any simplex τ in X one of the following two cases holds:

(1) the values of f at the vertices of τ are all equal;
(2) the set of values of f at the vertices of τ consists of two natural numbers that differ by 1.

This observation shows that we can extend f gradually to higher dimensional skeleta of X, using the functions from $\mathscr{F}_{\mathscr{S}}$.

By the construction of the functions in $\mathscr{F}_{\mathscr{S}}$ we know that if for some simplex τ in X the above case (1) holds then a function f obtained as above is constant at τ. This implies that f is also constant at the spheres $S_i(\sigma, X)$, with values i, and thus it is a distance-like function for (X, σ), as required. This finishes the proof.

Proof of Proposition 16.2. — Let $\widetilde{X}$ be the universal cover of X with the lifted metric. Then girth(X) is equal to the infimum of the distances $d_{\widetilde{X}}(p_1, p_2)$ over all points $p \in X$ and all pairs p_1, p_2 of distinct lifts of p to $\widetilde{X}$. Fix a pair p_1, p_2 as above, and let σ be a simplex of $\widetilde{X}$ containing p_1. Observe that, if $m = sys_h(X)$, then p_2 lies outside the ball $B_{m-2}(\sigma, \widetilde{X})$. It follows that

$$d_{\widetilde{X}}(p_1, p_2) \geq \inf\{d_{\widetilde{X}}(p_1, q) : q \in S_{m-2}(\sigma, \widetilde{X})\}. \quad (\mathbf{16.2.1})$$

Let $M_{\mathscr{S}}$ be as in Lemma 16.4. Since $\mathrm{Shapes}(\widetilde{X}) \subset \mathscr{S}$, the same lemma implies that there is a distance-like function f for $(\widetilde{X}, \sigma)$ with $M_f \leq M_{\mathscr{S}}$. We clearly have $f(p_1) = 0$ and $f(q) = m - 2$ for any $q \in S_{m-2}(\sigma, \widetilde{X})$. Applying Lemma 16.3 we get

$$d_{\widetilde{X}}(p_1, q) \geq \frac{1}{M_f} \cdot (m-2) \geq \frac{1}{M_{\mathscr{S}}} \cdot (m-2) = \frac{1}{M_{\mathscr{S}}} \cdot (sys_h(X) - 2).$$

Combining this with the inequality (16.2.1) we get the proposition for $D_{\mathscr{S}} = 1/M_{\mathscr{S}}$.

We now shift our attention to piecewise constant curvature acute angled complexes. We will apply Proposition 16.2 together with the results and ideas of Section 15 to get the following.

16.5. *Proposition. — Let $\mathscr{S}_0$ be a finite set of isometry classes of acute angled spherical simplices, and denote by $\mathscr{S}$ its link completion, i.e. the union of $\mathscr{S}_0$ and the set of isometry classes of links at all faces for all simplices in $\mathscr{S}_0$. Let $D_{\mathscr{S}}$ be a constant as in Proposition 16.2, and k a natural number such that*

$$k \geq \max\left[6, \frac{7\pi}{2D_{\mathscr{S}}} + 2\right].$$

If X is a k-large piecewise spherical complex with $\mathrm{Shapes}(X) \subset \mathscr{S}_0$ *then X is* CAT(1).

By applying the characterization of the CAT(0) and CAT(−1) conditions in terms of the CAT(1) condition for links (see condition (4) in [BH, Theorem 5.4, p. 206]), Proposition 16.5 implies the following.

16.6. *Corollary. — Let $\mathscr{T}$ be a finite set of isometry classes of acute angled euclidean (respectively hyperbolic) simplices, and denote by $\mathscr{S}$ the set of isometry classes of links at all faces for all simplices in $\mathscr{T}$. Let $D_{\mathscr{S}}$ be a constant as in Proposition 16.2, and k a natural number such that*

$$k \geq \max\left[6, \frac{7\pi}{2D_{\mathscr{S}}} + 2\right].$$

If X is a k-systolic piecewise euclidean (respectively piecewise hyperbolic) complex with $\mathrm{Shapes}(X) \subset \mathscr{T}$ *then X is* CAT(0) *(respectively* CAT(−1)*).*

Proof of Proposition 16.5. — First note that if X is k-large then $sys_h(X) \geq k$ and $sys_h(X_\sigma) \geq k$ for all links X_σ of X. It follows then from Proposition 16.2 that $\mathrm{girth}(X) \geq 7\pi/2$ and $\mathrm{girth}(X_\sigma) \geq 7\pi/2$ for all links X_σ. Moreover, by acuteness, diameters of all simplices in X and in all links X_σ are less than $\pi/2$, so if δ is as in Corollary 15.3 for X or for X_σ respectively, we get

$$\mathrm{girth}(X) \geq \frac{7\pi}{2} \geq \pi + 5\delta \quad \text{and} \quad \mathrm{girth}(X_\sigma) \geq \frac{7\pi}{2} \geq \pi + 5\delta.$$

Now, using induction with respect to the dimension of complexes, based on Corollaries 15.2, 15.3 and on the above inequalities, we get that all links X_σ in X are CAT(1) and all small balls in them are π-convex. In the end of this inductive proof we get that X is CAT(1), hence the proposition.

In the next series of preparatory results we study piecewise spherical complexes composed of regular simplices with fixed side lengths. Such complexes occur as links in complexes with standard piecewise euclidean metrics. We define and study some functions on the regular spherical simplices. These functions allow to construct appropriate distance-like functions on the complexes as above and to calculate explicitly the constants $M_{\mathscr{S}}$ as in Lemma 16.3 in the situations under our interest.

Let Σ^n_L be the n-dimensional spherical (with constant curvature 1) regular simplex with side lengths L. This makes sense for $0 < L < 2\pi/3$, but we will be interested in the cases when $\pi/3 \leq L < \pi/2$. Let S^n be the sphere of radius 1 canonically embedded in the euclidean space E^{n+1}, and suppose that Σ^n_L is embedded in S^n. Denote by Δ^n_L the simplex in E^{n+1} affinely spanned by the vertices $v_1, \dots, v_{n+1}$ of Σ^n_L, with the induced regular euclidean metric in which the sides of Δ^n_L have lengths $2\sin(L/2)$. Consider also the radial projection $P^n_L : \Sigma^n_L \to \Delta^n_L$, in the direction of the center of S^n, which is clearly a diffeomorphism. For $j = 0, 1, \dots, n$ let $\lambda^n_{L,j}$ be the linear function on the simplex Δ^n_L with values 1 at the vertices $v_1, \dots, v_j$ and 0 at the remaining vertices. Finally, define functions $\varphi^n_{L,j} : \Sigma^n_L \to \mathbf{R}$ by putting $\varphi^n_{L,j} := \lambda^n_{L,j} \circ P^n_L$.

16.7. *Lemma. — For $j = 1, \dots, n$ let $H^n_{L,j}$ be the distance in Δ^n_L between the barycenters of opposite faces of dimensions $j-1$ and $n-j$. Denote also by β^n_L the distance in the simplex Σ^n_L between its barycenter and any of its vertices. Then*

$$\max\left\{\left\|\nabla\varphi^n_{L,j}(x)\right\| : x \in \Sigma^n_L\right\} \leq \frac{1}{H^n_{L,j} \cdot \cos\beta^n_L}$$

for $j = 1, \dots, n$.

Proof. — Note that, since the function λ^n_L is linear and takes the values 0 and 1 at the opposite faces of dimensions $j-1$ and $n-j$ respectively, we have

$$(\mathbf{16.7.1}) \qquad \left\|\nabla\lambda^n_{L,j}(y)\right\| = \frac{1}{H^n_{L,j}} \quad \text{for each } y \in \Delta^n_L.$$

Since $\varphi^n_{L,j} = \lambda^n_{L,j} \circ P^n_L$, we may use the following estimate for gradient length of a pulled back function, which follows directly from the chain rule.

Fact. — Let M_1, M_2 be riemannian manifolds, $f : M_1 \to R$ a smooth function and $q : M_2 \to M_1$ a smooth map. Then for any $x \in M_2$ we have

$$\|\nabla(f \circ q)(x)\| \leq \|\nabla f(q(x))\| \cdot \|dq_x\|, \tag{16.7.2}$$

where $\|dq_x\|$ is the norm of the differential $dq_x : T_xM_2 \to T_{f(x)}M_1$ with respect to riemannian norms at tangent spaces.

To apply the above fact in our proof we need to estimate the norms $\|(d\varphi^n_{L,j})_x\|$ for $x \in \Sigma^n_L$. View again Σ^n_L as embedded in $S^n \subset E^{n+1}$, and Δ^n_L as affinely spanned in E^{n+1} by the vertices of Σ^n_L. The riemannian lengths of vectors tangent to Σ^n_L and Δ^n_L coincide then with the ordinary euclidean lengths of these vectors in E^{n+1}. Fix any $x \in \Sigma^n_L$ and any vector V tangent to σ^n_L at x. Put $y = P^n_L(x) \in \Delta^n_L$ and note that the differential $(dP^n_{L,j})_x : T_x\Sigma^n_L \to T_y\Delta^n_{L,j}$ is the restriction of the differential $dP_x : T_xE^{n+1} \to T_y\Delta^n$ of the radial projection in E^{n+1} (with respect to the center of S^n) from an open neighbourhood U of Σ^n_L to the hyperplane containing Δ^n_L. Let $V = V_r + V_p$, where V_r is the radial component of V in E^{n+1} (parallel to the radius of S^n through x) and V_p is its component parallel to Δ^n_L. Since clearly $dP_x(V_r) = 0$ and $dP_x(V_p) = a \cdot V_p$, where $a \leq 1$ is the ratio of the distances from the center of S^n of the points y and x respectively, we get $(dP^n_L)_x(V) = a \cdot V_p$. To estimate the length of the component V_p, denote by α_x the angle between the radii in S^n through the barycenter of σ^n_L and through x. Since α_x is also the dihedral angle between the hyperplane tangent to Σ^n_L at x and the hyperplane containing Δ^n_L, we get $\|V_p\| \leq \|V\|/\cos\alpha_x$. But in our case we have $\alpha_x \leq \beta^n_L$ and we obtain an estimate

$$\left\|\left(dP^n_L\right)_x(V)\right\| = \|a \cdot V_p\| \leq \frac{a}{\cos\beta^n_L}\|V\| \leq \frac{1}{\cos\beta^n_L}\|V\|.$$

This shows that

$$\left\|\left(dP^n_L\right)_x\right\| \leq \frac{1}{\cos\beta^n_L} \quad \text{for each } x \in \Sigma^n_L.$$

By combining this with (16.7.1) and (16.7.2) the lemma follows.

16.8. *Corollary.* — *If* $\pi/3 \leq L \leq \pi/2$ *then*

$$\max\left\{\left\|\nabla\varphi^n_{L,j}(x)\right\| : x \in \Sigma^n_L\right\} \leq \frac{(n+1)\sqrt{2}}{2}$$

for $j = 1, \ldots, n$.

Proof. — Note that the size of the simplex Δ^n_{L} increases with the increase of L and hence for $\mathrm{L} \geq \pi/3$ we have $\mathrm{H}^n_{\mathrm{L},j} \geq \mathrm{H}^n_{\pi/3,j}$. A direct computation in the simplex $\Delta^n_{\pi/3}$ (which has side lengths 1) shows that

$$\mathrm{H}^n_{\pi/3,j} \geq \frac{\sqrt{2}}{\sqrt{n+1}}$$

for any $1 \leq j \leq n$. On the other hand, if $\mathrm{L} \leq \pi/2$, we clearly have $\beta^n_{\mathrm{L}} \leq \beta^n_{\pi/2}$. By a direct computation in the right-angled spherical simplex $\Sigma^n_{\pi/2}$ we get that $\cos \beta^n_{\pi/2} = 1/\sqrt{n+1}$ which implies that

$$(\mathbf{16.8.2}) \qquad \cos \beta^n_{\mathrm{L}} \geq \frac{1}{\sqrt{n+1}}.$$

Combining the inequalities (16.8.1) and (16.8.2) with the inequality from Lemma 16.7 finishes the proof.

Proof of Theorem 16.1. — Note that, due to the definition of the functions $\varphi^n_{\mathrm{L},j}$ in terms of linear functions and radial projections, the restriction of any such function to a face $\Sigma^{n'}_{\mathrm{L}}$ in Σ^n_{L} is either constant equal to 1 or coincides with the appropriate function $\varphi^{n'}_{\mathrm{L},j'}$. Thus, the functions obtained from the functions $\varphi^n_{\mathrm{L},j}$ (for all n and j) by adding natural constants are sufficient to construct distance-like functions as in the proof of Lemma 16.4 for metric complexes with all simplices spherical regular of side length L. Denoting by $\mathscr{S}^n_{\mathrm{L}}$ the set of (isometry classes of) simplices Σ^i_{L} with $0 \leq i \leq n$, and assuming that $\pi/3 \leq \mathrm{L} \leq \pi/2$, we get from Corollary 16.8 that $\mathrm{M}_{\mathscr{S}^n_{\mathrm{L}}} = (n+1)\sqrt{2}/2$ works in Lemma 16.4 for $\mathscr{S} = \mathscr{S}^n_{\mathrm{L}}$.

Let $\mathscr{T}$ be the set of isometry classes of the standard regular euclidean simplices of dimensions $\leq n$. Then the set $\mathscr{S}$, as in Corollary 16.8, of isometry classes of links at all faces for all simplices in $\mathscr{T}$ can be expressed as the union

$$\mathscr{S} = \cup_{i=0}^{n-2} \mathscr{S}^{n-1-i}_{\mathrm{L}_i},$$

where each of the sets $\mathscr{S}^{n-1-i}_{\mathrm{L}_i}$ consists of links at i-dimensional faces and the numbers $\mathrm{L}_i = \arccos(1/(k+2))$ are the side lengths in such links, as a direct calculation shows. Since $\pi/3 \leq \mathrm{L}_i < \pi/2$, the argument above shows that we can take $\mathrm{M}_{\mathscr{S}} = n\sqrt{2}/2$ in the conclusion of Lemma 16.4 for $\mathscr{S}$ as above. Consequently, by referring to the end of proof of Proposition 16.2, we can take $\mathrm{D}_{\mathscr{S}} = 1/\mathrm{M}_{\mathscr{S}} = \sqrt{2}/n$ in the conclusion of Corollary 16.6, for $\mathscr{T}$ and $\mathscr{S}$ as above, hence the theorem.

17. Locally 6-large simplices of groups

In this section we recall and adapt to our needs some notions and facts related to simplices of groups and simple complexes of groups. We will use them in the construction described in Section 18. Since both simplices of groups and simple complexes of groups are special cases of complexes of groups, some parts of this section repeat the exposition of Section 6 in these special cases. However, the exposition here is more detailed, more self-contained, and free from several technicalities. For example, we do not mention twisting elements $g_{\sigma\tau\rho}$, since they are all assumed to be trivial, i.e. equal to the units in the corresponding groups. On the other hand, we discuss explicitly the notions related to developability. We also change slightly the notation to make it more convenient for our purposes. The reader is advised to consult Section 12 in Part II of [BH] as a standard reference.

For a simplex Δ, denote by P_Δ the poset of all nonempty faces of Δ, including Δ itself, and denote by $<$ the relation of being a proper (sub)face. A *simplex of groups* $\mathcal{G}$ over Δ is a family $G_\sigma : \sigma \in P_\Delta$ of groups, together with a family of injective homomorphisms $\psi_{\sigma\tau} : G_\tau \to G_\sigma$ for any pair $\sigma < \tau$, such that $\psi_{\sigma\tau} \circ \psi_{\tau\rho} = \psi_{\sigma\rho}$ whenever $\sigma < \tau < \rho$. We will call groups G_σ *local groups* of $\mathcal{G}$ and homomorphisms $\psi_{\sigma\tau}$ *structure homomorphisms* of $\mathcal{G}$.

A *morphism* $m : \mathcal{G} \to F$ from a simplex of groups $\mathcal{G}$ over Δ to a group F is a family $m_\sigma : \sigma \in P_\Delta$ of homomorphisms $m_\sigma : G_\sigma \to F$ which agree with the structure homomorphisms of $\mathcal{G}$ in the sense that $m_\tau = m_\sigma \circ \psi_{\sigma\tau}$ whenever $\sigma < \tau$. Given a simplex of groups $\mathcal{G}$, denote by $\hat{\mathcal{G}}$ the *direct limit* of $\mathcal{G}$, i.e. the quotient group of the free product of the groups $G_\sigma : \sigma \in P_\Delta$ by the normal subgroup generated by relations of form $g = \psi_{\sigma\tau}(g)$ for all structure homomorphisms $\psi_{\sigma\tau}$ and all $g \in G_\tau$. Denote by $i_{\mathcal{G}} : \mathcal{G} \to \hat{\mathcal{G}}$ the canonical morphism to the direct limit. This morphism has (or can be characterized by) the universal property saying that any morphism $m : \mathcal{G} \to F$ factors through $i_{\mathcal{G}}$, i.e. there is the homomorphism $\hat{m} : \hat{\mathcal{G}} \to F$ such that $m = \hat{m} \circ i_{\mathcal{G}}$. The homomorphism $\hat{m}$ is unique and we call it *the homomorphism induced by* m.

A morphism $m : \mathcal{G} \to F$ is *locally injective* if all its homomorphisms m_σ are injective. It is *surjective* if the target group F is generated by the union $\bigcup_{\sigma \in P_\Delta} m_\sigma(G_\sigma)$. A simplex of groups is *developable* if it admits a locally injective morphism (equivalently, if its canonical morphism to the direct limit is locally injective). Locally injective and surjective morphisms can be characterized in terms of the direct limit as being identical to the compositions $q \circ i_{\mathcal{G}}$, where $q : \hat{\mathcal{G}} \to \hat{\mathcal{G}}/N$ is the quotient homomorphism and $N \subset \hat{\mathcal{G}}$ is a normal subgroup such that $N \cap (i_{\mathcal{G}})_\sigma(G_\sigma) = \{1\}$ for any $\sigma \in P_\Delta$.

Given a locally injective morphism $m : \mathcal{G} \to F$ of a simplex of groups $\mathcal{G}$ over Δ, we define the *development* $D(\mathcal{G}, m)$ of $\mathcal{G}$ with respect to m as follows.

First, identify the local groups G_σ with their images $m_\sigma(G_\sigma) \subset F$, and the structure homomorphisms $\psi_{\sigma\tau}$ with the inclusions of the corresponding subgroups of F. Define an equivalence relation $\sim$ on the set $\Delta \times F$ by

$$(x, g) \sim (y, h) \quad \text{iff} \quad x = y \in \sigma \text{ and } g^{-1}h \in G_\sigma \text{ for some face } \sigma \text{ of } \Delta.$$

Let $[x, g]$ be the equivalence class of (x, g), $[\sigma, g] := \{[x, g] : x \in \sigma\}$, and put

$$D(\mathcal{G}, m) = \Delta \times F / \sim.$$

We obtain then a multi-simplicial complex with the faces $[\sigma, g]$. These are injective images of $\sigma \times \{g\}$ under the quotient map of $\sim$. This complex is multi-simplicial and not just simplicial since the intersection of its faces is in general a union of faces and not just a single face. This construction is called the Basic Construction in [BH, II.12]. We insist on using a coarser simplicial structure than [BH] (who use the barycentric subdivisions of our faces).

Most of the simplices of groups in this paper will satisfy the property that the local group $\mathcal{G}_\Delta$ (where Δ is the underlying simplex of $\mathcal{G}$) is trivial, i.e. $G_\Delta = \{1\}$. We will call such simplices of groups ∂-*supported*. The next proposition gathers general and well known properties of developments. We present these properties in the restricted context of ∂-supported simplices of groups, which simplifies formulations and is sufficient for the purposes of this paper. These results (including their proofs) can be found in [BH, II.12] (compare also [JS1, Proposition 3.2]).

17.1. *Proposition. — Let $\mathcal{G}$ be a ∂-supported simplex of groups over a simplex Δ, and let $m : \mathcal{G} \to F$ be a locally injective morphism.*

(1) The formula $h[x, g] = [x, hg]$ defines an action of the group F on $D(\mathcal{G}, m)$ by automorphisms. The quotient map of this action is equal to the map induced by the projection $\Delta \times F \to \Delta$. The action is without inversions, i.e. a face preserved by an automorphisms is fixed pointwise. The stabilizer of a face $[\sigma, g]$ is a conjugation $G_\sigma^g := gG_\sigma g^{-1}$ (we still view the local groups of $\mathcal{G}$ as subgroups of F, via m).

(2) $D(\mathcal{G}, m)$ is finite (as a complex) if and only if F is a finite group.

(3) $D(\mathcal{G}, m)$ is locally finite if and only if the groups G_σ for all faces σ of Δ are finite. In fact, for local finiteness it is sufficient to require that the groups G_v for all vertices v of Δ are finite.

(4) $D(\mathcal{G}, m)$ is connected if and only if the morphism m is surjective.

(5) $D(\mathcal{G}, m)$ is a pure complex, i.e. it is the union of its top dimensional faces.

(6) $D(\mathcal{G}, m)$ is gallery connected if and only if the subgroups G_s for all codimension-1 faces s of Δ generate F. Recall that gallery connected means that any two top dimensional faces are connected by a finite sequence of top dimensional faces such that consecutive faces have common codimension-1 subface.

(7) $D(\mathscr{G}, m)$ *is a pseudomanifold if and only if in addition to (3) and (6) the local groups* G_s *of* $\mathscr{G}$ *are isomorphic to* Z_2 *for all codimension-1 faces* s *of* Δ.

(8) $D(\mathscr{G}, m)$ *is an orientable pseudomanifold if and only if in addition to (7) there is a homomorphism* $\rho : F \to Z_2$ *whose restriction* $\rho_s : G_s \to Z_2$ *is an isomorphism for all codimension-1 faces* s *of* Δ *(equivalently,* $\rho \circ m_s : G_s \to Z_2$ *is an isomorphism for any such* s*).*

The next proposition describes the fundamental group of the development of a surjective morphism, in terms of the direct limit. Recall that we denote by $\hat{m} : \hat{\mathscr{G}} \to F$ the homomorphism induced by a morphism $m : \mathscr{G} \to F$. Here we do not need to assume that $\mathscr{G}$ is ∂-supported.

17.2. *Proposition. — Let* $\mathscr{G}$ *be a developable simplex of groups and let* $m : \mathscr{G} \to F$ *be a locally injective and surjective morphism. Then* $\pi_1(D(\mathscr{G}, m)) = \ker(\hat{m} : \hat{\mathscr{G}} \to F)$. *In particular,* $D(\mathscr{G}, m)$ *is simply connected iff* $F = \hat{\mathscr{G}}$ *and* $m = i_{\mathscr{G}}$.

We will call development $D(\mathscr{G}, i_{\mathscr{G}})$ the *universal development* of a developable simplex of groups $\mathscr{G}$ (or the universal covering of $\mathscr{G}$), and denote it shortly by $\widetilde{\mathscr{G}}$.

We now turn to discussion of links in developments. Given a simplex Δ and its face σ, the *link* Δ_σ of Δ at σ is the spherical simplex composed of the unit vectors tangent to Δ and orthogonal to σ at a fixed interior point of σ. The face poset P_{Δ_σ} of Δ_σ canonically identifies with the subposet $(P_\Delta)_\sigma$ in P_Δ consisting of all faces τ such that τ properly contains σ.

If K is a multi-simplicial complex, and σ is its face, then the link K_σ is a union of the links τ_σ for all faces τ of K that properly contain σ, glued together into a multi-simplicial complex according to the equivalence relation on the disjoint union induced by the natural inclusions $\tau_\sigma \subset \tau'_\sigma$ for all pairs $\tau \subset \tau'$.

Given a simplex of groups $\mathscr{G}$ over Δ and a face σ of Δ, consider the restriction $\mathscr{G}_\sigma := \mathscr{G}|_{(\mathscr{P}_\Delta)_\sigma}$ and view it as a simplex of groups over the link simplex Δ_σ. Put also $i_\sigma := \{\psi_{\sigma\tau} : \tau \in (\mathscr{P}_\Delta)_\sigma\}$ and note that $i_\sigma : \mathscr{G}_\sigma \to G_\sigma$ is a morphism. Observe that since all the homomorphisms $\psi_{\sigma\tau}$ are injective, i_σ is a locally injective morphism.

17.3. *Proposition. — Let* $\mathscr{G}$ *be a simplex of groups over* Δ *and let* $m : \mathscr{G} \to F$ *be a locally injective morphism. Then, given a face* $[\sigma, g]$ *in the development* $D(\mathscr{G}, m)$*, the link* $D(\mathscr{G}, m)_{[\sigma,g]}$ *is isomorphic to the development* $D(\mathscr{G}_\sigma, i_\sigma)$. *Moreover, this isomorphism is equivariant with respect to the action of the stabilizing subgroup* $\mathrm{Stab}(F, [\sigma, g])$ *on* $D(\mathscr{G}, m)_{[\sigma,g]}$ *and the action of* G_σ *on* $D(\mathscr{G}_\sigma, i_\sigma)$.

We will call $D(\mathscr{G}_\sigma, i_\sigma)$ the *local development* (or the link) of $\mathscr{G}$ at σ. This coincides with the notion of the link $L(\mathscr{G}, \sigma)$ as defined in Section 6.

Following Definition 6.2, we say that a simplex of groups is *locally k-large* if all of its local developments are k-large (in particular truly simplicial, not just multi-simplicial). Theorem 6.1 implies then the following.

17.4. *Corollary. — For $k \geq 6$, any locally k-large simplex of groups is developable.*

17.5. *Remark.* — Note that if the homotopical systole of a development of a locally k-large simplex of groups is ≥ 3 (which is a nontrivial condition for a multi-simplicial complex), then it is simplicial. To see this, observe first that a multi-simplicial complex X with simplicial links, which is not simplicial, must have a double edge (i.e. two edges with both endpoints coinciding). Second, note that the cycle consisting of these two edges is homotopically nontrivial in X. This follows from the fact that the universal covering of X is simplicial since, being locally 6-large, it can be obtained as the union of a sequence of small extensions (see Section 4), starting from a single simplex, and all these extensions together with their union are simplicial. This implies that the homotopical systole of X is 2, justifying the initial statement.

Our last goal in this section is to recall terminology related to the so called simple complexes of groups (examples of which are simplices of groups), and to formulate some results which extend the already mentioned results for simplices of groups. We will need these concepts and facts in the next section, in the proof of Proposition 18.3.

Let X be a simplicial complex and let F be a group acting on X by automorphisms. A subcomplex $K \subset X$ is a *strict fundamental domain* of this action if the restricted quotient map $K \to F\backslash X$ is an isomorphism of simplicial complexes. Given an action of F that admits a strict fundamental domain K, we associate to any face σ of K a group $G_\sigma := \mathrm{Stab}(F, \sigma)$, the stabilizer of σ in F. In fact, due to the existence of a strict fundamental domain, the stabilizer G_σ fixes the simplex σ pointwise. We have obtained a system $\{G_\sigma\}$ of groups with inclusions $G_\tau \subset G_\sigma$ whenever $\sigma \subset \tau$. We call this system the *simple complex of groups associated to the action of* F.

A *simple complex of groups* $\mathcal{G}$ over a simplicial complex Q is a system of groups G_σ associated to the faces of Q, equipped with a system of injective homomorphisms $\psi_{\sigma\tau} : G_\tau \to G_\sigma$ for all pairs $\sigma \subset \tau$, such that $\psi_{\sigma\tau} \circ \psi_{\tau\rho} = \psi_{\sigma\rho}$ whenever $\sigma \subset \tau \subset \rho$.

The notions of a morphism to a group, local injectivity and surjectivity of a morphism, developability of $\mathcal{G}$ and development $D(\mathcal{G}, m)$ associated to a locally injective morphism $m : \mathcal{G} \to F$ have straightforward extensions from the case of $\mathcal{G}$ being a simplex of groups to that of a simple complex of groups. It is then clear that if $m : \mathcal{G} \to F$ is a locally injective morphism then $\mathcal{G}$ is equivalent (isomorphic as a simple complex of groups) to the simple complex of groups associated to the action of F on the development $D(\mathcal{G}, m)$. Thus, developability of $\mathcal{G}$ can be characterized geometrically by saying that $\mathcal{G}$ is isomorphic to a simple complex of groups associated to an action. Moreover, the obvious analogue of Proposition 17.2 holds if the underlying complex Q of a simple complex of groups $\mathcal{G}$ is connected.

Now we extend the notion of the local development, as defined above for simplices of groups, to arbitrary simple complexes of groups. Let $\mathcal{G}$ be a simple complex of groups over Q and let σ be a face of Q. Consider the link Q_σ of Q at σ, and for any face τ in Q_σ denote by $\bar{\tau}$ the corresponding face of Q properly containing σ. Define then a simple complex of groups $\mathcal{G}_\sigma = (\{G'_\tau\}, \{\psi'_{\tau\rho}\})$ over Q_σ by putting $G'_\tau := G_{\bar{\tau}}$ and $\psi'_{\tau\rho} := \psi_{\bar{\tau}\bar{\rho}}$. Define also a locally injective morphism $i_\sigma : \mathcal{G}_\sigma \to G_\sigma$ consisting of homomorphisms $(i_\sigma)_\tau : G'_\tau \to G_\sigma$ given by $(i_\sigma)_\tau := \psi_{\sigma\bar{\tau}}$. The development $D(\mathcal{G}_\sigma, i_\sigma)$, equipped with the action of G_σ, is then called the *local development* of $\mathcal{G}$ at σ (or the link of $\mathcal{G}$ at σ). If $\mathcal{G}$ is developable then the local developments of $\mathcal{G}$ occur as links in the developments of $\mathcal{G}$ for all injective morphisms. More precisely, if $m : \mathcal{G} \to F$ is an injective morphism, and $[\sigma, g]$ a face in the corresponding development $D(\mathcal{G}, m)$, then the link $D(\mathcal{G}, m)_{[\sigma,g]}$, with the induced action of the stabilizing subgroup of F, is equivariantly isomorphic to the local development $D(\mathcal{G}_\sigma, i_\sigma)$.

A simple complex of groups over Q is *locally* 6-*large* if all of its local developments are 6-large. Clearly, by Theorem 6.1, every locally 6-large simple complex of groups is developable.

18. Extra-tilability

In this section we introduce a condition called extra-tilability which allows one to construct, inductively with respect to the dimension, simplices of groups admitting finite k-large developments (for arbitrary $k \geq 6$). A construction of such developments is presented in Section 19. In this section we indicate various useful consequences of the introduced condition.

18.1. *Definition. — A simplicial complex* X *equipped with an action of a group* G *by simplicial automorphisms is* extra-tilable *if the following conditions are satisfied:*

(1) the action is simply transitive on top-dimensional simplices of X *and its quotient is a simplex (equivalently, any top-dimensional simplex is a strict fundamental domain for this action);*

(2) X *is 6-large;*

(3) for any face σ *of* X *the ball* $B_1(\sigma, X)$ *is a strict fundamental domain for the restricted action of a subgroup of* G *on* X.

A simplex of groups $\mathcal{G}$ is *locally extra-tilable* if local developments of $\mathcal{G}$ equipped with actions of the corresponding local groups are all extra-tilable.

18.2. *Example.*

(1) The Coxeter (or dihedral) group $D_n = \langle s_1, s_2 | s_1^2, s_2^2, (s_1 s_2)^n \rangle$ with $n = 6k$ or $n = \infty$, with its canonical action on the corresponding Coxeter complex (i.e. a division of S^1 into $2n$ segments), is obviously extra-tilable.
(2) Let X be the Coxeter complex of the triangle Coxeter group $(6, 6, 6)$, which may be viewed as a triangulation of the hyperbolic plane by regular triangles with angles $\pi/6$. It follows from Poincare's Theorem that the action of this group on X is extra-tilable.
(3) The quotient simplex of groups associated to the action in (2) is locally extra-tilable.

Note that condition (1) in Definition 18.1 implies that the complex of groups associated to the action of G on X is a ∂-supported simplex of groups. Consequently, X is equivariantly isomorphic to a development of this simplex of groups. For this reason, we will often speak of *extra-tilable developments* of ∂-supported simplices of groups (rather than of extra-tilable complexes).

The reader can easily verify that if the pair X, G is extra-tilable then links of X equipped with the actions of the corresponding stabilizers in G are extra-tilable. Consequently, a simplex of groups that admits an extra-tilable development is locally extra-tilable. The next proposition provides the converse of this statement, together with a much stronger property that will be crucial in our later arguments.

18.3. *Proposition. — Let $\mathcal{G}$ be a locally extra-tilable simplex of groups. Then the action of the direct limit $\hat{\mathcal{G}}$ on the universal development $\widetilde{\mathcal{G}} = D(\mathcal{G}, i_{\mathcal{G}})$ has the following property: each n-ball $B = B_n(\sigma, \widetilde{\mathcal{G}})$ in $\widetilde{\mathcal{G}}$, for any natural number n, is a strict fundamental domain for the action of a unique subgroup H_B of $\hat{\mathcal{G}}$. In particular, $\widetilde{\mathcal{G}}$ equipped with the action of $\hat{\mathcal{G}}$ is extra-tilable.*

To prove Proposition 18.3 we need the following.

18.4. *Lemma. — Let $\mathcal{G}$ be a ∂-supported locally 6-large simplex of groups over a simplex Δ, and let $m : \mathcal{G} \to G$ be a locally injective and surjective morphism. Suppose that for some simplex $\sigma \subset D(\mathcal{G}, m)$ the ball $B := B_1(\sigma, D(\mathcal{G}, m))$ is a strict fundamental domain for the action of a subgroup $H < G$. Denote by $\mathcal{H}$ the simple complex of groups over B associated to the action of H on $D(\mathcal{G}, m)$, and by $\nu : \mathcal{H} \to H$ the associated morphism. Then*

(1) ν is surjective, i.e. H is generated by the union of the images $\nu_\sigma(H_\sigma)$ of the local groups H_σ of $\mathcal{H}$;
(2) B determines the subgroup H uniquely.

Proof. — To prove part (1), note first that the development $D(\mathcal{G}, m)$ is, by surjectivity of m, connected. Since any simple complex of groups with connected development is surjective, we get surjectivity of $\mathcal{H}$ by the fact that $D(\mathcal{H}, \nu) = D(\mathcal{G}, m)$.

The proof of (2) goes by induction on $n = \dim \Delta$. Let $H' < G$ be another subgroup for which B is a strict fundamental domain. Denote by $\mathcal{H}'$ the simple complex of groups over B associated to the action of H' on $D(\mathcal{G}, m)$, and by H'_σ its local groups at simplices σ of B.

Suppose first that $\dim \Delta = 1$. Since $\mathcal{G}$ is ∂-supported, the local groups of both $\mathcal{H}$ and $\mathcal{H}'$ at edges are all trivial. We will show that for every vertex v of B the local groups H_v and H'_v coincide. By applying (1), this property implies that $H = H'$, hence (2).

The equality $H_v = H'_v$ is obvious for vertices v from the interior of B (i.e. vertices of the central simplex σ), since then both groups are trivial. For the remaining vertices v both these groups coincide with the stabilizer of G at v, which one easily deduces from the fact that there is exactly one edge in B adjacent to v (and from simple transitivity of G on the edges of $D(\mathcal{G}, m)$).

In the general case, note that for any simplex σ of B both groups H_σ, H'_σ act on the link $[D(\mathcal{G}, m)]_\sigma = D(G_\sigma, m_\sigma)$ with the strict fundamental domain B_σ. The inductive assumption implies that $H_\sigma = H'_\sigma$, and again the proof is concluded by applying (1).

Proof of Proposition 18.3. — Note that since $\mathcal{G}$ is locally extra-tilable, it is in particular locally 6-large. Thus, by Corollary 17.4, $\mathcal{G}$ is developable and hence it makes sense to speak of the universal development $\widetilde{\mathcal{G}} = D(\mathcal{G}, i_{\mathcal{G}})$. By Proposition 17.2, $\widetilde{\mathcal{G}}$ is simply connected, and hence it is a systolic complex. For any ball B in $\widetilde{\mathcal{G}}$ consider a simple complex of groups $\mathcal{H} = (\{H_\sigma\}, \{\phi_{\tau\sigma}\})$ over B defined as follows. For any face σ of B consider the link $(\widetilde{\mathcal{G}})_\sigma$ and the action of the stabilizer $\mathrm{Stab}(\widehat{\mathcal{G}}, \sigma)$ on it. Let σ_0 be the image of σ under the quotient map $\widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}/\widehat{\mathcal{G}} = \Delta$. Then the action of $\mathrm{Stab}(\widehat{\mathcal{G}}, \sigma)$ on $(\widetilde{\mathcal{G}})_\sigma$ is equivariantly isomorphic to the action of the local group G_{σ_0} of $\mathcal{G}$ on the local development $D(\mathcal{G}_{\sigma_0}, i_{\sigma_0})$ and hence it is extra-tilable. By strict convexity of balls (Corollary 7.9.2), the link B_σ either coincides with $(\widetilde{\mathcal{G}})_\sigma$ or has a form $B_1(\tau, (\widetilde{\mathcal{G}})_\sigma)$ for some simplex $\tau \subset (\widetilde{\mathcal{G}})_\sigma$. In any case, by local extra-tilability of $\mathcal{G}$, B_σ is a strict fundamental domain for the action of a subgroup of $\mathrm{Stab}(\widehat{\mathcal{G}}, \sigma)$ on $(\widetilde{\mathcal{G}})_\sigma$. Moreover, due to Lemma 18.4, this subgroup is unique, and we take it as the local group H_σ in $\mathcal{H}$. Note that if $\sigma \subset \tau$ then $H_\tau \subset H_\sigma$. In fact, H_τ can be identified as a subgroup of H_σ more precisely as follows. Denote by τ' the face in the link $(\widetilde{\mathcal{G}})_\sigma$ corresponding to τ. Then, viewing H_σ as acting on $(\widetilde{\mathcal{G}})_\sigma$, H_τ is equal to the stabilizer of τ' in this action. We take as the structure homomorphism $\phi_{\sigma\tau}$ for $\mathcal{H}$ the inclusion homomorphism from H_τ to H_σ, for any relevant pair σ, τ of simplices in B.

Consider the morphism $j : \mathscr{H} \to \hat{\mathscr{G}}$ given by the inclusions of the local groups H_σ in $\hat{\mathscr{G}}$, and denote by $\hat{j} : \hat{\mathscr{H}} \to \hat{\mathscr{G}}$ the corresponding homomorphism between the direct limits. Since j is locally injective, $\mathscr{H}$ is developable and we denote by $\widetilde{\mathscr{H}}$ the universal development of $\mathscr{H}$. The ball B, identified with the subcomplex $[B, 1]$ in $\widetilde{\mathscr{H}}$, is clearly a strict fundamental domain for the action of $\hat{\mathscr{H}}$ on $\widetilde{\mathscr{H}}$. To prove the proposition, we will show that there is a $\hat{j}$-equivariant isomorphism between $\widetilde{\mathscr{G}}$ and $\widetilde{\mathscr{H}}$ that is identical on B. If this is the case, B is a strict fundamental domain for the subgroup $H_B := \hat{j}(\hat{\mathscr{H}}) < \hat{\mathscr{G}}$.

Let $J : \widetilde{\mathscr{H}} \to \widetilde{\mathscr{G}}$ be a simplicial map given by $J([x, g]) := \hat{j}(g) \cdot x$ for any $x \in B$ (where x on the right lies in $B \subset \widetilde{\mathscr{G}}$). From what was said above about local groups of $\mathscr{H}$, it follows that the local development of $\mathscr{H}$ at a face σ of B is equivariantly isomorphic to the link $(\widetilde{\mathscr{G}})_\sigma$ acted upon by the group H_σ. This implies that the map J induces isomorphisms at links of all simplices in $\widetilde{\mathscr{H}}$, and hence it is a covering. Since both complexes $\widetilde{\mathscr{H}}$ and $\widetilde{\mathscr{G}}$ are connected and simply connected, it follows that J is an isomorphism as required.

It remains to prove uniqueness of H_B. In view of the strong convexity of B (see Corollary 7.12), this follows by the argument as in the proof of Lemma 18.4.2. Thus the proposition follows.

The above arguments yield in fact the following more general result.

18.5. *Proposition. — Let $\mathscr{G}$ be a locally extra-tilable simplex of groups and let* Q *be any strongly convex subcomplex in the universal development* $\widetilde{\mathscr{G}} = D(\mathscr{G}, i_{\mathscr{G}})$. *Then* Q *is a strict fundamental domain for the action of a unique subgroup* H_Q *of the direct limit* $\hat{\mathscr{G}}$.

In the next proposition only part (2) is important for further applications. We include part (1) to indicate the relationship of the phenomena that we obtain with residual finiteness of involved groups.

18.6. *Proposition. — Let $\mathscr{G}$ be a locally extra-tilable simplex of finite groups. Then*

(1) the direct limit group $\hat{\mathscr{G}}$ is residually finite;
(2) for any natural k there is an injective morphism $m : \mathscr{G} \to F$ into a finite group F such that we have $sys_h[D(\mathscr{G}, m)] \geq k$.

Proof. — Let Δ be the underlying simplex of $\mathscr{G}$. To prove (1), recall that a group G is residually finite if for any $g \in G$ with $g \neq 1$ there is a normal subgroup $N < G$ of finite index such that $g \notin N$. Let $g \in \hat{\mathscr{G}}$, $g \neq 1$. Consider a ball B in the universal development $\widetilde{\mathscr{G}}$ centered at $[\Delta, 1]$ and containing $[\Delta, g]$. By Proposition 18.3, there is a subgroup $H_B < \hat{\mathscr{G}}$ for which B is a strict fundamental domain. Note that $g \notin H_B$ because each orbit of H_B intersects B only once.

Moreover, since $\widetilde{\mathscr{G}}$ is locally finite, B is finite (as a complex), and since $\hat{\mathscr{G}}$ acts simply transitively on top-dimensional faces of $\widetilde{\mathscr{G}}$, it follows that H_B is a finite index subgroup of $\hat{\mathscr{G}}$. Thus the normalization $N = \bigcap_{h\in\hat{\mathscr{G}}} hH_Bh^{-1}$ has also finite index in $\hat{\mathscr{G}}$, and clearly $g \notin N$. This finishes the proof of part (1).

To prove (2), consider the face $[\Delta, 1]$ in $\widetilde{\mathscr{G}}$ and the ball $B = B_k([\Delta, 1], \widetilde{\mathscr{G}})$ centered at this face. Observe that a polygonal path in $\widetilde{\mathscr{G}}$ connecting a vertex of $[\Delta, 1]$ with a vertex outside B has length greater than k (i.e. consists of more than k edges). Let H_B be the subgroup of $\hat{\mathscr{G}}$ for which B is a strict fundamental domain, and let $N = \bigcap_{h\in\hat{\mathscr{G}}} hH_Bh^{-1}$. As before, N is a finite index subgroup in $\hat{\mathscr{G}}$.

Recall that for each vertex v of Δ the local group G_v is identified with the stabilizer of $\hat{\mathscr{G}}$ at $[v, 1]$ (in its action on $\widetilde{\mathscr{G}}$). Thus, since $\mathscr{G}$ is ∂-supported, we have $G_v \cap H_B = \{1\}$, and hence also $G_v \cap N = \{1\}$. It follows that the composition $\mathscr{G} \to \hat{\mathscr{G}} \to \hat{\mathscr{G}}/N$ is a locally injective morphism to a finite group $\hat{\mathscr{G}}/N$. We take this morphism as m and the quotient $\hat{\mathscr{G}}/N$ as F.

We now estimate from below the homotopical systole of the development $D(\mathscr{G}, m)$. Since N is a subgroup of H_B, the orbit of a vertex v of $[\Delta, 1]$ under the action of N on $\widetilde{\mathscr{G}}$ intersects B only at v. Thus the polygonal distance between v and any other vertex from this orbit is greater than k (in fact, this distance is even $\geq 2k$, but we don't need this sharper estimate). It follows that any homotopically nontrivial closed polygonal path in $D(\mathscr{G}, m)$ passing through a vertex of $[\Delta, 1]$ has length $> k$. On the other hand, $D(\mathscr{G}, m)$ is acted upon by the quotient group $\hat{\mathscr{G}}/N$ and this action is transitive on top-dimensional faces. Thus any homotopically nontrivial path in $D(\mathscr{G}, m)$ can be mapped by an automorphism of $D(\mathscr{G}, m)$ to a path that intersects $[\Delta, 1]$. Thus, the homotopical systole of $D(\mathscr{G}, m)$ is greater than k, which finishes the proof of part (2).

We say that a locally injective morphism $m : \mathscr{G} \to F$ from a simplex of groups $\mathscr{G}$ is *extra-tilable* if the development $D(\mathscr{G}, m)$ acted upon by the group F is extra-tilable. Obviously, to have a extra-tilable morphism, a simplex of groups has to be locally extra-tilable. The next proposition, a culmination of the results in this section, will be the key technical tool in the arguments involved in the main construction presented in the next section.

18.7. *Proposition. — Let $\mathscr{G}$ be a locally k-large simplex of finite groups, for some $k \geq 6$, and suppose $\mathscr{G}$ is locally extra-tilable. Then $\mathscr{G}$ admits an extra-tilable morphism $\mu : \mathscr{G} \to E$ to a finite group E such that the development $D(\mathscr{G}, \mu)$ is k-large.*

Proof. — Since it follows from our assumptions that $\mathscr{G}$ is locally 6-large, let $m : \mathscr{G} \to F$ be a locally injective morphism to a finite group F as prescribed by Proposition 18.6.2, i.e. such that $sys_h[D(\mathscr{G}, m)] \geq k$. Then $D(\mathscr{G}, m)$ is clearly

k-large (see Corollary 1.5). Denote by $\mathrm{K} = \ker \hat{m}$ the kernel of the homomorphism $\hat{m} : \hat{\mathscr{G}} \to \mathrm{F}$ induced by m, and note that K has finite index in $\hat{\mathscr{G}}$.

For any face $\sigma \subset \widetilde{\mathscr{G}}$ consider the ball $\mathrm{B}^\sigma := \mathrm{B}_1(\sigma, \widetilde{\mathscr{G}})$ and the subgroup $\mathrm{H}^\sigma < \hat{\mathscr{G}}$ for which B^σ is a strict fundamental domain. Clearly, H^σ is a finite index subgroup for each σ. Consider the intersection $\mathrm{K} \cap \bigcap_{\sigma \subset [\Delta,1]} \mathrm{H}^\sigma$, which is still of finite index in $\hat{\mathscr{G}}$, and normalize it to get a finite index normal subgroup N of $\hat{\mathscr{G}}$. Put $\mathrm{E} := \hat{\mathscr{G}}/\mathrm{N}$ and denote by μ the natural morphism from $\mathscr{G}$ to E. Since $\mathrm{N} \subset \mathrm{K}$, the development $\mathrm{D}(\mathscr{G}, \mu)$ is a covering of the development $\mathrm{D}(\mathscr{G}, m)$ and, since the latter is k-large, the former is k-large too. It remains to prove that μ is extra-tilable.

By the fact that $[\Delta, 1]$ is a fundamental domain for the action of $\hat{\mathscr{G}}$ on $\widetilde{\mathscr{G}}$, for any face $\sigma \subset \widetilde{\mathscr{G}}$ there is a face $\sigma_0 \subset [\Delta, 1]$ and an element $g \in \hat{\mathscr{G}}$ such that $\mathrm{H}_\sigma = g\mathrm{H}_{\sigma_0}g^{-1}$. In particular, since the subgroup N is contained in H_{σ_0} and normal, it is also contained in H_σ. Denote by $p : \widetilde{\mathscr{G}} \to \mathrm{D}(\mathscr{G}, \mu)$ the covering map induced by the quotient homomorphism $\hat{\mathscr{G}} \to \mathrm{E} = \hat{\mathscr{G}}/\mathrm{N}$. It follows that the image $p(\mathrm{B}^\sigma)$ is a strict fundamental domain for the action of the subgroup $\mathrm{H}^\sigma/\mathrm{N} \subset \mathrm{E}$ on $\mathrm{D}(\mathscr{G}, \mu)$. Since the images $p(\mathrm{B}^\sigma)$ for all simplices σ in $\widetilde{\mathscr{G}}$ exhaust the balls of radius 1 centered at faces in $\mathrm{D}(\mathscr{G}, \mu)$, the action of E on $\mathrm{D}(\mathscr{G}, \mu)$ is extra-tilable, and the proposition follows.

19. Existence of k-large developments

In this section we give a rather general construction of finite k-large developments of simplices of groups in arbitrary dimension. This construction allows to get examples of complexes with various interesting properties. Our main result is the following.

19.1. *Proposition. — Let Δ be a simplex and suppose that for any codimension 1 face s of Δ we are given a finite group A_s. Then for any $k \geq 6$ there exists a ∂-supported simplex of finite groups $\mathscr{G} = (\{\mathrm{G}_\sigma\}, \{\psi_{\sigma\tau}\})$ and a locally injective and surjective morphism $m : \mathscr{G} \to \mathrm{F}$ to a finite group F such that $\mathrm{G}_s = \mathrm{A}_s$ for any codimension 1 face s of Δ and*

(1) $\mathscr{G}$ is locally extra-tilable;
(2) the development $\mathrm{D}(\mathscr{G}, m)$ is (finite and) k-large.

Proof. — We will construct appropriate groups G_σ inductively with respect to the codimension of σ in Δ. Here we will view F as $\mathrm{G}_\emptyset$, the group associated to the "empty face" $\emptyset$ of Δ of codimension $\dim(\Delta) + 1$.

By the requirements of the proposition, we have to put $\mathrm{G}_\Delta = \{1\}$ and $\mathrm{G}_s = \mathrm{A}_s$ for all faces s of codimension 1. This gives the starting point for our induction. Suppose that finite groups G_σ are already defined for all faces σ of codimension $\leq i$, together with injective homomorphisms $\psi_{\sigma\tau}$ as required. Suppose also that for all such σ the following condition (which will be an additional part of the inductive hypothesis) is satisfied: The groups $G_\tau : \sigma \subset \tau$ form a simplex of groups $\mathcal{G}^\sigma$ over the link Δ_σ and the homomorphisms $\psi_{\sigma\tau}$ form a locally injective and surjective morphism $m^\sigma : \mathcal{G}^\sigma \to G_\sigma$ such that the development $D(\mathcal{G}^\sigma, m^\sigma)$ is k-large and locally extra-tilable. Note that for $i = 1$ these inductive assumptions are fulfilled. For any face ρ of codimension $i+1$ in Δ consider the simplex of groups $\mathcal{G}^\rho$ over the link Δ_ρ formed of the groups $G_\sigma : \rho \subset \sigma$. By the inductive assumptions, this gives a locally k-large ∂-supported and locally extra-tilable simplex of groups. By Proposition 18.7, there is a surjective extra-tilable morphism $\mu : \mathcal{G}^\rho \to E$ to a finite group E such that the development $D(\mathcal{G}^\rho, \mu)$ is k-large. By putting $G_\rho := E$ and $\psi_{\rho\sigma} := \mu_\sigma$ we get the inductive hypothesis for $i+1$. This finishes the proof.

19.2. *Corollary. — For each natural n and each $k \geq 6$ there exists an n-dimensional compact simplicial pseudomanifold that is k-large. Moreover, this pseudomanifold can be obtained to be orientable.*

Proof. — In view of Proposition 17.1.7, the first statement in the corollary follows from Proposition 19.1 by putting $A_s = Z_2$ for all codimension 1 faces s.

To ensure orientability, we need to modify slightly constructions in the proofs of Propositions 19.1 and 18.7. Recall from Proposition 17.1.8 that a necessary condition for the development $D(\mathcal{G}, m)$ associated to a morphism $m : \mathcal{G} \to F$ to be an orientable pseudomanifold is the existence of a homomorphism $r : F \to Z_2$ such that the composed morphism $r \circ m$ maps the local groups G_s at codimension 1 faces s isomorphically to Z_2. Thus, when constructing local groups G_σ, we need to have additional homomorphisms $r_\sigma : G_\sigma \to Z_2$, forming together a morphism from $\mathcal{G}$ to Z_2, such that the compositions $r_\sigma \circ \psi_{\sigma s} : G_s \to Z_2$ are isomorphisms. By the inductive assumption concerning this property, there is always a homomorphism $\hat{r}_\sigma : \hat{\mathcal{G}}^\sigma \to Z_2$ from the direct limit of the simplex of groups $\mathcal{G}^\sigma$, with the desired property. Thus, to have the appropriate r_σ, it is necessary that the normal subgroup N giving G_σ as the quotient $\hat{\mathcal{G}}^\sigma/N$ is contained in the kernel of $\hat{r}_\sigma$. Since this can be obtained by passing to a finite index subgroup in the previously chosen N, the corollary follows.

We mention further consequences of Corollary 19.2.

19.3. *Corollary.*

(1) For each natural n there exists a developable simplex of groups whose fundamental group is Gromov-hyperbolic, virtually torsion-free, and has virtual cohomological dimension n.

(2) *For each natural n there exists an n-dimensional compact simplicial orientable pseudomanifold whose universal cover is* CAT(0) *with respect to the standard piecewise euclidean metric.*

(3) *For each natural n and each real number* $d > 0$ *there exists an n-dimensional compact simplicial orientable pseudomanifold whose universal cover is* CAT(-1) *with respect to the piecewise hyperbolic metric for which the simplices are regular hyperbolic with edge lengths* d.

Proof. — By Corollary 19.2, for every natural n there exists an n-dimensional compact simplicial orientable pseudomanifold X which is 7-large. It is obtained as a development of a certain simplex of finite groups $\mathcal{G}$. The fundamental group Γ of X is a subgroup of finite index in the fundamental group of $\mathcal{G}$, and it is torsion-free. To see this, note that X is aspherical (Theorem 4.1.1), and hence it is a classifying space for Γ. Since X is finite dimensional, Γ cannot contain a finite subgroup. Moreover, since (being a compact pseudomanifold of dimension n) X has nontrivial cohomology in dimension n, the group Γ has cohomological dimension equal to n. Finally, by Corollary 2.2, the group Γ is Gromov hyperbolic. This proves (1).

Parts (2) and (3) follow from Corollary 19.2 in view of Theorem 14.1.

Parts (2) and (3) of the above corollary give an affirmative answer, in arbitrary dimension n, to a question raised by D. Burago [Bu, p. 292]. The answer for $n = 3$ has been given in [BuFKK].

As one more application we note that Corollary 19.2 allows an alternative approach to the main result of our paper [JS1] stating that for each natural n there exists a Gromov hyperbolic Coxeter group with virtual cohomological dimension n. As we have shown in [JS1], to construct such a group it is sufficient to construct a compact orientable n-dimensional pseudomanifold which satisfies "flag-no-square" condition (which is equivalent to 5-largeness). Since k-largeness for $k \geq 6$ implies 5-largeness, we get such pseudomanifolds by the construction of Proposition 19.1 (improved as in the proof of orientability in Corollary 19.2), which is different from the construction in [JS1].

20. Non-positively curved branched covers

In this section we use the idea of extra-tilability to show the existence of nonpositively curved finite branched covers for a class of compact piecewise euclidean pseudomanifolds that contains all manifolds. This answers a question of M. Gromov. Using the same method, we show that any finite complex K is homotopy equivalent to the classifying space for proper G-bundles of a CAT(-1) (hence Gromov hyperbolic) group G. This answers a question of I. Leary.

We start by recalling some terminology. A *chamber* in a simplicial pseudomanifold is any of its top-dimensional faces. A simplicial pseudomanifold is *normal* if all of its links are gallery-connected (See Proposition 17.1.6). We borrow the term "normal" from M. Goresky and R. MacPherson [GMcP]). The property of being normal does not depend on a triangulation of a pseudomanifold (see Section 4.1 in [GMcP]). Manifolds are obviously normal.

A *branched covering* of a simplicial pseudomanifold X is a simplicial pseudomanifold Y equipped with a nondegenerate simplicial map $p : Y \to X$ which is a covering map outside codimension 2 skeleta.

The main results in this section are the following two theorems.

20.1. *Theorem. — Let* X *be a compact connected normal simplicial pseudomanifold with a piecewise euclidean (respectively, piecewise hyperbolic) metric. Then* X *has a compact branched covering* Y *which is nonpositively curved (respectively, has curvature* $\kappa \leq -1$*) with respect to the induced piecewise constant curvature metric.*

20.2. *Theorem. — For any finite complex* K *there is a* CAT(-1) *space* X *and a group* G *acting properly discontinuously and cocompactly by isometries on* X, *so that the quotient* $G\backslash X$ *is homotopy equivalent to* K.

Both theorems above are corollaries to a stronger technical result contained in Proposition 20.3. To formulate this proposition we need more definitions. We say that a simplex of groups $\mathcal{G}$ over a simplex Δ, with local groups G_s at all codimension 1 faces s isomorphic to Z_2, is *symmetric* if it satisfies the following conditions:

(1) the local groups G_σ are generated by their local subgroups at faces of codimension 1 (i.e. at those codimension 1 faces s which contain σ);
(2) any automorphism f of the underlying simplex Δ extends to an automorphism φ of $\mathcal{G}$.

Note that, due to condition (1), automorphisms φ from condition (2) are uniquely determined by automorphisms f.

A morphism $m : \mathcal{G} \to F$ is a *symmetric morphism* if m is surjective, $\mathcal{G}$ is a symmetric simplex of groups, and for any automorphism φ of $\mathcal{G}$ as in (2) there is an automorphism a_φ of F such that $m \circ \varphi = a_\varphi \circ m$. Note that, due to surjectivity of m, automorphisms a_φ are uniquely determined by automorphisms φ.

A *symmetric development* is the development associated to a symmetric morphism.

20.3. *Proposition.* — *Given any $k \geq 6$, every finite family of compact connected normal simplicial pseudomanifolds $\{X_i\}$ of the same dimension has a common compact branched covering Y which is an extra-tilable symmetric development of a simplex of finite groups, and which is k-large. Moreover, for each X_i there is a group Γ_i of simplicial automorphisms of the universal cover $\widetilde{Y}$ of Y such that X_i is isomorphic to the quotient $\Gamma_i \backslash \widetilde{Y}$.*

Before giving a proof of the proposition, we show how it implies Theorem 20.1. The proof of Theorem 20.2, together with discussion of its consequences, occupies the last part of the section (after the proof of Proposition 20.3).

Proof of Theorem 20.1 (using Proposition 20.3). — Given a metric pseudomanifold as in the theorem, denote by Π the set of all shapes of simplices of X and note that Π is finite. Clearly, any branched covering Y of X equipped with the lifted metric satisfies the condition Shapes(Y) $\subset \Pi$. Let $k \geq 6$ be a natural number associated to Π as in the assertion of Theorem 14.1. By Proposition 20.3, X has a compact branched covering Y which is k-large, and hence also locally k-large. By Theorem 14.1, Y is then nonpositively curved (respectively, has curvature $\kappa \leq -1$), as required.

Proof of Proposition 20.3. — We use induction with respect to the dimension n of pseudomanifolds X_i.

For $n = 1$, each X_i is a triangulation of the circle, and we denote by l_i the number of edges in X_i. Let L be a common multiple of all numbers l_i and 12. Put Y to be the triangulation of the circle consisting of L edges. Then Y is as asserted in the proposition. To see this, note that due to divisibility of L by 12, Y is an extra-tilable development of the ∂-supported edge of groups with groups Z_2 at vertices. The other assertions of the proposition are in this case obvious.

We now pass to the case of arbitrary dimension n. Consider the family $\mathscr{X}$ of all links at vertices in all pseudomanifolds X_i. Due to compactness of X_i's, this family is finite. Moreover, since links of normal pseudomanifolds are normal, $\mathscr{X}$ consists of compact connected normal pseudomanifolds of the same dimension $n - 1$. By applying inductive hypothesis to the family $\mathscr{X}$, we obtain an extra-tilable symmetric morphism $m : \mathscr{G} \to F$ from an $(n-1)$-dimensional simplex of finite groups $\mathscr{G}$ to a finite group F such that the development $D(\mathscr{G}, m)$ satisfies all assertions of the proposition relative to $\mathscr{X}$. Let $\mathscr{H}$ be an n-dimensional simplex of groups described as follows. For local groups at faces of codimension $< n$ take the local groups of $\mathscr{G}$ at faces of the same codimension (which are all isomorphic due to symmetry of $\mathscr{G}$). For local groups at vertices take the group F. Symmetry of $\mathscr{G}$ and m allows to take as structure homomorphisms for $\mathscr{H}$ the homomorphisms occurring in $\mathscr{G}$ and in the morphism m. The so obtained simplex of finite groups $\mathscr{H}$ is clearly symmetric, locally k-large and locally extra-tilable. Since, being locally

k-large, $\mathscr{H}$ is developable, consider its universal development $\widetilde{\mathscr{H}}$. Our next aim is to show that $\widetilde{\mathscr{H}}$ is a common branched covering of pseudomanifolds X_i. However, since $\widetilde{\mathscr{H}}$ is not compact, this will not yet finish the proof.

Fix one of the pseudomanifolds X_i, a chamber C in it, and any isomorphism $p_0 : D_0 \to C$ of some chamber D_0 of $\widetilde{\mathscr{H}}$ with C. We will show that p_0 can be extended to a branched covering $p : \widetilde{\mathscr{H}} \to X_i$. For this, note that any gallery γ in $\widetilde{\mathscr{H}}$ starting at the chamber D_0 determines uniquely the map $p_\gamma : D \to X_i$ from the final chamber D in γ, by means of unfolding γ on X_i starting with p_0. Then, since (by Proposition 17.1.6) $\widetilde{\mathscr{H}}$ is gallery connected, we define p separately on each chamber D in $\widetilde{\mathscr{H}}$ by putting $p|_D = p_\gamma$ for some choice of a gallery γ connecting D_0 to D. To see that p is well defined we need to show that $p_\gamma : D \to X_i$ does not depend on the choice of γ. Equivalently, we need to show that for any gallery γ starting and terminating at D_0 we have $p_\gamma = p_0$.

Since $\widetilde{\mathscr{H}}$ is simply connected, γ can be expressed, up to cancellation of back and forth subpaths, as the concatenation of *elementary closed galleries* started at D_0, i.e. galleries of form

$$D_0, D_1, ..., D_l, D_l^1, D_l^2, ..., D_l^m, D_l, D_{l-1}, ..., D_0$$

with chambers $D_l, D_l^1, ..., D_l^m$ contained in the residue of a single vertex of $\widetilde{\mathscr{H}}$. Clearly, it is then sufficient to show that $p_\gamma = p_0$ for any elementary closed gallery γ started at D_0. This however follows directly from the fact that links of $\widetilde{\mathscr{H}}$ at vertices, which are all isomorphic to the development $D(\mathscr{G}, m)$, are symmetric branched coverings of the links of X_i at vertices. Thus, the map p is well defined, and the fact that it is a branched covering follows easily from its definition.

Denote by $\mathrm{Sym}(\widetilde{\mathscr{H}})$ the full group of simplicial automorphisms of $\widetilde{\mathscr{H}}$. Due to symmetry of $\widetilde{\mathscr{H}}$, and rigidity implied by the fact that $\widetilde{\mathscr{H}}$ is a pseudomanifold, this group is a semidirect extension of the direct limit $\hat{\mathscr{H}}$ by the group of automorphisms of the underlying simplex of $\mathscr{H}$. We will now show that for each X_i there exists a subgroup $\Gamma_i < \mathrm{Sym}(\widetilde{\mathscr{H}})$ such that the quotient $\Gamma_i \backslash \widetilde{\mathscr{H}}$ is isomorphic to X_i.

Consider the set $p^{-1}(C)$ of all chambers in $\widetilde{\mathscr{H}}$ which are mapped through p on C. Clearly, this set contains our distinguished chamber D_0. For any chamber $D \in p^{-1}(C)$ consider the isomorphism $u_D : D_0 \to D$ such that $p \circ u_D = p_0$. Clearly, u_D can be extended uniquely to an automorphism of $\widetilde{\mathscr{H}}$, and we denote this automorphism by g_D. Moreover, each automorphism g_D obviously commutes with p. Consequently, the set $\{g_D : D \in p^{-1}(C)\}$ coincides with the group of all automorphisms of $\widetilde{\mathscr{H}}$ that commute with p. We denote this group by Γ_i and note that it acts simply transitively on the set $p^{-1}(C)$. Furthermore, for any chamber C' adjacent to C along a codimension 1 face, and for any chamber $D \in p^{-1}(C)$, there is exactly one chamber $D' \in p^{-1}(C')$ adjacent to C'. Moreover, the assignment $D \to D'$ establishes 1-1 correspondence between the sets of chambers $p^{-1}(C)$ and $p^{-1}(C')$. It follows that the group Γ_i acts simply transitively on the set $p^{-1}(C')$. Since X_i is gallery connected, the same argument gives the same conclusion for the set $p^{-1}(C'')$, for any chamber C'' of X_i. This implies that the map $P : \Gamma_i \backslash \widetilde{\mathscr{H}} \to X_i$ induced by p is an isomorphism, as required. It is also important to note that, since each X_i is compact, each of the groups Γ_i has finite index in $\mathrm{Sym}(\widetilde{\mathscr{H}})$.

We want now to find a compact development of $\mathscr{H}$ which will be k-large and which still be a branched covering of all X_i's. Since $\mathscr{H}$ is locally k-large and locally extra-tilable, by Proposition 18.7 there exists an extra-tilable morphism $\mu : \mathscr{H} \to E$ to a finite group E such that the development $D(\mathscr{H}, \mu)$ is k-large. Denote by K the kernel of the induced homomorphism $i_\mu : \hat{\mathscr{H}} \to E$. Take the intersection $K \cap \bigcap_i \Gamma_i$ and normalize it in $\mathrm{Sym}(\widetilde{\mathscr{H}})$ to get a normal subgroup N in $\hat{\mathscr{H}}$ for which the induced morphism $\mu : \mathscr{H} \to \hat{\mathscr{H}}/N$ is symmetric (due to normalization in $\mathrm{Sym}(\widetilde{\mathscr{H}})$), still extra-tilable, and whose development $D(\mathscr{H}, \mu)$ is still k-large (last two properties due to the inclusion $N < K$). Since, due to the inclusions $N < \Gamma_i$, $D(\mathscr{H}, \mu)$ is still a common branched covering of X_i's, the proposition follows.

Proof of Theorem 20.2. — Let Z be a compact simplicial manifold with boundary having the same homotopy type as the complex K. It can be obtained for example by embedding K in R^N, and taking its regular neighbourhood in a sufficiently fine triangulation. Denote by X the double of Z, i.e. the closed manifold obtained by gluing two copies of Z by the identity map of their boundaries. It follows from Proposition 20.3 that for any $k \geq 6$ there is a k-systolic pseudomanifold $\widetilde{Y}$ and a group Γ acting simplicially, properly discontinuously, and cocompactly on it, such that X is isomorphic to the quotient $\Gamma \backslash \widetilde{Y}$. By Theorem 14.1, taking k sufficiently large, we can arrange that $\widetilde{Y}$ is CAT(-1) with respect to some piecewise hyperbolic metric with regular simplices, and then Γ acts by isometries.

Denote by $i : X \to X$ the involution which exchanges the copies of Z, used in the construction X, fixing their common boundary. We claim that, if $\widetilde{Y}$ is taken to be the universal development $\widetilde{\mathscr{H}}$ as in the proof of Proposition 20.3, then i can be lifted to an isomorphism $\tilde{i}$ of $\widetilde{Y}$. To see that, fix a chamber C in X and consider lifts D and D' of C and $i(C)$ respectively, to $\widetilde{\mathscr{H}}$. Now, take as $\tilde{i}$ the isomorphism from the group $\mathrm{Sym}(\widetilde{\mathscr{H}})$ induced by the map $i_0 : D \to D'$ such that i_0 commutes with i through the covering $\widetilde{\mathscr{H}} \to X$. Due to rigidity implied by the fact that we deal with gallery-connected pseudomanifolds, $\tilde{i}$ is a lift of i as required. Using $\tilde{i}$ we get the extension G of Γ, of index 2, whose action on $\widetilde{\mathscr{H}}$ projects to the action of Z_2 generated by i on X. Consequently, the quotient $G \backslash \widetilde{\mathscr{H}}$ is isomorphic to $Z = Z_2 \backslash X$, and the theorem follows.

Theorem 20.2 has interesting corollaries. We refer to [LN] for the background on Corollary 20.4.

20.4. *Corollary. — Any finite complex* K *is homotopy equivalent to the classifying space for proper* G*-bundles of a* CAT(−1) *(hence Gromov hyperbolic) group* G.

20.5. *Corollary. — Any homotopy type of a finite complex occurs as the quotient* $G\backslash R_d(G)$ *of the Rips' complex* $R_d(G)$ *(with sufficiently large d) of some Gromov hyperbolic group* G.

Corollary 20.5 follows from Proposition 20.3 in view of the following observation. Given a CAT(−1) space X and a group G acting on X properly discontinuously cocompactly by isometries, for sufficiently large d, the action of G on the Rips' complex $R_d(G)$ is equivariantly homotopy equivalent to the action on X. This follows from the fact that if G is Gromov hyperbolic then, for sufficiently large d, the quotient of the Rips' complex $G\backslash R_d(G)$ is the classifying space for proper G-bundles, and the latter is uniquely determined up to homotopy equivalence (see [MS]).

Corollaries 20.4 and 20.5 give answer to questions of Ian Leary (see [QGGT, Question 1.24] and [L]).

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